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by

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MASTER

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SUMMARY

The study of electromagnetic wave scattering and propagation characteristics in inhomogeneous media has recently attracted increased attention in basic and applied science, particularly where applied to plasma sheaths.

In this paper a solution to several classes of plasma problems has been found by applying a technique similar to that used in nonrelativistic quantum mechanics when studying particle scattering by partial waves. In both types of problems it is desired to find out how a varying index of refraction affects the propagation of incident waves. The solutions are represented as sums or integrals of Fourier components which represent the partial waves. When the inhomogeneous medium is removed, the partial waves are known for many cases of interest; when the spatially-varying medium is reintroduced, each of these partial waves will change. It is convenient to define the partial wave phase shifts as the natural logarithm of the ratio of the new to the old partial waves. These phase shifts will in general be complex numbers.

The general mathematics of this approach is similar for both the quantum and electromagnetic problems. A scalar equation is under consideration in both approaches since the electromagnetic vector wave equation has been reduced to a scalar equation by restricting the form of the conducting surface and the kind of antenna allowed. The principal differences between the two types of problems are contained in the boundary conditions imposed in each case. In addition to these differences, it must be remembered that in the quantum problem the incident wave is generated outside the varying region. In the electromagnetic problem, one considers restriction from an antenna on a missile surface, the scattering properties of the sheath, and the effect of the sheath on reception from an external source.

By using the approach described above, an analytic solution to several problems of interest is obtained in the form of an infinite series where successive terms are defined by an integral recursion relation. It should be mentioned that no restrictions are necessary with regard to near- and far-fields and with regard to the thickness of the medium layer.

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CHAPTER I

INTRODUCTION

The solution of the wave equation with a continually-varying wave number has until recently been of primary interest in the field of quantum mechanics, as applied to the Schroedinger equation with a varying potential. Current interest in the wave equation has arisen also in the propagation of electromagnetic waves in plasma media on conducting surfaces where the plasma permittivity can be treated as a continuous (complex) function. A common approach to this problem has been to approximate the continuous variation by a constant average of ϵ, and to obtain a solution by applying the usual electromagnetic boundary conditions between regions with different average values. However, this approximation may cause the solution to differ markedly from the physically correct solution.[1]

A more recent approach has been to formulate a power series solution[2] or a WKB-like solution for the fields.[3] The power series approach requires the use of large computers, and the value of an analytic solution is lost. The WKB solution is not valid for a plasma thickness of the same order of magnitude

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[1] The analysis with homogeneous sheaths in References 10, 12, 14-16, in the Annotated Bibliography may be compared with the analysis using inhomogeneous sheaths in References 1, 2, 5, 11, 20, 22, 23. The difference is also pointed out explicitly in Chapter VI of this paper.

[2] The power series technique has been applied in References 2, 11, 22, and 29 in the Annotated Bibliography.

[3] The WKB approach is utilized in References 11 and 23 in the Annotated Bibliography.

as the wavelength of the source, and this is the situation that prevails in many reentry problems.

In this paper, a different technique for handling the problem has been derived for several coordinate symmetries. In the radiation problems considered a constant-phase strip antenna is present on the conducting surface, while in the scattering and transmission problems a polarized wave is incident from infinity. By appropriate utilization of these restrictions, the vector wave equation can be reduced to a linear, second order partial differential equation that is separable. The boundary conditions on the electromagnetic fields at the conducting surface are then applied to this equation in such a way that one integration may be performed. A linear, first order differential equation is obtained and this can be integrated directly without further restriction.

The technique applied here is similar to that used in nonrelativistic quantum mechanics when studying particle scattering by partial waves;[4] in both problems we want to find out how a varying index of refraction affects the propagation of an incident wave. The solutions to both problems can be represented as a sum or integral of Fourier components, and to these components the name partial waves may be given. When the inhomogeneous region is removed the partial waves are completely known; when the varying region is replaced it is convenient to express the new set of Fourier components in terms of the old, and this is done by defining the partial wave phase shifts in terms of the ratio of the new to the old components. The general treatment of this approach is similar for both the quantum and electromagnetic problems.

However, the problems are quite different when viewed from other aspects. Perfectly conducting boundaries will always exist in the electromagnetic problem, so that both the regular and irregular solutions must be

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[4] The quantum mechanical partial wave technique is discussed thoroughly in References 4 and 8 in the Annotated Bibliography.

retained. In the quantum problem, this would correspond to allowing an infinite potential to exist. Since the method of solution in this paper depends on the presence of the boundary, the mathematical techniques employed are quite different from those of the quantum case. There is also a difference in the basic wave equations to be solved; the quantum problem involves a scalar wave equation in three dimensions while in the electromagnetic case a vector wave equation must be solved and both two- and three-dimensional models are considered. In addition to these differences, it must be remembered that in the quantum problem the incident wave is generated outside the varying region. In the first case, radiation from an antenna on the missile surface is considered, while in the second case we are concerned either with the scattering properties of the sheath or with the effect of the sheath on reception from an external source.

The solution is formulated in three steps. First, we are given a particular problem we wish to solve, described by a differential equation and certain boundary conditions. This problem will in general be very difficult to solve, so we must approach it indirectly.

In the second step, we generate an entirely new, workable problem designed to represent a first-order approximation to the actual problem. In order to generate this approximate problem, we first define an average value of the permittivity such that

$$\mathrm{Re}\ [\text{average value of } \epsilon] = \frac{1}{2}\left[\begin{matrix}\text{maximum value of} \\ \text{the Re } \epsilon \text{ across} \\ \text{the sheath}\end{matrix} + \begin{matrix}\text{minimum value of} \\ \text{Re } \epsilon \text{ across the} \\ \text{sheath}\end{matrix}\right]$$

$$\mathrm{Im}\ [\text{average value of } \epsilon] = \frac{1}{2}\left[\begin{matrix}\text{maximum value of} \\ \text{the Im } \epsilon \text{ across} \\ \text{the sheath}\end{matrix} + \begin{matrix}\text{minimum value of} \\ \text{Im } \epsilon \text{ across the} \\ \text{sheath}\end{matrix}\right]$$

This definition is chosen to minimize the maximum difference between the actual value of ϵ and the approximate value across the sheath. The new workable problem is generated by replacing the region of varying ϵ in the actual problem by a region with a constant, average value of ϵ. The approximate fields can then be found by applying the known electromagnetic boundary conditions to the known solutions.

In the third part of the solution, the fields for the actual problem are described in terms of those for the approximate, solvable problem. The difference between these solutions is expressed as an (infinite) set of differences between the Fourier components of the fields themselves. To these Fourier components the name partial waves has been designated. The differences between the actual and approximate partial waves is then expressed by the (infinite) set of partial wave phase shifts. This nomenclature is in anology with that used in quantum-mechanical scattering problems.

Using the mathematical approach discussed above, an analytic solution is obtained in the form of an infinite series where successive terms are defined by an integral recursion relation. For many cases of interest this series will converge rapidly.

Three two-dimensional models have been analyzed. The first two are a conducting cylinder or wedge clad in an inhomogeneous medium either in the presence of an electric or magnetic strip source or an incident polarized wave from infinity. The third is a conducting plane clad in a linearly-inhomogeneous medium with periodic electric or magnetic strip sources or an appropriate incident wave. The two three-dimensional models considered are a conducting sphere or cone in a varying medium in the presence of a circumferential strip antenna or an incident polarized wave from infinity. Elliptical and parabolic coordinate systems cannot be utilized since the metrical coefficients[5] in these cases are functions of more than one coordinate variable and the equations are not separable.

The iterative method of solution is derived in general form in Chapter II, and the mathematical similarities and differences between the

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[5] J. A. Stratton, Electromagnetic Theory, McGraw-Hill Book Company, Inc., 1941, Chapter I.

quantum and electromagnetic problems are explicitly demonstrated. Once the general method of solution has been developed, it is necessary to show that the results are applicable to the plasma problem.

The problems with planar, cylindrical, and spherical conducting surfaces are considered in Chapters III and IV. In the former the vector wave equations are reduced to linear second order differential equations for each case, and in the latter it is shown that the general method can be applied to obtain a solution. The wedge and cone problems have been placed separately in Chapter V since they are found to be only partially solvable.

Once the applicability of the method has been established, it is desirable to work out an example to illustrate the usefulness and accuracy of the results. This is done for a simple problem in Chapter VI, where the limitations on the WKB and step function solutions are presented.

The appendices are intended to provide the mathematical and physical references that are necessary in order to utilize the results of this work.

CHAPTER II

METHOD OF SOLUTION

A. General Development

In order to study electromagnetic wave propagation in inhomogeneous media, one must obtain solutions differential equations of the form

$$\frac{d^2 F_n^I}{d\xi^2} + X_1[n, \xi, \epsilon_r(\xi)]\frac{dF_n^I}{d\xi} + X_2[n, \xi, \epsilon_r(\xi)]F_n^I = 0 \qquad (2\text{-}1)$$

where F_n^I is the n^{th} Fourier component[6] of an electric field E or magnetic field H, ξ is one of the coordinate variables in the problem of interest, and $\epsilon_r(\xi)$ is the relative permittivity in the region over which equation (2-1) must hold:

$$\xi_i \le \xi \le \xi_e \quad [\text{Region I}]. \qquad (2\text{-}2)$$

When $\epsilon_r(\xi)$ is set equal to a constant value ϵ_a, the medium is homogeneous, and in many cases of interest the solutions $\overline{F}_n^I$ are known:

$$\frac{d^2 \overline{F}_n^I}{d\xi^2} + X_1[n, \xi, \epsilon_a]\frac{d\overline{F}_n^I}{d\xi} + X_2[n, \xi, \epsilon_a]\overline{F}_n^I = 0. \qquad (2\text{-}3)$$

If ϵ_a is chosen to be average value of $\epsilon_r(\xi)$ over the range $\xi_i \le \xi \le \xi_e$,

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[6] It will be assumed for simplicity that n takes on only discrete values. If a summation is involved, the radiation far fields may be found by simply letting $\rho \to \infty$. If an integral is involved, however, the integration must first be performed before letting $\rho \to \infty$ since $\lim_{\rho\to\infty} \int F(\rho)d\rho \neq \int [\lim_{\rho\to\infty} F(\rho)]d\rho$.

then $\overline{F}^{I}_{n}$ is an approximation to the field F^{I}_{n}. Although many important features of the actual problem may not appear in the approximate one, it represents a useful starting point. The following analysis utilizes the approximate field $\overline{F}^{I}_{n}$ to obtain a solution for the actual field F^{I}_{n}.

In the scattering problems, a perfectly-conducting surface is assumed to exist at the inner boundary $\xi = \xi_i$, and in the radiation problems, a specified strip antenna on an otherwise perfectly-conducting surface is assumed to exist at this boundary. We will assume that the same boundary conditions apply to both the actual and approximate solutions[7], and one of the following cases will always hold:

$$\text{Case I} \qquad F_n^{I}(\xi_i) = \overline{F}_n^{I}(\xi_i) = \begin{cases} 0 & \text{(scattering problem)} \\ \text{known function} & \text{(radiation problem)} \end{cases} \tag{2-4}$$

$$\text{Case II} \qquad \left.\frac{dF_n^{I}}{d\xi}\right|_{\xi_i} = K_i \left.\frac{d\overline{F}_n^{I}}{d\xi}\right|_{\xi_i} = \begin{cases} 0 & \text{(scattering problem)} \\ \text{known function} & \text{(radiation problem)} \end{cases} \tag{2-5}$$

In all the radiation problems to be treated, free space exists everywhere outside Region I:

$$\epsilon_r(\xi) = 1 \quad \text{for all} \quad \xi \geq \xi_e \quad [\text{Region II}]. \tag{2-6}$$

Only outgoing waves will exist in Region II, so that the n^{th} Fourier component of the exact field will be equal to a complex number multiplied times the n^{th} component of the approximate field. This number will be written:

$$e^{i\delta_n} \qquad (\delta_n \text{ complex}). \tag{2-7}$$

---

7 This approximation has also been noted in Reference 29 in the Annotated Bibliography.

The Fourier components of the field will be called the partial waves, and δ_n the n^{th} partial wave phase shift.

In scattering problems, one or more sources will exist outside Region I and these sources will be assumed distant enought from ξ_e so that reflections off of them may be neglected.

$$\epsilon_r(\xi) = 1 \qquad \xi \geq \xi_e \qquad \text{(except for sources)} \qquad \text{[Region II]} \tag{2-8}$$

Both incoming and outgoing waves will now exist in Region II. The components of the incoming waves will be the same in both the exact and approximate problems, while the outgoing wave components will differ by a multiplied complex constant, as before. This constant will be of the form (2-7), where δ_n is again the n^{th} partial wave phase shift.

The boundary conditions joining Regions I and II will depend on the continuity of the tangential E and H fields:

$$\bar{F}_n^{I}(\xi_e) = \bar{F}_n^{II}(\xi_e) \qquad \frac{d\bar{F}_n^{I}}{d\xi}(\xi_e) = \frac{d\bar{F}_n^{II}}{d\xi}(\xi_e)\ \bar{K}_e$$

$$F_n^{I}(\xi_e) = F_n^{II}(\xi_e) \qquad \frac{dF_n^{I}}{d\xi}(\xi_e) = \frac{dF_n^{II}}{d\xi}(\xi_e)\ K_e\ . \tag{2-9}$$

For future reference, we note that for the radiation problems, if $k_e = \bar{k}_e$,

$$\left(F_n^{I} \frac{d\bar{F}_n^{I}}{d\xi} - \bar{F}_n^{I} \frac{dF_n^{I}}{d\xi} \right)_{\xi_e} = 0. \tag{2-10}$$

Also for future reference, we note that in the scattering problems, if $k_e = \bar{k}_e$,

$$\left[\frac{F_n^{I} \frac{d\bar{F}_n^{I}}{d\xi} - \bar{F}_n^{I} \frac{dF_n^{I}}{d\xi}}{F_n^{I} - \bar{F}_n^{I}} \right]_{\xi_e} = K_e \left[\frac{d\bar{I}_n}{d\xi} - \frac{\bar{I}_n}{\bar{O}_n} \frac{d\bar{O}_n}{d\xi} \right]_{\xi_e} \tag{2-11}$$

where

$$\bar{F}_n^{II} = \bar{I}_n + \bar{O}_n \qquad I_n = \text{incoming wave}$$

$$F_n^{II} = \bar{I}_n + \bar{O}_n e^{i\delta_n} \qquad O_n = \text{outgoing wave} \quad .$$

The first step toward obtaining a solution is to put equations (2-1) and (2-3) into the standard forms

$$\frac{d^2 y_n}{d\xi^2} + I[n, \xi, \epsilon_r(\xi)]\, y_n = 0 \tag{2-13}$$

$$\frac{d^2 \bar{y}_n}{d\xi^2} + I[n, \xi, \epsilon_a]\, \bar{y}_n = 0 \quad . \tag{2-14}$$

This can be accomplished by choosing[8]

$$y_n = F_n^{I} e^{+\frac{1}{2}\int_a^{\xi} X_1[n, t, \epsilon_r(t)]\,dt} \tag{2-15}$$

$$I_n[n, \xi, \epsilon_r(\xi)] = X_2[n, \xi, \epsilon_r(\xi)] - \frac{1}{4} X_1^2[n, \xi, \epsilon_r(\xi)] - \frac{1}{2}\frac{d}{d\xi} X_1[n, \xi, \epsilon_r(\xi)] \tag{2-16}$$

$$\bar{y}_n = \bar{F}_n^{I} e^{+\frac{1}{2}\int_a^{\xi} X_1[n, t, \epsilon_a]\,dt} \tag{2-17}$$

$$I_n[n, \xi, \epsilon_a] = X_2[n, \xi, \epsilon_a] - \frac{1}{4} X_1^2[n, \xi, \epsilon_a] - \frac{1}{2}\frac{d}{d\xi} X_1[n, \xi, \epsilon_a] \tag{2-18}$$

where a is an arbitrary constant that must be the same in both cases so that $y_n = \bar{y}_n$ when $F_n = \bar{F}_n$ and $\epsilon_r(\xi) = \epsilon_a$. For simplicity, define

$$I[n, \xi, \epsilon_r(\xi)] = \phi \tag{2-19}$$

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8 See Annotated Bibliography, Reference 27.

$$I[n, \xi, \epsilon_a] = \phi_0 \tag{2-20}$$

and the equations to be solved become

$$\frac{d^2 y_n}{d\xi^2} = -\phi y_n \tag{2-21}$$

$$\frac{d^2 \overline{y}_n}{d\xi^2} = -\phi_0 \overline{y}_n \,. \tag{2-22}$$

Now multiply (2-21) by $\overline{y}_n$ from the left, (2-22) by y_n from the left and substract the resulting equations:

$$\begin{aligned} \overline{y}_n \frac{d^2 y_n}{d\xi^2} - y_n \frac{d^2 \overline{y}_n}{d\xi^2} &= (\phi_0 - \phi)\, y_n \overline{y}_n \\ &= \frac{d}{d\xi}\left(\overline{y}_n \frac{dy_n}{d\xi} - y_n \frac{d\overline{y}_n}{d\xi} \right). \end{aligned} \tag{2-23}$$

Integrating between arbitrary limits A and B,

$$\left(\overline{y}_n \frac{dy_n}{d\xi} - y_n \frac{d\overline{y}_n}{d\xi} \right)_A^B = \int_A^B (\phi_0 - \phi)\, y_n \overline{y}_n \, d\xi \,. \tag{2-24}$$

Substituting from (2-15) and (2-17)

$$\begin{aligned} &\left\{ G(\xi) \left[\overline{F}_n^{\,I} \frac{dF_n^{\,I}}{d\xi} - F_n^{\,I} \frac{d\overline{F}_n^{\,I}}{d\xi} - \tfrac{1}{2} F_n^{\,I} \overline{F}_n^{\,I} \left(X_1[n, \xi, \epsilon_a] - X_1[n, \xi, \epsilon_r(\xi)] \right) \right] \right\}_A^B \\ &= \int_A^B (\phi_0 - \phi)\, F_n^{\,I} \overline{F}_n^{\,I}\, G(\xi)\, d\xi \end{aligned} \tag{2-25}$$

where

$$G(\xi) = e^{+\frac{1}{2}\int_\alpha^\xi \left(X_1[n, t, \epsilon_a] + X_1[n, t, \epsilon_r(t)] \right) dt} \,.$$

B. Derivation of the Recursion Integral

B.1 Scattering

For the scattering case, choose

$$
\begin{aligned}
&A = \xi_i \\
&B = \xi' \text{ (arbitrary)} \\
&X_1[n, \xi_i, \epsilon_a] = X_1[n, \xi_i, \epsilon_r(\xi_i)] \text{ for case II, equation (2-5) only} \\
&X_1[n, \xi_e, \epsilon_a] = X_1[n, \xi_e, \epsilon_r(\xi_e)] \text{ for both cases}
\end{aligned}
\tag{2-26}
$$

and from (2-4) and (2-5) the left-hand side of (2-25) will always vanish at ξ_i:

$$
\left[\bar{F}_n^{I} \frac{dF_n^{I}}{d\xi} - F_n^{I} \frac{d\bar{F}_n^{I}}{d\xi} - \frac{1}{2} F_n^{I} \bar{F}_n^{I} \left(X_1[n, \xi, \epsilon_a] - X_1[n, \xi, \epsilon_r(\xi)]\right)\right]_{\xi'}
= \frac{1}{G(\xi')} \int_{\xi_i}^{\xi'} (\phi_0 - \phi) F_n^{I} \bar{F}_n^{I} G(\xi)\, d\xi .
\tag{2-27}
$$

Since only F_n^{I} and $dF_n^{I}/d\xi$ are unknown functions, this is a linear first order differential equation in F_n^{I}.

$$
\frac{dF_n^{I}}{d\xi} - \left\{\frac{1}{2}\left(X_1[n, \xi, \epsilon_a] - X_1[n, \xi, \epsilon_r(\xi)]\right) - \frac{1}{\bar{F}_n^{I}} \frac{d\bar{F}_n^{I}}{d\xi}\right\} F_n^{I}
- \frac{1}{\bar{F}_n^{I} G(\xi)} \int_{\xi_i}^{\xi} (\phi_0 - \phi) F_n^{I} \bar{F}_n^{I} G(\xi')\, d\xi' = 0 .
\tag{2-28}
$$

(for $\xi \neq \xi_i$ when $\bar{F}_n^{I}$ represents an E field)[9]

---

[9] $\bar{F}_n^{I}(\xi)$ can only be zero at ξ_i. If it were zero at any other value $\xi = \xi_m$, a perfectly-conducting surface could be placed at ξ_m without altering the fields. In this case, no waves would propagate into or out of the region $\xi_i \leq \xi \leq \xi_m$, and this possibly can be avoided by requiring $\epsilon_r(\xi)$ to be finite everywhere.

The solution to any equation in the form

$$\frac{dy}{dx} + \alpha(x)\,y + \beta(x) = 0 \tag{2-29}$$

is

$$y = -e^{-\int \alpha(x)dx}\left[\int \beta(x)\, e^{\int \alpha(x)\,dx}\, dx - C\right] \tag{2-30}$$

so that (2-24) may be directly integrated:

$$F_n^{I} = \bar{\bar{F}}_n^{I} P(\xi)\left[\int^{\xi} \frac{d\xi'}{(\bar{\bar{F}}_n^{I})^2 P(\xi')\, G(\xi')} \int_{\xi_i}^{\xi'} (\phi_0 - \phi)\, F_n^{I} \bar{\bar{F}}_n^{I}\, G(\xi'')\,d\xi'' + C\right] \tag{2-31}$$

where

$$P(\xi) = e^{+1/2 \int_{\alpha}^{\xi} \left(X_1[n, t, \epsilon_a] - X_1[n, t, \epsilon_I(t)]\right)dt}$$

$$e^{\int \frac{1}{\bar{\bar{F}}_n^{I}} \frac{d\bar{\bar{F}}_n^{I}}{d\xi} d\xi} = \bar{\bar{F}}_n^{I}$$

$$e^{-\int \frac{1}{\bar{\bar{F}}_n^{I}} \frac{d\bar{\bar{F}}_n^{I}}{d\xi} d\xi} = (\bar{\bar{F}}_n^{I})^{-1}. \tag{2-32}$$

The boundary condition at ξ_i has already been applied, so now we must use the known boundary conditions at ξ_e. To evaluate C, we first find from (2-27) that

$$\left[\frac{\bar{\bar{F}}_n^{I}\frac{dF_n^{I}}{d\xi} - F_n^{I}\frac{d\bar{\bar{F}}_n^{I}}{d\xi}}{F_n^{I} - \bar{\bar{F}}_n^{I}}\right]_{\xi_e} = \frac{\left[(\bar{\bar{F}}_n^{I})^2 G(\xi) P(\xi)\right]_{\xi_e}^{-1} \int_{\xi_i}^{\xi_e} (\phi_0 - \phi)\, F_n^{I} \bar{\bar{F}}_n^{I}\, G(\xi'')\,d\xi''}{\int^{\xi_e} \frac{d\xi'}{(\bar{\bar{F}}_n^{I})^2 P(\xi') G(\xi')} \int_{\xi_i}^{\xi'} (\phi_0 - \phi)\, F_n^{I} \bar{\bar{F}}_n^{I}\, G(\xi'')\,d\xi''} \tag{2-33}$$

and from (2-11),

$$C = \frac{1}{P(\xi_e)} - \int^{\xi_e} \frac{d\xi'}{(\bar{F}_n^I)^2 P(\xi') G(\xi')} \int_{\xi_i}^{\xi'} (\phi_0-\phi) F_n^I \bar{F}_n^I G(\xi'') d\xi''$$

$$+ \frac{1}{B(\xi_e)} \int_{\xi_i}^{\xi_e} (\phi_0-\phi) F_n^I \bar{F}_n^I G(\xi'') d\xi'' \tag{2-34}$$

where

$$B(\xi_e) = K_e \bar{F}_n^I(\xi_e) P^2(\xi_e) G(\xi_e) \left(\frac{d\bar{I}_n}{d\xi} - \frac{\bar{I}_n}{\bar{O}_n} \frac{d\bar{O}_n}{d\xi} \right)_{\xi_e} . \tag{2-35}$$

Substituting (2-34) into (2-31), we obtain

$$F_n^I = \bar{F}_n^I P(\xi) \left[\frac{1}{P(\xi_e)} - \int_{\xi}^{\xi_e} \frac{d\xi'}{(\bar{F}_n^I)^2 P(\xi') G(\xi')} \int_{\xi_i'}^{\xi'} (\phi_0-\phi) F_n^I \bar{F}_n^I G(\xi'') d\xi'' \right.$$

$$\left. + \frac{1}{B(\xi_e)} \int_{\xi_i'}^{\xi_e} (\phi_0-\phi) F_n^I \bar{F}_n^I G(\xi'') d\xi'' \right] \tag{2-36}$$

(scattering)

for the scattering problem.

