10-1-1966

Adaptive Two-Step Detection of Radar Signals, UARI Research Report No. 38

R. J. Polge

UARI Research Report No. 38

ADAPTIVE TWO-STEP DETECTION OF RADAR SIGNALS

by

R. J. Polge

Final Technical Report

This research work was supported by
the U. S. Army Missile Command under
Grant No. DA-AMC-01-021-64-G1
"Task A"

University of Alabama Research Institute
Huntsville, Alabama

October, 1966

UARI Research Report No. 38

ADAPTIVE TWO-STEP DETECTION OF RADAR SIGNALS

by

R. J. Polge

Final Technical Report

This research work was supported by
the U. S. Army Missile Command under
Grant No. DA-AMC-01-021-64-G1
"Task A"

University of Alabama Research Institute
Huntsville, Alabama

October, 1966

ACKNOWLEDGEMENTS

This research work was supported by the U. S. Army, Grant Number DA-AMC-01-021-64-G1, Task A.

The author and Principal Investigator, Dr. R. J. Polge, would like to express his gratitude to Mrs. Rolf Duerr for the numerical computations and to Mrs. Sherridean Erbskorn for typing the report.

TABLE OF CONTENTS

ABSTRACT

This report studies the two-steps detection of search radar pulses in the presence of noise. The two informations, location and existence of a target, are obtained simultaneously in a conventional radar receiver, but successively in a two-steps radar receiver. The location of all the significant pulses, i.e. targets and large noise, is obtained first by a low level continuous threshold detection. The first threshold detector unlatches a second threshold detector (once for each significant pulse) which decides whether the significant pulse is a target pulse or a noise pulse.

Once the location of the pulses in known, the problem of radar becomes a problem of telemetry and the adaptive decision technique developed in the UARI Report No. 33 [1] for the detection of pulse code modulated signals is applicable. The noise just before the unknown signal, determined by sampling, and the noise during the interval of detection are more or less correlated; therefore, using correlation techniques, it is possible to predict the noise anywhere during the interval of detection. The probability of error in the detection is considerably reduced in an adaptive scheme where the noisy signal is corrected by substracting the predicted noise. The mathematics are more difficult for the adaptive two-threshold detection than for the adaptive PCM detection because of the complexity of the joint probability density after envelope detection.

The probability of false alarm for an adaptive two-steps detection, P_{fat}, is expressed by a double integral. P_{fat} is expanded as a power series in powers of ρ^2 where ρ is the autocorrelation coefficient between sampled and detected signal; the series is valid only for small values of ρ. Finally, P_{fat} is computed with a digital computer for typical value of ρ and D_C (level of the second threshold detector). It is shown that the probability of false alarm is smaller for an adaptive two-steps detection than for a conventional detection, especially for large values of ρ.

LIST OF SYMBOLS COMMONLY USED

D_c	Constant Threshold Level
${}_1F_1(a; b; z)$	Hypergeometric Function
$I_0(z)$	Modified Bessel Function of Order Zero
$I_1(z)$	Modified Bessel Function of Order One
$L_n^{\alpha}(z)$	Generalized Laguerre Polynomial
$n(t)$	Noise at the input of the Envelope Detector
$p_0(r)$	Probability density at the Output of the Envelope Detector for Noise Alone
$p_1(r)$	Probability Density at the Output of the Envelope Detector Signal and Noise
P_d	Probability of Detection
P_{fa}	Probability of False Alarm for Conventional Detection
P_{fat}	Probability of False Alarm for Adaptive Two-Steps Detection
$P_{fat}(r^*)$	Conditional Probability of False Alarm for Adaptive Two-Steps Detection knowing the Sampled Value r^*
P_m	Probability of Miss
$r(t)$	Output of the Envelope Detector to Normal Noise
$s(t)$	Noisy Signal at the Input of the Envelope Detector
ρ	Autocorrelation Coefficient Between Sampled Signal and Detected Signal
σ^2	Variance of the Noise Before Detection
$\wedge$	As a superscript means Hilbert Transform
$*$	As a superscript indicates the sampling time
—	Above a random variable means average
/	r/r^* means r knowing that r^* exists; $p(r/r^*)$ conditional probability of r knowing r^*

Chapter I

INTRODUCTION

The pulse radar emits a series of modulated narrow pulses at the repetition frequency and receives a mixture of delayed pulses and of noise. Every time a radar pulse hits a target, a reflected pulse is returned to the radar. The time interval between the transmitted and the received pulse is a measure of the range of the object. The reflected pulses are contaminated by random noise, this produces false detections and misses. The signal detection for a search radar consists primarily in determining the location and the existence of a target, the pulse shape being secondary.

The detection of pulse code modulated signals in the presence of noise or jamming signals is analyzed in the UARI Report No. 33 [1]. A decision circuit compares the noisy signal to a threshold to determine whether or not a pulse was sent. The average probability of error is reduced if an adaptive threshold is used instead of a constant threshold. The noisy signal at time t_1^* just before the interval of detection and the noisy signal during the interval of detection are more or less correlated; therefore, using correlation techniques, it is possible to predict the noise anywhere between t_1^* and t_1. The noisy signal can be corrected by substracting the predicted noise (and also the residual voltage due to the previous pulses). In the threshold detection, it is exactly equivalent to (1) compare the corrected signal to a constant threshold or (2) compare the signal to a corrected threshold; both techniques are used. Consider the threshold detection or rectangular pulses in presence of normal noise, the use of an adaptive threshold has the effect of an increase in the Signal-to-Noise ratio by $(\frac{1}{1-\rho^{*2}})$ where ρ^{*2} is the autocorrelation coefficient between the sampled signal and the detected signal. More generally, the largest is ρ^*, the more the probabilities of error are reduced by using an adaptive threshold. The average probability of error

is expressed in integral form and minimized directly; the maximum Signal-to-Noise ratio does not necessarily correspond to the minimum average probability of error.

In telemetry, the unknown signal is located in a known interval of time so that sampling can be used to determine the noise before detection and the threshold varied accordingly. This technique is not applicable in radar detection because the signal can appear everywhere and any sample would be a mixture of signal and noise. In order to transform the problem of radar detection into a problem of telemetry, a new two-step detection technique is proposed. The first step is a conventional radar detection but with a very low threshold level so that the probability of miss is negligible; of course the probability of false alarm is very high. The first detection eliminates the points where there is positively no echo, leaving a discrete number of unknown points which are either noise or echo. This is now a problem of telemetry because the location of the unknown signal is known and an adaptive technique is applicable. While the computation of the average probability of error after a two-step detection is complex, a considerable improvement can be expected.

In Chapter II, the analysis of a conventional threshold detection is reviewed. Chapter III explains the principle and shows a block diagram of a two-steps threshold detection. The probability of false alarm for a two-steps detection, P_{fat}, is obtained as a double integral in Chapter IV. Chapter V is the series expansion of P_{fat} in power of ρ^2. In Chapter VI, P_{fat}, is computed with a digital computer for typical values of ρ^2 and D_C. Chapter VII is the conclusion.

Chapter II

CONVENTIONAL SEARCH RADAR DETECTION

II-1 Block Diagram of a Conventional Pulse Radar.

The block diagram of a standard pulse radar is shown in Fig. II-1. The function of the emitter is to transmit short modulated pulses; the function of the receiver is to detect the pulses reflected on the targets; the time interval between a transmitted and a reflected pulse is a measure of the range.

The emitter consists of a timer, a modulator, a transmitter, an anti-transmit-receive switch (ATR). The radar receiver shown is of the superheterodyne type. It consists of a RF amplifier, a mixer, a local oscillator, a narrowband IF amplifier, an envelope detector, a threshold detector, and an indicator. The transmit-receive switch (TR) and the antenna are common to both emitter and receiver. The modulator, the transmitter, the ATR, the TR and the antenna are high power devices. The timer is a low power device with just enough power to control the modulator. The elements of the receiver are low power devices since the energy of the reflected pulses is several order of magnitude smaller than the energy of the transmitted pulses.

The timer, which is also called the trigger generator, or the synchronizer, generates a series of narrow pulses (trigger pulses) at the pulse repetition frequency. These timing pulses turn on the modulator which pulses the transmitter. The modulated RF pulse generated by the transmitter travels along the transmission line to the antenna, where it is radiated into space. The TR switch is a fast acting switch which disconnects the receiver during transmission; if the receiver were not disconnected, it might be damaged by the transmitter power. After transmission of the RF modulated pulse, the TR switch reconnects the receiver to the antenna. All the power reflected by the targets must enter the receiver; the ATR switch, which is closed during the transmission and opened during reception, avoids that a portion of the reflected power be wasted in the transmitter.

