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Scattering from Bodies of Revolution by Point-Matching UARI Research Report No. 31

H. Y. Yee

Master

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SCATTERING FROM BODIES OF REVOLUTION BY POINT-MATCHING

by

H. Y. Yee

This research work was supported by
the National Aeronautics and Space Administration
under research grant NsG-381

UNIVERSITY OF ALABAMA RESEARCH INSTITUTE
Huntsville, Alabama

December 1965

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Attention: Dr. T. L. K. Smull

Dear Dr. Smull:

Enclosed are twenty-five (25) copies of the University of Alabama Research Institute Research Report No. 31 entitled, "Scattering From Bodies Of Revolution By Point-Matching", by H. Y. Yee. The research for this report was supported by the National Aeronautics & Space Administration Grant NsG-381.

Sincerely yours,

W. P. Watts
Administrative Manager

(CT)

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Director, Research Division, National Aeronautics & Space Administration (1)

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SUMMARY:

The application of the point-matching method to three-dimensional problems is formulated in this report. Investigations are started from the scattering of plane waves by accoustically soft and hard bodies of revolution. Due to the symmetry of the scatterer, the incident wave is expanded in Fourier series of azimuth angle. A number of systems of simultaneous inhomogeneous algebraic equations is necessary to obtain a solution. Each of these systems of equations is quite similar to that of the two dimensional problem. The accoustic formulation is extended to obtain the solution for scattering of electromagnetic plane waves by perfectly conducting rotational symmetric bodies. In this case, the boundary conditions consist of two tangential components of the electric field which vanish at the surface of the scatterer. However, the resultant equations are in the same form as in the accoustic problem.

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1. INTRODUCTION:

The point-matching method has been applied in many areas of engineering science during recent years[1-6]. Most of these applications are two-dimensional eigenvalue problems. A finite number of points around the periphery of the boundary in question are chosen such that these points describe the boundary contour approximately[5,6]. By utilizing a computer, this method can easily produce many practical solutions to the eigenvalue problem and similar types of problems. For example, all hollow-piped waveguides within the limitations posted in Ref. [6] can be solved by the same computer program. However, for three-dimensional problems, many more points are necessary to describe the surface of the body under consideration. The huge number of algebraic equations required in some systems may be beyond the capacity of the present computers. But, this difficulty can be overcome for rotational symmetric bodies which are frequently confronted in practical applications. The problem of scattering by bodies of revolution will be investigated by the point-matching method in this paper. The three-dimensional problem is reduced to several problems similar to those of two-dimensional cases. This reduction is due to the rotational symmetry of the scatterer and the resolution of the incident wave in terms of cylindrical modes.

In the following considerations, the scalar plane wave scattering by accoustically soft and hard bodies of revolution will be investigated first. This case is quite similar to the two-dimensional scattering problem[2-4]. From accoustic formulations it is easy to see the basic principles of the point-matching method in three-dimensional problems. Practical approximate solutions can be obtained if this method is applied to rotationally symmetric bodies of smooth contour which are not a gross deviation from a semi-circle. The scattering of electromagnetic wave by perfectly conducting bodies of revolution will be considered next. The formulation is more complicated, however, the basic principle and the applicability of the method are similar to those of accoustics. The scattering properties are determined by the polarization and the propagation direction of the incident plane wave as well as the shape of the scatterer. Since any arbitrarily polarized plane wave can be resolved into two components namely: the transverse electric (TE) and the transverse magnetic (TM) polarizations with respect to the symmetric axis of the body, the scattering of TE and TM waves can be considered separately without loss of the generality. The total scattered fields of an arbitrarily polarized plane wave are simply the superposition of the TE and the TM solutions.

In all cases, the method takes its starting point in the resolution of the incident wave functions into cylindrical modes which are the terms of the Fourier expansion with respect to the azimuth angle of the incident wave function. The scattered field is expressed in the form of the general spherical harmonic solution with specification of outgoing wave. The point-matching technique is applied to the boundary conditions for terms of the same azimuth variation. Then, the problem is approximated by a finite number of systems of linear inhomogeneous algebraic equations which can be solved by a computer in the same routine. With the knowledge of the scattered fields, all scattering properties can be evaluated easily. The expressions for the total scattered power and the total scattering cross section are quite simple. In the case of nose on scattering, the problem reduces to only one system of equations.