B.2 Radiation

For the radiation problem, let

$$\begin{aligned}
A &= \xi' \ (\text{arbitrary}) \\
B &= \xi_e \\
X_1[n, \xi_i, \epsilon_a] &= X_1[n, \xi_i, \epsilon_r(\xi_i)] \\
X_2[n, \xi_e, \epsilon_a] &= X_1[n, \xi_e, \epsilon_r(\xi_e)] .
\end{aligned} \tag{2-37}$$

From equation (2-10), the left-hand side of (2-25) will vanish at ξ_e.

$$-\left[\bar{F}_n^{I} \frac{dF_n^{I}}{d\xi} - F_n^{I} \frac{d\bar{F}_n^{I}}{d\xi} + \frac{1}{2} F_n^{I} \bar{F}_n^{I} \left(\chi_1[n, \xi, \epsilon_a] - \chi_1[n, \xi, \epsilon_r(\xi)] \right) \right]_{\xi'}$$

$$= \frac{1}{G(\xi')} \int_{\xi'}^{\xi_e} (\phi_0 - \phi) F_n^{I} \bar{F}_n^{I} G(\xi'') d\xi''. \qquad (2\text{-}38)$$

Integrating as before in steps (2-28) to (2-31),

$$F_n^{I} = \bar{F}_n^{I} P(\xi) \left[- \int^{\xi} \frac{d\xi'}{(\bar{F}_n^{I})^2 P(\xi') G(\xi')} \int_{\xi'}^{\xi_e} (\phi_0 - \phi) F_n^{I} \bar{F}_n^{I} G(\xi'') d\xi'' + C \right]. \qquad (2\text{-}39)$$

The boundary conditions at ξ_e have already been applied, so the boundary conditions at ξ_i must now be utilized. To evaluate C, we note from (2-4) and (2-5) that one of the following cases will always hold:

Case I $\qquad F_n^{I}(\xi_i) = \bar{F}_n^{I}(\xi_i) = \text{known function.} \qquad (2\text{-}4)$

Case II $\qquad \left.\frac{dF_n^{I}}{d\xi}\right|_{\xi_i} = K_i \left.\frac{d\bar{F}_n^{I}}{d\xi}\right|_{\xi_i} = \text{Known function.} \qquad (2\text{-}5)$

Combining these results with (2-37) and (2-39), we obtain

Case I $$C = \frac{1}{P(\xi_i)} + \int^{\xi_i} \frac{d\xi'}{(\bar{F}_n^{I})^2 P(\xi') G(\xi')} \int_{\xi'}^{\xi_e} (\phi_0 - \phi) F_n^{I} \bar{F}_n^{I} G(\xi'') d\xi'' \qquad (2\text{-}40)$$

Case II $$C = \frac{K_i}{P(\xi_i)} + \int^{\xi_i} \frac{d\xi'}{(\bar{F}_n^{I})^2 P(\xi') G(\xi')} \int_{\xi'}^{\xi_e} (\phi_0 - \phi) F_n^{I} \bar{F}_n^{I} G(\xi'') d\xi'' \qquad (2\text{-}41)$$

$$+ \left[\bar{F}_n^{I} P(\xi) G(\xi) \frac{d\bar{F}_n^{I}}{d\xi} \right]_{\xi_i}^{-1} \int_{\xi_i}^{\xi_e} (\phi_0 - \phi) F_n^{I} \bar{F}_n^{I} G(\xi'') d\xi''.$$

Substituting into (2-39),

Case I
$$F_n^I = \bar{F}_n^I P(\xi)\left[\frac{1}{P(\xi_i)} - \int_{\xi_i}^{\xi}\frac{d\xi'}{(\bar{F}_n^I)^2 P(\xi')G(\xi')}\int_{\xi'}^{\xi_e}(\phi_0-\phi)F_n^I\bar{F}_n^I G(\xi'')d\xi''\right] \tag{2-42}$$

Case II
$$F_n^I = \bar{F}_n^I P(\xi)\left[\frac{K_i}{P(\xi_i)} - \int_{\xi_i}^{\xi}\frac{d\xi'}{(\bar{F}_n^I)^2 P(\xi')G(\xi')}\int_{\xi'}^{\xi_e}(\phi_0-\phi)F_n^I\bar{F}_n^I G(\xi'')d\xi'' + \left[\bar{F}_n^I\frac{d\bar{F}_n^I}{d\xi}G(\xi)P(\xi)\right]_{\xi_i}^{-1}\int_{\xi_i}^{\xi_e}(\phi_0-\phi)F_n^I\bar{F}_n^I G(\xi'')d\xi''\right].$$

(2-43)

C. The Iterative Solution

The complete solutions to (2-1) are

$$F_n^I = \bar{F}_n^I P(\xi)\left[\frac{1}{P(\xi_e)} - \int_{\xi}^{\xi_e}\frac{d\xi'}{(\bar{F}_n^I)^2 P(\xi')G(\xi')}\int_{\xi_i}^{\xi'}(\phi_0-\phi)F_n^I\bar{F}_n^I G(\xi'')d\xi'' + \frac{1}{B(\xi_e)}\int_{\xi_i}^{\xi_e}(\phi_0-\phi)F_n^I\bar{F}_n^I G(\xi'')d\xi''\right]$$

(2-36)

(scattering)

$$F_n^I = \bar{F}_n^I P(\xi)\left[\frac{1}{P(\xi_i)} - \int_{\xi_i}^{\xi}\frac{d\xi'}{(\bar{F}_n^I)^2 P(\xi')G(\xi')}\int_{\xi'}^{\xi_e}(\phi_0-\phi)F_n^I\bar{F}_n^I G(\xi'')d\xi''\right]$$

(2-42)

(radiation, Case I)

$$F_n^{I} = \bar{F}_n^{I} P(\xi)\left[\frac{K_i}{P(\xi_i)} - \int_{\xi_i}^{\xi} \frac{d\xi'}{(\bar{F}_n^{I})^2 P(\xi') G(\xi')} \int_{\xi'}^{\xi_e} (\phi_o - \phi) F_n^{I} \bar{F}_n^{I} G(\xi'') d\xi''\right.$$

$$\left. + \left[\bar{F}_n^{I} P(\xi) G(\xi) \frac{d\bar{F}_n^{I}}{d\xi}\right]_{\xi_i}^{-1} \int_{\xi_i}^{\xi_e} (\phi_o - \phi) F_n^{I} \bar{F}_n^{I} G(\xi'') d\xi''\right] . \quad \text{(2-43)}$$

(radiation, Case II)

Now write F^{I}_{n} in terms of an infinite series, and substitute into the appropriate equation.

$$F_n^{I} = \bar{F}_n^{I} \frac{P(\xi)}{P(\xi_e)} \left(1 + \Delta_{n_1} + \Delta_{n_2} + \dots\right) \quad \text{(2-44)}$$

(scattering)

$$F_n^{I} = \bar{F}_n^{I} \frac{P(\xi)}{P(\xi_i)} \left(1 + \Delta_{n_1} + \Delta_{n_2} + \dots\right) \quad \text{(2-45)}$$

(radiation, Case I)

$$F_n^{I} = \bar{F}_n^{I} K_i \frac{P(\xi)}{P(\xi_i)} \left(1 + \Delta_{n_1} + \Delta_{n_2} + \dots\right) \quad \text{(2-46)}$$

(radiation, Case II)

with the resulting equations

$$\Delta_{n_1} + \Delta_{n_2} + \dots = -\int_{\xi}^{\xi_e} \frac{d\xi'}{\Gamma(\xi')} \int_{\xi_i}^{\xi'} (\phi_o - \phi) \Gamma(\xi'') \left(1 + \Delta_{n_1} + \Delta_{n_2} + \dots\right) d\xi''$$

$$+ \frac{1}{B(\xi_e)} \int_{\xi_i}^{\xi_e} (\phi_o - \phi) \Gamma(\xi'') \left(1 + \Delta_{n_1} + \Delta_{n_2} + \dots\right) d\xi'' \quad \text{(2-47)}$$

(scattering)

$$\Delta_{n_1} + \Delta_{n_2} + \ldots = -\int_{\xi_i}^{\xi} \frac{d\xi'}{\Gamma(\xi')} \int_{\xi'}^{\xi_e} (\phi_0 - \phi)\Gamma(\xi'')(1 + \Delta_{n_1} + \Delta_{n_2} + \ldots)\, d\xi''$$

(2-48)

(radiation, Case I)

$$\Delta_{n_1} + \Delta_{n_2} + \ldots = -\int_{\xi_i}^{\xi} \frac{d\xi'}{\Gamma(\xi')} \int_{\xi'}^{\xi_e} (\phi_0 - \phi)\Gamma(\xi'')(1 + \Delta_{n_1} + \Delta_{n_2} + \ldots)\, d\xi''$$

$$+ \frac{1}{\Gamma(\xi_i)\left(\frac{1}{F_n^I}\frac{dF_n^I}{d\xi}\right)_{\xi_i}} \int_{\xi_i}^{\xi_e} (\phi_0 - \phi)\Gamma(\xi'')(1 + \Delta_{n_1} + \Delta_{n_2} + \ldots)\, d\xi''$$

(2-49)

(radiation, Case II)

where $$\Gamma(\xi) = (F_n^I)^2 P(\xi) G(\xi) . \tag{2-50}$$

These equations can be satisfied by choosing

$$\left[\begin{aligned}
\Delta_{n_1} &= -\int_{\xi}^{\xi_e} \frac{d\xi'}{\Gamma(\xi')} \int_{\xi_i}^{\xi'} (\phi_0 - \phi)\Gamma(\xi'')\, d\xi'' + \frac{1}{B(\xi_e)} \int_{\xi_i}^{\xi_e} (\phi_0 - \phi)\Gamma(\xi'')\, d\xi'' \\
\Delta_{n_2} &= -\int_{\xi}^{\xi_e} \frac{d\xi'}{\Gamma(\xi')} \int_{\xi_i}^{\xi'} (\phi_0 - \phi)\Gamma(\xi'')\Delta_{n_1}\, d\xi'' + \frac{1}{B(\xi_e)} \int_{\xi_i}^{\xi_e} (\phi_0 - \phi)\Gamma(\xi'')\Delta_{n_1}\, d\xi'' \\
&\ \ \vdots
\end{aligned}\right.$$

(2-51)

(scattering)

$$\left[\begin{array}{l} \Delta_{n_1} = -\int_{\xi_i}^{\xi} \frac{d\xi'}{\Gamma(\xi')} \int_{\xi'}^{\xi_e} (\phi_0 - \phi)\,\Gamma(\xi'')\,d\xi'' \\ \Delta_{n_2} = -\int_{\xi_i}^{\xi} \frac{d\xi'}{\Gamma(\xi')} \int_{\xi'}^{\xi_e} (\phi_0 - \phi)\,\Gamma(\xi'')\,\Delta_{n_1}\,d\xi'' \\ \quad\vdots \end{array}\right.$$

(2-52)

(radiation, Case I)

$$\left[\begin{array}{l} \Delta_{n_1} = \int_{\xi_i}^{\xi} \frac{d\xi'}{\Gamma(\xi')} \int_{\xi'}^{\xi_e} (\phi_0 - \phi)\,\Gamma(\xi'')\,d\xi'' + \dfrac{1}{\left(\dfrac{\Gamma(\xi)}{\bar{F}_n^{I}}\,\dfrac{d\bar{F}_n^{I}}{d\xi}\right)_{\xi_i}} \int_{\xi_i}^{\xi_e} (\phi_0 - \phi)\,\Gamma(\xi'')\,d\xi'' \\ \Delta_{n_2} = \int_{\xi_i}^{\xi} \frac{d\xi'}{\Gamma(\xi')} \int_{\xi'}^{\xi_e} (\phi_0 - \phi)\,\Gamma(\xi'')\,\Delta_{n_1}\,d\xi'' + \dfrac{1}{\left(\dfrac{\Gamma(\xi)}{\bar{F}_n^{I}}\,\dfrac{d\bar{F}_n^{I}}{d\xi}\right)_{\xi_i}} \int_{\xi_i}^{\xi_e} (\phi_0 - \phi)\,\Gamma(\xi'')\,\Delta_{n_1}\,d\xi'' \\ \quad\vdots \end{array}\right.$$

(2-53)

(radiation, Case II)

The infinite series in each case is essentially a power series in $(\phi_o - \phi)$ and the series will converge rapidly if $(\phi_o - \phi)$ is small. It is for this case that the solution will be most useful.

The phase shifts are given directly by the evaluation of the Δ_n at ξ_e. If we keep only first order terms,

$$\Delta_{n_1}(\xi_e) = \frac{1}{B(\xi_e)} \int_{\xi_i}^{\xi_e} (\phi_0 - \phi)\,\Gamma(\xi'')\,d\xi''$$

(2-54)

(scattering)

$$\Delta_{n_1}(\xi_e) = -\int_{\xi_i}^{\xi_e} \frac{d\xi'}{\Gamma(\xi')} \int_{\xi'}^{\xi_e} (\phi_0 - \phi)\,\Gamma(\xi'')\,d\xi''$$

(2-55)
(radiation, Case I)

$$\Delta_{n_1}(\xi_e) = -\int_{\xi_i}^{\xi_e} \frac{d\xi'}{\Gamma(\xi')} \int_{\xi'}^{\xi_e} (\phi_0 - \phi)\,\Gamma(\xi'')\,d\xi''$$

$$+ \frac{1}{\left(\dfrac{\Gamma(\xi)}{F_n^I}\,\dfrac{dF_n^I}{d\xi}\right)_{\xi_i}} \int_{\xi_i}^{\xi_e} (\phi_0 - \phi)\,\Gamma(\xi'')\,d\xi'' \,.$$

(2-56)
(radiation, Case II)

CHAPTER III

DERIVATION OF THE BASIC DIFFERENTIAL EQUATIONS

In this chapter the basic propagation equations are derived for problems with specific rectangular, cylindrical and spherical symmetries. The derivation begins with Maxwells equations in vector form and leads in each case to a linear, second order differential equation. The derivations are carried out in detail so that the Chapter will be useful for general reference. The rectangular case is considered first, and the appropriate model compatible with the symmetry restrictions is shown in Figure 1. Then the cylindrical case is considered for two different symmetries, illustrated in Figures 2 and 3.

In the last section, the spherical case is developed and the model for this problem is shown in Figure 4.

Maxwell's curl equations for propagation in a medium with a constant permeability μ and a general, anisotropic, inhomogeneous permittivity can be written

$$K_0 \nabla \times \bar{H}(x_1 x_2 x_3 t) = \epsilon(x_1 x_2 x_3) \frac{\partial}{\partial t} \bar{E}(x_1 x_2 x_3 t) + \sigma(x_1 x_2 x_3) \bar{E}(x_1 x_2 x_3 t)$$

$$K_0 \nabla \times \bar{E}(x_1 x_2 x_3 t) = -\mu \frac{\partial}{\partial t} \bar{H}(x_1 x_2 x_3 t)$$

where $\bar{E}$ and $\bar{H}$ are functions of position and time, ϵ and σ are functions of position, and the factor k_o has been intorduced to make the coordinates dimensionless. All distances will be measured in terms of $k_o = \frac{2\pi}{\lambda_0} = \omega\sqrt{\mu\epsilon_0}$.

Assuming harmonic time dependence, the curl equations become

$$\begin{aligned} K_0 \nabla \times \bar{H}(x_1 x_2 x_3) &= [i\omega\epsilon(x_1 x_2 x_3) + \sigma(x_1 x_2 x_3)]\,\bar{E}(x_1 x_2 x_3) \\ &= i\omega\epsilon'(x_1 x_2 x_3)\,\bar{E}(x_1 x_2 x_3) \qquad (3\text{-}1) \end{aligned}$$

$$K_0 \nabla \times \overline{E}(x_1 x_2 x_3) = -i\omega\mu\, \overline{H}(x_1 x_2 x_3) \tag{3-2}$$

where ϵ' has been defined to be complex. Hereafter the prime will be dropped. We will now derive the vector wave equations, starting with these two relations. Multiplying both sides of (3-1) by ϵ^{-1} from the left, we obtain

$$K_0 \epsilon^{-1} \nabla \times \overline{H} = i\omega \overline{E} .$$

Taking the curl of both sides and substituting (3-2),

$$K_0^2 \epsilon \nabla \times \epsilon^{-1} \nabla \times \overline{H} = \mu \epsilon \omega^2 \overline{H} \tag{3-3}$$

which is the vector equation for $\overline{H}$. Taking the curl of both sides of (3-2), and substituting (3-1),

$$K_0^2 \nabla \times \nabla \times \overline{E} = \mu \epsilon \omega^2 \overline{E}, \tag{3-4}$$

the vector equation for $\overline{E}$. Utilizing the following vector identities[10]

$$\nabla \times \phi \overline{A} = \phi \nabla \times \overline{A} + \nabla \phi \times \overline{A}$$

$$\nabla \cdot \phi \overline{A} = \phi \nabla \cdot \overline{A} + \nabla \phi \cdot \overline{A}$$

where ϕ is any scalar function and $\overline{A}$ an arbitrary vector, we can complete the derivation of the wave equations. Now note the two Maxwell divergence equations

$$\nabla \cdot \overline{B} = 0 \tag{3-5}$$

$$\nabla \cdot \overline{D} = \rho \tag{3-6}$$

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[10] J. Reitz and F. Milford, Foundations of Electromagnetic Theory, Addison-Wesley, (1960), p 18.

[11] See for instance Richmond, IEEE Transactions on Antennas and Propagation, (May 1962), pp 300-305 (Appendix).

Assume that the medium to be studied is electrically neutral, i.e., that $\rho = 0$.[11] Then (3-5) and (3-6), combined with the above vector relations, can be expressed in the form

$$\nabla \cdot \mu \overline{H} = \mu \nabla \cdot \overline{H} = 0 \tag{3-7}$$

$$\nabla \cdot \epsilon \overline{E} = \epsilon \nabla \cdot \overline{E} + \nabla \epsilon \cdot \overline{E} = 0, \quad \nabla \cdot \overline{E} = -\epsilon^{-1} \nabla \epsilon \cdot \overline{E}. \tag{3-8}$$

Now expand equations (3-3) and (3-4) for $\overline{E}$ and $\overline{H}$ to the form

$$\nabla \times \nabla \times \overline{H} + \epsilon \nabla \epsilon^{-1} \times \nabla \times \overline{H} - \epsilon/\epsilon_0 \, \overline{H} = 0 \tag{3-9a}$$

$$\nabla \times \nabla \times \overline{E} - \epsilon/\epsilon_0 \, \overline{E} = 0. \tag{3-10a}$$

Introducing a new operator $\nabla \cdot \nabla \overline{H}$ by the definition[12]

$$\nabla \times \nabla \times \overline{H} = \nabla \nabla \cdot \overline{H} - \nabla \cdot \nabla \overline{H}$$

the curl curl operators can be eliminated

$$\nabla \nabla \cdot \overline{H} - \nabla \cdot \nabla \overline{H} + \epsilon \nabla \epsilon^{-1} \times \nabla \times \overline{H} - \epsilon/\epsilon_0 \, \overline{H} = 0$$

$$\nabla \nabla \cdot \overline{E} - \nabla \cdot \nabla \overline{E} - \epsilon/\epsilon_0 \, \overline{E} = 0.$$

Substituting from (3-7) and (3-8)

$$\nabla \cdot \nabla \overline{H} - \epsilon \nabla \epsilon^{-1} \times \nabla \times \overline{H} + \epsilon/\epsilon_0 \, \overline{H} = 0 \tag{3-9b}$$

$$\nabla \cdot \nabla \overline{E} + \nabla \left(\epsilon^{-1} \nabla \epsilon \cdot \overline{E} \right) + \epsilon/\epsilon_0 \, \overline{E} = 0. \tag{3-10b}$$

In rectangular coordinates only, we have

$$\nabla \cdot \nabla \overline{H} = \nabla^2 \overline{H}, \quad \nabla \cdot \nabla \overline{E} = \nabla^2 \overline{E}.$$

The vector wave equations (3-9) and (3-10), combined with the appropriate boundary conditions, will completely define the solutions to a given problem in which inhomogeneous media are present. In the cases treated below,

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12 J. Stratton, <u>Electromagnetic Theory</u>, McGraw-Hill, (1941), p 49.

specific solutions to the wave equations will be derived in rectangular, cylindrical and spherical coordinate systems. In later applications, conducting boundaries with rectangular, cylindrical and spherical symmetries respectively will be considered. In each symmetry case, ϵ will be chosen to be isotropic, and the inhomogeneity represented by the variation in ϵ and the sources will be chosen so that either $\overline{E}$ or $\overline{H}$ will have only one nonzero component. In the problems in which only one $\overline{E}$ component exists, the fields will be described as $\overline{E}$-polarized in the direction of the component. In problems in which only one $\overline{H}$ component exists, the fields will be described as $\overline{H}$-polarized in the direction of the component. The solutions can be characterized as follows:

<u>H-polarized in the $\hat{\xi}$ direction</u>

Consistent with an infinite, constant-phase magnetic line source along an arbitrary coordinate direction $\hat{\xi}$, $\overline{E}$ is pure transverse ($E_{\xi} = 0$). For all the symmetries considered below, H_{ξ} will be the only nonzero component of $\overline{H}$.

<u>E-polarized in the $\hat{\xi}$ direction</u>

Consistent with an infinite, constant-phase electric line source along an arbitrary coordinate direction $\hat{\xi}$, $\overline{H}$ is pure transverse ($H_{\xi} = 0$). For all the symmetries considered below, E_{ξ} will be the only nonzero component of $\overline{E}$.

A. <u>Rectangular Coordinates</u>

The first case to be considered is the solution of (3-9) and (3-10) in a rectangular coordinate system, these solutions later to be applied to problems with a planar conducting boundary. (Figure 1).

A.1 H-polarized in the $\hat{z}$ direction

For this solution, choose the arbitrary direction $\hat{\xi}$ to be along the z axis

$$\hat{\xi} = \hat{z}$$

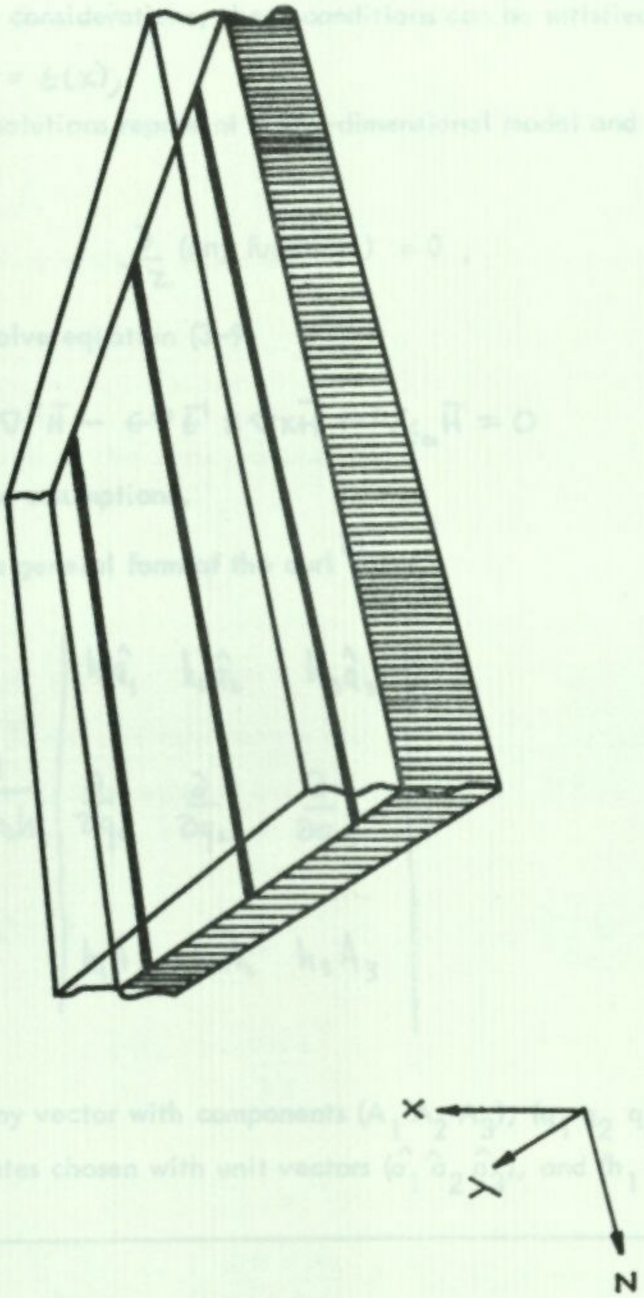

Figure 1. Conducting Plane with Axial Antennas

so that $\overline{E}$ has no z component

$$\overline{E} = \hat{x}\, E_x\,(x,y) + \hat{y}\, E_y\,(x,y)$$

and $\overline{H}$ has only a z component.

$$\overline{H} = \hat{z}\, H_z\,(x,y).$$

From symmetry considerations, these conditions can be satisfied if

$$\epsilon = \epsilon(x),$$

and since the solutions represent a two-dimensional model and are independent of z,

$$\frac{\partial}{\partial z}\,(\text{any function}) = 0\ .$$

We will now solve equation (3-9)

$$\nabla^2\overline{H} - \epsilon\nabla\epsilon^{-1} \times \nabla\times\overline{H} + \epsilon/\epsilon_0\,\overline{H} = 0$$

subject to these assumptions.

From the general form of the curl[13]

$$\nabla\times\overline{A} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{a}_1 & h_2\hat{a}_2 & h_3\hat{a}_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix} \tag{3-11}$$

where $\overline{A}$ is any vector with components $(A_1\ A_2\ A_3)$, $(q_1\ q_2\ q_3)$ are the three coordinates chosen with unit vectors $(\hat{a}_1\ \hat{a}_2\ \hat{a}_3)$, and $(h_1\ h_2\ h_3)$ are

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[13] J. Stratton, op. cit., Chapter 1.

the metrical coefficients. For the problem now under consideration, take

$$(q_1 q_2 q_3) = (x y z) \qquad (\hat{a}_1 \hat{a}_2 \hat{a}_3) = (\hat{x}\ \hat{y}\ \hat{z})$$

$$h_1 = h_2 = h_3 = 1\,. \tag{3-12}$$

The curl of H becomes

$$\nabla \times \overline{H} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & 0 \\ 0 & 0 & H_z \end{vmatrix} = \hat{x}\frac{\partial H_z}{\partial y} - \hat{y}\frac{\partial H_z}{\partial x} \tag{3-13}$$

and since ϵ is a function of x only,

$$\epsilon \nabla \epsilon^{-1} = \hat{x}\,\epsilon \frac{d\epsilon^{-1}}{dx} = -\hat{x}\,\frac{1}{\epsilon}\frac{d\epsilon}{dx}\,. \tag{3-14}$$

The general form for the cross product is[13]

$$\overline{A} \times \overline{B} = \begin{vmatrix} \hat{a}_1 & \hat{a}_2 & \hat{a}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} \tag{3-15}$$

where $\overline{A}$ and $\overline{B}$ are arbitrary vectors and $(\hat{a}_1\ \hat{a}_2\ \hat{a}_3)$ the unit vectors in the appropriate coordinate system. Let $\overline{A} = \epsilon\nabla\epsilon^{-1}$, $\overline{B} = \nabla \times \overline{H}$, so that

$$\epsilon\nabla\epsilon^{-1} \times \nabla \times \overline{H} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -\dfrac{1}{\epsilon}\dfrac{d\epsilon}{dx} & 0 & 0 \\ \dfrac{\partial H_z}{\partial y} & -\dfrac{\partial H_z}{\partial x} & 0 \end{vmatrix} = \hat{z}\,\frac{1}{\epsilon}\frac{d\epsilon}{dx}\frac{\partial H_z}{\partial x} \tag{3-16}$$

and the differential equation for H_z is

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} - \frac{1}{\epsilon}\frac{d\epsilon}{dx}\frac{\partial H_z}{\partial x} + \epsilon/\epsilon_0\, H_z = 0\,. \tag{3-17}$$

The only other nonzero fields, E_x and E_y, can be directly derived from H_z. From (3-1),

$$K_0 \nabla \times \overline{H} = i\omega\epsilon \overline{E}$$

we find that

$$E_x = \frac{K_0}{i\omega\epsilon}\frac{\partial H_z}{\partial y} \qquad E_y = \frac{-K_0}{i\omega\epsilon}\frac{\partial H_z}{\partial x}, \tag{3-18}$$

so that all the fields can be found once (3-17) is solved and H_z is determined. If $\epsilon = \epsilon_o \epsilon_a$ equation (3-17) reduces to the well-known form[14]

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + \epsilon_a H_z = 0 \tag{3-18a}$$

(where the factor $k_o{}^2$ has been absorbed into the variables x and y). This equation can easily be solved by separation of variables, and through the choices of symmetry made in the previous derivation, (3-17) is also solvable by the separation of variables method. Let

$$H_z(x,y) = L(x)\,M(y) \tag{3-19}$$

where L and M are arbitrary functions of one coordinate each. Substituting into (3-17),

$$M\frac{d^2L}{dx^2} + L\frac{d^2M}{dy^2} - \frac{1}{\epsilon}\frac{d\epsilon}{dx}M\frac{dL}{dx} + \frac{\epsilon}{\epsilon_0}LM = 0 \tag{3-20}$$

and

$$\frac{1}{L}\frac{d^2L}{dx^2} - \frac{1}{\epsilon L}\frac{d\epsilon}{dx}\frac{dL}{dx} + \frac{\epsilon}{\epsilon_0} = -\frac{1}{M}\frac{d^2M}{dy^2} = u^2, \; M = e^{\pm iuy} \tag{3-21}$$

since the whole y dependence has been separated out. The only unknown function is $L_u(x)$:

$$\frac{d^2L_u}{dx^2} - \frac{1}{\epsilon}\frac{d\epsilon}{dx}\frac{dL_u}{dx} + (\epsilon_r - u^2)L_u = 0 \tag{3-22}$$

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[14] R. Harrington, <u>Time-Harmonic Electromagnetic Fields</u>, McGraw-Hill, (1961), p 143.

where we have defined $\epsilon \equiv \epsilon_o \epsilon_r(x)$. The partial differential equation (3-17) has now been separated into two independent ordinary differential equations. One of these equations is immediately solvable, but the one for L_u can be solved directly only for a few, special choices of $\epsilon(x)$.

The general solution for H_z is obtained by forming a linear combination of the solutions obtained by letting u take on all possible values. This procedure is exactly analogous to the solution of partial differential equations by Fourier transform techniques.