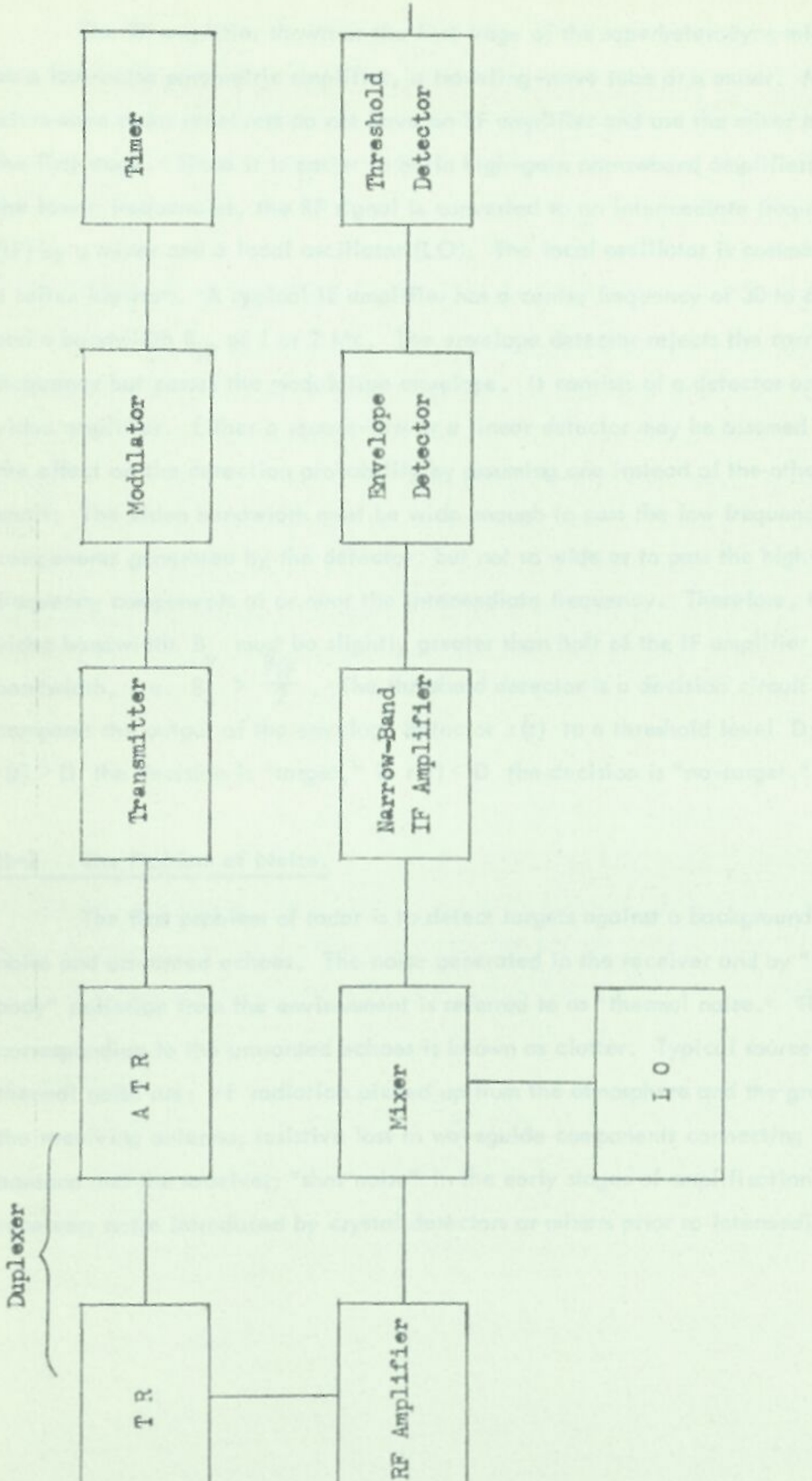

Fig. II-1 Block diagram of a conventional pulse radar

The RF amplifier shown at the first stage of the superheterodyne might be a low-noise parametric amplifier, a traveling-wave tube or a maser. Many microwave radar receivers do not have an RF amplifier and use the mixer as the first stage. Since it is easier to build high-gain narrowband amplifiers at the lower frequencies, the RF signal is converted to an intermediate frequency (IF) by a mixer and a local oscillator (LO). The local oscillator is commonly a reflex klystron. A typical IF amplifier has a center frequency of 30 to 60 Mc and a bandwidth B_{IF} of 1 or 2 Mc. The envelope detector rejects the carrier frequency but passes the modulation envelope. It consists of a detector and a video amplifier. Either a square-law or a linear detector may be assumed since the effect on the detection probability by assuming one instead of the other is small. The video bandwidth must be wide enough to pass the low frequency components generated by the detector, but not so wide as to pass the high-frequency components at or near the intermediate frequency. Therefore, the video bandwidth B_v must be slightly greater than half of the IF amplifier bandwidth, i.e. $B_v > \frac{B_{IF}}{2}$. The threshold detector is a decision circuit which compares the output of the envelope detector $r(t)$ to a threshold level D; if $r(t) > D$ the decision is "target," if $r(t) < D$ the decision is "no-target."

II-2 The Problem of Noise.

The first problem of radar is to detect targets against a background of noise and unwanted echoes. The noise generated in the receiver and by "black body" radiation from the environment is referred to as "thermal noise." The noise corresponding to the unwanted echoes is known as clutter. Typical sources of thermal noise are: r f radiation picked up from the atmosphere and the ground by the receiving antenna; resistive loss in waveguide components connecting the antenna and the receiver; "shot noise" in the early stages of amplification in the receiver; noise introduced by crystal detectors or mixers prior to intermediate

frequency amplification. In this report, all the various noises are assumed normal. The RF amplifier, the mixer, and the IF amplifier are linear components. Since a normal probability density remains normal under linear transformation or under linear combination, the probability density at the input of the envelope detector is normal.

II-3 The Envelope Detection.

The threshold detector makes a decision about whether or not a target is present. The probability of error in the decision is a function of the threshold level and of the probability density at the input of the threshold detector. Therefore, it is necessary to find the probability density at the output of the envelope detector, knowing that the probability density at the input is normal.

Following Dugundji [2], the envelope of a function $u(t)$ is the absolute value of its pre-envelope. The pre-envelope is a complex valued function $u(t) + i\,\hat{u}(t)$, where $\hat{u}(t)$ denotes the Hilbert transform of $u(t)$,

$$\hat{u}(t) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{u(x)}{t - x}\, dx \qquad \text{(II-1)}$$

The proof is based upon the properties of $\hat{u}(t)$: $\hat{u}(t)$ has the same power spectrum as $u(t)$ and is uncorrelated with u at the same time instant; the autocorrelation of the pre-envelope of $u(t)$ is twice the pre-envelope of the autocorrelation of $u(t)$. By using the pre-envelope technique, the envelope of the output of a linear filter is easily calculated; then, the first probabilities density for the envelope of the output of an arbitrary linear filter when the input is an arbitrary signal plus normal noise.

Assuming the noise negligible, the output of the IF amplifier would be zero in the absence of a target, and a modulated pulse in the presence of a target. The modulated pulse has a width equal to the radar pulse width and a carrier frequency equal to the If frequency. More generally, the output of the IF amplifier is a

normal noise if no target is present and the sum of a normal noise and a sine wave if a target is present.

The signal $s(t)$ at the input of the envelope detector is the sum of a message $m(t)$ and of a noise $n(t)$, i.e.

$$s(t) = m(t) + n(t) \tag{II-2}$$

where $m(t) = 0$ if no echo is received

$= A \sin(\omega t - \phi(t))$ if an echo is received and $n(t)$ is a normal noise of mean zero and of variance σ^2. Let $r(t)$ denote the output of the envelope detector. The probability density of r is [3].

$$p_0(r) = \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} \tag{II-3}$$

if no target is present, and

$$p_1(r) = \frac{r}{\sigma^2} e^{-\frac{r^2+A^2}{2\sigma^2}} I_0\left(\frac{rA}{\sigma^2}\right) \tag{II-4}$$

where I_0 is the Bessel function of zero order and purely imaginary agreement, if a target is present. Figure II-2 shows the graphs of $p_0(r)$ and $p_1(r)$ for typical values of σ and A.

Consider two envelope values r_1 and r_2 separated by a time interval τ, i.e. $r(t) = r_1$ and $r(t+\tau) = r_2$. The joint probability density of two successive envelope values of a normal process is [4],

$$p_0(r_1, r_2; \tau) = \frac{r_1 r_2}{\sigma^4(1-\rho^2)} I_0\left[\frac{\rho r_1 r_2}{\sigma^2(1-\rho^2)}\right] e^{-\left(\frac{r_1^2 + r_2^2}{2\sigma^2(1-\rho^2)}\right)} \tag{II-5}$$

where ρ is the autocorrelation coefficient of the normal noise before envelope detection for an interval τ between two samples.

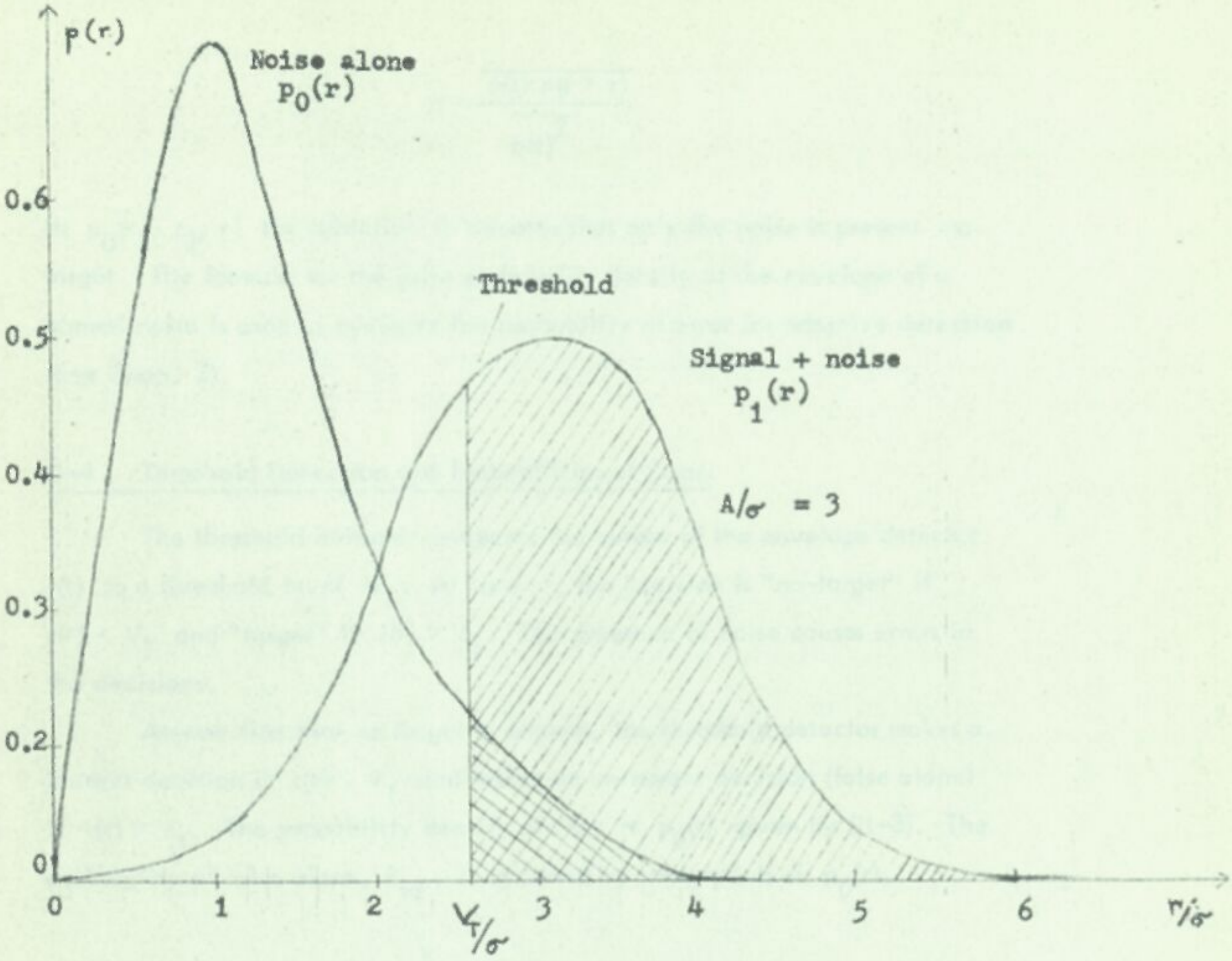

Fig. II-2 : Probability density after envelope detection
A signal amplitude - σ rms noise voltage - V_T threshold level

$$\rho = \frac{\overline{n(t)\; n(t+\tau)}}{\overline{n(t)^2}}$$

In $p_0(r_1, r_2; \tau)$ the subscript 0 denotes that only the noise is present no target. The formula for the joint probability density of the envelope of a normal noise is used to evaluate the probability of error for adaptive detection (See Chap. 3).

II-4 Threshold Detection and Probabilities of Error.

The threshold detector compares the output of the envelope detector $r(t)$ to a threshold level V_T. At time t, the decision is "no-target" if $r(t) < V_T$ and "target" if $r(t) > V_T$. The presence of noise causes errors in the decisions.

Assume first that no target is present, the threshold detector makes a correct decision if $r(t) < V_T$ and makes an uncorrect decision (false alarm) if $r(t) > V_T$. The probability density of $r(t)$ is $p_0(r)$ given by (II-3). The probability of false alarm, P_{fa}, is obtained by integration of $p_0(r)$,

$$P_{fa} = \int_{V_T}^{\infty} p_0(r)\, dr$$

$$= \int_{V_T}^{\infty} \frac{r}{\sigma^2}\, e^{-\frac{r^2}{2\sigma^2}}\, dr = e^{-\frac{V_T^2}{2\sigma^2}} \qquad \text{(II-6)}$$

Assume next that a target is present, the threshold detector makes a correct decision (detection) if $r(t) > V_T$ and makes an uncorrect decision (miss) if $r(t) < V_T$. The probability density of $r(t)$ is $p_1(r)$ given by (II-4). The probability of detection P_d is obtained by integration of $p_1(r)$,

$$P_d = \int_{V_T}^{\infty} p_1(r)\, dr$$

$$= \int_{V_T}^{\infty} \frac{r}{\sigma^2} e^{-\frac{r^2+A^2}{2\sigma^2}} I_0\left(\frac{rA}{\sigma^2}\right) dr \qquad \text{(II-7)}$$

The probability of miss, P_m, is the probability of losing a signal. P_m is the complement of P_d.

$$P_m = 1 - P_d = \int_0^{V_T} p_1(r)\, dr \qquad \text{(II-8)}$$

Equation (II-7) has been evaluated by W. R. Bennet and S. O. Rice and has been plotted by Rice [3]. Rice has also derived a series approximation for P_d valid when $RA/\sigma^2 \gg 1$, $A \gg |r-A|$ and terms in A^{-3} and beyond can be neglected

$$P_d = \frac{1}{2}\left(1 - \operatorname{erf} \frac{V_T - A}{\sqrt{2\sigma^2}}\right) + \frac{e^{-\frac{(V_T-A)^2}{2\sigma^2}}}{2\sqrt{2\pi}\left(\frac{A}{\sigma}\right)} \left[1 - \frac{V_T - A}{4A} + \frac{\frac{1+(V_T-A)^2}{\sigma^2}}{8\frac{A^2}{\sigma^2}} + \ldots\right] \qquad \text{(II-9)}$$

where the error function is defined as

$$\operatorname{erf} z = \frac{2}{\sqrt{\pi}} \int_0^z e^{-u^2}\, du$$

The conventional threshold detector can be represented by the block diagram of Fig. II-3 which shows the probability of detection and of false alarm.