2. BACKGROUND:

The scatterer under consideration is a body of revolution which encloses the origin of the coordinate system as shown in Fig. 1. In accoustics, the body is either soft or hard. For scattering of electromagnetic waves, the body is made of perfectly conducting material. Without loss of generality, the incident wave may be assumed to be propagating in the direction $\theta = \theta_o$, and $\phi = 0$; where θ and ϕ are the spherical coordinates with z axis colinear with the symmetric axis of the body. The incident accoustic plane wave is a scalar quantity while the incident electromagnetic plane wave is a vector. Any arbitrarily polarized plane wave can be resolved into two components: one polarized with electric field perpendicular to the plane of incidence; and the other polarized with electric field parallel to the plane of incidence. With respect to the z-axis, the symmetric axis of the body, the perpendicular polarization is a transverse electric (TE) wave while the parallel polarization is a transverse magnetic (TM) wave. For simplicity, the scattering of the TE and the TM waves are considered separately. The scattering properties of the arbitrarily polarized plane waves are simply the superposition of these two solutions.

To simplify the analysis, the unit vectors $\hat{u}$, $\hat{v}$, and $\hat{\phi}$ (see Fig. 2) of the orthogonal coordinate system of the scatterer will be introduced. Their relationship to the Cartesian unit vectors are as follows:

$$\begin{aligned} \hat{u} &= \hat{x} \sin \alpha \cos \phi + \hat{y} \sin\alpha \sin \phi + \hat{z} \cos \alpha \\ \hat{v} &= \hat{x} \cos \alpha \cos \phi + \hat{y} \cos\alpha \sin \phi - \hat{z} \sin \alpha \\ \hat{\phi} &= -\hat{x} \sin \phi + \hat{y} \cos \phi \end{aligned} \tag{1}$$

where $\alpha = \cos^{-1} (\hat{u} \cdot \hat{z})$ is the angle between unit vectors $\hat{u}$ and $\hat{z}$. Using the conventional symbols (r, θ, ϕ) for the spherical coordinate system, it is easy to show that

$$\begin{aligned} \hat{r} &= \hat{u} \cos(\theta - \alpha) + \hat{v} \sin(\theta - \alpha) \\ \hat{\theta} &= -\hat{u} \sin(\theta - \alpha) + \hat{v} \cos(\theta - \alpha) \end{aligned} \tag{2}$$

Note that when $\alpha = \theta$, $\hat{u}$ and $\hat{v}$ are reduced to be $\hat{r}$ and $\hat{\theta}$, and the scatterer becomes a sphere. By means of relationships, (1) and (2), transformations of the coordinate components of a vector among the Cartesian, spherical and the (u, v, ϕ) coordinate systems are given by

$$\begin{aligned} A_u &= A_x \sin\alpha \cos\phi + A_y \sin\alpha \sin\phi + A_z \cos\alpha \\ &= A_r \cos(\theta - \alpha) - A_\theta \sin(\theta - \alpha) \\ A_v &= A_x \cos\alpha \cos\phi + A_y \cos\alpha \sin\phi - A_z \sin\alpha \\ &= A_r \sin(\theta - \alpha) + A_\theta \cos(\theta - \alpha) \\ A_\phi &= -A_x \sin\phi + A_y \cos\phi \end{aligned} \tag{3}$$

Before going into the detail analysis, it is convenient to discuss the expansions of the incident wave functions in terms of cylindrical modes. This is done by expanding the factors, exp. $(j k_x x)$, $\cos\phi$ exp. $(j k_x x)$, and $\sin\phi$ exp. $(j k_x x)$ in a Fourier series of ϕ, where $k_x = k \sin\theta_o$, k is the propagation constant of the medium, and $j = \sqrt{-1}$. It is known that[7]

$$\text{exp.}\,(j k_x x) = \sum_{m=o}^{\infty} \epsilon_m j^m J_m (k_x \rho) \cos m\phi \tag{4}$$

where m is an integer, ϵ_m is the Neumann's number, i.e. $\epsilon_m = 1$ for $m = 0$, $\epsilon_m = 2$ for $m > 0$, J_m is the first kind Bessel function of order m , and $x = \rho \cos\phi$. By utilizing Eq.(4), it can be shown

$$\cos\phi \;\text{exp.}(j k_x x) = - \sum_{m=o}^{\infty} \epsilon_m j^{m+1} J_m{}' (k_x \rho) \cos m\phi \tag{5}$$

$$\sin\phi \;\text{exp.}(j k_x x) = - \sum_{m=o}^{\infty} \epsilon_m j^{m+1} (m/k_x \rho) J_m(k_x \rho) \sin m\phi \tag{6}$$

where the prime denotes the derivative of a function with respect to the argument. Observe that when $\theta_o = 0$, the right hand side of Eqs. (4) - (6) are reduced to unity, $\cos\phi$, and $\sin\phi$, respectively. This is the condition for nose on scattering. By virture of Eqs. (4) - (6), the incident plane waves can be resolved into hamonics of the azimuth angle ϕ.