In the completely unrestricted case, u can take on a continuous, infinite range of values, and H_z can be written

$$H_z(x,y) = \int_{-\infty}^{+\infty} du\, e^{iuy} \left[a_u^{I} L_u^{I}(x) + a_u^{II} L_u^{II}(x) \right] \tag{3-23}$$

where L^{I}_u and L^{II}_u are any two linearly independent solutions of (3-22) and a^{I}_u and a^{II}_u are arbitrary functions of the transform variables to be determined from boundary conditions.

In the subsequent development, however, the following restrictions of periodicity is made on H_z:

$$H_z(x,y) = H_z(x, y+\tau), \tag{3-24}$$

and since any such periodic function can be represented in the form[15]

$$H_z(x,y) = \sum_{n=-\infty}^{+\infty} H_n(x) e^{in\bar{\omega}y} \qquad \bar{\omega} = \frac{2\pi}{k_0\tau} \tag{3-25}$$

we must restrict u to the discrete values

$$u = n\bar{\omega} = \frac{2\pi n}{k_0\tau} = n\frac{\lambda_0}{\tau} \tag{3-26}$$

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[15] W. Kaplan, Operational Methods for Linear Systems, Addison-Wesley, (1962), p 186, 236.

so that (3-23) becomes

$$H_z(x,y) = \sum_{n=-\infty}^{+\infty} e^{in\lambda_0/\tau y}\left[a_n^{I} L_n^{I}(x) + a_n^{II} L_n^{II}(x)\right]. \qquad (3\text{-}27)$$

A.2 <u>E-polarized in the $\hat{z}$ Direction</u>

Now that the method of solution has been derived for rectangular coordinates with fields H-polarized in the $\hat{z}$ direction, we now apply similar techniques to fields E-polarized in the $\hat{z}$ direction. Again we choose our vector $\hat{\xi}$ in the $\hat{z}$ direction,

$$\hat{\xi} = \hat{z}.$$

We specify that only an E_z field exists

$$\bar{E} = \hat{z}\, E_z(x,y)$$

and only H_x and H_y fields exist.

$$\bar{H} = \hat{x}\, H_x(x,y) + \hat{y}\, H_y(x,y).$$

These symmetries are obtained by taking

$$\epsilon = \epsilon(x)$$

and result in the two-dimensional nature

$$\frac{\partial}{\partial z}\,(\text{any function}) = 0.$$

These restrictions will now be applied to (3-10):

$$\nabla^2 \bar{E} + \nabla(\epsilon^{-1}\nabla\epsilon\cdot\bar{E}) + \frac{\epsilon}{\epsilon_0}\bar{E} = 0.$$

Since ϵ is a function of x only,

$$\nabla\epsilon = \hat{x}\frac{d\epsilon}{dx}, \quad \nabla\epsilon\cdot\bar{E} = 0 \qquad (3\text{-}28)$$

and the differential equation for E_z is

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \epsilon/\epsilon_0\, E_z = 0. \qquad (3\text{-}29)$$

To obtain the other nonzero fields from E_z, note that from (3-2)

$$K_o \nabla \times \bar{E} = -i\omega\mu \bar{H}$$

we obtain

$$H_x = \frac{-K_o}{i\omega\mu} \frac{\partial E_z}{\partial y} \qquad H_y = \frac{-K_o}{i\omega\mu} \frac{\partial E_z}{\partial x}$$

giving all fields in terms of E_z as desired. If $\epsilon_r \equiv \epsilon_a$, (3-29) reduces to the usual form noted before.[16]

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \epsilon_a E_z = 0. \tag{3-31}$$

However, (3-29) will also separate immediately without this assumption. We choose α, β to be two arbitrary functions of one coordinate each

$$E_z = \alpha(x)\,\beta(y) \tag{3-32}$$

and by direct substitution into (3-22),

$$\frac{1}{\alpha}\frac{d^2\alpha}{dx^2} + \frac{\epsilon}{\epsilon_o} = -\frac{1}{\beta}\frac{d^2\beta}{dy^2} = u^2 \quad , \quad \beta = e^{\pm i u y}$$

$$\frac{d^2\alpha_u}{dx^2} + (\epsilon_r - u^2)\alpha_u = 0 \tag{3-33}$$

so that the only integration not directly performable is the one for α (x). The most general form for E_z is obtained by summing over $u = n\,\lambda_o/\tau$

$$E_z(x,y) = \sum_{n=-\infty}^{+\infty} e^{in\lambda_o/\tau\, y}\left[b_n^{I}\alpha_n^{I}(x) + b_n^{II}\alpha_n^{II}(x)\right]. \tag{3-34}$$

α^{I}_n and α^{II}_n are the linearly independent solutions to (3-24) and b^{I}_n and

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16 R. Harrington, op. cit., p 143.

b''_n are arbitrary functions of the transform variables, to be detemined by the boundary conditions.

B. Cylindrical Coordinates

We now choose a new set of coordinates in which to solve (3-9) and (3-10). The choice of cylindrical coordinates is made for application to problems with the appropriate symmetry. (Figures 2 and 3).

B.1. H-polarized in the $\hat{z}$ Direction

For the first solution, consider equation (3-9b)

$$\nabla\cdot\nabla\bar{H} - \epsilon\nabla\epsilon^{-1}\times\nabla\times\bar{H} + \frac{\epsilon}{\epsilon_0}\bar{H} = 0. \tag{3-9b}$$

This time, we again choose our $\hat{\xi}$ vector to lie along the z axis

$$\hat{\xi} = \hat{z}$$

and allow only the $\bar{E}$ fields

$$\bar{E} = \hat{\rho}\, E_\rho(\rho,\theta) + \hat{\theta}\, E_\theta(\rho,\theta)$$

and the single $\bar{H}$ field

$$\bar{H} = \hat{z}\, H_z(\rho,\theta)$$

to exist. Such a symmetry is obtained by letting

$$\epsilon = \epsilon(\rho)$$

and requires that

$$\frac{\partial}{\partial z}\text{(any function)} = 0.$$

From the definition of the curl (3-11) with the new identifications

$$(q_1 q_2 q_3) = (\rho,\theta,z) \qquad (\hat{a}_1\hat{a}_2\hat{a}_3) = (\hat{\rho}\ \hat{\theta}\ \hat{z})$$

$$h_1 = h_3 = 1, \quad h_2 = \rho$$

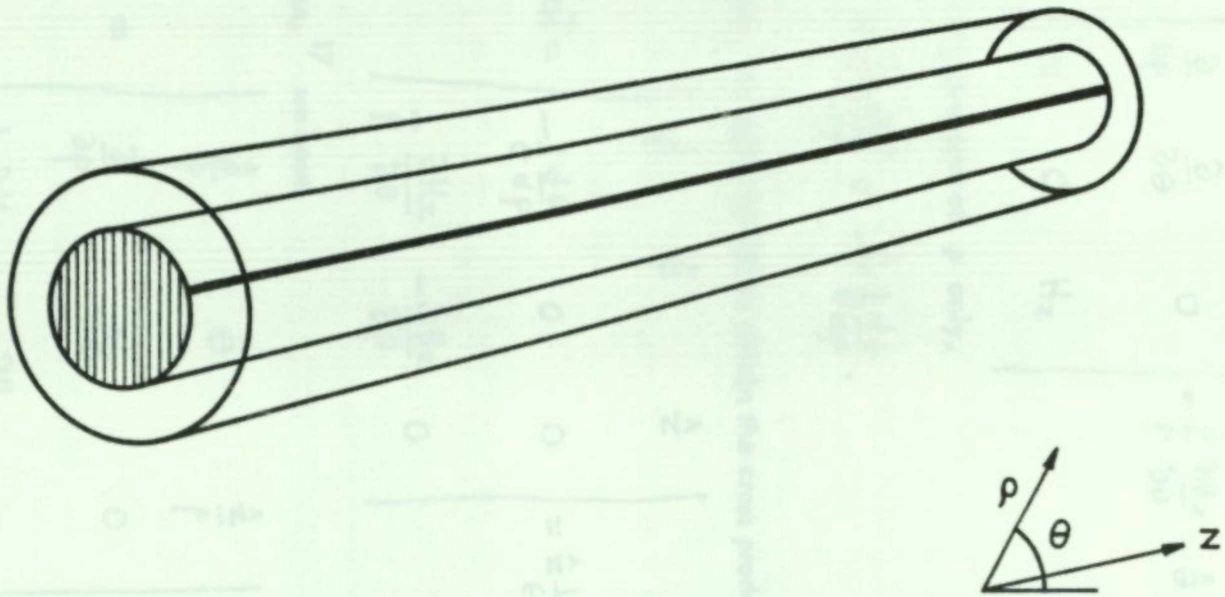

Figure 2. Conducting Cylinder with Axial Antenna

the curl of $\overline{H}$ for the present case is

$$\nabla \times \overline{H} = \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\theta} & \hat{z} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \theta} & 0 \\ 0 & 0 & H_z \end{vmatrix} = \frac{\hat{\rho}}{\rho}\frac{\partial H_z}{\partial \theta} - \hat{\theta}\frac{\partial H_z}{\partial \rho} \tag{3-35}$$

and since ϵ is a function of ρ only,

$$\epsilon \nabla \epsilon^{-1} = \hat{\rho}\,\epsilon \frac{d\epsilon^{-1}}{d\rho} = -\hat{\rho}\,\frac{1}{\epsilon}\frac{d\epsilon}{d\rho}\,. \tag{3-36}$$

As done before, we apply (3-15) to obtain the cross product

$$\epsilon \nabla \epsilon^{-1} \times \nabla \times \overline{H} = \begin{vmatrix} \hat{\rho} & \hat{\theta} & \hat{z} \\ -\frac{1}{\epsilon}\frac{d\epsilon}{d\rho} & 0 & 0 \\ \frac{1}{\rho}\frac{\partial H_z}{\partial \theta} & -\frac{\partial H_z}{\partial \rho} & 0 \end{vmatrix} = \hat{z}\,\frac{1}{\epsilon}\frac{d\epsilon}{d\rho}\frac{\partial H_z}{\partial \rho}\,. \tag{3-37}$$

From Stratton,[17] we have

$$\nabla \times \nabla \times \overline{H} = \begin{vmatrix} \frac{\hat{\rho}}{\rho} & \hat{\theta} & \frac{\hat{z}}{\rho} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \theta} & 0 \\ \frac{1}{\rho}\frac{\partial H_z}{\partial \theta} & -\rho\frac{\partial H_z}{\partial \rho} & 0 \end{vmatrix} \tag{3-38}$$

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[17] J. Stratton, op. cit., p 50.

$$\nabla \times \nabla \times \bar{H} = -\hat{z}\left[\frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\frac{\partial H_z}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 H_z}{\partial \theta^2}\right] = -\nabla \cdot \nabla \bar{H} \qquad (3\text{-}39)$$

so that, substituting into (3-9b), the differential equation for H_z becomes

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\frac{\partial H_z}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 H_z}{\partial \theta^2} - \frac{1}{\epsilon}\frac{d\epsilon}{d\rho}\frac{\partial H_z}{\partial \rho} + \frac{\epsilon}{\epsilon_0}H_z = 0. \qquad (3\text{-}40)$$

To obtain the other nonzero fields, we see from (3-1)

$$K_0 \nabla \times \bar{H} = i\omega\epsilon\bar{E}$$

that

$$E_\theta = -\frac{K_0}{i\omega\epsilon}\frac{\partial H_z}{\partial \rho} \qquad E_\rho = \frac{K_0}{i\omega\epsilon\rho}\frac{\partial H_z}{\partial \theta} \qquad (3\text{-}41)$$

All fields are known when H_z is known. If for comparison with standard texts we again take H_z to be time harmonic with no z dependence and $\epsilon = \epsilon_0 \epsilon_a$, (3-40) reduces to

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\frac{\partial H_z}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 H_z}{\partial \theta^2} + \epsilon_a H_z = 0 \qquad (3\text{-}42)$$

in agreement with the literature.[18] But as before, equation (3-40) can be solved as it stands by separating variables without the necessity for further restrictions. Let

$$H_z = A(\rho)\,B(\theta) \qquad (3\text{-}43)$$

as explained earlier. Substituting into (3-40),

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[18] R. Harrington, op. cit., p 198.

$$\frac{\rho}{A}\left(\rho\frac{d^2A}{d\rho^2}+\frac{dA}{d\rho}\right)-\frac{\rho^2}{A\epsilon}\frac{d\epsilon}{d\rho}\frac{dA}{d\rho}+\frac{\epsilon}{\epsilon_0}\rho^2=-\frac{1}{B}\frac{d^2B}{d\theta^2}=m^2,\ B=e^{\pm im\theta} \quad (3\text{-}44)$$

where now m must be taken as an integer so that the solution $H_z(\rho, \theta)$ will be single-valued.[19] The equation for $A_m(\rho)$ becomes

$$\frac{d^2A_m}{d\rho^2}+\left(\frac{1}{\rho}-\frac{1}{\epsilon}\frac{d\epsilon}{d\rho}\right)\frac{dA_m}{d\rho}+\left(\epsilon_r-\frac{m^2}{\rho^2}\right)A_m=0. \quad (3\text{-}45)$$

This is the only one of the two ordinary differential equations not directly solvable. The general fom of H_z can be written, as discussed earlier,

$$H_z(\rho,\theta)=\sum_{m=-\infty}^{+\infty}e^{im\theta}\left[a_m^{I}A_m^{I}(\rho)+a_m^{II}A_m^{II}(\rho)\right] \quad (3\text{-}46)$$

where A_m^{I} and A_m^{II} are the linearly independent solutions of (3-45) and a_m^{I} and a_m^{II} the arbitrary constants to be determined by boundary conditions.

B.2. E-polarized in the $\hat{z}$ Direction

For the E-symmetric case in cylindrical coordinates, we take

$$\hat{\xi}=\hat{z}$$

again, and now seek a solution to (3-10a)

$$\nabla\times\nabla\times\bar{E}-\epsilon/\epsilon_0\,\bar{E}=0 \quad (3\text{-}10a)$$

with the assumptions that the only nonzero electric field is E_z,

$$\bar{E}=\hat{z}\,E_z(\rho,\theta)$$

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[19] R. Harrington, op. cit., p 200.

and the only nonzero magnetic fields are H_ρ and H_θ :

$$\bar{H} = \hat{\rho} H_\rho(\rho,\theta) + \hat{\theta} H_\theta(\rho,\theta).$$

These restrictions can be satisfied by the choice of symmetry

$$\epsilon = \epsilon(\rho)$$

and the requirement

$$\frac{\partial}{\partial z}(\text{any function}) = 0.$$

Again from Stratton,

$$\nabla\times\nabla\times\bar{E} = \begin{vmatrix} \frac{\hat{\rho}}{\rho} & \hat{\theta} & \frac{\hat{z}}{\rho} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\theta} & 0 \\ \frac{1}{\rho}\frac{\partial E_z}{\partial\theta} & -\rho\frac{\partial E_z}{\partial\rho} & 0 \end{vmatrix} \tag{3-47}$$

$$= -\hat{z}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial E_z}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 E_z}{\partial\theta^2}\right]$$

and substituting into (3-10a) directly, the differential equation for E_z is

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial E_z}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 E_z}{\partial\theta^2} + \frac{\epsilon}{\epsilon_0}E_z = 0. \tag{3-48}$$

From equation (3-2)

$$k_0\nabla\times\bar{E} = -i\omega\mu\bar{H}$$

we find

$$H_\theta = \frac{k_0}{i\omega\mu}\frac{\partial E_z}{\partial\rho} \qquad H_\rho = \frac{-k_0}{i\omega\mu\rho}\frac{\partial E_z}{\partial\theta} \tag{3-49}$$

giving the rest of the fields. If $\epsilon = \epsilon_o \epsilon_a$, (3-48) also reduces to the well-known homogeneous form.[20] To solve (3-48) directly by separating variables, let

$$E_z(\rho,\theta) = M(\rho)N(\theta) \tag{3-50}$$

where M and N are the arbitrary functions of one coordinate each. Then we obtain

$$\frac{\rho^2}{M}\frac{d^2M}{d\rho^2} + \frac{\rho}{M}\frac{dM}{d\rho} + \frac{\epsilon}{\epsilon_0}M = -\frac{1}{N}\frac{d^2N}{d\theta^2} = m^2, \quad N = e^{\pm im\theta} \tag{3-51}$$

and

$$\frac{d^2M_m}{d\rho^2} + \frac{1}{\rho}\frac{dM_m}{d\rho} + \left(\epsilon_r - \frac{m^2}{\rho^2}\right)M_m = 0. \tag{3-52}$$

Only (3-52) cannot be directly solved. As before, E_z can be generally written in the form

$$E_z(\rho,\theta) = \sum_{m=-\infty}^{+\infty} e^{im\theta}\left[b_m^{I} M_m^{I}(\rho) + b_m^{II} M_m^{II}(\rho)\right] \tag{3-53}$$

where M_m^{I} and M_m^{II} are the linearly independent solutions of (3-52) and b_m^{I} and b_m^{II} are the arbitrary constants.

B.3 <u>H-polarized in the $\hat{\theta}$ Direction</u>

Now choose $\hat{\xi} = \hat{\theta}$ and consider solutions of (3-9a)

$$\nabla\times\nabla\times\overline{H} + \epsilon\nabla\epsilon^{-1}\times\nabla\times\overline{H} - \epsilon/\epsilon_0\,\overline{H} = 0$$

with the $\overline{H}$ field

$$\overline{H} = \hat{\theta}H_\theta(\rho, z)$$

and the $\overline{E}$ field

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[20] See Equation (3-42)

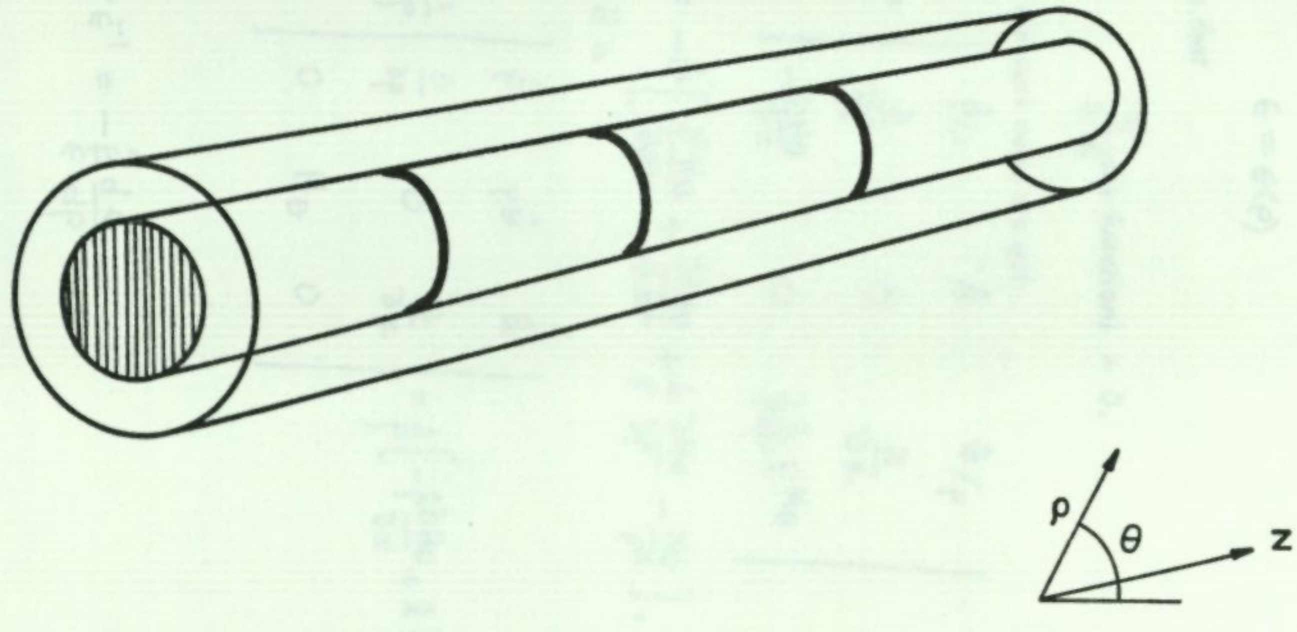

Figure 3. Conducting Cylinder with Circumferential Antennas

$$\bar{E} = \hat{\rho}\, E_\rho(\rho, z) + \hat{z}\, E_z(\rho, z).$$

We must take

$$\epsilon = \epsilon(\rho)$$

and require that

$$\frac{\partial}{\partial \theta}(\text{any function}) = 0.$$

Evaluating the curl curl, we get:

$$\nabla \times \nabla \times \bar{H} = \begin{vmatrix} \hat{\rho}/\rho & \hat{\theta} & \hat{z}/\rho \\ \frac{\partial}{\partial \rho} & 0 & \frac{\partial}{\partial z} \\ -\frac{\partial H_\theta}{\partial z} & 0 & \frac{1}{\rho}\frac{\partial}{\partial \rho}\rho H_\theta \end{vmatrix} \tag{3-54}$$

$$= -\hat{\theta}\left[\frac{\partial^2 H_\theta}{\partial \rho^2} + \frac{\partial^2 H_\theta}{\partial z^2} + \frac{1}{\rho}\frac{\partial H_\theta}{\partial \rho} - \frac{H_\theta}{\rho^2}\right].$$

The curl of $\bar{H}$ is

$$\nabla \times \bar{H} = \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\theta} & \hat{z} \\ \frac{\partial}{\partial \rho} & 0 & \frac{\partial}{\partial z} \\ 0 & H_\theta & 0 \end{vmatrix} = \frac{1}{\rho}\left[-\hat{\rho}\frac{\partial H_\theta}{\partial z} + \hat{z}\frac{\partial H_\theta}{\partial \rho}\right] \tag{3-55}$$

and

$$\epsilon \nabla \epsilon^{-1} = -\frac{\hat{\rho}}{\epsilon}\frac{d\epsilon}{d\rho}$$

so that

$$\epsilon \nabla \epsilon^{-1} \times \nabla \times \overline{H} = \begin{vmatrix} \hat{\rho} & \hat{\theta} & \hat{z} \\ -\frac{1}{\epsilon}\frac{d\epsilon}{d\rho} & 0 & 0 \\ -\frac{1}{\rho}\frac{\partial H_\theta}{\partial z} & 0 & \frac{\partial H_\theta}{\partial \rho} \end{vmatrix} = \hat{\theta}\frac{1}{\epsilon}\frac{d\epsilon}{d\rho}\frac{\partial H_\theta}{\partial \rho} \tag{3-56}$$

and the equation for H_θ is

$$\frac{\partial^2 H_\theta}{\partial \rho^2} + \frac{\partial^2 H_\theta}{\partial z^2} + \left(\frac{1}{\rho} - \frac{1}{\epsilon}\frac{d\epsilon}{d\rho}\right)\frac{\partial H_\theta}{\partial \rho} + \left(\epsilon_r - \frac{1}{\rho^2}\right) H_\theta = 0. \tag{3-57}$$

If $\epsilon_r = \epsilon_a$, we obtain the homogeneous form

$$\frac{\partial^2 H_\theta}{\partial \rho^2} + \frac{\partial^2 H_\theta}{\partial z^2} + \frac{1}{\rho}\frac{\partial H_\theta}{\partial \rho} + \left(\epsilon_a - \frac{1}{\rho^2}\right) H_\theta = 0. \tag{3-58}$$

From equation (3-2)

$$K_0 \nabla \times \overline{H} = i\omega\epsilon\overline{E}$$

the other fields are given in the form

$$E_\rho = \frac{-K_0}{i\omega\epsilon\rho}\frac{\partial H_\theta}{\partial z} \qquad E_z = \frac{K_0}{i\omega\epsilon\rho}\frac{\partial H_\theta}{\partial \rho}. \tag{3-59}$$

We assume that

$$H_\theta = A(z)\,B(\rho)$$

and derive the equations

$$\frac{1}{B}\frac{d^2 B}{d\rho^2} + \frac{1}{A}\frac{d^2 A}{dz^2} + \left(\frac{1}{\rho} - \frac{1}{\epsilon}\frac{d\epsilon}{d\rho}\right)\frac{1}{B}\frac{dB}{d\rho} + \left(\epsilon_r - \frac{1}{\rho^2}\right) = 0 \tag{3-60}$$

$$\frac{1}{A}\frac{d^2 A}{dz^2} = -\nu^2, \qquad A = e^{\pm i\nu z} \tag{3-61}$$

$$\frac{d^2 B_\nu}{d\rho^2} + \left(\frac{1}{\rho} - \frac{1}{\epsilon}\frac{d\epsilon}{d\rho}\right)\frac{dB_\nu}{d\rho} + \left(\epsilon_r - \frac{1}{\rho^2} - \nu^2\right) B_\nu = 0. \tag{3-62}$$

We will assume that H_θ is periodic in the z direction with period τ

$$H_\theta(z) = H_\theta(z+\tau) \tag{3-63}$$

so that μ can only take on the values

$$\mu = n\lambda_0/\tau .$$

The total solution then becomes

$$H_\theta = \sum_{n=-\infty}^{+\infty} e^{in\lambda_0/\tau z}\left[a_n^{I} B_n^{I}(\rho) + a_n^{II} B_n^{II}(\rho)\right] \tag{3-64}$$

where a_n^{I} and a_n^{II} are arbitrary constants.

B.4 E-polarized in the $\hat{\theta}$ Direction

Choose $\hat{\xi} = \hat{\theta}$

and consider solutions of (3-10a)

$$\nabla\times\nabla\times\overline{E} - \epsilon/\epsilon_0\,\overline{E} = 0$$

with the $\overline{E}$ field

$$\overline{E} = \hat{\theta}\,E_\theta(\rho, z)$$

and the $\overline{H}$ field

$$\overline{H} = \hat{\rho}\,H_\rho(\rho, z) + \hat{z}\,H_z(\rho, z).$$

We must take

$$\epsilon = \epsilon(\rho)$$

and require that

$$\frac{\partial}{\partial\theta}\,(\text{any function}) = 0 .$$

Evaluating the curl curl, we get

$$\nabla\times\nabla\times\overline{E} = \begin{vmatrix} \hat{\rho}/\rho & \hat{\theta} & \hat{z}/\rho \\ \frac{\partial}{\partial\rho} & 0 & \frac{\partial}{\partial z} \\ -\frac{\partial E_\theta}{\partial z} & 0 & \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho E_\theta \end{vmatrix} \tag{3-65}$$

$$\nabla\times\nabla\times\bar{E} = -\hat{\theta}\left[\frac{\partial^2 E_\theta}{\partial\rho^2} + \frac{\partial^2 E_\theta}{\partial z^2} + \frac{1}{\rho}\frac{\partial E_\theta}{\partial\rho} - \frac{E_\theta}{\rho^2}\right] \tag{3-66}$$

and the equation for E_θ becomes

$$\frac{\partial^2 E_\theta}{\partial\rho^2} + \frac{\partial^2 E_\theta}{\partial z^2} + \frac{1}{\rho}\frac{\partial E_\theta}{\partial\rho} - \frac{E_\theta}{\rho^2} + \epsilon_r E_\theta = 0. \tag{3-67}$$

For the homogeneous case $\epsilon_r = \epsilon_a$ this takes the same form as (3-58). The other fields are found from (3-2):

$$K_0 \nabla\times\bar{E} = -i\omega\mu\bar{H}$$

$$H_\rho = \frac{K_0}{i\omega\mu\rho}\frac{\partial E_\theta}{\partial z} \qquad H_z = \frac{-K_0}{i\omega\mu\rho}\frac{\partial E_\theta}{\partial\rho}. \tag{3-68}$$

To separate variables, we choose

$$E_\theta = C(\rho)\,D(z)$$

and obtain the equations

$$\frac{1}{C}\frac{d^2C}{d\rho^2} + \frac{1}{D}\frac{d^2D}{dz^2} + \frac{1}{\rho C}\frac{dC}{d\rho} - \frac{1}{\rho^2} + \epsilon_r = 0 \tag{3-69}$$

$$\frac{1}{D}\frac{d^2D}{dz^2} = -\nu^2, \qquad D = e^{\pm i\nu z} \tag{3-70}$$

$$\frac{d^2C_\nu}{d\rho^2} + \frac{1}{\rho}\frac{dC_\nu}{d\rho} + \left(\epsilon_r - \frac{1}{\rho^2} - \nu^2\right)C_\nu = 0 \tag{3-71}$$

where D has been assumed periodic in the z direction with period τ as done in the last section:

$$\nu = n\lambda_0/\tau.$$

The general solution for E_θ is

$$E_\theta(\rho,z) = \sum_{n=-\infty}^{+\infty} e^{in\pi_0/\ell\, z}\left[a_n^{I} C_n^{I}(\rho) + a_n^{II} C_n^{II}(\rho)\right] \tag{3-72}$$

where a_n^{I} and a_n^{II} are arbitrary constants.