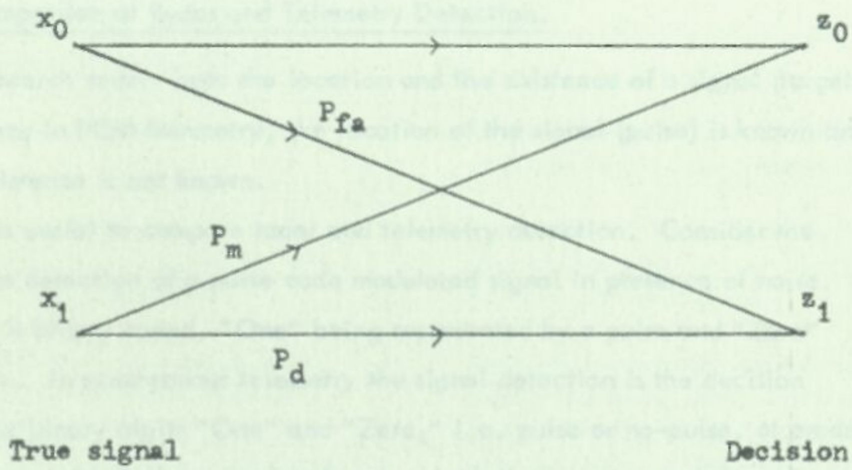

Fig. II-3 Statistical diagram of a conventional threshold detector

x_0 no target received, z_0 no target detected

x_1 target received, z_1 target detected

Chapter III

PRINCIPLE AND BLOCK DIAGRAM OF THE TWO-STEPS THRESHOLD DETECTION

III-1 Comparison of Radar and Telemetry Detection.

In search radar, both the location and the existence of a signal (target) are unknown; in PCM telemetry, the location of the signal (pulse) is known and only its existence is not known.

It is useful to compare radar and telemetry detection. Consider the synchronous detection of a pulse code modulated signal in presence of noise. The signal is binary coded, "One" being represented by a pulse and "Zero" by no-pulse. In synchronous telemetry the signal detection is the decision between the binary digits "One" and "Zero," i.e. pulse or no-pulse, at predetermined intervals of time. In search radar, both the location and the existence of a target, i.e. a pulse, must be determined. The differences between synchronous pcm detection and radar detection are tabulated in Fig. III-1.

III-2 Principle of Two-Step Detection.

The two informations, location and existence of a target, are obtained simultaneously in a conventional radar receiver, but successively in a two-steps radar receiver. The purpose of the two-steps detection is to transform the problem of radar into a problem of telemetry so that an adaptive detection technique can be used. The adaptive technique is based on correlation between the noise before the unknown signal and the noise during the unknown signal; it requires the knowledge of the location of the unknown signal.

In the two-steps detection, the first detection is a continuous low level threshold detection which determines the location of the targets. Since the threshold level of the first detection is very low, practically all the targets are

	Synchronous PCM Detection	Standard Search Radar Detection
(1)	A priori probability of a pulse is known; quite often $P(0) \approx P(1)$.	A priori probability of a target unknown.
(2)	All pulses have the same amplitude.	The amplitude of the reflected pulses depends upon the target (size, distance, etc.)
(3)	A digit is transmitted every T sec.; the interval between pulses is mT where m is an integer.	Nothing is known on the space interval between targets, i.e. on the time interval between reflected pulses.
(4)	All the information is contained in the small time intervals defined by $mT < t < mT + \Delta$ where $\Delta \ll T$. Since the threshold detector knows a priori where the digits are and needs only to find what was sent (pulse or no-pulse), it is latched except during the information intervals. One decision is made every T sec.	The distance between targets being completely unknown, the reflected pulses may be anywhere. Since the threshold detector must find both, where signal are received and what they are (target or noise), it is unlatched all the time. Decisions are made continuously; the number of decisions per sec. is $(1/\tau)$ where τ is the radar pulse width.
(5)	Two consecutive digits are independent (except when redundancy is desired.)	The antenna sends several pulses on the same target. Therefore, several echoes are received for one target and the probability of detection is increased by integration before or after envelope detection.

Fig. III-1 Table for Comparison of Telemetry and Radar.

detected but many pulses of noise are mistaken for a target; i.e. the probability of miss is negligible and the probability of false alarm is very high. The first detection reduces the ambiguity about the location of the targets from a non denumerable continuous set (any range is possible a priori) to a denumerable discrete set (a target can be only at one of the locations determined by the first detection.)

The first threshold detector transforms the problem of radar detection into a problem of telemetry detection. Denote as "unknown pulses" the pulses determined as possible targets by the first threshold detector. The second threshold detector makes the decision "no-target" or "target" among the unknown pulses. The second threshold detection is essentially the same as a telemetry detection except for two unimportant differences: (1) the unknown pulses are equally spaced in synchronous telemetry detection but not in the two-step radar detection and (2) the decision is "zero" or "one" in telemetry but "target" or "no-target" in the two-step radar detection. The technique of adaptive threshold detection is applicable. The second threshold detector is latched except for an instant during the interval of the unknown pulse; the noise just before the unknown pulse is determined by sampling and the noise inside the unknown interval can be predicted using correlation techniques. The predicted noise is the most probable value of the noise, if it is substracted from the unknown pulse the probability of error in the detection is reduced (See [1].)

In the synchronous detection of pulse code modulated signals with adaptive decision circuits [1], precise timing is provided by a pilot clock which divides the pseudo period T into tenths and serves as a time reference. In the two-steps threshold detection, the first detector serves as a clock for the sampler-holder and the second detector, as explained in Par. III-4.

III-3 Components of the Two-Steps Detector Block Diagram.

The block diagram of the two-steps detector is shown in Fig. III-2. It consists of two threshold detectors (TD1 and TD2), two delay-lines (D1 and D2),

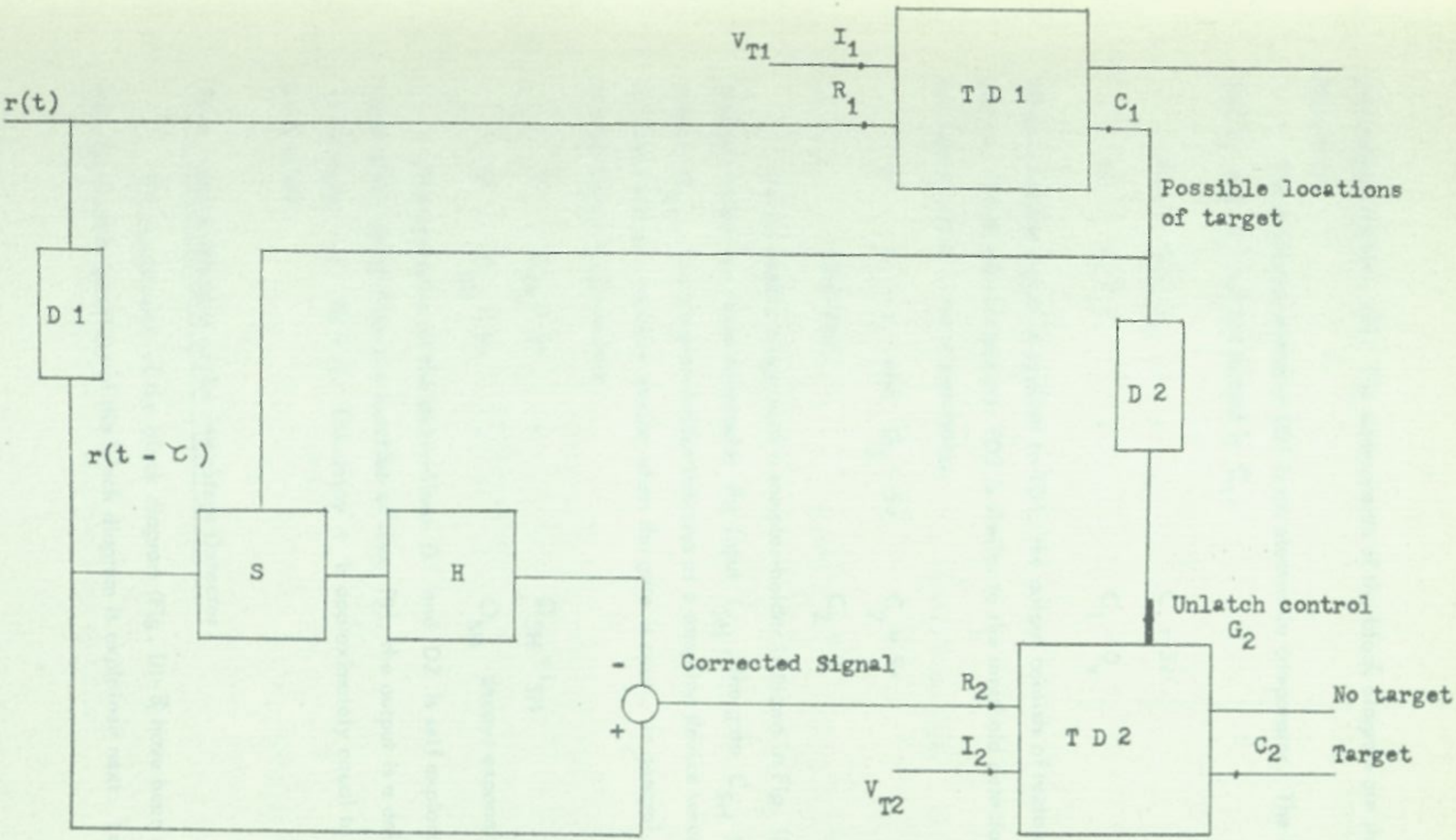

Fig. III-2 Block diagram of the two-step detector

D1, D2 delay lines ; S H sampler holder ; TD1 TD2 comparator

one sampler-holder, SH. The components of the block diagram are shown in Fig. III-3.

The threshold detector TD1 is an electronic comparator. The inputs are R_1 and I_1 and the output is C_1.

$$\begin{array}{llll} \text{If} & R_1 \geq I_1 & , & C_1 = 5v \\ \text{If} & R_1 < I_1 & , & C_1 = 0_v \end{array} \qquad \text{(III-1)}$$

When a random signal is applied to TD1, the output consists of rectangular random pulses. The threshold detector TD2 is similar to the threshold detector TD1 except for a gate control. More precisely,

$$\begin{array}{lll} \text{If} & R_1 \geq I_1 \text{ and } G_2 = 5v, & C_2 = 5v \\ & \text{otherwise,} & C_2 = 0 \end{array} \qquad \text{(III-2)}$$

The schematic diagram of a sampler-holder is shown in Fig. III-3. The sampler-holder has three terminals: the input I_{SH}, the gate C_{SH} and the output O_{SH}. The sampler-holder behaves as a sampling device when the gate is closed and as a memory device when the gate is open. In general, an exponential decay is convenient.

$$\begin{array}{llll} \text{If} & C_{SH} > 5v & , & O_{SH} = I_{SH} \\ \text{If} & C_{SH} < 5v & , & O_{SH} \text{ decays exponentially.} \end{array} \qquad \text{(III-3)}$$

The operation of the delay-lines D1 and D2 is self explanatory; if the input of the delay-line is a function of time f(t), the output is a delayed replica of the input, i.e. $f(t - \tau)$. The delay τ is approximately equal to the radar pulse width.