C. Spherical Coordinates (Figure 4).

C.1 H-polarized in $\hat{\phi}$ Direction

We must now take $\hat{\xi}$ in the $\hat{\phi}$ direction,

$$\hat{\xi} = \hat{\phi}.$$

We now will consider the solutions of (3-9b)

$$\nabla\cdot\nabla\overline{H} - \epsilon\nabla\epsilon^{-1}\times\nabla\times\overline{H} + \epsilon/\epsilon_0\,\overline{H} = 0$$

with the restrictions that the only $\overline{H}$ field is

$$\overline{H} = \hat{\phi}\, H_\phi(r,\theta)$$

and the only $\overline{E}$ field components that exist are

$$\overline{E} = \hat{r}\,E_r(r,\theta) + \hat{\theta}\,E_\theta(r,\theta).$$

To satisfy these requirements, we may take

$$\epsilon = \epsilon(r)$$

and require that

$$\frac{\partial}{\partial\phi}(\text{any function}) = 0.$$

Again, a two-dimensional separable problem has been obtained. From the definition of the curl (3-11) with the identification

$$(q_1 q_2 q_3) = (r, \theta, \phi) \qquad (\hat{a}_1 \hat{a}_2 \hat{a}_3) = (\hat{r}\,\hat{\theta}\,\hat{\phi})$$

$$h_1 = 1\ ,\ h_2 = r,\ \ h_3 = r\sin\theta \tag{3-73}$$

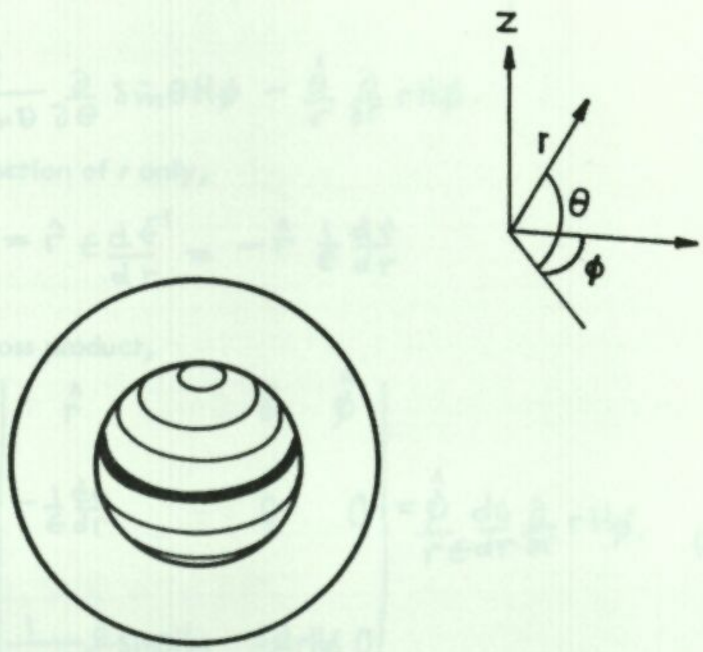

Figure 4. Conducting Sphere with Circumferential Antenna

the curl of $\overline{H}$ is

$$\nabla \times \overline{H} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & 0 \\ 0 & 0 & r\sin\theta H\phi \end{vmatrix} \tag{3-74}$$

$$= \frac{\hat{r}}{r\sin\theta}\frac{\partial}{\partial\theta}\sin\theta H\phi - \frac{\hat{\theta}}{r}\frac{\partial}{\partial r} rH\phi .$$

Since ϵ is a function of r only,

$$\epsilon\nabla\epsilon^{-1} = \hat{r}\,\epsilon\frac{d\epsilon^{-1}}{dr} = -\hat{r}\,\frac{1}{\epsilon}\frac{d\epsilon}{dr}$$

and taking the cross product,

$$\epsilon\nabla\epsilon^{-1} \times \nabla\times\overline{H} = \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{\phi} \\ -\frac{1}{\epsilon}\frac{d\epsilon}{dr} & 0 & 0 \\ \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\sin\theta H\phi & -\frac{1}{r}\frac{\partial}{\partial r} rH\phi & 0 \end{vmatrix} = \frac{\hat{\phi}}{r\epsilon}\frac{d\epsilon}{dr}\frac{\partial}{\partial r} rH\phi . \tag{3-75}$$

From the general equation for $\nabla\times\nabla\times\overline{H}$,

$$\nabla\times\nabla\times\overline{H} = \begin{vmatrix} \frac{\hat{r}}{r^2\sin\theta} & \frac{\hat{\theta}}{r\sin\theta} & \frac{\hat{\phi}}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & 0 \\ \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\sin\theta H\phi & -\frac{\partial}{\partial r} rH\phi & 0 \end{vmatrix} \tag{3-76}$$

$$= -\hat{\phi}\left[\frac{1}{r}\frac{\partial^2}{\partial r^2} rH\phi + \frac{1}{r^2}\frac{\partial}{\partial\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta H\phi\right] = -\nabla\cdot\nabla\overline{H}$$

and the equation for H_ϕ becomes

$$\frac{1}{r}\frac{\partial^2}{\partial r^2} rH_\phi + \frac{1}{r^2}\frac{\partial}{\partial\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta H_\phi - \frac{1}{r\epsilon}\frac{d\epsilon}{dr}\frac{\partial}{\partial r} rH_\phi + \frac{\epsilon}{\epsilon_0}H_\phi = 0 \quad (3\text{-}77)$$

or in another form,

$$\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial H_\phi}{\partial r} + \frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial H_\phi}{\partial\theta} - \frac{H_\phi}{\sin^2\theta}\right]$$

$$-\frac{1}{r\epsilon}\frac{d\epsilon}{dr}\frac{\partial}{\partial r} rH_\phi + \epsilon/\epsilon_0 H_\phi = 0. \quad (3\text{-}78)$$

If $\epsilon_r = \epsilon_a$, this becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial H_\phi}{\partial r} + \frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial H_\phi}{\partial\theta} - \frac{H_\phi}{\sin^2\theta}\right] + \epsilon_a H_\phi = 0, \quad (3\text{-}79)$$

the homogeneous case. Using (3-1) as done before,

$$K_0 \nabla\times\overline{H} = i\omega\epsilon\overline{E}$$

the other fields are given by

$$E_r = \frac{K_0}{i\omega\epsilon r}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta H_\phi$$

$$(3\text{-}80)$$

$$E_\theta = \frac{-K_0}{i\omega\epsilon r}\frac{\partial}{\partial r} rH_\phi$$

once H_ϕ has been determined. Equation (3-79) can be solved by separation of variables, letting

$$H_\phi = F(\theta)G(r)$$

and we obtain two ordinary differential equations:[21]

---

[21] We must have l an integer so that F will be finite at $\theta = 0$ and $\theta = \pi$.

$$-\frac{1}{F}\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{dF}{d\theta}-\frac{F}{\sin^2\theta}\right]=\ell(\ell+1) \tag{3-81}$$

$$F=P_\ell^{\,1}(\cos\theta)$$

$$\frac{d^2G_\ell}{dr^2}+\left(\frac{2}{r}-\frac{1}{\epsilon}\frac{d\epsilon}{dr}\right)\frac{dG_\ell}{dr}+\left(\epsilon_r-\frac{\ell(\ell+1)}{r^2}-\frac{1}{r\epsilon}\frac{d\epsilon}{dr}\right)G_\ell=0, \tag{3-82}$$

where $P_l^{\,1}$ $(\cos\,\theta)$ is an associated Legendre polynomial. The general solution for H_ϕ is then

$$H_\phi=\sum_\ell P_\ell^{\,1}(\cos\theta)\left[a_\ell^{\,I}G_\ell^{\,I}(r)+a_\ell^{\,II}G_\ell^{\,II}(r)\right] \tag{3-83}$$

where $a_l^{\,I}$ and $a_l^{\,II}$ are the arbitrary constants to be determined by boundary conditions.

C.2 <u>E-polarized in the $\hat{\phi}$ Direction</u>

In analogy to the H-polarized case, we seek solutions to the differential equation

$$\nabla\times\nabla\times\bar{E}+\epsilon/\epsilon_0\,\bar{E}=0$$

with the vector $\hat{\xi}$ taken to lie in the $\hat{\phi}$ direction:

$$\hat{\xi}=\hat{\phi}.$$

The only electric field is E_ϕ

$$\bar{E}=\hat{\phi}\,E_\phi(r,\theta)$$

and the only $\bar{H}$ fields are

$$\bar{H}=\hat{r}H_r(r,\theta)+\hat{\theta}\,H_\theta(r,\theta).$$

Then ϵ must have the functional form

$$\epsilon = \epsilon(r)$$

and the condition

$$\frac{\partial}{\partial \phi}(\text{any function}) = 0$$

must be satisfied. From these assumptions,

$$\nabla \times \nabla \times \overline{E} = \begin{vmatrix} \frac{\hat{r}}{r^2 \sin\theta} & \frac{\hat{\theta}}{r\sin\theta} & \frac{\hat{\phi}}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & 0 \\ \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\sin\theta E_\phi & -\frac{\partial}{\partial r} r E_\phi & 0 \end{vmatrix} \tag{3-84}$$

$$= -\hat{\phi}\left[\frac{1}{r}\frac{\partial^2}{\partial r^2} r E_\phi + \frac{1}{r^2}\frac{\partial}{\partial\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta E_\phi\right]$$

and the equation to be solved becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial E_\phi}{\partial r} + \frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial E_\phi}{\partial\theta} - \frac{E_\phi}{\sin^2\theta}\right] + \epsilon_r E_\phi = 0. \tag{3-85}$$

In the homogeneous case $\epsilon_r = \epsilon_a$, this also reduces to the form (3-79). From the definition of the curl and (3-2),

$$\nabla \times \overline{E} = -i\omega\mu \overline{H}$$

we obtain

$$H_r = \frac{-K_0}{i\omega\mu r \sin\theta}\frac{\partial}{\partial\theta}\sin\theta E_\phi \qquad H_\theta = \frac{1}{i\omega\mu r}\frac{\partial}{\partial r} r E_\phi, \tag{3-86}$$

giving all the nonzero fields. To solve (3-85) by separation of variables, let

$$E_\phi = L(r)M(\theta)$$

and two ordinary differential equations are obtained:[22]

$$\frac{1}{L}\frac{d}{dr}r^2\frac{dL}{dr} + \frac{\epsilon}{\epsilon_0}r^2 = -\frac{1}{M}\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{dM}{d\theta} - \frac{M}{\sin^2\theta}\right] = l(l+1)$$

$$M = P_l^1(\cos\theta)$$

$$\frac{d^2L_l}{dr^2} + \frac{2}{r}\frac{dL_l}{dr} + \left(\epsilon_r - \frac{l(l+1)}{r^2}\right)L_l = 0. \qquad (3\text{-}87)$$

The θ dependence is again obtained in Legendre polynomials, and in terms of the solutions L_l^{I} and L_l^{II} to (3-87), E_ϕ is given in general as

$$E_\phi(r,\theta) = \sum_l P_l^1(\cos\theta)\left[b_l^{I}L_l^{I}(r) + b_l^{II}L_l^{II}(r)\right].$$

As before, (3-87) and (3-88) represent the complete solution, with b_l^{I} and b_l^{II} arbitrary constants that must be chosen to satisfy the given boundary conditions.

D. Summary of Derived Equations

For each of the problems discussed in Sections 3A - 3C, a differential equation in the form of equation (2-1) has been derived. In each case, the solution to the appropriate wave propagation problem can be obtained only by obtaining the solution to this differential equation. A method for obtaining such a solution has been outlined in Chapter II, so that the only aspect of each problem still undefined is the choice of boundary symmetries compatible with the assumptions already made. The differential equations that have been derived are summarized below.

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22 See Equation (3-81) for reference.

D.1 Rectangular Coordinates

H-polarized in the $\hat{z}$ direction

$$\frac{d^2 L_n}{dx^2} - \frac{1}{\epsilon}\frac{d\epsilon}{dx}\frac{dL_n}{dx} + (\epsilon_r - n^2\frac{\lambda_0^2}{\tau^2})L_n = 0.$$

E-polarized in the $\hat{z}$ direction

$$\frac{d^2\alpha_n}{dx^2} + (\epsilon_r - n^2\frac{\lambda_0^2}{\tau^2})\alpha_n = 0.$$

D.2 Cylindrical Coordinates

H-polarized in the $\hat{z}$ direction

$$\frac{d^2 A_m}{d\rho^2} + \left(\frac{1}{\rho} - \frac{1}{\epsilon}\frac{d\epsilon}{d\rho}\right)\frac{dA_m}{d\rho} + (\epsilon_r - \frac{m^2}{\rho^2})A_m = 0.$$

E-polarized in the $\hat{z}$ direction

$$\frac{d^2 M_m}{d\rho^2} + \frac{1}{\rho}\frac{dM_m}{d\rho} + (\epsilon_r - \frac{m^2}{\rho^2})M_m = 0.$$

H-polarized in the $\hat{\theta}$ direction

$$\frac{d^2 B_m}{d\rho^2} + \left(\frac{1}{\rho} - \frac{1}{\epsilon}\frac{d\epsilon}{d\rho}\right)\frac{dB_m}{d\rho} + (\epsilon_r - \frac{1}{\rho^2} - n^2\frac{\lambda_0^2}{\tau^2})B_m = 0.$$

E-polarized in the $\hat{\theta}$ direction

$$\frac{d^2 C_m}{d\rho^2} + \frac{1}{\rho}\frac{dC_m}{d\rho} + (\epsilon_r - \frac{1}{\rho^2} - n^2\frac{\lambda_0^2}{\tau^2})C_m = 0.$$

D.3 Spherical Coordinates

H-polarized in the $\hat{\phi}$ direction

$$\frac{d^2 G_l}{dr^2} + \left(\frac{2}{r} - \frac{1}{\epsilon}\frac{d\epsilon}{dr}\right)\frac{dG_l}{dr} + (\epsilon_r - \frac{l(l+1)}{r^2} - \frac{1}{r\epsilon}\frac{d\epsilon}{dr})G_l = 0.$$

E-polarized in the $\hat{\phi}$ direction

$$\frac{d^2 L_l}{dr^2} + \frac{2}{r}\frac{dL_l}{dr} + \left(\epsilon_r - \frac{l(l+1)}{r^2}\right) L_l = 0.$$

CHAPTER IV

SOLVABLE PROBLEMS

A. Plane

A.1 H-polarized in the $\hat{z}$ Direction

The differential equation to be solved is (3-22)

$$\frac{d^2 L_n}{dx^2} - \frac{1}{\epsilon_r}\frac{d\epsilon_r}{dx}\frac{dL_n}{dx} + \left(\epsilon_r - n^2\frac{\lambda_0^2}{\tau^2}\right)L_n = 0$$

subject to the requirements that

$$\begin{aligned}
&1.\ \frac{\partial}{\partial z}\ \text{(any function)} = 0\\
&2.\ \text{all fields are periodic in } y \text{ with period } \tau \\
&3.\ \text{the only } \overline{H} \text{ field component is } H_z\\
&4.\ \text{the only } \overline{E} \text{ field components are } E_x \text{ and } E_y\\
&5.\ \epsilon = \epsilon(x)\,.
\end{aligned} \qquad (4\text{-}1)$$

These conditions can be satisfied by placing the conducting boundary at the plane $x = 0$ with H-field strip antennas (magnetic strip sources) in the z direction, a distance τ apart so that

$$E_y(x=0) = \begin{cases} 0 & \text{scattering} \\ \sum_n \Gamma_n e^{in\frac{\lambda_0}{\tau}y} & \text{radiation} \end{cases} \qquad (4\text{-}2)$$

where the Γ_n are determined by the choice of antenna. From (3-18)

$$E_y = -\frac{k_0}{i\omega\epsilon}\frac{\partial H_z}{\partial x}$$

and from the orthogonality relations derived in Appendix 1,

$$(E_y)_n = \frac{-K_o}{i\omega\epsilon}\left(\frac{\partial H_z}{\partial x}\right)_n$$

so that the boundary conditions (4-1) apply to the derivative of L_n. We can let $L_n = F_n$ [See equation (2-1)] and Case II [equation (2-5)] will hold when obtaining a solution. From (2-10), (2-11), (2-26), and (2-37), we must demand that

$$X_1[n, x, \epsilon_r(x)]_{x=0} = X_1[n, x, \epsilon_a]_{x=0}$$

$$X_1[n, x, \epsilon_r(x)]_{x=x_e} = X_1[n, x, \epsilon_a]_{x=x_e} \qquad (4\text{-}3)$$

$$K_e = \bar{K}_e .$$

By comparing (2-1) and (3-22), we have

$$X_1[n, x, \epsilon_r(x)] = -\frac{1}{\epsilon_r}\frac{d\epsilon_r}{dx} \qquad X_2[n, x, \epsilon_r(x)] = \epsilon_r - n^2\frac{\lambda_o^2}{\tau^2}$$

$$X_1[n, x, \epsilon_a] = 0 \qquad X_2[n, x, \epsilon_a] = \epsilon_a - n^2\frac{\lambda_o^2}{\tau^2} \qquad (4\text{-}4)$$

and from (2-9) and (3-18), $k_e = \bar{k}_e$ if

$$\epsilon_r(x_e) = \epsilon_a .$$

Combining all of these conditions, $\epsilon_r(x)$ must satisfy all the following relations:

$$\epsilon_r(x_e) = \epsilon_a \qquad \left.\frac{d\epsilon_r}{dx}\right|_{x=0} = 0 \qquad \left.\frac{d\epsilon_r}{dx}\right|_{x=x_e} = 0 . \qquad (4\text{-}5)$$

From (4-4)

$$\phi(x) = \epsilon_r - n^2\frac{\lambda_0^2}{\lambda^2} - \frac{1}{4}\left(\frac{1}{\epsilon_r}\frac{d\epsilon_r}{dx}\right)^2 + \frac{1}{2}\frac{d}{dx}\frac{1}{\epsilon_r}\frac{d\epsilon_r}{dx}$$

$$\phi_0(x) = \epsilon_a - n^2\frac{\lambda_0^2}{\lambda^2}$$

$$P(x) = \sqrt{\frac{\epsilon_r(x)}{\epsilon_r(d)}} \tag{4-6}$$

$$G(x) = \sqrt{\frac{\epsilon_r(d)}{\epsilon_r(x)}}$$

$$\Gamma(x) = (\bar{F}_n^{\,I})^2$$

and from equation (3-18a), with $\epsilon_r = \epsilon_a$,

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + \epsilon_a H_z = 0$$

we have

$$\bar{F}_n^{\,I} = e^{in\frac{\lambda_0}{\lambda}y}\left[a_n^I e^{i\sqrt{\epsilon_a - n^2\frac{\lambda_0^2}{\lambda^2}}\,x} + a_n^{II} e^{-i\sqrt{\epsilon_a - n^2\frac{\lambda_0^2}{\lambda^2}}\,x}\right]. \tag{4-7}$$

Equations (2-41) - (2-43) now give the solution immediately.

A.2 E-polarized in the $\hat{z}$ direction

The differential equation to be treated is (3-29),

$$\frac{d^2\alpha_n}{dx^2} + \left(\epsilon_r - n^2\frac{\lambda_0^2}{\lambda^2}\right)\alpha_n = 0$$

subject to the requirements (4-1) with the roles of E and H reversed. These conditions can be satisfied by placing the conducting boundary at the plane x = o with E-field strip antennas (electric strip sources) in the z direction a distance τ apart so that

$$E_z(x=0) \quad = \begin{cases} 0 & \text{scattering} \\ \sum_n \lambda_n e^{in\frac{\lambda_0}{\tau}y} & \text{radiation} \end{cases} \tag{4-8}$$

where the λ_η are determined by the choice of antenna. Since the boundary conditions (4-8) are applied to the variable in equation (3-29), we can let $F_n = a_n$, and Case I [equation (2-4)] will apply. The conditions (4-3) must again be satisfied, and by comparing (2-1) and (3-29) we have

$$\begin{aligned} X_1[n, x, \epsilon_r(x)] &= X_1[n, x, \epsilon_a] = 0 \\ X_2[n, x, \epsilon_r(x)] &= \epsilon_r - n^2\frac{\lambda_0^2}{\tau^2} \\ X_2[n, x, \epsilon_a] &= \epsilon_a - n^2\frac{\lambda_0^2}{\tau^2} \end{aligned} \tag{4-9}$$

From (2-9) and (3-30), we will have

$$k_e = \overline{k_e}$$

automatically, so that none of the conditions (4-3) will restrict ϵ_r at any point. From (4-9),

$$\begin{aligned} \phi(x) &= \epsilon_r - n^2\frac{\lambda_0^2}{\tau^2} \\ \phi_0(x) &= \epsilon_a - n^2\frac{\lambda_0^2}{\tau^2} \\ P(x) &= G(x) = 1 \\ \Gamma(x) &= (F_n^{I})^2 \end{aligned} \tag{4-10}$$

and for $\epsilon_r = \epsilon_a$, $\overline{F}_n^{\,1}$ is given by (4-7).

B. Cylinder

B.1 H-polarized in the $\hat{z}$ Direction

The differential equation to be solved now is (3-45),

$$\frac{d^2A_m}{d\rho^2} + \left(\frac{1}{\rho} - \frac{1}{\epsilon}\frac{d\epsilon}{d\rho}\right)\frac{dA_m}{d\rho} + \left(\epsilon_r - \frac{m^2}{\rho^2}\right)A_m = 0$$

subject to the requirements that

1. $\frac{\partial}{\partial z}$ (any function) $= 0$
2. the only H-field component is H_z (4-11)
3. the only E-field components are E_ρ and E_θ .

These conditions can be satisfied by placing a conducting cylinder of radius ρ_i in the z direction with an H-field strip antenna (magnetic strip source) in the z direction so that

$$E_\theta(\rho=\rho_i) = \begin{cases} = 0 & \text{scattering} \\ \sum_m S_m e^{im\theta} & \text{radiation} \end{cases} \tag{4-12}$$

where the S_m are determined by the antenna. From (3-41)

$$E_\theta = \frac{-k_0}{i\omega\epsilon}\frac{\partial H_z}{\partial \rho}$$

and from the orthogonality relations in Appendix 1,

$$(E_\theta)_m = \frac{-k_0}{i\omega\epsilon}\left(\frac{\partial H_z}{\partial \rho}\right)_m$$

so that the boundary conditions (4-12) apply to the derivative of A_m. We can let $F_m = A_m$ [from equation (2-1)] and Case II will hold. Conditions (4-3) must also hold, and by comparing (2-1) and (3-45), we have

$$X_1[n,\rho,\epsilon_r(\rho)] = \frac{1}{\rho} - \frac{1}{\epsilon}\frac{d\epsilon}{d\rho} \qquad X_1[n,\rho,\epsilon_a] = \frac{1}{\rho}$$

$$X_2[n,\rho,\epsilon_r(\rho)] = \epsilon_r - \frac{m^2}{\rho^2} \qquad X_2[n,\rho,\epsilon_a] = \epsilon_a - \frac{m^2}{\rho^2} \tag{4-13}$$

and so that $k_e = k_e{}'$,

$$\epsilon_r(\rho_e) = \epsilon_a .$$

Combining these conditions, we must have

$$\epsilon_r(\rho_e) = \epsilon_a$$

$$\left.\frac{d\epsilon_r}{d\rho}\right|_{\rho_i} = 0 \qquad \left.\frac{d\epsilon_r}{d\rho}\right|_{\rho_e} = 0 . \tag{4-14}$$

From (4-13),

$$\phi(\rho) = \epsilon_r - \frac{m^2}{\rho^2} - \frac{1}{4}\left(\frac{1}{\rho} - \frac{1}{\epsilon}\frac{d\epsilon}{d\rho}\right)^2 - \frac{1}{2}\frac{d}{d\rho}\left(\frac{1}{\rho} - \frac{1}{\epsilon}\frac{d\epsilon}{d\rho}\right)$$

$$\phi_o(\rho) = \epsilon_a - \frac{m^2 - 1/4}{\rho^2} \tag{4-15}$$

$$P(\rho) = \sqrt{\frac{\epsilon_r(\rho)}{\epsilon_r(\alpha)}}$$

$$Q(\rho) = \left(\frac{\alpha}{\rho}\right)^{-1}\sqrt{\frac{\epsilon_r(\alpha)}{\epsilon_r(\rho)}}$$

$$\Gamma(\rho) = \frac{\rho}{\alpha}\left(F_n^{I}\right)^2$$

and from equation (3-42) with $\epsilon_r = \epsilon_a$,

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial H_z}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 H_z}{\partial\theta^2} + \epsilon_a H_z = 0$$

we have

$$\bar{F}_m^{\,I} = e^{im\theta}\left[b_m^{\,I} J_m(\rho) + b_m^{\,II} Y_m(\rho) \right] \tag{4-16}$$

where J_m and Y_m are Bessel functions. Equations (2-41) - (2-43) now give the solution.

B.2 <u>E-polarized in the $\hat{z}$ Direction</u>

The equation to be solved is (3-52)

$$\frac{d^2 M_m}{d\rho^2} + \frac{1}{\rho}\frac{dM_m}{d\rho} + \left(\epsilon_r - \frac{m^2}{\rho^2}\right) M_m = 0$$

subject to the requirements (4-11) with E and H interchanged. The conditions can be satisfied by placing a conducting cylinder of radius ρ_i in the z direction with an $\bar{E}$-field strip antenna (electric strip source) in the z direction, so that

$$E_z(\rho=\rho_i) = \begin{cases} = 0 & \text{scattering} \\ \sum_m t_m e^{im\theta} & \text{radiation} \end{cases} \tag{4-17}$$

where the t_m are determined by the antenna. Since (4-17) applies directly to the field variable in (3-52) we can take $F_m = M_m$ and apply Case I. Conditions (4-3) must hold, and by comparing (2-1) and (3-52),

$$X_1[n, \rho, \epsilon_r(\rho)] = X_1[n, \rho, \epsilon_a] = 1/\rho$$

$$X_2[n, \rho, \epsilon_r(\rho)] = \epsilon_r - m^2/\rho^2 \qquad X_2[n, \rho, \epsilon_a] = \epsilon_a - m^2/\rho^2 \tag{4-18}$$

and the condition $k_e = \bar{k}_e$ is automatically satisfied from (3-49).

We see that $\epsilon_r(\rho)$ is not restricted by any of these conditions. From (4-18),

$$
\begin{aligned}
\phi &= \epsilon_r - \frac{m^2 - 1/4}{\rho^2} \\
\phi_0 &= \epsilon_a - \frac{m^2 - 1/4}{\rho^2} \\
P(\rho) &= 1 \\
G(\rho) &= \rho/\alpha \\
\Gamma(\rho) &= \rho/\alpha \left(\bar{F}_n^{I}\right)^2
\end{aligned}
\tag{4-19}
$$

and when $\epsilon_r = \epsilon_a$, the solution to (3-52) is given by (4-16).

B.3 <u>H-polarized in the $\hat{\theta}$ Direction</u>

The differential equation that has to be solved is (3-62)

$$\frac{d^2 B_\nu}{d\rho^2} + \left(\frac{1}{\rho} - \frac{1}{\epsilon}\frac{d\epsilon}{d\rho}\right)\frac{dB_\nu}{d\rho} + \left(\epsilon_r - \frac{1}{\rho^2} - \nu^2\right)B_\nu = 0 \tag{4-20}$$

subject to

1. $\frac{\partial}{\partial\theta}$ (any function) = 0
2. the only H field component is H_θ
3. the only E field components are E_ρ and E_z
4. H_θ must be periodic in the z direction with period τ.

These conditions can be satisfied by placing a conducting cylinder of radius ρ_i in the z direction, with H-field strip antennas in the θ direction a distance τ apart in the z direction. The boundary conditions will be

$$E_z(\rho = \rho_i) = \begin{cases} 0 & \text{scattering} \\ \sum_m \beta_m e^{im\theta} & \text{radiation} \end{cases} \tag{4-21}$$

where β_m determines the choice of antenna. From (3-59) and the orthogonality relations in Appendix 1,

$$(E_z)_\mu = \frac{K_0}{i\omega\epsilon\rho}\left(\frac{\partial H_\theta}{\partial\rho}\right)_\mu$$

so that we can let $F_\mu = B_\mu$ and apply Case II, equation (2-5). Conditions (4-3) must hold, and by comparing (2-1) and (3-62),

$$\begin{aligned} &X_1[n,\rho,\epsilon_r(\rho)] = 1/\rho - 1/\epsilon \, d\epsilon/d\rho \qquad X_1[n,\rho,\epsilon_a] = 1/\rho \\ &X_2[n,\rho,\epsilon_r(\rho)] = \epsilon_r - 1/\rho^2 - \mu^2 \\ &X_2[n,\rho,\epsilon_a] = \epsilon_a - 1/\rho^2 - \mu^2 \end{aligned} \tag{4-22}$$

and to make $k_e = \overline{k}_e$, we must take

$$\epsilon_r(\rho_e) = \epsilon_a$$

from (3-59). Combining the restrictions, we obtain

$$\left.\frac{d\epsilon_r}{d\rho}\right|_{\rho_i} = 0 \qquad \left.\frac{d\epsilon_r}{d\rho}\right|_{\rho_e} = 0 \qquad \epsilon_r(\rho_e) = \epsilon_a . \tag{4-23}$$

From (4-22),

$$\begin{aligned} &\phi = \epsilon_r - 1/\rho^2 - \mu^2 - 1/4\left(\frac{1}{\rho} - \frac{1}{\epsilon}\frac{d\epsilon}{d\rho}\right)^2 - \frac{1}{2}\frac{d}{d\rho}\left(\frac{1}{\rho} - \frac{1}{\epsilon}\frac{d\epsilon}{d\rho}\right) \\ &\phi_0 = \epsilon_r - \frac{3/4}{\rho^2} - \mu^2 \\ &P(\rho) = \sqrt{\frac{\epsilon_r(\rho)}{\epsilon_r(\alpha)}} \\ &G(\rho) = \frac{\rho}{\alpha}\sqrt{\frac{\epsilon_r(\alpha)}{\epsilon_r(\rho)}} \\ &\Gamma(\rho) = (F_n{}^{I})^2 \rho/\alpha \end{aligned} \tag{4-24}$$

and from equation (3-58), we have

$$\overline{F}_n^{\,I} = e^{i\mu\frac{\lambda_0}{2}z}\left[a_\mu J_1\left(\sqrt{\epsilon_2-\mu^2}\,\rho\right) + b_\mu Y_1\left(\sqrt{\epsilon_2-\mu^2}\,\rho\right)\right] \tag{4-25}$$

where J_1 and Y_1 are Bessel functions.

B.4 E-polarized in the $\hat{\theta}$ direction

The differential equation is (3-71)

$$\frac{d^2 C_\mu}{d\rho^2} + \frac{1}{\rho}\frac{dC_\mu}{d\rho} + \left(\epsilon_r - 1/\rho^2 - \mu^2\right)C_\mu = 0$$

subject to conditions (4-20) with E and H interchanged. The conditions can be satisfied by placing a conducting cylinder of radius ρ_i in the z direction with E-field strip antennas in the θ direction a distance τ apart in the z direction. The boundary conditions will be

$$E_\theta = \begin{cases} 0 & \text{scattering} \\ \sum_m \gamma_m e^{im\theta} & \text{radiation} \end{cases} \tag{4-26}$$

where γ_m characterizes the antenna. We can let $C_\mu = F_\mu$ [see equation (2-1)] since the boundary conditions apply directly to the variable. Comparing (2-1) and (3-71),

$$X_1[n, \rho, \epsilon_r(\rho)] = X_1[n, \rho, \epsilon_2] = 1/\rho$$

$$X_2[n, \rho, \epsilon_r(\rho)] = \epsilon_r - 1/\rho^2 - \mu^2 \tag{4-27}$$

$$X_2[n, \rho, \epsilon_2] = \epsilon_2 - 1/\rho^2 - \mu^2$$

and $k_e = \overline{k}_e$ automatically from (3-68). There are no restrictions on $\epsilon_r(\rho)$.

From (4-26),

$$\begin{aligned}
\phi &= \epsilon_r - \frac{3/4}{\rho^2} - \mu^2 \\
\phi_o &= \epsilon_a - \frac{3/4}{\rho^2} - \mu^2 \\
P(\rho) &= 1 \\
G(\rho) &= \rho/\alpha \\
\Gamma(\rho) &= \rho/\alpha \left(\overline{F}_n^{\,I}\right)^2
\end{aligned} \tag{4-28}$$

and $\overline{F}_n^{\,I}$ is given by (4-25).

C. <u>Sphere</u>

C.1 <u>H-polarized in the $\hat{\phi}$ direction</u>

The appropriate differential equation is (3-82)

$$\frac{d^2 G_\ell}{dr^2} + \left(\frac{2}{r} - \frac{1}{\epsilon}\frac{d\epsilon}{dr}\right)\frac{dG_\ell}{dr} + \left(\epsilon_r - \frac{\ell(\ell+1)}{r^2} - \frac{1}{r\epsilon}\frac{d\epsilon}{dr}\right)G_\ell = 0$$

subject to requirements that

1. $\frac{\partial}{\partial \phi}$ (any function) $= 0$
2. the only $\overline{H}$ field component is H_ϕ (4-29)
3. the only $\overline{E}$ field components are E_r and E_θ
4. $\epsilon = \epsilon(r)$.

These conditions can be satisfied by placing a conducting sphere of radius r_i about the origin, with a circumferential H-field strip antenna in the ϕ direction on the surface. Therefore, we will have

$$E_\theta(r=r_i) = \begin{cases} 0 & \text{scattering} \\ \sum_\ell P_\ell^1(\cos\theta) g_\ell & \text{radiation} \end{cases} \tag{4-29a}$$

where the g_l specify the antenna. From (3-80) and the Appendix 1 orthogonality relations,

$$(E_\theta)_l = \frac{-k_0}{i\omega\epsilon r}\left(\frac{\partial}{\partial r} rH_\phi\right)_l$$

so that the variable F in equation (2-1) must be chosen equal to the product rG_l. Under this transformation, (3-82) becomes

$$\frac{d^2F_l^I}{dr^2} - \frac{1}{\epsilon}\frac{d\epsilon}{dr}\frac{dF_l^I}{dr} + \left(\epsilon_r - \frac{l(l+1)}{r^2}\right)F_l^I = 0 \qquad (4\text{-}30)$$

where Case II now applies [equation (2-5)]. Conditions (4-3) must hold, and by comparing (2-1) and (4-30) we see that

$$X_1[n, r; \epsilon_r(r)] = -\frac{1}{\epsilon}\frac{d\epsilon}{dr} \qquad X_1[n, r; \epsilon_a] = 0$$

$$X_2[n, r, \epsilon_r(r)] = \epsilon_r - \frac{l(l+1)}{r^2} \qquad (4\text{-}31)$$

$$X_2[n, r, \epsilon_a] = \epsilon_a - \frac{l(l+1)}{r^2}$$

and from (3-80) we must have

$$\epsilon_r(r_e) = \epsilon_a$$

so that $k_e = \bar{k}_e$. In summary, the relations that must be satisfied are

$$\left.\frac{d\epsilon_r}{dr}\right|_{r_i} = 0 \qquad \left.\frac{d\epsilon_r}{dr}\right|_{r_e} = 0 \qquad \epsilon_r(r_e) = \epsilon_a\,. \qquad (4\text{-}32)$$

From (4-30)

$$\phi(r) = \epsilon_r - \frac{\ell(\ell+1)}{r^2} - \frac{1}{4}\left(\frac{1}{\epsilon}\frac{d\epsilon}{dr}\right)^2 - \frac{1}{2}\frac{d}{dr}\frac{1}{\epsilon}\frac{d\epsilon}{dr}$$

$$\phi_0(r) = \epsilon_a - \frac{\ell(\ell+1)}{r^2}$$

$$P(r) = \sqrt{\frac{\epsilon_r(r)}{\epsilon_r(a)}} \tag{4-33}$$

$$G(r) = \sqrt{\frac{\epsilon_r(a)}{\epsilon_r(r)}}$$

$$\Gamma(r) = \left(\bar{F}_n^{I}\right)^2$$

and from equations (3-79) and (4-30), we have for $\epsilon_r = \epsilon_a$,

$$\frac{d^2\bar{F}_\ell^{I}}{dr^2} + \left(\epsilon_a - \frac{\ell(\ell+1)}{r^2}\right)\bar{F}_\ell^{I} = 0$$

with the solution

$$r\bar{G}_\ell = \bar{F}_\ell^{I} = \sqrt{r}\, P_\ell^1(\cos\theta)\left[s_\ell J_{\sqrt{\frac{1}{4}-\ell(\ell+1)}}(\sqrt{\epsilon_a}\, r) + t_\ell Y_{\sqrt{\frac{1}{4}-\ell(\ell+1)}}(\sqrt{\epsilon_a}\, r)\right] \tag{4-34}$$

where J and Y are Bessel functions.

C.2 E-polarized in the $\hat{\phi}$ Direction

The differential equation to be solved is (3-87)

$$\frac{d^2 L_\ell}{dr^2} + \frac{2}{r}\frac{dL_\ell}{dr} + \left(\epsilon_r - \frac{\ell(\ell+1)}{r^2}\right)L_\ell = 0$$

subject to conditions (4-29) with E and H interchanged. The conditions can be satisfied by placing a conducting sphere of radius r_i about the origin, with a circumferential E-field strip antenna in the ϕ direction on the surface. The boundary condition will be

$$E_\phi(r=r_i) = \begin{cases} 0 & \text{scattering} \\ \sum_\ell h_\ell P_\ell^1(\cos\theta) & \text{radiation} \end{cases} \tag{4-35}$$

where h_l determines the antenna. Since the boundary condition (4-35) applies to L_1 directly, we may choose $F_1 = L_1$, and Case I applies. Conditions (4-3) again must be satisfied, and by comparing (4-35) and (2-1) we find that

$$X_1[n, r, \epsilon_r(r)] = 2/r \qquad X_2[n, r, \epsilon_r(r)] = \epsilon_r - \frac{\ell(\ell+1)}{r^2}$$
$$X_1[n, r, \epsilon_a] = 2/r \qquad X_2[n, r, \epsilon_a] = \epsilon_a - \frac{\ell(\ell+1)}{r^2} \tag{4-36}$$

and $k_e = \overline{k}_e$ automatically, from (3-86). Equations (4-3) are satisfied identically, and do not restrict the choice of ϵ_r.

From (4-87)

$$\phi(r) = \epsilon_r - \frac{\ell(\ell+1)+2}{r^2}$$

$$\phi_0(r) = \epsilon_a - \frac{\ell(\ell+1)+2}{r^2}$$

$$P(r) = 1$$

$$G(r) = r^2/\alpha^2$$

$$\Gamma(r) = r^2/\alpha^2 \left(\overline{F}_\ell^{\,I}\right)^2$$

and from (3-29), $\overline{F}_1^{\,I}$ is also given by (4-34).

CHAPTER V

PARTIALLY SOLVABLE PROBLEMS

The wedge and cone are considered in this chapter, and it is shown that complete solutions cannot be obtained using the boundary value techniques of this paper. The wedge configuration is shown in Figure 5 and the cone in Figure 6.

A. Wedge

In all the prior problems in cylindrical coordinates, ϵ_r has been a function of ρ and the boundary conditions have been specified by choosing the value of one of the fields at a constant radius. We have shown in Appendix 1 that for $\epsilon_r(\rho)$ the θ dependent functions are orthogonal for different values of the spearation constant m. We will now consider the difficulties encountered when either of these conditions are changed. Consider first the H-field polarized in the $\hat{z}$ direction. If ϵ_r is allowed to be a function of both ρ and θ, (3-37) must be modified to the form

$$\epsilon\nabla\epsilon^{-1}\times\nabla\times\overline{H} = \begin{vmatrix} \hat{\rho} & \hat{\theta} & \hat{z} \\ -\frac{1}{\epsilon}\frac{\partial\epsilon}{\partial\rho} & -\frac{1}{\epsilon\rho}\frac{\partial\epsilon}{\partial\theta} & 0 \\ \frac{1}{\rho}\frac{\partial H_z}{\partial\theta} & -\frac{\partial H_z}{\partial\rho} & 0 \end{vmatrix} = \hat{z}\left[\frac{1}{\epsilon}\frac{\partial\epsilon}{\partial\rho}\frac{\partial H_z}{\partial\rho} + \frac{1}{\rho^2\epsilon}\frac{\partial\epsilon}{\partial\theta}\frac{\partial H_z}{\partial\theta}\right]. \quad (5\text{-}1)$$

The expression for the $\nabla\times\nabla\times\overline{H}$ is unchanged, so from (3-39) the partial differential equation for H_z is

$$\rho\frac{\partial}{\partial\rho}\rho\frac{\partial H_z}{\partial\rho} + \frac{\partial^2 H_z}{\partial\theta^2} - \frac{\rho^2}{\epsilon}\frac{\partial\epsilon}{\partial\rho}\frac{\partial H_z}{\partial\rho} - \frac{1}{\epsilon}\frac{\partial\epsilon}{\partial\theta}\frac{\partial H_z}{\partial\theta} + \frac{\epsilon\rho^2}{\epsilon_0}H_z = 0. \quad (5\text{-}2)$$

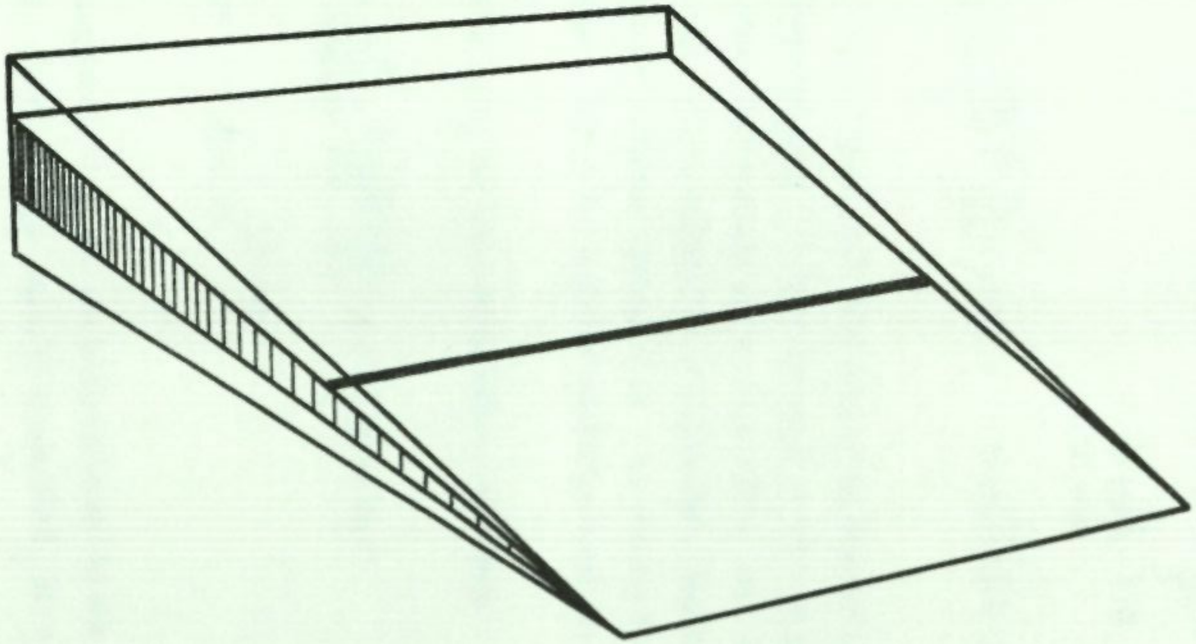

Figure 5. Conducting Wedge with Axial Antenna

This equation is separable if $\epsilon = \epsilon_1 = \epsilon_0 \dfrac{\epsilon_r(\theta)}{\rho^2}$ or if $\epsilon = \epsilon_2 = \epsilon_0\,\epsilon_r(\rho)$. For $\epsilon = \epsilon_1$, and letting $H_z = A(\rho)\,B(\theta)$, we obtain

$$\frac{\rho}{A}\frac{d}{d\rho}\rho\frac{dA}{d\rho} + \frac{2\rho}{A}\frac{dA}{d\rho} = (ik-\lambda)^2 + 2(ik-\lambda) \qquad A = \frac{1}{\rho^\lambda}e^{ik\ln\rho} \tag{5-3}$$

λ real, > 0

K real

$$\frac{1}{B}\frac{d^2B}{d\theta^2} - \frac{1}{B\epsilon_r(\theta)}\frac{d\epsilon_r(\theta)}{d\theta}\frac{dB}{d\theta} + \epsilon_r(\theta) = -(ik-\lambda)^2 - 2(ik-\lambda) \tag{5-4}$$

where K and λ are not quantized. The form of the separation constant has been determined by requiring A to be finite at $\rho = \infty$. A similar requirement cannot be made at the origin since $\epsilon(\rho = 0) = \infty$. If an attempt is made to set $\lambda = 0$, an inconsistent result is obtained. The functions $A(\rho)$ are not orthogonal for different choices of $(ik - \lambda)$, and the functions $B(\theta)$ will not be orthogonal either for different separation constants if $\epsilon_r(\theta)$ is allowed to exist.

If we let $\epsilon = \epsilon_2$ and separate as before, we obtain

$$\frac{\rho}{A}\frac{d}{d\rho}\rho\frac{dA}{d\rho} - \frac{\rho^2}{A\epsilon_r(\rho)}\frac{d\epsilon_r(\rho)}{d\rho}\frac{dA}{d\rho} + \rho^2\epsilon_r(\rho) = m^2 \tag{5-5}$$

$$\frac{1}{B}\frac{d^2B}{d\theta^2} = -m^2 \qquad B = e^{\pm im\theta} \tag{5-6}$$

The separation constant in this case has been quantized by the requirement $B(\theta) = B(\theta + 2\pi)$. For the wedge, we wish to specify E_ρ at $\pm\,\theta_0$, and for $\epsilon = \epsilon_2$,

$$E(\rho,\theta_0) = F(\rho) = \sum_m e^{im\theta_0} a_m^{\mp} A_m^{\mp}(\rho)$$

$$E(\rho,-\theta_0) = G(\rho) = \sum_m e^{-im\theta_0} a_m^{\mp} A_m^{\mp}(\rho) \tag{5-7}$$

where $A^{II}{}_m(\rho)$ representing the irregular solution has been dropped since $\rho = 0$ is included in the region of interest. A new difficulty is now encountered, however. When $F(\rho)$ and $G(\rho)$ have been specified, the coefficients $a_m^I e^{im\theta_0}$ and $a_m^I e^{-im\theta_0}$ cannot be determined. In order to evaluate the constants, the functions $A^I_m(\rho)$ and $A^I_{m'}(\rho)$ must be orthogonal over some range of integration of ρ:

$$\int_A^B A_m^I(\rho)\, A_{m'}^I(\rho)\, W(\rho)\, d\rho \overset{?}{=} 0 \qquad m \neq m'$$

$$\overset{?}{=} 1 \qquad m = m' \tag{5-8}$$

[where some known weighting function $W(\rho)$ might be included], so that we may write

$$\int_A^B d\rho\, F(\rho)\, A^I_{m'}(\rho)\, W(\rho) = e^{im'\theta_0} a^I_{m'}$$

$$\int_A^B d\rho\, G(\rho)\, A^I_{m'}(\rho)\, W(\rho) = e^{-im'\theta_0} a^I_{m'} \,. \tag{5-9}$$

However, the functions $A_m^I(\rho)$ are unknowns, and the integrals (5-9) cannot be evaluated even if (5-8) can be shown to exist. For $\epsilon = \epsilon_1$, the functions $A(\rho)$ are not orthogonal, and the same difficulty is met in applying the boundary conditions.

Consider now the E-field polarized in the $\hat{z}$ direction. The partial differential equation for E_z is

$$\rho \frac{\partial}{\partial \rho} \rho \frac{\partial H_z}{\partial \rho} + \frac{\partial^2 H_z}{\partial \theta^2} + \frac{\epsilon \rho^2}{\epsilon_0} H_z = 0 \,. \tag{5-10}$$

This equation can be separated if $\epsilon = \epsilon_1 = \epsilon_0 \dfrac{\epsilon_r(\theta)}{\rho^2}$ or if $\epsilon = \epsilon_2 = \epsilon_0 \epsilon_r(\rho)$:

$$\rho \frac{\partial}{\partial \rho} \rho \frac{\partial H_z}{\partial \rho} + \frac{\partial^2 H_z}{\partial \theta^2} + \epsilon_r(\theta) H_z = 0 \qquad \epsilon = \epsilon_1 \tag{5-11}$$

$$\rho \frac{\partial}{\partial \rho} \rho \frac{\partial H_z}{\partial \rho} + \frac{\partial^2 H_z}{\partial \theta^2} + \rho^2 \epsilon_r(\rho) H_z = 0 \qquad \epsilon = \epsilon_2 . \tag{5-12}$$

Equation (5-12) leads to exactly the same difficulties encountered in equations (5-7) to (5-9). Equation (5-11), however, is similar to (5-3) and (5-4). Separating variables,

$$H_z = A(\rho)\, B(\theta)$$

we obtain

$$\frac{1}{A} \rho \frac{d}{d\rho} \rho \frac{dA}{d\rho} + \frac{1}{B} \frac{d^2 B}{d\theta^2} + \epsilon_r(\theta) = 0 \tag{5-13}$$

with the resulting equations

$$\frac{1}{B} \frac{d^2 B}{d\theta^2} + \epsilon_r(\theta) = -(ik - \lambda)^2 \tag{5-14}$$

$$\frac{\rho}{A} \frac{d}{d\rho} \rho \frac{dA}{d\rho} = (ik - \lambda)^2 \qquad A = \frac{1}{\rho^\lambda} e^{ik \ln \rho} \qquad \begin{array}{l} \lambda \text{ real}, > 0 \\ k \text{ real} . \end{array}$$

When an attempt is made to solve the problem of a cylinder clad in a medium $\epsilon_1 = \epsilon_0 \dfrac{\epsilon_r(\theta)}{\rho^2}$ with a strip antenna in the z direction, the boundary conditions can again not be applied since the θ dependent solutions are unknown functions and are nonorthogonal for different separation constants.

The problem of strip antennas in the ρ direction on the surface of a wedge is not solvable since neither $\overline{E}$ nor $\overline{H}$ will have only one nonzero component. The same difficulty is encountered with a cylinder clad in a medium with a θ dependence and strip antennas in the θ direction.

B. <u>Cone</u>

Consider first the $\overline{H}$-polarized in the $\hat{\phi}$ direction, where ϵ_r is

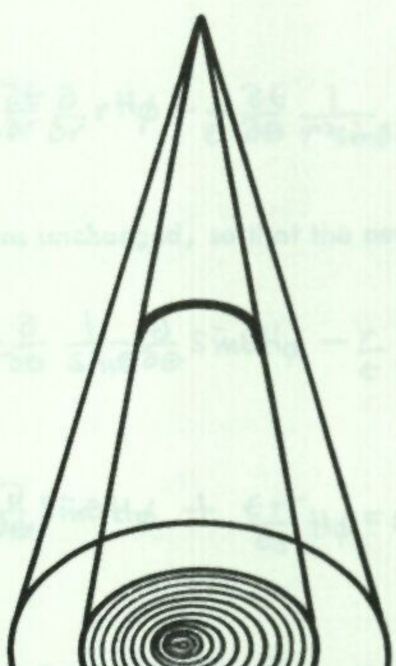

Figure 6. Conducting Cone with Circumferential Antenna

allowed to be a function of r and θ. Equation (3-75) becomes

$$\epsilon\nabla\epsilon^{-1}\times\nabla\times\bar{H} = \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{\phi} \\ -\frac{1}{\epsilon}\frac{\partial\epsilon}{\partial r} & -\frac{1}{r\epsilon}\frac{\partial\epsilon}{\partial\theta} & 0 \\ \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\sin\theta H\phi & -\frac{1}{r}\frac{\partial}{\partial r}rH\phi & 0 \end{vmatrix}$$

$$= \hat{\phi}\left[\frac{1}{r\epsilon}\frac{\partial\epsilon}{\partial r}\frac{\partial}{\partial r}rH\phi + \frac{1}{\epsilon}\frac{\partial\epsilon}{\partial\theta}\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta H\phi\right]. \qquad (5\text{-}15)$$

The curl curl remains unchanged, so that the new equation for $H\phi$ is

$$\frac{\partial}{\partial r}r^2\frac{\partial H\phi}{\partial r} + \frac{\partial}{\partial\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta H\phi - \frac{r}{\epsilon}\frac{\partial\epsilon}{\partial r}\frac{\partial}{\partial r}rH\phi$$

$$-\frac{1}{\epsilon}\frac{\partial\epsilon}{\partial\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta H\phi + \frac{\epsilon r^2}{\epsilon_0}H\phi = 0. \qquad (5\text{-}16)$$

This will separate if $\epsilon = \epsilon_1 = \epsilon_0\frac{\epsilon_r(\theta)}{r^2}$ or if $\epsilon = \epsilon_2 = \epsilon_0\epsilon_r(r)$. The choice $\epsilon = \epsilon_2$ leads to difficulties encountered in equations (5-7) to (5-9). If $\epsilon = \epsilon_1$, we obtain

$$H\phi = A(r)B(\theta)$$

$$\frac{1}{A}\frac{d}{dr}r^2\frac{dA}{dr} + \frac{2}{A}\frac{d}{dr}rA = (ik-\lambda+1)(ik-\lambda+2) \qquad A = \frac{1}{r^\lambda}e^{ik\ln r}$$

$$\lambda \text{ real}, >0$$
$$k \text{ real}$$

$$\frac{1}{B}\frac{d}{d\theta}\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta B + \epsilon_r(\theta) - \frac{1}{\epsilon_r(\theta)}\frac{\partial\epsilon_r(\theta)}{\partial\theta}\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta B \qquad (5\text{-}17)$$

$$= -(ik-\lambda+1)(ik-\lambda+2)$$

where the form of $(ik - \lambda)$ is chosen so that A is finite at $\rho = \infty$. This problem is again unsolvable in all cases since neither the functions $A(\rho)$ or $B(\theta)$ are orthogonal with their same member for different separation constants. For the fields E-polarized in the $\hat{\phi}$ direction, the $\epsilon = \epsilon_2$ case is unsolvable for the same reasons already given. For $\epsilon = \epsilon_1$, we obtain

$$\frac{1}{A}\frac{d}{dr} r^2 \frac{dA}{dr} = (ik-\lambda)(ik-\lambda+1) \qquad A = \frac{1}{r^\lambda} e^{ik \ln r} \qquad \begin{array}{l}\lambda \text{ real}, > 0 \\ k \text{ real}\end{array}$$

$$\frac{1}{B}\frac{d}{d\theta}\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta B + \epsilon_r(\theta) = -(ik-\lambda)(ik-\lambda+1) \tag{5-18}$$

again leading to nonorthogonal functions.