III-4 Block Diagram of the Two-Steps Detector.

The components of the block diagram (Fig. III-3) have been described in Par. III-3, the operation of the block diagram is explained next. Let r(t) denote

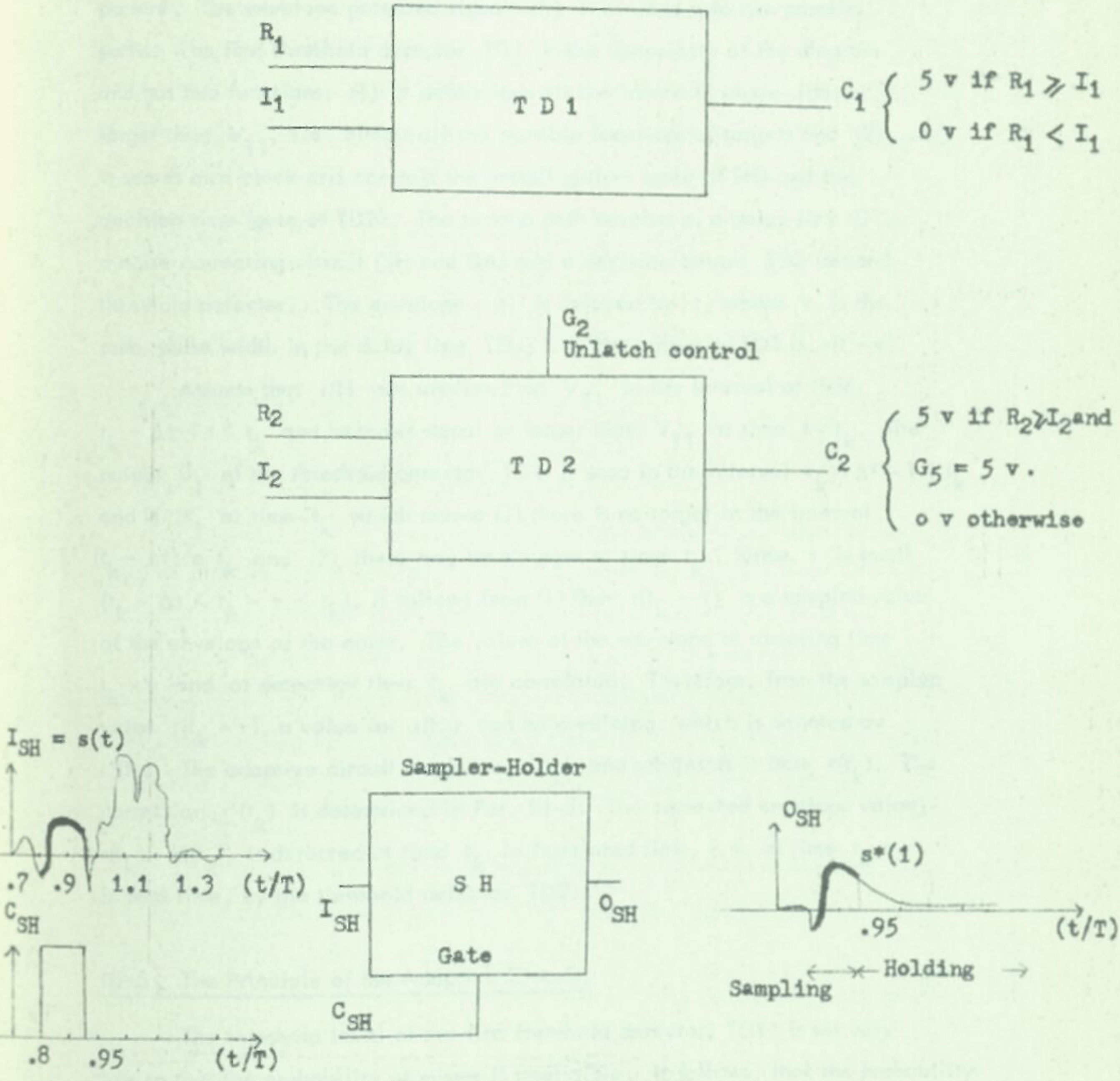

Fig. III-3 : Components of the two-step detector

the output of the envelope detector when a target may be present and let $r_n(t)$ denote the output of the envelope detector when only the noise is present. The envelope detected signal $r(t)$ is divided into two parallel paths. The first threshold detector TD1 is the upper path of the diagram and has two functions: (1) it determines all the intervals where $r(t)$ is larger than V_{T1}, i.e. almost all the possible locations of targets and (2) it serves as a clock and controls the sampling time (gate of SH) and the decision time (gate of TD2). The second path consists of a delay line D1, a noise correcting circuit (SH and DA) and a decision circuit TD2 (second threshold detector). The envelope $r(t)$ is delayed by τ, where τ is the radar pulse width in the delay line TD1; i.e. the output of TD1 is $r(t-\tau)$.

Assume that $r(t)$ was smaller than V_{T1} in the interval of time $t_k - \Delta t < t < t_k$ and becomes equal or larger than V_{T1} at time $t = t_k$. The output 0_1 of the threshold detector TD1 is zero in the interval $t_k - \Delta t < t < t_k$ and is V_1 at time t_k which means (1) there is no target in the interval $(t_k - \Delta t)$ to t_k and (2) there may be a target at time t_k. Since τ is small $(t_k - \Delta t < t_k - \tau < t_k)$, it follows from (1) that $r(t_k - \tau)$ is a sampled value of the envelope of the noise. The values of the envelope at sampling time $t_k - \tau$ and at detection time t_k are correlated. Therefore, from the sampled value $r(t_k - \tau)$, a value for $r(t_k)$ can be predicted, which is denoted by $r'(t_k)$. The adaptive circuit computes $r'(t_k)$ and substracts it from $r(t_k)$. The correction $r'(t_k)$ is determined in Par. III-5. The corrected envelope value, $r(t_k) - r'(t_k)$, is detected at time t_k in translated time, i.e. at time $t_k + \tau$ in real time, by the threshold detector TD2.

III-5 The Principle of the Adaptive Circuit.

The threshold level of the first threshold detector, TD1, is set very low so that the probability of misses is negligible. It follows, that the probability of false alarm is very high. The main purpose of the second threshold detector TD2 is to reduce this probability of false alarm.

A false alarm in the two-steps threshold detection is the joint event: false alarm for first threshold, false alarm for second threshold. However, since the first threshold level is very low, one can consider the joint false alarm as the false alarm of the second threshold detector.

The probability of false alarm is obtained by integration of the probability density of the envelope of the noise. Let $r(t_k - \tau) = r^*$ and $r(t_k) = r$ denote the values of the envelope at the time of sampling and detection, respectively. In the adaptive scheme r^* is known; therefore, the conditional probability density $p(r \mid r^*)$, must be used for r. The expected value of r knowing r^* is

$$\overline{r \mid r^*} = \int_0^\infty r\, p(r \mid r^*)\, dr$$

while the expected value of r not knowing r^* is

$$\overline{r} = \int_0^\infty r\, p(r)\, dr$$

The noise can be corrected by substracting its expected value. Two adaptive techniques are applicable: (1) "adaptive signal correction" where the signal is corrected by substracting the expected value of the noise and (2) "adaptive threshold detection" where the threshold is corrected by adding up the expected value of the noise. Since the threshold detector decision is based upon the comparison of signal and threshold, the two adaptive techniques are exactly equivalent. The study is made for "adaptive signal correction" following Fig. III-2.

The threshold level accounts implicitly for $\overline{r}$ in the conventional detection. Therefore, the correction for the "adaptive signal correction" is only $r' = (\overline{r \mid r^*} - \overline{r})$, where r' is $r'(t_k)$ of Par. III-4.

In conclusion, the signal is corrected before the second threshold detection, the corrected signal is:

$$r - r' = r + \overline{r} - \overline{r \mid r^*} \qquad \text{(III-6)}$$

and the second-threshold level is D_c. It is equivalent to correct the second threshold instead of the signal, i.e. the envelope r is detected with the adaptive threshold D_A:

$$D_A = D_C + \overline{r|r^*} - \overline{r} \tag{III-7}$$

where $\overline{r}$ and $\overline{r|r^*}$ are given by (4-9) and (4-20), respectively.

Chapter IV

PROBABILITY OF FALSE ALARM FOR A TWO-STEPS RADAR DETECTION

IV-1 Probability of False Alarm for Two-Steps Detection.

Consider the two-steps detection of the envelope of a radar signal at time t_k shown in Fig. III-2. Let, as before, $r(t_k) = r$ and $r(t_k - \tau) = r^*$. In order that $r(t_k) = r$ produces a false alarm in the two-steps detection, it must be detected by the two threshold detectors TD1 and TD2. Therefore, the conditional probability of false alarm for a given r^*, $P_{fat}(r^*)$, is the joint probability of the event $r > V_{T1}$ and $r - (\overline{r|r^*} - \bar{r}) > V_{T2}$, knowing r^*:

$$P_{fat}(r^*)^{\dagger} = P(r > V_{T1}, (r > V_{T2} + \overline{r|r^*} - \bar{r}) | r^*)$$

Since the first threshold level has been selected very low, with a good approximation

$$P_{fat}(r^*) \approx P((r > V_{T2} + \overline{r|r^*} - \bar{r}) | r^*) \tag{IV-1}$$

The conditional probability of false alarm $P_{fat}(r^*)$, is a function of r^*. The average probability of false alarm P_{fat} is obtained by averaging $P_{fat}(r^*)$ with respect to r^*.

$$P_{fat} = \int_0^\infty p(r^*)\, P_{fa}(r^*)\, dr^* \tag{IV-2}$$

Let $p_0(r|r^*)$ be the conditional probability density of the envelope of the noise at time t_k, knowing the sampled value at time $t_k - \tau$; the subscript 0 denotes the absence of a target. The probability of false alarm can be written in integral form,

$$P_{fat}(r^*) = \int_{V_{T2} + \overline{r|r^*} - \bar{r}}^{\infty} p_0(r|r^*)\, dr \tag{IV-3}$$

† P_{fat} is the probability of false alarm for adaptive two-steps detections.

$$P_{fat} = \int_0^{\infty} \int_{V_{T2} + \overline{r|r^*} - \overline{r}}^{\infty} p_0(r, r^*)\, dr\, dr^*$$

Substitution of r and r^* for r_1 and r_2 in (II-5) yields

$$p_0(r, r^*; \tau) = \frac{r\, r^*}{\sigma^4 (1 - \rho^2)}\, I_0 \left[\frac{\rho\, r\, r^*}{\sigma^2 (1 - \rho^2)}\right] e^{-\frac{r^2 + r^{*2}}{2\sigma^2 (1 - \rho^2)}} \qquad \text{(IV-4)}$$

From which the conditional probability $p_0(r\,|\,r^*)$ can be obtained,

$$p_0(r\,|\,r^*) = \frac{p_0(r, r^*; \tau)}{p_0(r^*)} \qquad \text{(IV-5)}$$

From (II-3)

$$p_0(r^*) = \frac{r^*}{\sigma^2}\, e^{-\frac{r^{*2}}{2\sigma^2}} \qquad \text{(IV-6)}$$

After combination of (IV-4), (IV-5) and (IV-6),

$$p_0(r\,|\,r^*) = \frac{r}{\sigma^2 (1 - \rho^2)}\, e^{-\frac{(r^2 - \rho^2 r^{*2})}{2\sigma^2 (1 - \rho^2)}}\, I_0 \left[\frac{\rho\, r\, r^*}{\sigma^2 (1 - \rho^2)}\right] \qquad \text{(IV-7)}$$

The expected value of r, $\overline{r}$, and the expected value of r knowing r^*, $\overline{r|r^*}$, are evaluated in the next paragraph.