The sphere clad in a medium $\epsilon_1 = \epsilon_0 \dfrac{\epsilon_r(\theta)}{r^2}$ with a strip antenna in the ϕ direction is not solvable since the θ dependent functions are unknown and nonorthogonal.

The problem of a strip antenna in the $\hat{r}$ direction on the surface of a cone is not solvable since $\bar{E}$ and $\bar{H}$ will both have more than one nonzero component. This difficulty is encountered also if ϵ_r is allowed to be a function of ϕ .

CHAPTER VI

DETAILED ANALYSIS OF THE PLANAR CASE

A. E-polarized in the $\hat{z}$ Direction

1. The Iterative Solution

Consider now the case of an electric strip source on a conducting plane (fields E-polarized in the $\hat{z}$ direction) with a variation in ϵ_r represented by the equation[23]

$$\epsilon_r(x) = 1 + \lambda e^{-2x/t} \left(1 + 2x/t + \beta x^2\right). \tag{6-1}$$

Above the antenna we may take $\beta = 0$. This relation has the properties that

$$\left.\frac{d\epsilon_r}{dx}\right|_0 = 0 \qquad \left.\frac{d\epsilon_r}{dx}\right|_{x_e} \approx 0 \quad \text{if} \quad x_e >> t$$

$$\left.\begin{aligned} \epsilon_r\big|_{max} &= \epsilon_r(0) = 1 + \lambda \quad \text{if } \beta = 0. \\ \epsilon_r(t) &\approx 1 + \lambda/e . \end{aligned}\right\} \tag{6-2}$$

We will choose λ small enough so that only first order terms need be considered. From Equation (2-55), we find that

$$\Delta_{n_1}(x_e) = -\int_0^{x_e} \frac{dx'}{\Gamma(x')} \int_{x'}^{x_e} (\phi_0 - \phi)\, \Gamma(x)\, dx$$

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[23] Refer to Figures 7-10. Figure 8 represents the ϵ variation close to the missile nose, Figure 9 the variation at the nose-body junction and Figure 10 the variation above the antenna.

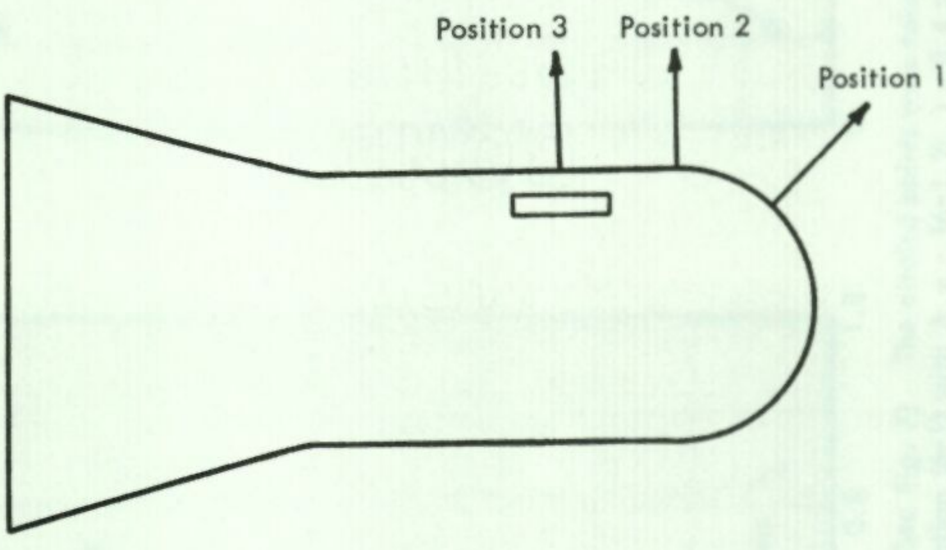

Figure 7. Shape of SCOUT Missile

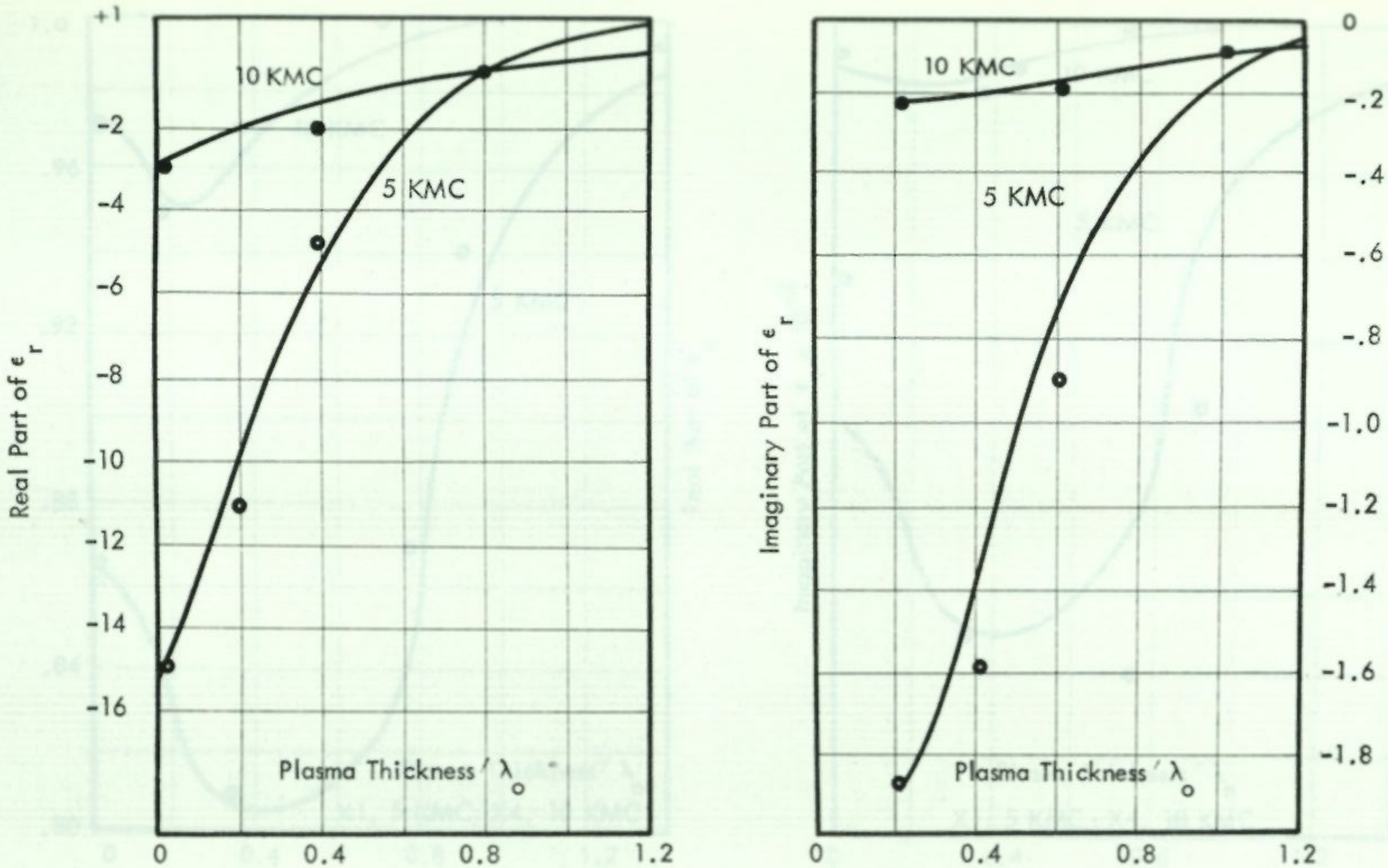

Figure 8. Graph of ϵ_r, Position 1 (See Fig. 7). The circled points were taken from AFCRL Report 87 (See Appendix 7). The solid lines are a plot of Equation (6-1) with $\lambda = -16-1.9i$, $t = 0.4$ for $f = 5$ KMC; $\lambda = -4 -.24i$, $t = 1.0$ for $f = 10$ KMC. $\beta = 0$ for both cases.

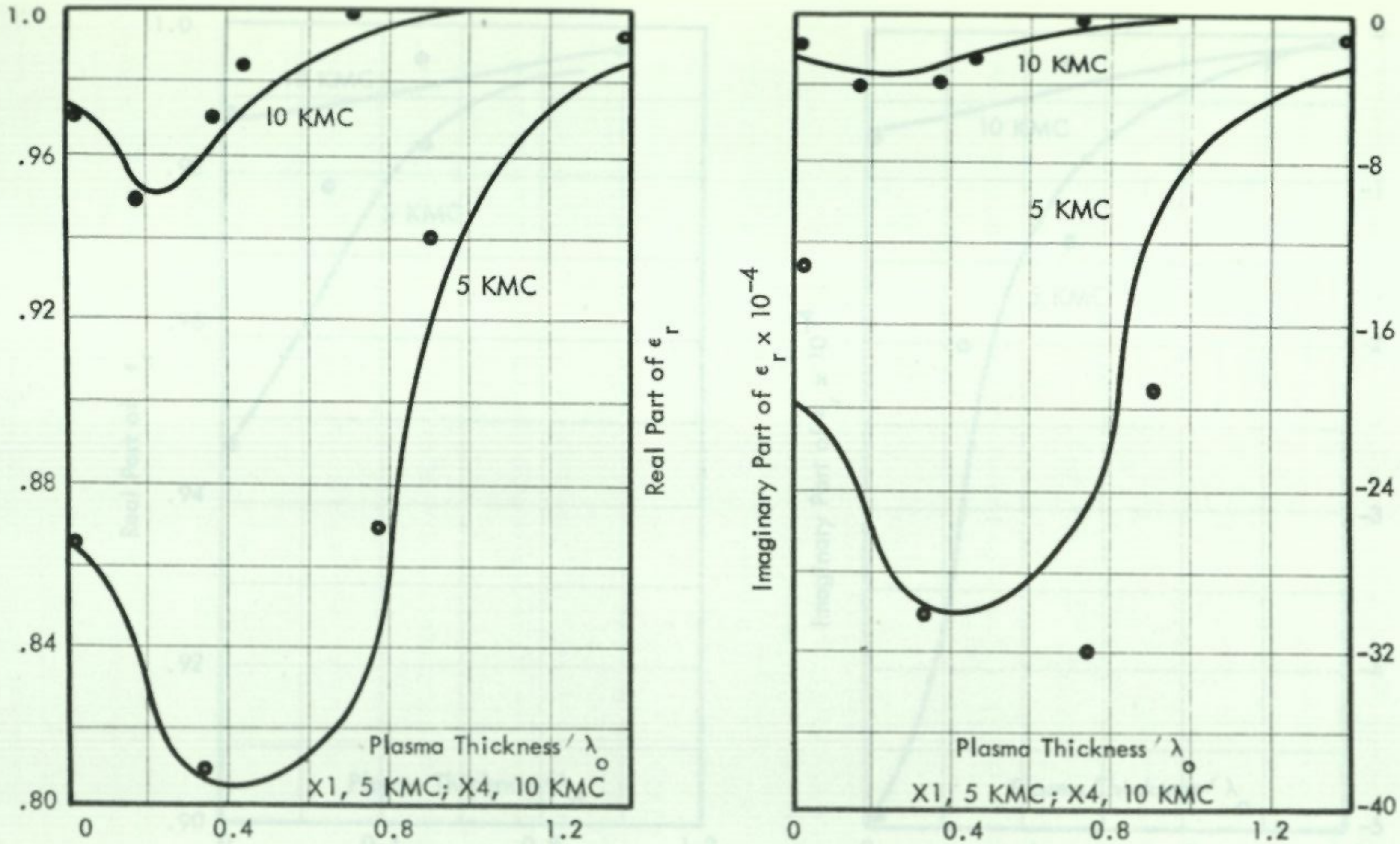

Figure 9 . Graph of ϵ_r, Position 2 (See Fig. 7). The circled points were taken from AFCRL Report 87 (See Appendix 7). The solid lines are a plot of Equation (6-1) with $\lambda = -.13 - i19.5 \times 10^{-4}$, $t = 0.4$, $\beta = 50$ for f = 5 KMC; $\lambda = -.028 - i2 \times 10^{-4}$, $t = 1.0$, $\beta = 10$ for f = 10 KMC.

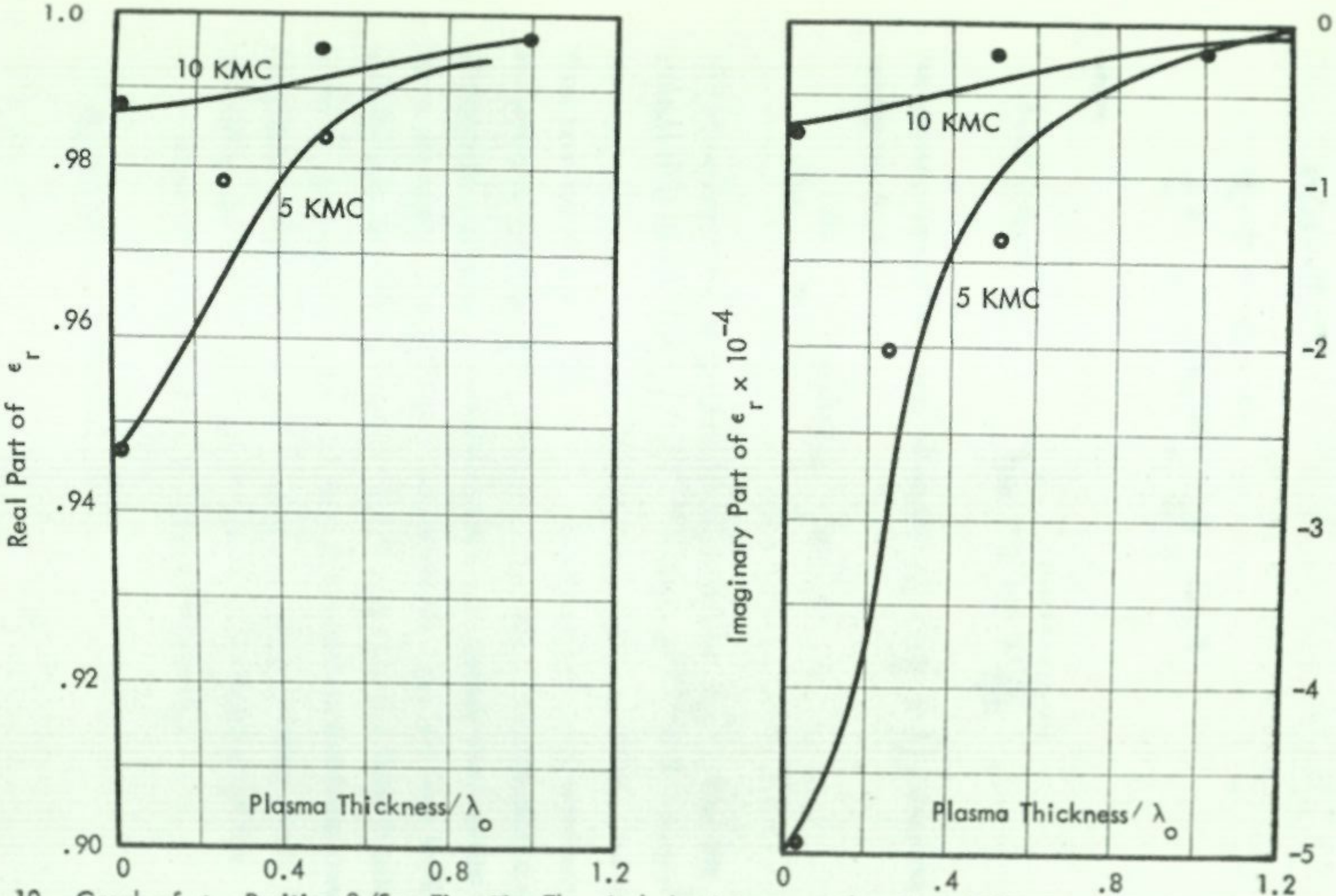

Figure 10. Graph of ϵ_r, Position 3 (See Fig. 7). The circled points were taken from AFCRL Report 87 (See Appendix 7). The solid lines are a plot of Equation (6-1) with $\lambda = -.053-5 \times 10^{-4}i$, $\epsilon = 0.4$, $\beta = 0$ for $f = 5$ KMC; $\lambda = -.013-.67 \times 10^{-4}i$, $t = 0.8$, $\beta = 0$ for $f = 10$ KMC.

and from (4-7) and (4-10),

$$\Gamma(x) = (\overline{F}_n^{\,I})^2$$

$$\phi_0 - \phi = \epsilon_a - \epsilon_r$$

$$\overline{F}_n^{\,I} = a_n^{I} e^{ik_{nx}x} + a_n^{II} e^{-ik_{nx}x} \tag{6-3}$$

where

$$k_{nx} = \sqrt{\epsilon_a - n^2 \frac{\lambda_0^2}{\tau^2}} \,, \quad k_{nx}^{0} = \sqrt{1 - n^2 \frac{\lambda_0^2}{\tau^2}} \,. \tag{6-4}$$

We are only interested in real values of k_{nx}^{0} $\left(n\frac{\lambda_0}{\tau} \leq 1\right)$, since the fields outside the sheath

$$Ez_n^{II} = D_n e^{in\lambda_0/\tau y} \; e^{ik_{nx}^{0} x}$$

will be exponentially attenuated for all imaginary K_{nx}^{o}. From the orthogonality of the functions $e^{in\lambda_0/\tau y}$ and $e^{im\lambda_0/\tau y}$, each n^{th} term must individually satisfy appropriate boundary conditions, and only those terms for which $n\lambda_0/\tau \leq 1$ inside the sheath will contribute to the radiation fields outside the sheath. The term n = 0 represents a wave propagating without any y dependence; i.e., a plane wave moving in the x direction. From symmetry considerations, this term must be the same for both fields $\overline{H}$-polarized in the $\hat{z}$ direction and fields $\overline{E}$-polarized in the $\hat{z}$ direction. The term $n\lambda_0/\tau = 1$ is a wave propagating along the surface of the sheath in the y direction and is a surface wave. The terms for which $0 \leq n\frac{\lambda_0}{\tau} \leq 1$ are a mixture of the above situations.

If we define the antenna by the coefficients γ_n,

$$Ez^{I}(x=0) = \sum_n \gamma_n e^{in\lambda_0/\tau y} \tag{6-5}$$

then by the straightforward application of boundary value techniques, we find that

$$a_n^{I} = \frac{\gamma_n e^{-ik_{nx}x_e}}{e^{-ik_{nx}x_e} + \left(\frac{k_{nx}-k_{nx}^{o}}{k_{nx}+k_{nx}^{o}}\right) e^{+ik_{nx}x_e}} \tag{6-6}$$

$$a_n^{II} = \frac{\gamma_n e^{+ik_{nx}x_e}\left(\frac{k_{nx}-k_{nx}^{o}}{k_{nx}+k_{nx}^{o}}\right)}{e^{-ik_{nx}x_e} + \left(\frac{k_{nx}-k_{nx}^{o}}{k_{nx}+k_{nx}^{o}}\right) e^{+ik_{nx}x_e}} \;. \tag{6-7}$$

We will choose $\epsilon_2 = 1$ so that we have

$$\bar{F}_n^{I} = \gamma_n e^{ik_{nx}^{o} x} \tag{6-8}$$

and

$$\Delta_{n_1}^{IT} = i\delta_n = \frac{\lambda t^2(3+k_{nx}^{o2}t^2)}{4(1+k_{nx}^{o2}t^2)^2} + \frac{i\lambda k_{nx}^{o} t^3(2+k_{nx}^{o2}t^2)}{(1+k_{nx}^{o2}t^2)^2} \;. \tag{6-9}$$

In order to better understand and analyze (6-9), the following computations will be made for $n\frac{\lambda_0}{2} << 1$:

(i) Find $\Delta_{n1}(x_e)$ to first order in λ for a sheath of width $x_e = t$ with a constant value of $\epsilon = 1 + \lambda$. This result should show why the step-function approach is not an adequate way to treat inhomogeneous media.

This problem can be worked by rigorous boundary value techniques, by the WKB method and by the iterative solution derived in this paper. All these results should agree and a check on prior computations may be obtained.

(ii) Find $\Delta_{n_1}(x_e)$ to first order in λ using the WKB technique. This result should agree with Equation (6-9) for large values of t (which will make ϵ slowly varying over a wavelength). This problem will again provide a way of checking the equations derived in this paper.

2. The Step Function Solution

a. Rigorous method

From Equations (6-3), (6-5), and (6-6), the field at x_e is given by

$$E_{z_n}(t) = \gamma_n e^{in\lambda_0/\tau y}\left[\frac{1+\left(\frac{K_{nx}-K_{nx}^0}{K_{nx}+K_{nx}^0}\right)}{e^{-iK_{nx}t}+\left(\frac{K_{nx}-K_{nx}^0}{K_{nx}+K_{nx}^0}\right)e^{+iK_{nx}t}}\right]. \quad (6\text{-}10)$$

If we let $\epsilon = 1 + \lambda$,

$$K_{nx} = K_{nx}^0\left(1+\frac{\lambda}{2K_{nx}^{0\,2}}\right) \quad \text{for} \quad \frac{n\lambda_0}{\tau} \ll 1 \quad (6\text{-}11)$$

$$\frac{K_{nx}-K_{nx}^0}{K_{nx}+K_{nx}^0} = \frac{\lambda}{4K_{nx}^{0\,2}} \quad (6\text{-}12)$$

and we obtain

$$\Delta_{n_1}^{\text{STEP FCN.}-\text{RIG.}} = \frac{\lambda}{4K_{nx}^{0\,2}}\left(1-e^{2iK_{nx}^0 t}\right) + \frac{i\lambda t}{2K_{nx}^0}. \quad (6\text{-}13)$$

b. WKB method

The WKB method gives for the electric field:

$$E_{z_n} = \frac{e^{in\lambda_0/\pi y}}{\sqrt{K_{nx}}}\left[A_n e^{i\int K_{nx}\,dx} + B_n e^{-i\int K_{nx}\,dx}\right]. \tag{6-14}$$

If we take $\epsilon = 1 + \lambda f(x)$ and require $\left.\frac{df(x)}{dx}\right|_{x_e} = 0$, then

$$E_{z_n} = \gamma_n \sqrt{\frac{K_{nx}(0)}{K_{nx}(t)}} \left[\frac{e^{-i\int_x^t K_{nx}\,dx} + \left(\frac{K_{nx}(t) - K_{nx}^0}{K_{nx}(t) + K_{nx}^0}\right) e^{+i\int_x^t K_{nx}\,dx}}{e^{-i\int_0^t K_{nx}\,dx} + \left(\frac{K_{nx}(t) - K_{nx}^0}{K_{nx}(t) + K_{nx}^0}\right) e^{+i\int_0^t K_{nx}\,dx}}\right]. \tag{6-15}$$

At $x = x_e$,

$$E_{z_n} = \gamma_n \sqrt{\frac{K_{nx}(0)}{K_{nx}(t)}} \left[\frac{1 + \left(\frac{K_{nx}(t) - K_{nx}^0}{K_{nx}(t) + K_{nx}^0}\right)}{e^{-i\int_0^t K_{nx}\,dx} + \left(\frac{K_{nx}(t) - K_{nx}^0}{K_{nx}(t) + K_{nx}^0}\right) e^{+i\int_0^t K_{nx}\,dx}}\right]. \tag{6-16}$$

If we take $f(x) \equiv 1$, Equation (6-16) is identical with (6-10), a useful check. But without making this restriction, if we let λ be small and take only first order terms, we obtain

$$\Delta_{n_1}^{WKB} = -\frac{\lambda f(t)\, e^{2ik^0_{nx}t}}{4K_{nx}^{02}} + \frac{\lambda f(0)}{4K_{nx}^{02}} + \frac{i\lambda}{2K^0_{nx}}\int_0^t f(x)dx \; . \tag{6-17}$$

for f (x) ≡ 1, this solution is identical with the one obtained before, Equation (6-13).

c. Iterative method

From Equation (2-55)

$$\Delta_{n_1}(X_e) = -\int_0^{X_e} \frac{dx'}{\Gamma(x')}\int_{x'}^{X_e} (\phi_0 - \phi)\,\Gamma(x)\,dx$$

and Equations (6-3), we obtain by direct integration

$$\Delta_{n_1}^{STEP\ FCN - IT.} = \frac{\lambda}{4K_{nx}^{02}}\left(1 - e^{2iK^0_{nx}t}\right) + \frac{i\lambda t}{2K^0_{nx}} \; , \tag{6-18}$$

a result identical with those obtained by the other methods.

3. The WKB Solution

This solution is easily obtained from Equation (6-17) with $f(x) = e^{-2x/t}(1 + 2x/t)$:

$$\Delta_{n_1}^{WKB} = \frac{i\lambda t}{2K^0_{nx}} + \frac{\lambda}{4K_{nx}^{02}} \tag{6-19}$$

4. Comparison of the Different Methods

We have now derived the following equations:

$$\Delta_{n_1}^{IT} = \frac{\lambda t^2(3 + K_{nx}^{02}t^2)}{4(1 + K_{nx}^{02}t^2)^2} + \frac{i\lambda K^0_{nx}t^3(2 + K_{nx}^{02}t^2)}{2(1 + K_{nx}^{02}t^2)^2}$$

$$\Delta_{n_1}^{WKB} = \frac{i\lambda t}{2K_{nx}^{o}} + \frac{\lambda}{4K_{nx}^{o2}}$$

$$\Delta_{n_1}^{STEP\ FCN.} = \frac{\lambda}{4K_{nx}^{o2}}\left(1 - e^{2iK_{nx}^{o}t}\right) + \frac{i\lambda t}{2K_{nx}^{o}} \quad \text{(by all 3 methods).}$$

We first note that

$$\Delta_{n_1}^{IT.} \xrightarrow[\text{large } t]{} \Delta_{n_1}^{WKB} \tag{6-20}$$

as it must, since this is the region of validity for the WKB Method. It is also interesting to see that the step-function approximation has introduced a non-existent oscillating term into the answer, due to the properties of the step-function region acting as a resonant cavity. However, for small t, the step term is a much better approximation than in the WKB solution. All three solutions are graphed in Figure 11 for n = 0 and the iterative solution is graphed for several n for the case $\tau = 10\ \lambda_o$ in Figures (11-15).

B. <u>$\bar{H}$-polarized in the $\hat{z}$ Direction</u>

Now consider the case of a magnetic line source on a conducting plane (fields H-polarized in the z direction) with the same ϵ variation as before.

$$\epsilon_r(x) = 1 + \lambda e^{-2x/t}\,(1 + 2x/t), \tag{6-1}$$

From equation (2-56), we have

$$\Delta_{n_1}(x_e) = -\int_0^{x_e} \frac{dx'}{\Gamma(x')}\int_{x'}^{x_e} \Gamma(x'')(\phi_o - \phi)\,dx''$$

$$+ \left[\frac{\Gamma(x)\,dF_n^{I}}{F_n^{I}\,dx}\right]_0^{-1} \int_0^{x_e} \Gamma(x'')(\phi_o - \phi)\,dx'' \tag{6-21}$$

and from (4-6) and (4-7)

$$\Gamma(x) = (\bar{F}_n^{I})^2$$

$$\phi_0 - \phi = -\lambda e^{-2x/t}\left[(1-2t^2) + \frac{2}{t}(1+2t^2)x\right]$$

$$\bar{F}_n^{I} = C_n e^{iK^0_{nx} x} . \qquad (6\text{-}22)$$

We define the antenna coefficients γ_n as in (6-4) and find from direct integration that

$$\Delta_{n_1} = \frac{-t\lambda}{4(iK^0_{nx} - \frac{1}{t})^2} + \frac{\lambda(1+2t^2)(1+\frac{2i}{K^0_{nx}t})}{4(iK^0_{nx}-\frac{1}{t})^2} + \frac{\lambda(1-2t^2)}{4iK^0_{nx}(iK^0_{nx}-\frac{1}{t})} . \qquad (6\text{-}23)$$

From (2-46), we see that to first order in λ,

$$F_n^{I} = \bar{F}_n^{I}\left(1 + \Delta_{n_1} + \frac{\lambda}{2}\right) \qquad (6\text{-}24)$$

so that the total correction term is

$$\Delta'_{n_1} = \frac{F_n^{I}}{\bar{F}_n^{I}} = \frac{\lambda t^2\left[3(2K^{0\,2}_{nx}-1) + t^2(2K^{0\,4}_{nx}-1)\right]}{4(1+K^{0\,2}_{nx}t^2)^2} + i\lambda\,\frac{K^0_{nx}t^3(2+K^{0\,2}_{nx}t^2) + 2t\left(1/K^0_{nx} - K^0_{nx}\right)}{2(1+K^{0\,2}_{nx}t^2)^2} . \qquad (6\text{-}25)$$

From symmetry considerations, we must require that Equation (6-25) is equal to Equation (6-9) when $n = 0$, $k^0_{nx} = 1$. This equality does in fact exist, an important check on the whole situation.

We now wish to show graphically how each n^{th} term contributes to the total field intensity at a given observation point and how the change in the field intensity at this point varies with the plasma thickness and the choice of antenna. In order to do this, we first note that the solution in

Region II (outside the plasma) is always of the form

$$F^{II} = \sum_{n=-\infty}^{+\infty} F_n^{II} = \sum_{n=-\infty}^{+\infty} a_n^{II} e^{i\bar{k}_n \cdot \bar{r}}$$

where $\bar{k}_n = k^o_{nx} \hat{x} + n \frac{\lambda_o}{\tau} \hat{y}$ and $\bar{r} = x \hat{x} + y \hat{y}$. When the plasma thickness is reduced to zero, the components are denoted as $\bar{F}_n^{II}$. The Fourier components for which k^o_{nx} is imaginary will be exponentially damped in the x direction and will not contribute to the radiation field. Therefore, if we restrict ourselves to consideration of the field intensity at large distances from the plane, we may write

$$F^{II} = \sum_{n \geq -\frac{\tau}{\lambda_o}}^{n \leq +\frac{\tau}{\lambda_o}} a_n^{II} e^{i\bar{k}_n \cdot \bar{r}} . \qquad \text{(radiation field)}$$

Each n^{th} term in this sum represents a plane wave of amplitude a_n^{II} traveling in the $\bar{k}_n$ direction. We can therefore describe each component F_n by a vector of length a_n^{II} in the direction of $\bar{k}_n$. There will be a finite number of terms contributing to the radiation field and all these terms will be directed into the upper-half plane above the plasma. The length of each vector will represent the maximum value this component will ever have. In order to find the actual radiation field intensity from the graphical plot described above the magnitude of each component must be multiplied by a complex phase factor, and the resultant scalars added together. These complex phases are of course dependent on the particular point of observation chosen.

The angular pattern may be presented in another form if we apply the symmetry relation F (y) = F (-y) and note $a_n^{II} = a_{-n}^{II}$. The field intensity may then be written in the form

$$F = \sum_{n=0}^{n \le +\lambda/\lambda_0} 2 a_n^{II} e^{i\dot{k}_{nx} x} \cos x \frac{n\lambda_0}{\tau} y .$$

In order for the vector representing F_n^{II} to have a length independent of the coordinates, we must take $y = 0$ once the sum has been reduced to positive n only. This restriction greatly reduces the importance of this particular form of F. We will return now to the prior, more general form and utilize it in all further discussion.

For convenience $\overline{F}_o^{II}$ is chosen to be a normalization factor and we define

$$r_n = \frac{\text{absolute value of } F_n^{II}}{\text{absolute value of } \overline{F}_o^{II}} \tag{6-26}$$

$$\theta_n = \tan^{-1} \frac{\text{x component of } \overline{k}}{\text{y component of } \overline{k}} . \tag{6-27}$$

Each point on the angular pattern is defined by its radius vector r_n and angle with respect to the conducting plane θ_n. Each Fourier component (or partial wave) F_n^{II} will give rise to a single vector on this graph.

In order to illustrate the results of the example worked out in this chapter, we choose $\tau = 10\,\lambda_o$. Therefore, n will run from -10 to +10 and 21 such vectors will exist. Equation (6-25) has been graphed for $n = 0, \pm 2, \pm 5, \pm 7, \pm 10$ in Figures (11 - 15), and the angular pattern in Figures 16 and 17 were plotted using these results, where from the general definition

$$r_n = \left| e^{\Delta'_{n_1}} \right| \tag{6-28}$$

$$\theta_n = \tan^{-1} \frac{t\, k^{o}_{nx}}{n \lambda_o} \tag{6-29}$$

The end-points of the vectors have been indicated by points and then these points have been joined together by a smooth curve.

For both E and H sources the t = 0 curve is a semicircle about the antenna, so that for this case the radius vector may be defined to be unity for all θ_n and the other patterns will then be shown with relative magnitudes. The electric line source is considered in Figure 16, and it will be noted that the partial waves with large y components contribute less and less to the total field intensity as the plasma thickness increases. On the other hand, when the magnetic line source pattern in Figure 17 is examined, it may be noted that the partial waves with large y components contribute more and more strongly to the total field intensity as the plasma thickness increases.

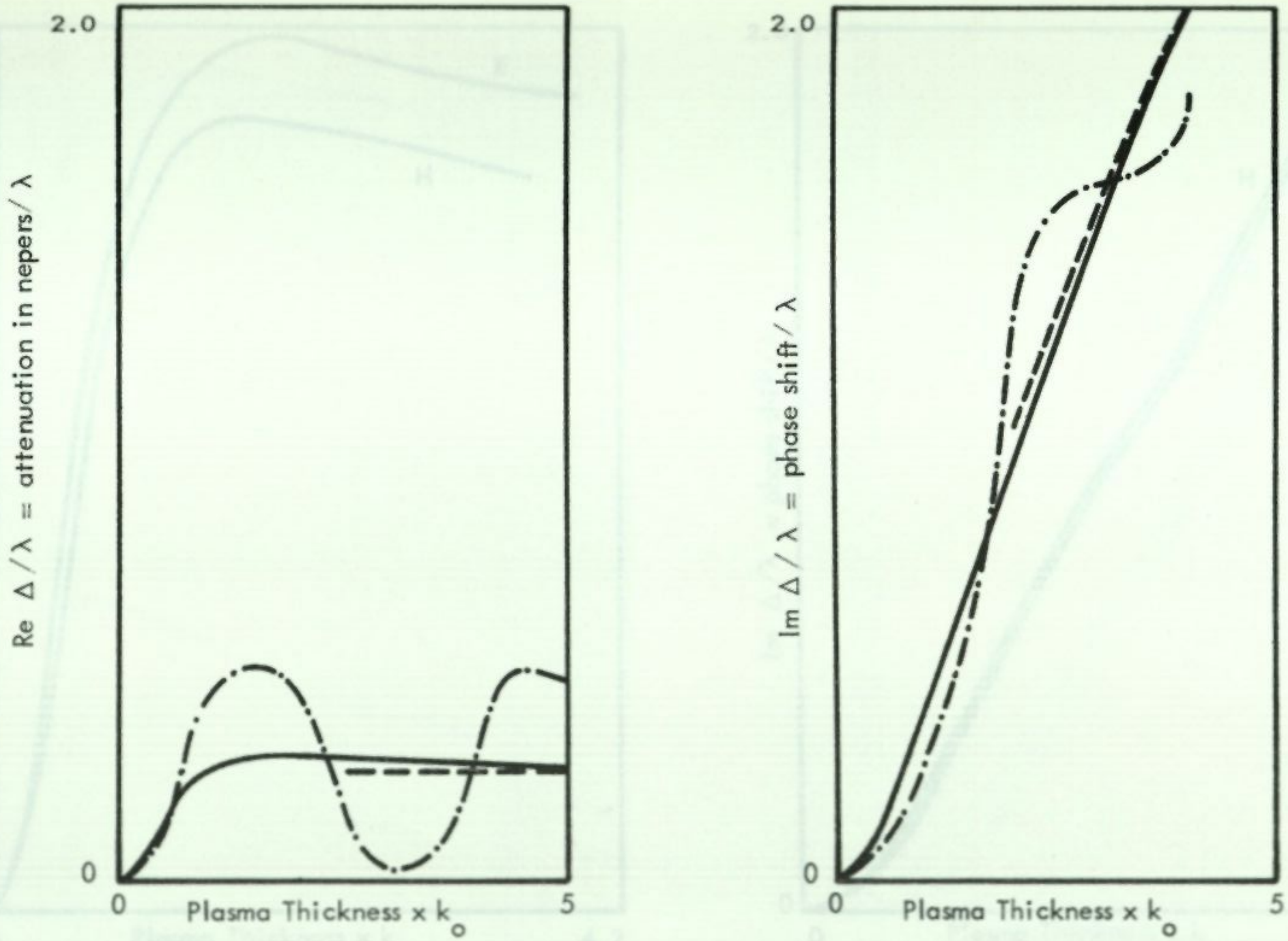

Figure 11. Phase shifts for $n = 0$ for sources both E and H polarized in the z direction. The solid line is the iterative solution described in this report, the dashed line the WKB solution and the dot-dash line the step function solution.

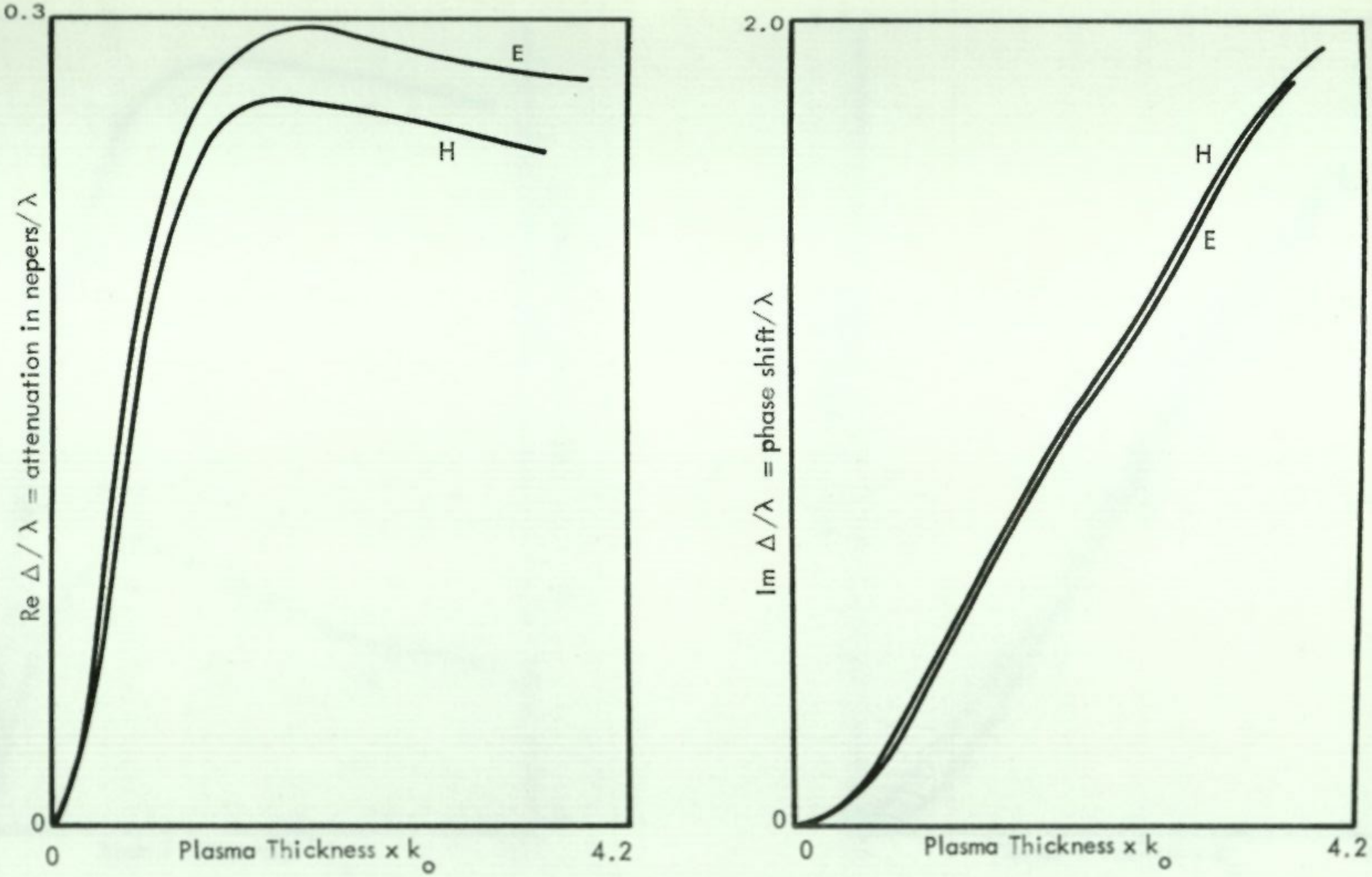

Figure 12. Phase shifts for $n = 2$. E refers to sources E-polarized in the z direction, H to sources H-polarized in the z direction.

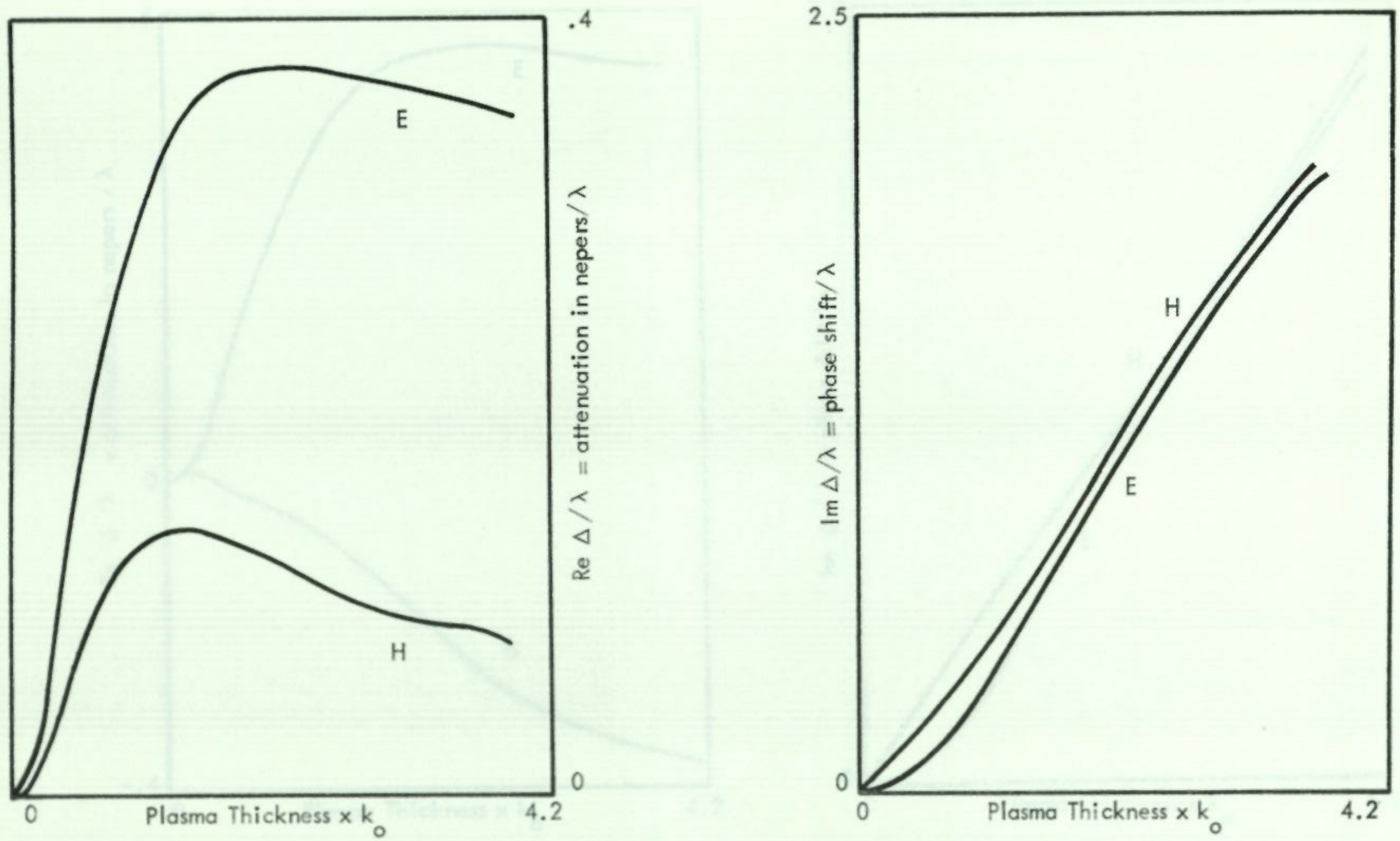

Figure 13. Phase shifts for n = 5. E refers to sources E-polarized in the z direction, H to sources H-polarized in the z direction.

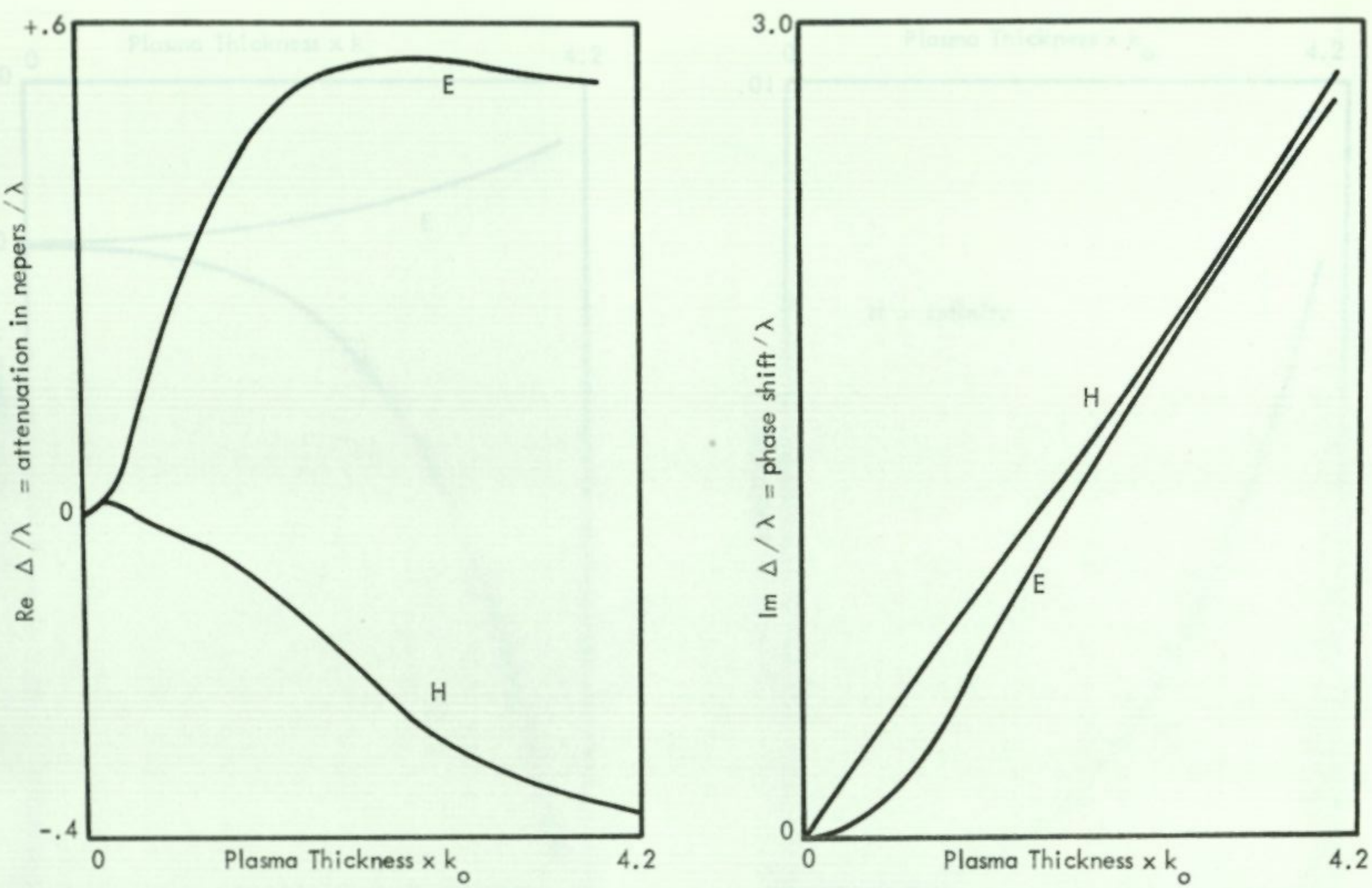

Figure 14. Phase shifts for n = 7. E refers to sources E-polarized in the z direction, H to sources H-polarized in the z direction.

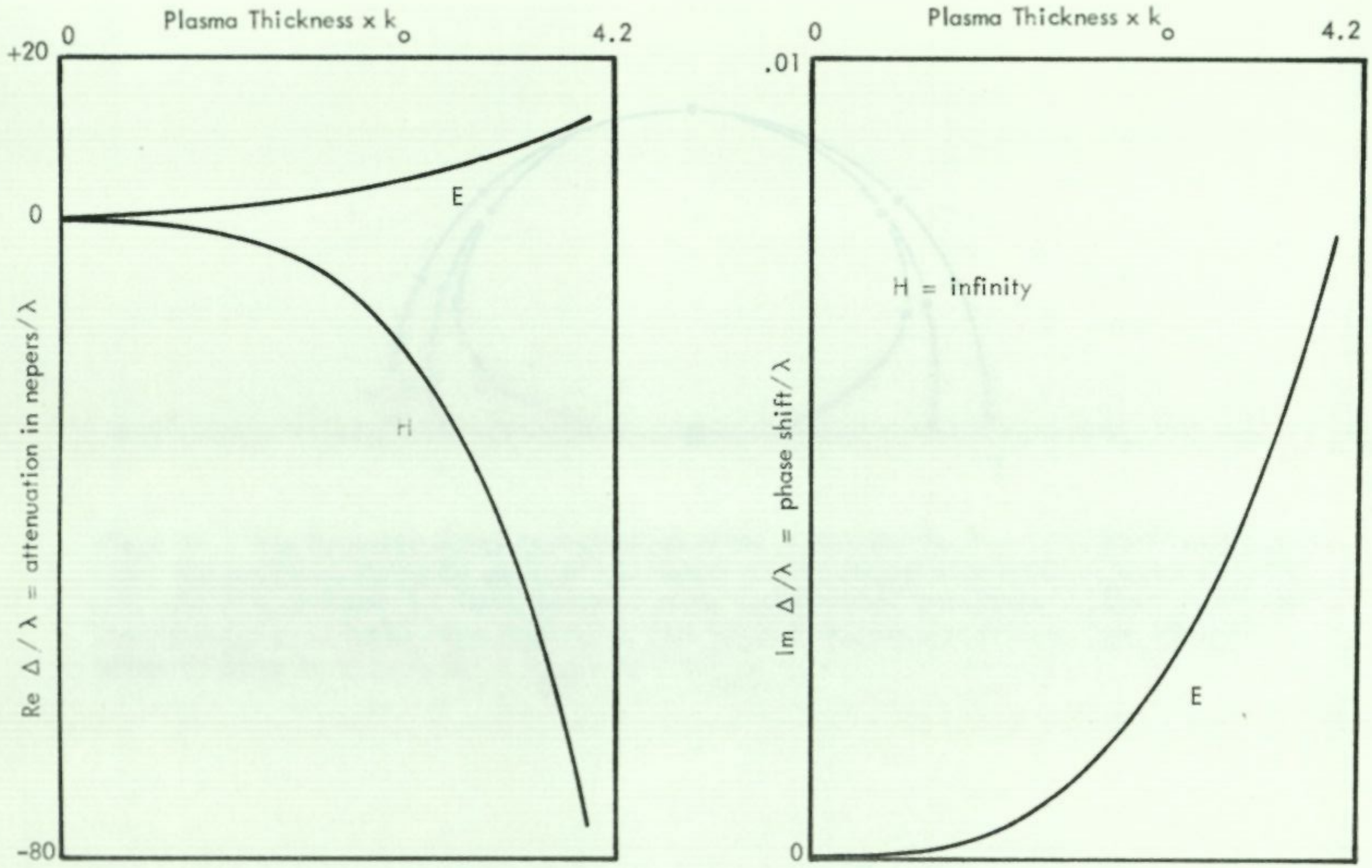

Figure 15. Phase shifts for n = 10. E refers to sources E-polarized in the z direction, H to sources H-polarized in the z direction.

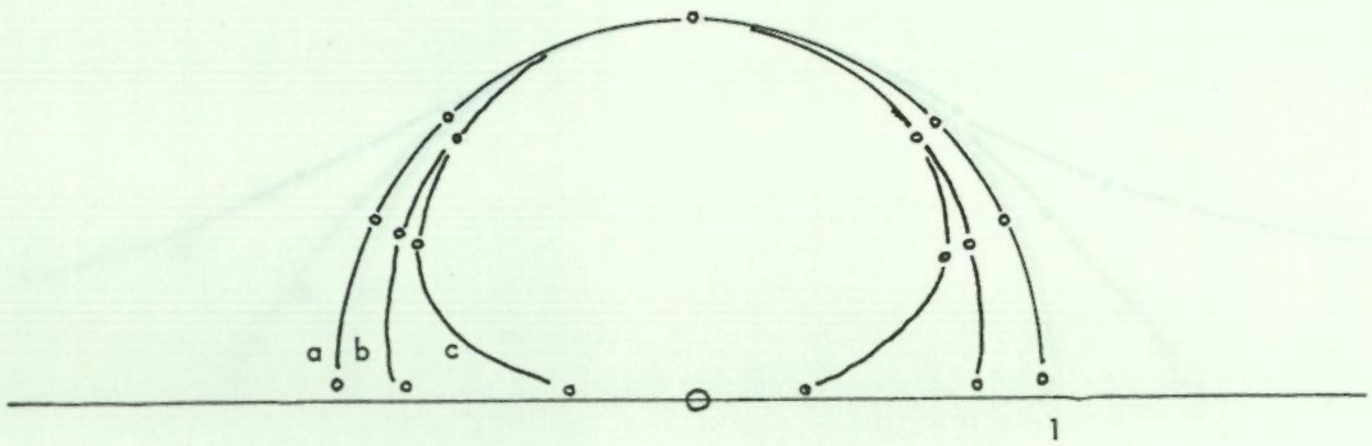

Figure 16. Angular pattern for source E-polarized in the z direction for $\lambda = -0.1$ (collision frequency $= 0$). The vectors describing the maximum contribution of each component have been computed for $n = 0, \pm 7, \pm 9, \pm 10$ (indicated by circles) and these points have then been connected by a smooth curve. The relative reduction of the maximum value of the field intensity components is shown for three choices of plasma thickness t: a) $t = 0$, b) $t = \lambda_{o/4}$, c) $t = \lambda_{o/2}$.