IV-2 Expected Value of the Envelope Voltage.

The expected value of the envelope of the noise at time t_k when no previous knowledge of the noise is available is $\overline{r(t_k)} = \overline{r}$, and

$$\bar{r} = \int_0^\infty r\, p_0(r)\, dr \tag{IV-8}$$

After substitution of (IV-6) and integration,

$$\bar{r} = \sqrt{\frac{\pi}{2}}\ \sigma \tag{IV-9}$$

The expected value of the envelope of the noise at time t_k knowing the sampled value $r^* = r(t_k - \tau)$ of the envelope of the noise at time $t_k - \tau$ is

$$\overline{r|r^*} = \int_0^\infty r\, p_0(r|r^*)\, dr \tag{IV-10}$$

After substitution of (IV-7) in (IV-10)

$$\overline{r|r^*} = \int_0^\infty \frac{r^2}{\sigma^2(1-\rho^2)}\ e^{\frac{-r^2-\rho^2 r^{*2}}{2\sigma^2(1-\rho^2)}}\ I_0\left[\frac{\rho\, r\, r^*}{\sigma^2(1-\rho^2)}\right] \tag{IV-11}$$

Change the variables r and r^* of (IV-11) into y and y^*, where

$$y = \frac{r}{\sigma\sqrt{1-\rho^2}} \qquad y^* = \frac{r^*}{\sigma\sqrt{1-\rho^2}} \tag{IV-12}$$

it follows,

$$\overline{r|r^*} = \sigma\sqrt{1-\rho^2} \int_0^\infty y^2 e^{-\frac{y^2}{2}}\ e^{-\frac{\rho^2 y^{*2}}{2}}\ I_0(\rho y y^*)\, dy \tag{IV-13}$$

Since this integral cannot be evaluated in closed form, the Bessel's function is expanded as a power's series:

$$I_0(\rho\, y\, y^*) = \sum_{n=0}^{\infty} \frac{(\rho\, y\, y^*)^{2n}}{2^{2n}\,(n!)^2} \tag{IV-14}$$

(IV-13) becomes

$$\overline{r|r^*} = \sigma\sqrt{1-\rho^2}\; e^{-\frac{\rho^2 y_*^2}{2}} \sum_{n=0}^{\infty} \frac{(\rho y^*)^{2n}}{2^{2n}(n!)^2} \int_0^{\infty} e^{-\frac{y^2}{2}} y^{2(n+1)}\, dy \qquad \text{(IV-15)}$$

Note that

$$\int_0^{\infty} e^{-\frac{y^2}{2}} y^{2(n+1)}\, dy = 1.\ 3.\ 5\ \ldots\ (2n+1)\sqrt{\frac{\pi}{2}} \qquad \text{(IV-16)}$$

Substitution of (IV-16) in (IV-15) yields,

$$\overline{r|r^*} = \sigma\sqrt{1-\rho^2}\sqrt{\frac{\pi}{2}}\; e^{-\frac{\rho^2 y_*^2}{2}} \sum_{n=0}^{\infty} \frac{3.5\ \ldots (2n+1)}{2^{2n}} \frac{1}{(n!)^2} (\rho y^*)^{2n} \qquad \text{(IV-17)}$$

The summation can be rearranged in the form

$$\sum_{n=0}^{\infty} \frac{3.\ 5.\ \ldots (2n+1)}{2^n} \frac{1}{n!} \frac{\left(\frac{\rho^2 y_*^2}{2}\right)}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(\frac{3}{2})_n}{(1)n} \frac{(\frac{\rho^2 y_*^2}{2})^n}{n!} \qquad \text{(IV-18)}$$

where ( )n is the standard notation for the factorial function. The summation (IV-17) is easily recognized as an hypergeometric function [5]

$$\sum_{n=0}^{\infty} \frac{(\frac{3}{2})_n}{(1)n} \frac{(\frac{\rho^2 y_*^2}{2})^n}{n!} = {}_1F_1(\tfrac{3}{2}; 1; \tfrac{\rho^2 y_*^2}{2}) \qquad \text{(IV-19)}$$

Combination of (IV-17), (IV-18), (IV-19) and (IV-12) yields,

$$\overline{r|r^*} = \sqrt{\frac{\pi}{2}}\; \sigma\sqrt{1-\rho^2}\; e^{\frac{\rho^2 r_*^2}{2\sigma^2(1-\rho^2)}}\; {}_1F_1\left(\frac{3}{2}; 1; \frac{\rho^2 r_*^2}{2\sigma^2(1-\rho^2)}\right) \qquad \text{(IV-20)}$$

By [5]

$$e^{-z}\ {}_1F_1(a;\ c;\ z) = {}_1F_1(c-a;\ c;\ -z) \tag{IV-21}$$

Substitution of (IV-21) in (IV-20) yields,

$$\overline{r|r^*} = \sigma\sqrt{1-\rho^2}\sqrt{\frac{\pi}{2}}\ {}_1F_1\left(-\frac{1}{2};\ 1;\ \frac{-\rho^2\ r^{*2}}{2\sigma^2(1-\rho^2)}\right) \tag{IV-22}$$

The expected value of r knowing r^* is a function of r^*, but the average value of $\overline{r|r^*}$ for all possible r^* must clearly be equal to $\bar{r}$. This can be checked readily.

$$\overline{\overline{r|r^*}} = \int_0^\infty p(r^*)\ \overline{r|r^*}\ dr^* =$$

$$\sigma\sqrt{1-\rho^2}\ \sqrt{\frac{\pi}{2}}\int_0^\infty \frac{r^*}{\sigma^2}\ e^{-\frac{r^{*2}}{2\sigma^2}}\ {}_1F_1\left(-\frac{1}{2};\ 1;\ \frac{-\rho^2\ r^{*2}}{2\sigma^2(1-\rho^2)}\right) dr^* \tag{IV-23}$$

After the change of variable $t = \dfrac{r^{*2}}{2\sigma^2}$ and expansion of the hypergeometric function,

$$\overline{\overline{r|r^*}} = \sigma\sqrt{1-\rho^2}\ \sqrt{\frac{\pi}{2}}\int_0^\infty e^{-t}\sum_{K=0}^{\infty}\frac{-\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdots(K-\frac{3}{2})}{K!}\left(\frac{-\rho^2}{1-\rho^2}\right)^K\frac{t^K}{K!}\,dt \tag{IV-24}$$

Since $\int_0^\infty e^{-t}\,t^K\,dt = K!$, (IV-24) becomes

$$\overline{\overline{r|r^*}} = \sigma\sqrt{1-\rho^2}\ \sqrt{\frac{\pi}{2}}\sum_{K=0}^{\infty} -\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdots(K-\frac{3}{2})\ \frac{\left(\frac{-\rho^2}{1-\rho^2}\right)^K}{K!}$$

$$= \sigma \sqrt{1-\rho^2} \sqrt{\frac{\pi}{2}} \; {}_1F_1(-1/2; -; \frac{-\rho^2}{1-\rho^2})$$

$$= \sigma \sqrt{1-\rho^2} \sqrt{\frac{\pi}{2}} \; (1 + \frac{\rho^2}{1-\rho^2})^{1/2} \qquad \text{(IV-25)}$$

$$= \sigma \sqrt{\frac{\pi}{2}} = \bar{r}$$

IV-3 Probability of False Alarm for Constant or Adaptive Threshold Detection.

In the case of constant threshold detection, there is no need for sampling before detection. However, the average probability of false alarm is the same, with or without sampling, if the sample is not used to correct the threshold or the signal. It is, therefore, possible to generalize (IV-3) the conditional probability of false alarm of an envelope r, knowing the predetection sample r^*, with a threshold level V_T, constant or adaptive is,

$$P_{fat}(r^*) = \int_{V_T}^{\infty} p_o(r|r^*)\, dr \qquad \text{(IV-26)}$$

The average probability of false alarm for a threshold V_T, constant or adaptive, is as before

$$P_{fat} = \int_0^{\infty} p_o(r^*)\, P_{fa}(r^*)\, dr^*$$

$$= \int_0^{\infty} \int_{V_T}^{\infty} p_o(r, r^*)\, dr\, dr^* \qquad \text{(IV-27)}$$

For constant threshold detection V_T is a constant; $V_T = D_C$; for adaptive two-steps detection V_T is obtained by combination of (III-7), (IV-9), and (IV-20):

$$V_T = D_C + \sigma \sqrt{\frac{\pi}{2}} \left(\sqrt{1-\rho^2} \; {}_1F_1 (-1/2; 1; \frac{-\rho^2 r^2}{2\sigma^2(1-\rho^2)}) - 1\right) \qquad \text{(IV-28)}$$

If V_T is a constant threshold, whether the signal is sampled or not before detection is irrelevant. In other words, the probability of false alarm for a constant threshold level V_T is either given by (IV-27) as the average of the conditional probability of false alarm for all possible samples or by (II-6) where sampling is not performed. The identity of (IV-27) and (II-6) for constant threshold is proved next.

Start with (IV-27),

$$P_{fat} = \int_0^\infty \int_{V_T}^\infty \frac{r^* r}{\sigma^4(1-\rho^2)} e^{-\frac{(r^2 - r^{*2})}{2\sigma^2(1-\rho^2)}} I_0\left(\frac{\rho r r^*}{\sigma^2(1-\rho^2)}\right) dr\, dr^* \qquad \text{(IV-29)}$$

Make the change of variables,

$$u = \frac{r^2}{2(1-\rho^2)\sigma^2} \qquad \text{and} \qquad v = \frac{r^{*2}}{2(1-\rho^2)\sigma^2} \qquad \text{(IV-30)}$$

$$P_{fat} = (1-\rho^2) \int_0^\infty \int_B^\infty e^{-u} e^{-v} I_0(2\rho\sqrt{uv})\, du\, dv \qquad \text{(IV-31)}$$

where

$$B = \frac{V_T^2}{2\sigma^2(1-\rho^2)} \qquad \text{(IV-32)}$$

Expand the modified Bessel function in series and substitute in (IV-31),

$$I_0(2\rho\sqrt{uv}) = \sum_{K=0}^\infty \frac{2\rho\sqrt{uv}}{2^K(K!)^2} = \sum_{K=0}^\infty \rho^{2K} \frac{u^K v^K}{K!\,K!} \qquad \text{(IV-33)}$$

$$P_{fat} = (1-\rho^2) \int_0^\infty \int_B^\infty e^{-u} e^{-v} \sum_{K=0}^\infty \rho^{2K} \frac{u^K v^K}{K!\,K!}\, du\, dv \qquad \text{(IV-34)}$$

Change the order of integration in (IV-34) and use

$$\int_0^\infty e^{-u} v^K\, dv = K! \qquad \text{(IV-35)}$$

It follows

$$P_{fat} = (1-\rho^2) \int_B^\infty e^{-u} \sum_{K=0}^\infty \frac{\rho^{2K} u^K}{K!} du \qquad \text{(IV-36)}$$

The summation is the series development of an exponential,

$$\sum_{K=0}^\infty \frac{\rho^{2K} u^K}{K!} = e^{\rho^2 u} \qquad \text{(IV-37)}$$

Substitution of (IV-37) in (IV-36) yields,

$$P_{fat} = (1-\rho^2) \int_B^\infty e^{-(1-\rho^2)u} du \qquad \text{(IV-38)}$$

$$= e^{-(1-\rho^2)B} \qquad \text{(IV-39)}$$

Substitution of (IV-32) in (IV-39), yields

$$P_{fat} = e^{-\frac{V_T^2}{2\sigma^2}} \qquad \text{(IV-40)}$$

(IV-40) which is derived from (IV-27) is identical to (II-6), as anticipated.