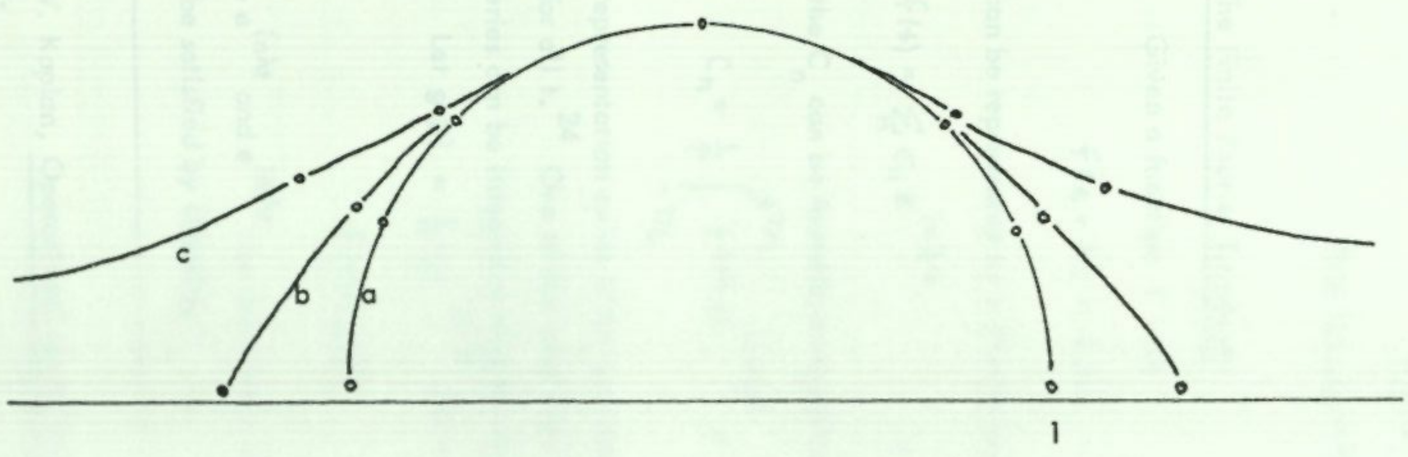

Figure 17. Angular pattern for source H-polarized in the z direction for $\lambda = \underset{\cdot}{-}\ 0.1$ (collision frequency = 0). The vectors describing the maximum contribution of each term have been computed for $n = 0, \pm 7, \pm 9, \pm 10$ (indicated by circles) and these points have been connected by a smooth curve. The relative increase in the maximum value of the field intensity components is shown for three choices of plasma thickness t: a) $t = 0$, b) $t = \lambda_{o/4}$, c) $t = \lambda_{o/2}$.

APPENDIX 1

THE FOURIER TRANSFORM

A. The Finite Fourier Transform

Given a function f with

$$f(t+\tau) = f(t) \qquad \text{(A1-1)}$$

that can be represented by a Fourier series

$$f(t) = \sum_n C_n e^{in\lambda t} \qquad \lambda = \frac{2\pi}{\tau} \qquad \text{(A1-2)}$$

then the C_n can be found by evaluating the integral

$$C_n = \frac{1}{\tau}\int_{-\tau/2}^{+\tau/2} f(t)\, e^{-in\lambda t}\, dt. \qquad \text{(A1-3)}$$

This representation exists if the summation (A1-2) converges uniformly to f (t) for all t.[24] One of the most important results of this requirement is the series can be integrated term by term.

Let $g(t) = \sum_n d_n e^{in\lambda t}$ be another function of t and require that

$$f(t) = g(t). \qquad \text{(A1-4)}$$

Since $e^{in\lambda t}$ and $e^{im\lambda t}$ are orthogonal functions, then Equation (A1-4) can only be satisfied by choosing $c_n = d_n$. This can be proven as follows:

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[24] W. Kaplan, Operational Methods for Linear Systems, Addison-Wesley, 1962.

Given

$$\sum_n C_n e^{in\lambda t} = \sum_m d_m e^{im\lambda t} \tag{A1-5}$$

Multiply both sides by $e^{-ip\lambda t}$

$$\sum_n C_n e^{i(n-p)\lambda t} = \sum_m d_m e^{i(m-p)\lambda t} \tag{A1-6}$$

and integrate from 0 to τ :

$$\sum_n C_n \int_0^\tau e^{i(n-p)\lambda t}\, dt = \sum_m d_m \int_0^\tau e^{i(m-p)\lambda t}\, dt. \tag{A1-7}$$

We know that

$$\int_0^\tau e^{at}\, dt = \begin{cases} \frac{1}{a}(e^{\tau a}-1) & a \neq 0 \\ \tau & a = 0 \end{cases} \tag{A1-8}$$

so that

$$\int_0^\tau e^{i(n-p)\lambda t} = \begin{cases} \frac{\tau}{i(n-p)2\pi}(e^{i2\pi(n-p)}-1) = 0 & n \neq p \\ \tau & n = p . \end{cases} \tag{A1-9}$$

Only the term n = p contributes to the left-hand side of (A1-7) and only the term m = p contributes to the right-hand side, i.e.,

$$C_p = d_p . \tag{A1-10}$$

B. The Fourier Transform

If we write $A_n = \frac{C_n}{\lambda}$, equations (A1-2) and (A1-3) become

$$f(t) = \sum_n A_n \lambda e^{in\lambda t} \tag{A1-11}$$

$$A_n = \frac{1}{2\pi} \int_{-\tau/2}^{+\tau/2} f(t)\, e^{-in\lambda t}\, dt. \tag{A1-12}$$

Now let $\omega_n = n\lambda$, so that

$$\lambda = \omega_{n+1} - \omega_n = \Delta\omega_n \tag{A1-13}$$

$$f(t) = \sum_n A_n e^{i\omega_n t}\, \Delta\omega_n \tag{A1-14}$$

$$A_n = \frac{1}{2\pi} \int_{-\tau/2}^{+\tau/2} f(t)\, e^{-i\omega_n t}\, dt. \tag{A1-15}$$

Now let $\lambda \to 0$, so that

$$f(t) = \int_{-\infty}^{+\infty} A(\omega)\, e^{i\omega t}\, d\omega \tag{A1-16}$$

$$A(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(t)\, e^{-i\omega t}\, dt. \tag{A1-17}$$

These are fundamental relations for Fourier integrals. Orthogonality relations similar to (A1-10) also exist for this case.

APPENDIX 2

OTHER METHODS OF SOLUTION

A. WKB Solution[25]

The Wentzel-Kramers-Brillouin, or WKB approximation, is applicable to situations in which the wave equation can be separated into one or more total differential equations, each of which involves a single independent variable.

The basic propagation equation considered can be written in the form

$$\frac{d^2u}{dx^2} + K^2(x)u = 0 \quad , \quad K^2(x) = \epsilon_r(x)K_o^2 > 0. \tag{A2-1}$$

Now make the change of variable

$$u(x) = Ae^{ik_o S(x)} \tag{A2-2}$$

and (A2-1) becomes

$$\frac{i}{K_o}\frac{d^2S}{dx^2} - \left(\frac{dS}{dx}\right)^2 + \epsilon_r(x) = 0. \tag{A2-3}$$

We substitute an expression of S in powers of k_o^{-1} in (A2-3) and equate equal powers of k_o:

$$S = S_o + \frac{1}{k_o}S_1 + \frac{1}{K_o^2}S_2 + \ldots \tag{A2-4}$$

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[25] L. Schiff, Quantum Mechanics, McGraw-Hill, 1955.

V. Ginzberg, Propagation of Electromagnetic Waves in Plasma, Gordon and Breach, 1960.

$$\left(\frac{i}{k_0}\frac{d^2S_0}{dx^2} + \frac{i}{k_0^2}\frac{d^2S_1}{dx^2} + \dots\right) - \left(\frac{dS_0}{dx} + \frac{1}{k_0}\frac{dS_1}{dx} + \dots\right)^2 + \epsilon_r(x) = 0 \tag{A2-5}$$

$$-\left(\frac{dS_0}{dx}\right)^2 + \epsilon_r(x) = 0 \tag{A2-6}$$

$$i\,\frac{d^2S_0}{dx^2} - 2\,\frac{dS_0}{dx}\frac{dS_1}{dx} = 0 \tag{A2-7}$$

$$\vdots$$

Integration of these equations gives

$$S_0 = \pm \int \sqrt{\epsilon_r(x)}\, dx \tag{A2-8}$$

$$S_1 = \frac{i}{2} \ln \sqrt{\epsilon_r(x)}\ , \tag{A2-9}$$

and we thus obtain to this order of approximation

$$u(x) = \frac{A e^{\pm i \int k(x) dx}}{\sqrt{k(x)}}\ . \tag{A2-10}$$

The WKB solution will be useful if

$$\left| \frac{\frac{1}{k_0}\frac{dS_1}{dx}}{\frac{dS_0}{dx}} \right| = \left| \frac{1}{4\epsilon_r^2}\frac{d\epsilon_r}{d(k_0 x)} \right| << 1 \tag{A2-11}$$

which means that the fractional change in ϵ_r over a wavelength must be small compared to unity.

B. Green's Function Solution

If $G(\bar{r}, \bar{r}_o)$ is a field at the observer's point $\bar{r}$ caused by a unit point source at $\bar{r}_o$, then the field at $\bar{r}$ caused by a source distribution $\rho(\bar{r}_o)$ is the integral of $G\rho$ over the whole range of $\bar{r}_o$ occupied by the source. The function G is called the Green's function. It is a solution to a given partial differential equation that is homogeneous everywhere except at one point. When the point is on a boundary, the Green's function may be used to satisfy inhomogeneous boundary conditions; when it is out in space, it may be used to satisfy the inhomogeneous equation.[26]

If the partial differential equation of interest is the Helmholtz equation

$$\bar{\nabla}^2 \psi + k^2 \psi = 0 \tag{A2-12}$$

then the required Green's function is the solution of the inhomogeneous Helmholtz equation

$$\bar{\nabla}^2 G_K(\bar{r}, \bar{r}_o) + k^2 G_K(\bar{r}, \bar{r}_o) = -4\pi \delta(\bar{r} - \bar{r}_o) \; . \tag{A2-13}$$

It can be shown that G_k is a symmetric function of $\bar{r}$ and $\bar{r}_o$, and from this requirement it follows that we must have

$$G_k(\bar{r}, \bar{r}_o) \xrightarrow[R \to 0]{} g_k(R) \quad , \; \bar{R} = \bar{r} - \bar{r}_o \; . \tag{A2-14}$$

To find the behavior of g_k for $R \to 0$, we integrate both sides of (A2-13) over a small sphere of radius ϵ about $\bar{r}_o$. This gives us

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26 An inhomogeneous boundary condition is one that requires the field or its derivative to have a specified, nonzero value on the boundary. An inhomogeneous equation contains a source term (a term not multiplied by the dependent variable or its derivatives).

$$\iiint \bar{\nabla}^2 G_K(\bar{r}, \bar{r}_0)\, dv + k^2 \iiint G_k(\bar{r}, \bar{r}_0)\, dv = -4\pi. \tag{A2-15}$$

The integral on the right-hand side equals -4π because of the properties of the delta function and become the sphere integrated over includes the point $\bar{r} = \bar{r}_o$. We assume that the first integral in (A2-15) will dominate as $R \rightarrow 0$.

$$\iiint \nabla^2 g_k(R)\, dv \longrightarrow -4\pi \quad \text{as} \quad \epsilon \rightarrow 0. \tag{A2-16}$$

The divergence theorem states that

$$\oint \bar{F} \cdot d\bar{A} = \iiint (\nabla \cdot \bar{F})\, dv$$

and applying this to (A2-16), since $\bar{\nabla}^2 = \nabla \cdot \nabla$:

$$-4\pi \longrightarrow \oint \nabla g_k \cdot d\bar{A} = \left(\frac{dg}{dR}\right)_{R=\epsilon} (4\pi\epsilon^2) \tag{A2-17}$$

or written another way,

$$\frac{dg}{dR}\, 4\pi R^2 \rightarrow -4\pi \quad \text{as} \quad R \rightarrow 0, \tag{A2-18}$$

so that

$$G_k(\bar{r}, \bar{r}_0) \rightarrow \frac{1}{R} \quad \text{as} \quad R = |\bar{r} - \bar{r}_0| \rightarrow 0 \tag{A2-19}$$

for the three-dimensional case. Similarly, for two dimensions

$$G_R(\bar{r},\bar{r}_0) \rightarrow -2\ln R \quad , \; R \rightarrow 0 . \tag{A2-20}$$

For one dimension, the Green's function G has a discontinuity in slope equal to -4π at $x = x_o$:

$$\left(\frac{dG}{dx}\right)_{x_o-\epsilon}^{x_o+\epsilon} = -4\pi \quad , \; \epsilon \rightarrow 0 . \tag{A2-21}$$

If the boundaries of a particular problem are at infinity, then

$$G_R(\bar{r},\bar{r}_0) = \frac{e^{\pm ikR}}{R} \quad \text{(3 dimensions)} \tag{A2-22}$$

$$G_R(\bar{r},\bar{r}_0) = i\pi H_0^{(1,2)}(kR) \quad \text{(2 dimensions)} \tag{A2-23}$$

$$G_R(\bar{r},\bar{r}_0) = \frac{2\pi i}{k} e^{ik|x-x_o|} \quad \text{(1 dimension)} . \tag{A2-24}$$

APPENDIX 3

PROPERTIES OF ORDINARY, LINEAR, SECOND ORDER DIFFERENTIAL EQUATIONS

The following definitions and statements can be found in many texts on differential equations:[27]

1. The <u>order</u> of a differential equation is the order of the highest-ordered derivative appearing in the equation.

2. An equation is <u>linear</u> if each term in the equation is either linear in all the dependent variables and their various derivatives or does not contain any of them.

3. An equation involving <u>ordinary</u> derivatives is called an ordinary differential equation.

4. Given the functions $f_1(x) \ldots, f_n(x)$ then if constants $c_1 \ldots, c_n$, not all zero, exist such that

$$c_1 f_1(x) + \ldots + c_n f_n(x) = 0$$

identically, the functions $f_1(x) \ldots f_n(x)$ are said to be linearly dependent. If no such relation exists, the functions are said to be linearly independent.

5. An ordinary differential equation of the n^{th} order has, in general, a solution containing n arbitrary constants. For a second order equation in y, the solution can be written

$$y = c_1 y_1 + c_2 y_2 .$$

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[27] For instance see E. Rainville, <u>Elementary Differential Equations</u>, MacMillian, 1957.

The functions y_1 and y_2 must be linearly independent or c_1 and c_2 degenerate to only one arbitrary constant.

APPENDIX 4

BOUNDARY CONDITIONS ON THE ELECTROMAGNETIC FIELDS

Maxwell's equations can be written in the form

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} \tag{A4-1}$$

$$\nabla \times \bar{H} = \frac{\partial \bar{D}}{\partial t} + \bar{J} \tag{A4-2}$$

$$\nabla \cdot \bar{B} = 0 \tag{A4-3}$$

$$\nabla \cdot \bar{D} = \rho \quad . \tag{A4-4}$$

In order to establish the boundary conditions on the fields, Equations (A4-1) - (A4-4) must be combined with the vector relations

$$\int_S \bar{A} \cdot \bar{n}\, da = \int_V \nabla \cdot \bar{A}\, dv \tag{A4-5}$$

$$\int_C \bar{A} \cdot d\bar{s} = \int_S (\nabla \times \bar{A}) \cdot \bar{n}\, da \tag{A4-6}$$

known respectively as the divergence theorem and Stokes' theorem.

From (A4-3), (A4-4) and (A4-5), we obtain

$$\int_S \bar{B} \cdot \bar{n}\, da = 0 \tag{A4-7}$$

$$\int_S \bar{D}\cdot\bar{n}\, da = \int \rho\, dv = q. \qquad \text{(A4-8)}$$

From (A4-1), (A4-2), and (A4-6), we obtain

$$\int_C \bar{E}\cdot d\bar{s} = -\int_S \frac{\partial \bar{B}}{\partial t}\cdot\bar{n}\, da \qquad \text{(A4-9)}$$

$$\int_C \bar{H}\cdot d\bar{s} = \int_S \left(\frac{\partial \bar{D}}{\partial t} + \bar{J}\right)\cdot\bar{n}\, da\ . \qquad \text{(A4-10)}$$

If a pillbox is constructed on S in the usual manner, we obtain from (A4-7) and (A4-8),

$$(\bar{B}\cdot\bar{n}_1 + \bar{B}\cdot\bar{n}_2)\,\Delta a = 0$$
$$(\bar{B}_1 - \bar{B}_2)\cdot\bar{n} = 0 \qquad \text{(A4-11)}$$

$$(\bar{D}\cdot\bar{n}_1 + \bar{D}\cdot\bar{n}_2)\,\Delta a = \omega\,\Delta a$$
$$(\bar{D}_1 - \bar{D}_2)\cdot\bar{n} = \omega \qquad \text{(A4-12)}$$

where ω is the surface change density. If a rectangular path C is drawn cutting S in the usual way, we obtain from (A4-9),

$$(\bar{E}\cdot\bar{\tau}_1 + \bar{E}\cdot\bar{\tau}_2)\,\Delta s = -\frac{\partial \bar{B}}{\partial t}\cdot\bar{n}_0\,\Delta s\,\Delta l \rightarrow 0$$
$$(n \times (\bar{E}_2 - \bar{E}_1)) = 0 \qquad \text{(A4-13)}$$

where $\bar{\tau}_1$ and $\bar{\tau}_2$ are in the direction of circulation and $\bar{n}_o$ perpendicular to the plane of path C. From (A4-10),

$$(\bar{H}\cdot\bar{\tau}_1 + \bar{H}\cdot\bar{\tau}_2)\,\Delta s = \left(\frac{\partial \bar{D}}{\partial t} + \bar{J}\right)\cdot\bar{n}_0\,\Delta s\,\Delta l$$
$$n \times (\bar{H}_2 - \bar{H}_1) = \bar{K} \qquad \text{(A4-14)}$$

where $\vec{K} = \lim_{\Delta \ell \to 0} \vec{J} \Delta \ell$

exists only if the conductivity of one medium becomes infinite.

APPENDIX 5

DESCRIBING A PLASMA SHEATH IN TERMS OF A VARYING ϵ[28]

The equation that governs the electron motion in a plasma is

$$m\frac{d\bar{v}}{dt} = -e\bar{E} - e\bar{v}\times\bar{B} - m\nu\bar{v} \tag{A5-1}$$

which will be recognized as the Lorentz force equation. $\bar{E}$ and $\bar{B}$ are the applied electric and magnetic fields, ν is the collision frequency for momentum transfer between electrons and atoms or ions, and a term multiplied by the pressure gradient has been assumed negligible.

We take $\bar{E}$ and $\bar{v}$ to be time harmonic, $E_{static} = B_{static} = 0$, and neglect B_{wave} since it is v/c times smaller than the electric force term. (A5-1) becomes

$$j\omega\bar{v}_0 = -\frac{e}{m}\bar{E}_0 - \nu\bar{v}_0 \tag{A5-2}$$

and solving for $\bar{v}_o$,

$$\bar{v}_0 = \frac{j\frac{e}{m}\bar{E}_0}{\omega - j\nu}\ . \tag{A5-3}$$

The current density $\bar{J}_o$ becomes

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28 This material was taken partly from "Outline of A Course in Plasma Physics", Part 2, American Journal of Physics, Vol. 31, Number 8, August 1963.

$$\bar{J}_0 = -Ne\bar{v}_0 = \frac{-j\,\frac{e^2 N}{m}\,\bar{E}_0}{\omega - j\nu} \qquad \text{(A5-4)}$$

(where N is the ionization density)

and substituting into Maxwell's equation, we obtain

$$\nabla \times \bar{H}_0 = j\omega\left(\epsilon_0 - \frac{\frac{Ne^2}{m\omega}}{\omega - j\nu}\right)\bar{E}_0$$

$$\nabla \times \bar{E}_0 = -j\omega\mu_0 \bar{H}_0 \ . \qquad \text{(A5-4)}$$

Comparing with Equations (3-1) and (3-2), we see that

$$\epsilon' = \epsilon_0 - \frac{Ne^2}{m\omega(\omega - j\nu)}$$

$$\epsilon' = \epsilon_0 - \frac{Ne^2}{m(\omega^2+\nu^2)} - j\,\frac{Ne^2}{m}\,\frac{\nu}{\omega(\omega^2+\nu^2)} \ . \qquad \text{(A5-5)}$$

APPENDIX 6

TYPICAL ϵ VARIATIONS AROUND A RE-ENTERING MISSILE

Computed values of the ionization density and collision frequency for a typical plasma sheath have been shown in several publications. The values used in the current example were taken from AFCRL Report 87,[29] and are computed for a SCOUT missile (Figure 7). This data, when substituted into Equations (A5-5) yields values of ϵ in the sheath (Figures 8-10).

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[29] W. Rotman and G. Meltz, Experimental Investigation of the Electromagnetic Effects of Re-Entry, Air Force Cambridge Research Laboratories, Bedford, Mass.

ANNOTATED BIBLIOGRAPHY

(1) Albini and Jahn, "Reflection and Transmission of Electromagnetic Waves at Electron Density Gradients," Journal of Applied Physics, (January, 1961), pp 75-82.

A TEM wave is incident on a linear ramp $\sigma = \sigma_0 \, z/z_0$ (or combination of ramps) terminating in a constant electron density and solutions to the wave equation in the ramp region are expressed in terms of Airy functions. Transmission and reflection coefficients were found to have a "strong, irregular dependence" on the ramp width. Further comments on the article are made by L. S. Taylor in a letter in the September, 1961 issue of JAP.

(2a) P. J. Wyatt, "Scattering of Electromagnetic Plane Waves from Inhomogeneous Spherically Symmetric Objects," Physical Review, (September, 1962), pp 1837-1843.

(2b) P. J. Wyatt, "interdependency of Plasma Frequency and Surface Diffusivity from Electromagnetic Scattering Analyses," Journal of Applied Physics, (July, 1963), pp 2078-2083.

A plane wave is incident on an inhomogeneous, spherically-symmetric scatterer composed of a medium whose optical properties are continuous everywhere except possibly at their outermost surfaces r = b. All of the important scattering quantities are expressed in terms of solutions to the following differential equations at b:

$$\frac{d^2W_\ell}{d\rho^2} - \frac{2}{n}\frac{dn}{d\rho}\frac{dW_\ell}{d\rho} + \left[n^2 - \frac{\ell(\ell+1)}{\rho^2}\right]W_\ell = 0$$

$$\frac{d^2G_\ell}{d\rho^2} + \left[n^2 - \frac{\ell(\ell+1)}{\rho^2}\right]G_\ell = 0.$$

An example is worked out, using computer techniques to generate the solutions.

(3) J. J. Brandstatter, An Introduction to Waves, Rays, and Radiation in Plasma Media, McGraw-Hill, (1963).

The material on inhomogeneous media is mainly concerned with ray tracing techniques.

(4) Mott and Massey, The Theory of Atomic Collisions, Oxford, (1933)

Chapter 2 presents a detailed development of quantum mechanical scattering using partial waves.

(5) J. R. Wait, "Reflection of Electromagnetic Waves Obliquely from An Inhomogeneous Medium," Journal of Applied Physics, (December, 1952), pp 1403-1404.

(Letter). A plane wave is incident obliquely on to the plane surface of a semi-infinite medium with a permittivity ϵ that varies in a direction perpendicular to the surface. An analytic solution is obtained for

$$\epsilon(z) = (\epsilon_1 - \epsilon_2) e^{\beta z} + \epsilon_2 \quad , \quad z \leq 0 .$$

(6) L. I. Schiff, "Scattering of Waves and Particles by Inhomogeneous Regions," Journal of the Optical Society of America, (February, 1962), pp 140-144.

Electromagnetic and Schroedinger waves are compared and approximation techniques for perturbations in optical path are discussed.

(7) Recent Developments in the Theory of Wave Propagation, A Seminar, New York University, (1949-1950).

A generalized solution to three-dimensional wave scattering is developed using S-matrix theory in Section III-C (p 30). Two possible definitions for S are discussed in detail. Schwinger's variational procedure for the solution of wave scattering problems is applied in Section III-D (p50). The remaining sections of the seminar notes are devoted to mathematical analysis of other important aspects of wave scattering.

(8) Powell and Craseman, Quantum Mechanics, Addison-Wesley, (1961).

Chapter 8 is completely devoted to the theory of scattering, including partial wave analysis and S-matrix theory.

(9) R. Shore and G. Meltz, "Anisotropic Plasma-Covered Magnetic Line Source," IRE Transactions on Antennas and Propagation, (January, 1962), pp 78-82.

Radiation from a magnetic line source in a ground plane covered by an anisotropic, homogeneous (lossy) plasma layer is considered.

(10) H. Raemer, "Radiation from Linear Electric or Magnetic Antennas Surrounded by a Spherical Plasma Shell," IRE Transactions on Antennas and Propagation, (January, 1962), pp 69-78.

The radiation from a linear electric or magnetic antenna surrounded by a spherical shell of homogeneous plasma is analyzed.

(11a) L. S. Taylor, "Electromagnetic Propagation in an Exponential Ionization Density," IRE Transactions on Antennas and Propagation, (September, 1961).

The propagation of a TE wave into a plane stratified medium in which the ionization density varies as exp z/z_o is investigated.

In a subsequent letter (March, 1962), J. R. Wait points out this result was obtained previously by a variety of others, who are referenced.

(11b) L. S. Taylor, "R. F. Reflectance of Plasma Sheaths," Proceedings of the IRE, (December, 1961), pp 1831-1836.

Series solution when the medium is bounded by a plane conducting wall.

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Solutions are found from step-by-step numerical integration.

(11d) J. Richmond, "The WKB Solution for Transmission Through Inhomogeneous Plane Layers," IRE Transactions on Antennas and Propagation, (July, 1962), pp 472-473.

(Letter). The WKB method is applied to the above problem and several examples evaluated.

(11e) R. Yamada, "Reflection of Electromagnetic Waves from a Stratified Inhomogeneity," IRE Transactions on Antennas and Propagation, (July, 1961), pp 364-370.

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(12) M. Plonus, "Diffraction of a Plane Wave by a Perfectly Conducting Sphere with a Concentric Shell," IRE Transactions on Antennas and Propagation, (November, 1961), pp 573-576.

Diffraction from a perfectly-conducting sphere with a homogeneous concentric shell spaced any distance from the sphere surface is presented.

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Green's function techniques are used to treat the propagation of electromagnetic waves in uniform, weakly interacting plasmas.

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A general plane wave is incident on a cylinder and solutions are found for the homogeneous sheath and for a thin sheath with

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Computer evaluations have been graphed.

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The solution is formulated for $\epsilon(K\rho) = \epsilon_0 [1 - \frac{N(K\rho)e^2}{m \epsilon_0 \omega^2}]$ and evaluated using series expansion and computer techniques.

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A one-dimensional variation in ϵ is assumed and WKB-type solutions are found. Also, a few rigorous solutions are obtained:

$$K(z) = (az+b)^{-2}$$

$$K(z) = (a_1 z + b_1)^{1/2} (a_2 z + b_2)^{-5/2}$$

$$K(z) = (a_1 b_2 - a_2 b_1)^2 (a_2 z + b_2)^{-4} [c^2 - (a_1 z + b_1)^2 (a_2 z + b_2)^2]$$

$$K(c) = K(-c) = 0.$$

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Numerical techniques are applied to the problem.

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