In conclusion:

Prob. of fal. alarm for two-steps non-adapt. detection	=	Prob. of fal. alarm for conventional ct. threshold
Prob. of fal. alarm for two-steps adaptive detection	<	Prob. of fal. alarm for conventional ct. threshold

Chapter V

SERIES EXPANSION OF THE PROBABILITY OF FALSE ALARM FOR TWO-STEPS DETECTION

V-1 Technique of Series Expansion.

The probability of false alarm, P_{fat}, is expressed by (IV-3) where $\bar{r} = \sqrt{\frac{\pi}{2}}\,\sigma$ and

$$\overline{r|r^*} = \sqrt{\frac{\pi}{2}}\,\sigma\sqrt{1-\rho^2}\;{}_1F_1\left(-\frac{1}{2}; 1; \frac{-\rho^2 r^2}{2\sigma^2(1-\rho^2)}\right) .$$

Therefore,

$$P_{fat} = \int_{o}^{\infty}\int_{D\sigma}^{\infty} \frac{r\,r^*}{(1-\rho^2)\sigma^4}\, e^{-\frac{r^2 - r^{*2}}{2(1-\rho^2)\sigma^2}}\, I_0\left(\frac{\rho\, r\, r^*}{(1-\rho^2)\sigma^2}\right) dr\, dr^* \qquad \text{(V-1)}$$

where

$$D\sigma = V_{T2} + \overline{r|r^*} - \bar{r} . \qquad \text{(V-2)}$$

(V-1) cannot be evaluated in closed form. In this chapter, P_{fat} is evaluated as a power series of ρ which converges rapidly for $\rho \leq .75$.

Perform the change of variables,

$$x = \frac{r^2}{2\sigma^2} ; \qquad y = \frac{r^{*2}}{2\sigma^2} ; \qquad t = \rho^2 \qquad \text{(V-3)}$$

(V-1) becomes,

$$P_{fat} = \int_{0}^{\infty}\int_{D2/2}^{\infty} \frac{1}{1-t}\, e^{-\left(\frac{x+y}{1-t}\right)}\, I_0\left(\frac{2\sqrt{txy}}{1-t}\right) dx\, dy \qquad \text{(V-4)}$$

The integrant of the double integral (V-4) can be transformed into a product of Laguerre Polynomials [6],

$$w(t x y) = (1-t)^{-1} e^{-\frac{(x+y)t}{1-t}} I_0\left(\frac{2\sqrt{txy}}{1-t}\right)$$

$$= \sum_{n=0}^{\infty} L_n(x) L_n(y) t^n \ , \qquad \text{(V-5)}$$

where $L_n(u)$ † is the Laguerre polynomial of zero order. The inside integral of (V-4) can be integrated using two well known integrals [7]:

$$\int_{\frac{D^2}{2}}^{\infty} e^{-x} L_n(x)\, dx = e^{-\frac{D^2}{2}} \quad \text{for } n = 0$$

$$\text{(V-6)}$$

$$\int_{\frac{D^2}{2}}^{\infty} e^{-x} L_n(x)\, dx = e^{-\frac{D^2}{2}} \left[L_n\left(\frac{D^2}{2}\right) - L_{n-1}\left(\frac{D^2}{2}\right)\right]$$

$$\text{for } n = 1, 2, 3 \dots ,$$

Furthermore, by [8]

$$L_n\left(\frac{D^2}{2}\right) - L_{n-1}\left(\frac{D^2}{2}\right) = L_n^{-1}\left(\frac{D^2}{2}\right) \qquad \text{(V-7)}$$

let

$$e^{-\frac{D^2}{2}} L_n^{-1}\left(\frac{D^2}{2}\right) = Z_n^{-1}\left(\frac{D^2}{2}\right) \qquad \text{(V-8)}$$

and substitute (V-6) and (V-8) in (V-4); it follows

$$P_{fat} = \sum_{n=0}^{\infty} \int_{0}^{\infty} e^{-y} L_n(y) Z_n^{-1}\left(\frac{D^2}{2}\right) t^n\, dy \qquad \text{(V-9)}$$

† The generalized Laguerre polynomials are denoted by $L_n^{\alpha}(x)$, but the superscript may be omitted when $\alpha = 0$, i.e. $L_n^0(x) \equiv L_n(x)$.

(V-9) cannot be integrated because $Z_n^{-1}(\frac{D^2}{2})$ is a complex function of D which is itself a complex function of y and t. The technique used here is to develop $Z_n^{-1}(\frac{D^2}{2})$ in powers of t, arrange (V-9) in order of increasing powers of t and integrate with respect to y.

The function Z has been defined to make the derivations easy, note that

$$\frac{d}{du} Z_m^{\alpha}(u) = - Z_m^{\alpha+1}(u) \qquad \text{(V-10)}$$

$$\frac{d}{dt} Z_m^{\alpha}(\frac{D^2}{2}) = - Z_m^{\alpha+1}(\frac{D^2}{2}) \, D \, \frac{dD}{dt} \qquad \text{(V-11)}$$

let

$$C_n^K = [\frac{d^K}{dt^K} Z_n^{-1}(\frac{D^2}{2})] \, . \qquad \text{(V-12)}$$

Note that C_K^n is a function of y. Then by Taylor's expansion,

$$Z_n^{-1}(\frac{D^2}{2}) = C_n^0 + t\,C_n^1 + \frac{t^2}{2!} C_n^2 + \frac{t^3}{3!} C_n^3 \ldots. \qquad \text{(V-13)}$$

$$= \sum_{K=0}^{\infty} \frac{C_n^K \, t^K}{K!} \qquad \text{(V-14)}$$

Substitution of (V-14) in (V-9) yields,

$$P_{fat} = \sum_{n=0}^{\infty} \sum_{K=0}^{\infty} \frac{t^{K+n}}{K!} \int_0^{\infty} e^{-y} L_n(y) \, C_n^K(y) \, dy \qquad \text{(V-15)}$$

The polynomials $C_n^K(y)$ are expressed as a linear combination of Laguerre polynomials; then, the integration can be performed using the orthogonal properties of Laguerre polynomials.

V-2 Evaluation of the Integrals in the Series.

From (V-2) and (V-3),

$$D = \frac{V_{T2}}{\sigma} + \sqrt{\frac{\pi}{2}}\ (\sqrt{1-t}\ {}_1F_1(-\tfrac{1}{2};\ 1;\ \tfrac{-ty}{1-t}) - 1) \tag{V-16}$$

Using Equation (3), p. 202 of Rainville [5], (V-16) becomes,

$$\begin{aligned} D &= \frac{V_{T2}}{\sigma} + \sqrt{\frac{\pi}{2}}\ \left(\sum_{n=0}^{\infty} \frac{(-1/2)_n\ L_n(y)}{n!}\ t^n - 1\right) \\ &= \frac{V_{T2}}{\sigma} + \sqrt{\frac{\pi}{2}} \sum_{n=1}^{\infty} \frac{(-1/2)_n\ L_n(y)}{n!}\ t^n \end{aligned} \tag{V-17}$$

from which the values of D and of its derivatives, $\frac{d^K D}{dt^K}$ can be obtained.

For short notation, let d_K be the value of $\frac{d^K D}{dt^K}$ at $t = 0$, i.e.

$d_K = \left.\frac{d^K D}{dt^K}\right|_{t=0}$; then,

$$\begin{aligned} d_0 &= \frac{V_{T2}}{\sigma} \\ d_1 &= \sqrt{\frac{\pi}{2}}\ (-1/2)_1\ L_1(y) \\ d_2 &= \sqrt{\frac{\pi}{2}}\ (-1/2)_2\ L_2(y) \\ &\cdots\cdots\cdots\cdots \\ d_K &= \sqrt{\frac{\pi}{2}}\ (-1/2)_K\ L_K(y) \end{aligned} \tag{V-18}$$

The first coefficients C_K^n are now computed :

$$C_n^0 = Z_n^{-1}\left(\frac{D^2}{2}\right)\Bigg|_{t=0} \quad ; \qquad \text{(V-19)}$$

from (V-8), (V-12) and (V-18), it follows,

$$C_n^0 = e^{-\frac{d_0^2}{2}} L_n^{-1}\left(\frac{d_0^2}{2}\right) . \qquad \text{(V-20)}$$

Using (V-11) with $\alpha = -1$ and (V-18),

$$C_n^1 = -e^{-\frac{d_0^2}{2}} L_n^0\left(\frac{d_0^2}{2}\right) d_0 \ d_1 . \qquad \text{(V-21)}$$

By differentiation of (V-11),

$$\frac{d^2}{dt^2} Z_n^{-1}\left(\frac{D^2}{2}\right) = Z_n^1\left(\frac{D^2}{2}\right) D^2\left(\frac{dD}{dt}\right)^2$$

$$- Z_n^0\left(\frac{D^2}{2}\right)\left(D\frac{d^2D}{dt^2} + \left(\frac{dD}{dt}\right)^2\right) . \qquad \text{(V-22)}$$

Let $t=0$ in (V-22),

$$C_n^2 = e^{-\frac{d_0^2}{2}} \left[L_n^1\left(\frac{d_0^2}{2}\right) d_0^2 \ d_1^2 - L_n^0\left(\frac{d_0^2}{2}\right)(d_0 \, d_2 + d_1^2)\right] , \qquad \text{(V-23)}$$

d_1^2 can be expressed linearly in terms of Laguerre polynomials, using

$$(L_1(y))^2 = 2\,L_2(y) - 2\,L_1(y) + L_0(y). \tag{V-24}$$

From (V-18) and (V-24), it follows

$$d_1^2 = \frac{\pi}{8}\,(2\,L_2(y) - 2\,L_1(y) + L_0(y))\,. \tag{V-25}$$

Let

$$I_n^K = \int_0^\infty e^{-y}\,L_n(y)\;C_n^K(y)\;dy\;, \tag{V-26}$$

(V-15) becomes

$$P_{fa} = \sum_{n=0}^{\infty}\;\sum_{n=0}^{\infty}\;\frac{t^{K+n}}{K\,!}\;I_n^K\;. \tag{V-27}$$

Let $$e^{-\frac{d_0^2}{2}}\;L_n^{\alpha}\left(\frac{d_0^2}{2}\right) = z_n^{\alpha}\;; \tag{V-28}$$

the orthogonality of Laguerre polynomials can be used to compute the I_n^K:

$$I_n^0 = \int_0^\infty e^{-y}\;L_n(y)\;z_n^{-1}\;dy$$

$$I_0^0 = z_0^{-1} = e^{-\frac{d_0^2}{2}} \tag{V-29}$$

$$I_n^0 = 0 \quad \text{for} \quad n \neq 0$$

$$I_n^1 = \int_0^\infty e^{-y}\,L_n(y)\,(-z_n^0)\,d_0\,\sqrt{\frac{\pi}{2}}\,\left(-\,\frac{1}{2}\right)\,L_1(y)\,dy \tag{V-30}$$

$$I_1^1 = \frac{\sqrt{\frac{\pi}{2}}\, d_0}{2} \quad z_1^0$$

$$I_n^1 = 0 \qquad \text{for } n \neq 1 \,. \tag{V-31}$$

Using (V-18), (V-23) and (V-25)

$$C_n^2 = (z_n^1\ d_0^2 - z_n^0)\, d_1^2 - z_n^0\ d_0\ d_z$$

$$= (z_n^1\ d_0^2 - z_n^0)\, \tfrac{\pi}{8}\, (2\, L_2(y) - 2\, L_1(y) + L_0(y)\,)$$

$$+ z_n^0\ d_0 \sqrt{\tfrac{\pi}{2}}\ \tfrac{1}{4}\, L_2(y) \tag{V-32}$$

$$= A_n^2\ L_2(y) + A_n^1\ L_1(y) + A_n^0\, L_0(y)$$

where $A_n^2 = (z_n^1\ d_0^2 - z_n^0)\, \frac{\pi}{4} + z_n^0\ \frac{d_0}{4} \sqrt{\frac{\pi}{2}}$

$$A_n^1 = -(z_n^1\ d_0^2 - z_n^0)\, \frac{\pi}{4} \tag{V-33}$$

$$A_n^0 = (z_n^1\ d_0^2 - z_n^0)\, \frac{\pi}{8}$$

I_n^2 is obtained as before using orthogonality:

$$I_0^2 = A_0^0 = (z_0^1\ d_0^2 - z_0^0)\, \frac{\pi}{8} \tag{V-34}$$

$$I_1^2 = A_1^1 = -(z_1^1 d_0^2 - z_1^0) \frac{\pi}{4} \tag{V-35}$$

$$I_2^2 = A_2^2 = (z_2^1 d_0^2 - z_2^0) \frac{\pi}{4} + z_2^0 \frac{d_0}{4} \sqrt{\frac{\pi}{2}} \tag{V-36}$$

$$I_n^2 = 0 \qquad \text{for} \qquad n \neq 0 \text{ or } 1 \text{ or } 2.$$

More generally, $I_n^K = 0$ for $n > K$ (V-37)

V-3 Series Approximation for P_{fa}.

The probability of false alarm is expressed by the infinite series (V-27). If ρ is not too large, $t = \rho^2$ is even smaller and the series converges rapidly.

For example, if $\rho < .75$, the series (V-14) can be approximated by its first three terms:

$$Z_n^{-1}\left(\frac{D^2}{2}\right) \approx \sum_{K=0}^{3} \frac{C_n^K \, t^K}{K!} \tag{V-38}$$

Substitution of (V-38) and (V-37) in (V-27) yields an approximation for the probability of false alarm:

$$P_{fat} \approx \sum_{K=0}^{2} \sum_{n=0}^{2} \frac{t^{K+n}}{K!} I_n^K \tag{V-39}$$

Since $I_1^0 = I_z^0 = 0$ and $I_0^1 = I_2^1 = 0$, (V-39) reduces to

$$P_{fat} \approx I_0^0 + I_1^1 t^2 + I_0^2 t^2 + I_1^2 \frac{t^3}{1} + I_2^2 \frac{t^4}{2!}$$

$$= I_0^0 + (I_1^1 + I_0^2) t^2 + I_1^2 t^3 + \frac{I_2^2}{2!} t^4 \tag{V-40}$$

Let $a = \frac{V_{T2}}{\sigma}$ † and $b = \frac{a^2}{2}$; then (V-18), (V-28), (V-29), (V-31), (V-34), (V-35) and (V-36) take a simpler form:

$$
\begin{aligned}
d_0 &= a \\
z_n^{\alpha} &= e^{-b} L_n^{\alpha}(b) \\
I_0^0 &= e^{-b} \\
I_1^1 &= e^{-b} \sqrt{\frac{\pi}{8}}\; a\, L_1^0(b) \\
I_0^2 &= e^{-b} \frac{\pi}{8} \left[2b\, L_0^1(b) - L_0^0(b) \right] \\
I_1^2 &= e^{-b} \frac{\pi}{4} \left[L_1^0(b) - 2b\, L_1^1(b) \right] \\
I_2^2 &= e^{-b} \left\{ \frac{\pi}{4} (2b\, L_2^1(b) - L_2^0(b)) + \sqrt{\frac{\pi}{2}}\, L_2^0(b) \frac{a}{4} \right\}
\end{aligned}
\qquad \text{(V-41)}
$$

The formulas for the Laguerre polynomials of (V-41) are:

$$
\begin{aligned}
L_0^K(x) &= 1 \\
L_1^K(x) &= 1 + K - x \\
L_2^K(x) &= \tfrac{1}{2}(1+K)(2+K) - (2+K)x + \tfrac{1}{2}x^2
\end{aligned}
\qquad \text{(V-42)}
$$

† a is the value of the second threshold level expressed in σ units.

It follows,

$$\begin{aligned} L_0^1(b) &= L_0^0(b) = 1 \\ L_1^0(b) &= 1 - b \\ L_1^1(b) &= 2 - b \\ L_2^1(b) &= 3 - 3b + \frac{b^2}{2} \\ L_2^0(b) &= 1 - 2b + \frac{b^2}{2} \end{aligned} \tag{V-43}$$

Substitution of (V-43) in (V-41) yields:

$$\begin{aligned} I_0^0 &= e^{-b} \\ I_1^1 &= e^{-b} \sqrt{\frac{\pi}{8}}\; a(1 - b) \\ I_0^2 &= e^{-b} \frac{\pi}{8}\; (2\,b - 1) \\ I_1^2 &= e^{-b} \frac{\pi}{4} (1 - 5b + 2\,b^2) \\ I_2^2 &= e^{-b} \{ \frac{\pi}{4} (-1 + 8b - 6b^2 + b^3 \\ &\quad + \sqrt{\frac{\pi}{2}}\; \frac{a}{4} (1 - 2b + \frac{b^2}{2})) \} \end{aligned} \tag{V-44}$$

After substitution of (V-44) in (V-40) and factorization of e^{-b}, it follows

$$P_{fat} = e^{-b} \{ 1 - (\frac{\pi}{8} - \sqrt{\frac{\pi}{8}}\; a + \sqrt{\frac{\pi}{8}}\; ab - \frac{\pi}{4} b)\; t^2$$

$$+ \frac{\pi}{4}(1 - 5b + 2b^2)\,t^3$$

$$+ [\frac{\pi}{4}(-1 + 8b - 6b^2 + b^3) + \sqrt{\frac{\pi}{2}}\ \frac{a}{4}(1 - 2b + \frac{b^2}{2})]\,t^4\} \qquad (V-45)$$

Note that the error for a constant threshold V_{T2} would be e^{-b}. Since the polynomial in t is less than 1, (V-45) shows that the probability of false alarm is reduced by an adaptive threshold.

If a is large, more terms are required in the approximation. The same technique can be applied using the identities

$$L_1(y)\,L_2(y) = 3\,L_3(y) - 4\,L_2(y) + 2\,L_1(y)$$

$$(L_1(y))^3 = 6\,L_3(y) - 12\,L_2(y) + 9\,L_1(y) - 2\,L_0(y)$$

. . . .

More work is required to determine the convergence of the series.

Chapter VI

DIGITAL COMPUTER EVALUATION OF THE PROBABILITY OF FALSE ALARM FOR A TWO-STEPS DETECTION.

VI-1 The Technique of Computation.

The probability of false alarm for a two-steps threshold detection cannot be evaluated in closed form. In Chapter V, a series expansion of P_{fat} valid when ρ and V_{T2} are not too large was obtained. Since the series expansion for P_{fat} does not converge very rapidly, it is useful to compute P_{fat} for typical value of ρ and V_{T2} using a digital computer.

The average probability of false alarm P_{fat} of a two-steps detection is expressed by (V-1); P_{fat} is a function of ρ (the autocorrelation coefficient of the noise before envelope detection for a time translation equal to the radar pulse width) and of V_{T2} (the second threshold level). After the change of variables (V-3), P_{fat} is given by (V-4). Formulas (V-3) and (V-4) are repeated for easy reference:

$$x = \frac{r^2}{2\sigma^2} \quad ; \qquad y = \frac{r^{*2}}{2\sigma^2} \quad ; \qquad t = \rho^2 \tag{VI-1}$$

$$P_{fat} = \int_0^\infty \int_{\frac{D^2}{2}}^\infty \frac{1}{1-t}\, e^{-\left(\frac{x+y}{1-t}\right)} I_0\left(\frac{2\sqrt{txy}}{1-t}\right) dx\, dy \tag{VI-2}$$

where

$$D = \frac{1}{\sigma}\left(V_{T2} + \overline{r|r^*} - \bar{r}\right) \tag{VI-3}$$

The purpose of the computer program is to evaluate (VI-2) for typical values of ρ and $\frac{V_{T2}}{\sigma}$.

(VI-2) must be transformed into an expression suitable for digital computation. The upper limits of the double integral can be replaced by a

finite value because the integrand decreases very rapidly for x and y. Therefore, it is a very good approximation to replace the upper limits by eight in (VI-2):

$$P_{fat} \approx \int_0^8 \int_{\frac{D^2}{2}}^8 \frac{1}{1-t}\, e^{-(\frac{x+y}{1-t})}\, I_0\left(\frac{2\sqrt{txy}}{1-t}\right) dx\, dy \qquad \text{(VI-4)}$$

The double integral (VI-4) can be transformed into a double sum:

$$P_{fat} \approx \Delta x\, \Delta y \sum_{j=0}^{j=8/\Delta y} \sum_{i=\frac{D^2}{z\Delta x}}^{i=8/\Delta x} \frac{1}{1-t}\, e^{-(\frac{i\Delta x + j\Delta y}{1-t})}\, I_0\left(\frac{2\sqrt{tij\Delta x\Delta y}}{1-t}\right) \qquad \text{(VI-5)}$$

The choice of the increments Δx and Δy is a compromise between accuracy and computer time; Δx and Δy are selected equal to 0.02. (VI-5) becomes

$$P_{fat} = 4\ 10^{-4} \sum_{j=0}^{400} \sum_{i=i_{min}}^{400} \frac{1}{1-t}\, e^{-\frac{0.02(i+j)}{1-t}}\, I_0\left(\frac{0.04\sqrt{tij}}{1-t}\right) \qquad \text{(VI-6)}$$

where $i_{min} = \text{Trunc.}\left[\frac{D^2}{2(0.02)}\right]$

$$= \text{Trunc.}\,[25\, D^2]^{\dagger} \qquad \text{(VI-7)}$$

After combination of (VI-1), (IV-22) and (VI-3):

$$D = \frac{V_{T2}}{\sigma} + \sqrt{\frac{\pi}{2}}\, [\sqrt{1-t}\ {}_1F_1\, (-\tfrac{1}{2};\, 1;\, \tfrac{-\,ty}{(1-t)}) - 1] \qquad \text{(VI-8)}$$

where $y = j\ 0.02$

† Trunc. means truncation; Trunc. $[25\, D^2]$ is the integer obtained by rounding up $25\, D^2$.

The hypergeometric function is computed in terms of the modified Bessel's functions of order zero and one, using [9]

$${}_1F_1(-\frac{1}{2}; 1; z) = e^{-\frac{z}{2}} [(1+z)\, I_0(\frac{z}{2}) + z\; I_1(\frac{z}{2})] \qquad \text{(VI-9)}$$

For z less than ten; the modified Bessel functions $I_0(z)$ and $I_1(z)$ are computed as power series; less than 16 terms are necessary for a very good approximation:

$$I_0(z) = \sum_{n=0}^{n<16} \frac{z^{2n}}{2^{2n}\,(n\,!)^2} \qquad \text{(VI-10)}$$

$$I_1(z) = \sum_{n=0}^{n<16} \frac{z^{2n+1}}{2^{2n+1}\,(n\,!)^2\,(n+1)} \qquad \text{(VI-11)}$$

For z larger than ten, the modified Bessel functions $I_0(z)$ and $I_1(z)$ are computed by using assymptotic approximations [10].

$$I_0(z) = .3989\ \frac{e^z}{\sqrt{z}} [1 + \frac{1}{8\,z} + \frac{9}{128z^2} + \frac{75}{1024z^3}] \qquad \text{(VI-12)}$$

$$I_1(z) = .3989\ \frac{e^z}{\sqrt{z}} [1 - \frac{3}{8\,z} - \frac{15}{128z^2} - \frac{105}{1024z^3}] \qquad \text{(VI-13)}$$

VI-2 The Computer Program.

The computer program consists of two parts: (1) tabulation of the modified Bessel functions $I_0(z)$ and $I_1(z)$ and of the Hypergeometric function ${}_1F_1(-\frac{1}{2}, 1, -z)$ for discrete values of z and (2) computation of the double summation (VI-6) for typical values of $t = \rho^2$ and D_C. The computation of the integrand and of the lower limit of the inner summation necessitates the

evaluation of $I_0(\frac{.04\sqrt{tij}}{1-t})$ and of ${}_1F_1(-\frac{1}{2}; 1; -\frac{jt\,0.02}{1-t})$, respectively; this is done by interpolation between the nearest values of $I_0(z)$ and ${}_1F_1(-\frac{1}{2}; 1; -z)$ computed in Par. VI-1.

VI-2 a. Tabulation of $I_0(z)$ and ${}_1F_1(-\frac{1}{2}; 1; -z)$.

The modified Bessel functions $I_0(z)$ and $I_1(z)$ are computed for discrete values of z. For $0 \leq z \leq 10$, $I_0(z)$ and $I_1(z)$ are computed with an increment equal to 0.005(i.e. $z = k_1\, 0.005$ where $k_1 = 0, 1 \ldots 2000$), using the series expansion (VI-10) and (VI-11). For $0 \leq z \leq 20$, $I_0(z)$ and $I_1(z)$ are computed with an increment equal to 0.005 (i.e. $z = k_2\, 0.005$ where $k_2 = 2001, 2002 \ldots 4001$) using the assymptotic expansions (VI-12) and (VI-13). For $20 \leq z \leq 80$, $I_0(z)$ is computed with an increment equal to 0.01 (i.e. $z = k_3\, 0.01$ where $k_3 = 2001, 2002 \ldots 8000$), using the assymptotic expansion (VI-12) and (VI-13). For $0 \leq z \leq 20$, the hypergeometric function ${}_1F_1(-\frac{1}{2}; 1; -z)$ is computed with an increment equal to 0.01, using (VI-9) and the tables for $I_0(\frac{z}{2})$ and $I_1(\frac{z}{2})$.

VI-2 b The Computation of the Double Sum.

The probability of false alarm for a two-steps detection, P_{fat}, is computed for typical values of $t = \rho^2$ and D_C:

$t = 0.8,\quad 0.6,\quad 0.4,\quad 0.2,\quad 0.01$

$D_C = 2,\quad 3,\quad 4,\quad 5$

Given t, D_C and a value of j, i_{min} is evaluated by truncation of $25\,D^2$. Keeping j fixed the first summation is performed by varying i from i_{min} to 400, for each j one value for the first summation, S_j, is obtained. The proba-

bility of false alarm for two-steps detection P_{fat} is then obtained by summation of all the S_i when i varies from 0 to 400. For each value of t, four values of P_{fat} are obtained, one for each value of D_C. The increment used in the tabulation of $I_0(z)$ is 0.005 for $0 \leq z \leq 20$ and 0.01 for $20 \leq z \leq 80$; therefore, there are two formulas of interpolations one valid for $0 \leq z \leq 20$ and one valid for $20 \leq z \leq 80$. For example, if $z = 7.813$, the interpolation for $I_0(z)$ is between $I_0(7.810)$ and $I_0(7.815)$ which are tabulated at the locations $\frac{7.810}{1005} + 1 = 1463$ and 1464, respectively. Let

$$I_0(7.810) = B(1463)$$

$$I_0(7.815) = B(1464)$$

then

$$I_0(7.810) = B(1463) + \frac{7.813 - 7.810}{.005}(B(1464) - B(1463)) \qquad \text{(VI-14)}$$

VI-3 Comparison Between Two-Steps Detection and Conventional Threshold Detection.

The probability of false alarm, P_{fa}, in the constant threshold detection of level D_C of the envelope of a normal noise of variance σ^2 is

$$P_{fa} = e^{-\frac{D_C^2}{2\sigma^2}}$$

as obtained in (II-6).

The probability of false alarm, P_{fat}, in the adaptive two-steps detection of the envelope of a normal noise of variance σ^2 is given by the computer program of Par. VI-2 for typical value of D_C and of the autocorrelation coefficient ρ.

Since P_{fat} and P_{fa} are very small, it is practical to compare $-\log P_{fat}$ to $-\log P_{fa}$, rather than P_{fat} to P_{fa}.

Figure VI-1 shows $-\log P_{fat}$ versus $\frac{D_C^2}{2\sigma^2}$ for various values of ρ^2, and $-\log P_{fa}$ versus $\frac{D_C^2}{2\sigma^2}$.

For a given value of ρ^2, $-\log P_{fat}$ increases when $\frac{D_C^2}{2\sigma^2}$ increases. For a given value of $\frac{D_C}{\sigma}$, $-\log P_{fat}$ decreases when ρ decreases and tends to $-\log P_{fa}$ when ρ tends to zero.

This means that the probability of false alarm for the adaptive two-steps detection is smaller than for the conventional threshold detection. The advantage is greater when ρ^2 is larger, i.e. when the autocorrelation coefficient between the sample and the detected signal is large. When ρ tends to zero, the noise at the time of detection cannot be predicted and the adaptive two-steps detection is not useful.

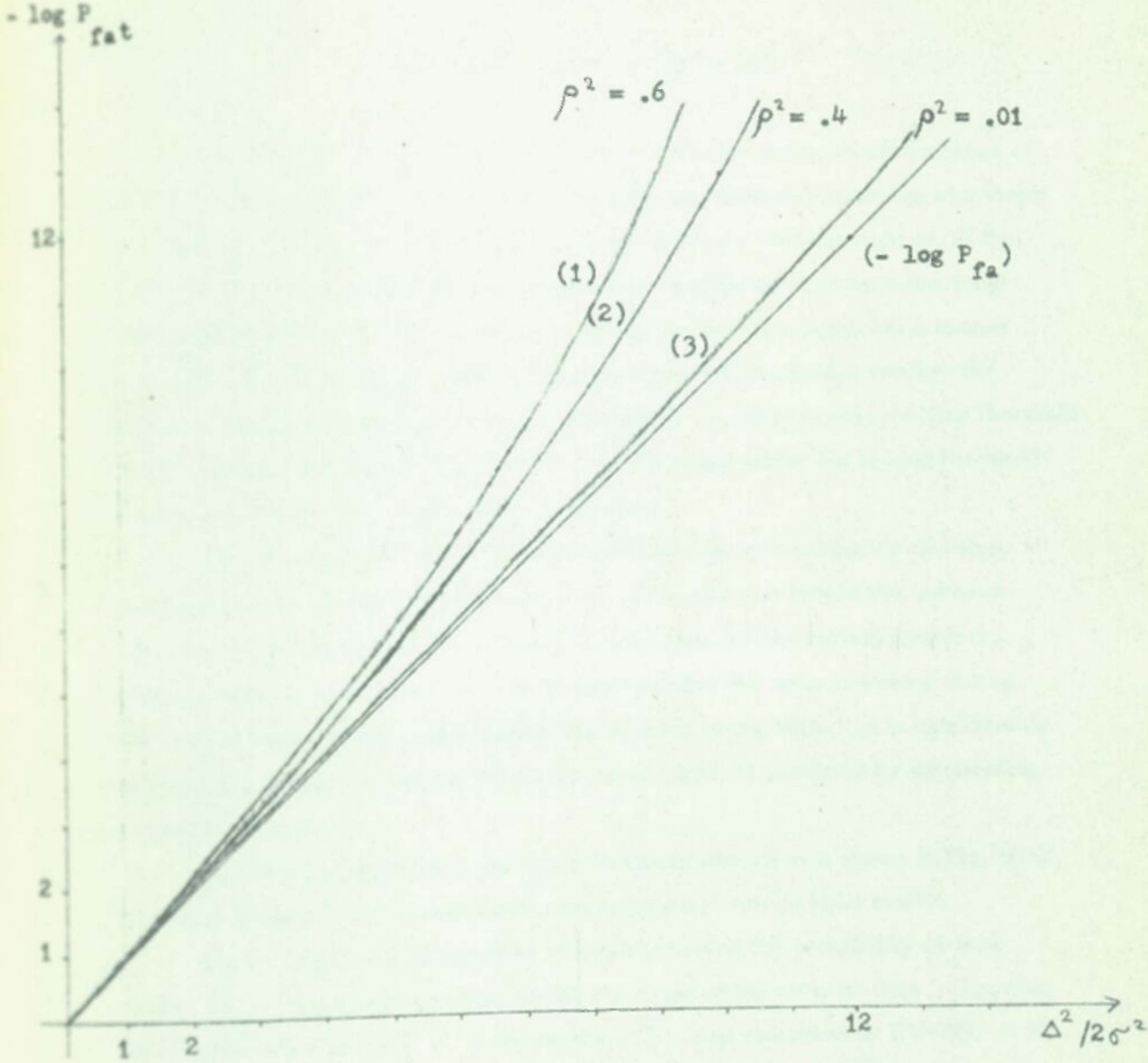

Fig. VI-1 Comparison between probability of false alarm for conventional and two-step detection

(1) P_{fat} at $\rho^2 = .6$, (2) P_{fat} at $\rho^2 = .4$, (3) P_{fat} at $\rho^2 = .01$

Chapter VII

CONCLUSION AND FUTURE WORK

The adaptive two-steps detection of search radar pulses in the presence of noise has been analyzed. The two informations, location and existence of a target are obtained successively in a two-steps radar receiver. The location of all the significant pulses, i.e. targets and large noise, is obtained first by a low level continuous threshold detection. The first threshold detector unlatches a second threshold detector (once for each significant pulse) which decides whether the significant pulse is a target pulse or a noise pulse. In other words, the first threshold detector determines the possible locations of the target while the second threshold detector determines the existence of the targets.

The two-steps detection makes possible the use of an adaptive decision technique similar to that used in telemetry. The noise just before the unknown signal and the noise during the interval of detection are correlated; therefore, using correlation techniques, it is possible to predict the noise anywhere during the interval of detection. The probability of error in the detection is considerably reduced in an adaptive scheme where the noisy signal is corrected by substracting the predicted noise.

The block diagram of a two-steps threshold detection is shown in Fig. III-2. The block diagram uses conventional components and can be built easily.

Most of the analysis concerns the evaluation of the probability of false alarm, P_{fat}. The expected value of the envelope of the noise at time t knowing the sampled value at time t^* is denoted by $\overline{r|r^*}$ and expressed by (IV-22). It is equivalent to compare the corrected envelope $r - r' = r + \overline{r} - \overline{r|r^*}$ to a constant threshold D_C or to compare the envelope r to an adaptive threshold $D_A = D_C + \overline{r|r^*} - \overline{r}$.

The probability of false alarm for an adaptive two-steps detection P_{fat} is given by the double integral (V-3) which cannot be evaluated in closed form. P_{fat} is expressed by a power series of ρ^2 valid for small value of ρ^2 in (V-45); P_{fat} is also computed with a digital computer for typical values of ρ^2 and D_C. It is shown that the probability of false alarm is smaller for a two-steps detection than for a conventional detection, especially when ρ is not too small.

Further work would be necessary to complete this study: (1) the assumption of P_{fat} being due to the second threshold alone is only an approximation, (2) the probability of detection of a two-steps detection has not been determined, and (3) a set up corresponding to the block diagram should be built and tested.