3. ACCOUSTIC SCATTERING BY BODY OF REVOLUTION

The model considered in this section is a scalor plane wave incident on a body of revolution. The scatterer is either soft or hard. Let the incident plane wave function be normalized and given by

$$\psi^{i} = \exp. \left[i(k_x x + k_z z) \right] \tag{7}$$

where $k_z = k \cos \theta$. The first exponential can be expressed as in Eq. (4). Physically, this means that the incident wave is the superposition of cylindrical modes. Therefore, the point-matching method can be applied to cases where the incident wave can be resolved into cylindrical modes. The scattered wave is governed by the Helmholtz's equation

$$\nabla^2 \psi^s + k^2 \psi^s = 0 \tag{8}$$

where ∇^2 is the Laplacian operator. The boundary conditions for the present problems are

$$\psi^i + \psi^s = 0 \qquad \text{at the surface of a soft scatterer}$$

and

$$\frac{\partial}{\partial u} (\psi^i + \psi^s) = 0 \qquad \text{at the surface of a hard scatterer}$$

where $\frac{\partial}{\partial u}$ is equivalent to the normal differentiation. Solutions of Eq. (8) are known only in a few coordinate systems where the variables can be separated. In general, the method of separation is not applicable. But in many cases, the scattered wave can be expressed by the general solution of one of the separable coordinate systems within practical acceptable approximation. Let the scattered wave be expressed by the general solution of Eq. (8) in spherical coordinate system with specification of outgoing waves, i.e.[8]

$$\psi^s = \sum_{\substack{m=0 \\ n=0}}^{\infty} \epsilon_m h_n^{(2)}(kr)\, p_n^m(\cos\theta) \left[A_{mn} \cos m\phi + B_{mn} \sin m\phi \right] \tag{9}$$

where A_{mn} and B_{mn} are constants determined by boundary conditions $h_n^{(2)}$ is the spherical Hankel function of the second kind, P_n^m is the Legende function of the first kind. From Eqs. (4) and (7), and the boundary conditions, it can be seen that $B_{mn} = 0$ for all m and n; and for each m,

$$j^m J_m(k_x \rho) \exp.(jk_z z) + \sum_{n=m}^{\infty} A_{mn} h_n^{(2)}(kr) P_n^m(\cos\theta) = 0 \text{ at } c \quad (10)$$

for a soft scatterer, and

$$\frac{\partial}{\partial u} [j^m J_m(k_x \rho) \exp.(jk_z z) + \sum_{n=m}^{\infty} A_{mn} h_n^{(2)}(kr) P_n^m(\cos\theta)] = 0 \text{ at } c \quad (11)$$

for a hard scatterer, where c denotes the meridian contour (the x-z plane or the y-z plane) of the scatterer. The expansion coefficients A_{mn} are determined by Eqs. (10) or (11). The operator $\frac{\partial}{\partial u}$ of Eq. (11) may be replaced by

$$\cos\gamma \frac{\partial}{\partial r} + \frac{\sin\gamma}{r} \frac{\partial}{\partial \theta}$$

where $\cos\gamma = \hat{u} \cdot \hat{r}$. To apply the point-matching method[1 - 6], it is assumed that only a finite number of terms of the series expression in Eq. (9) are necessary to retain for good approximation. The infinite summations of Eqs. (9) - (11) may then be replaced by $\sum_{n=m}^{N}$, where N is an integer. Similar to the point-matching method for two-dimensional problems, (N - m) points, namely: (r_1, θ_1) (r_2, θ_2), - - - - - (r_{N-m}, θ_{N-m}), are chosen around the meridian contour of the scatterer and at these points Eqs. (10) and (11) are satisfied. A system of inhomogeneous linear algebraic equations which can be solved for (N - m) expansion coefficients A_{mn} of each Eq. (10) and (11) is formed. That is

$$\left[\sum_{n=m}^{N} h_n^{(2)}(kr) P_n^m(\cos\theta) A_{mn} = -j^m J_m(k_x \rho) \exp.(jk_z z) \right]_{\substack{r = r_q \\ \theta = \theta_q}} \tag{12}$$

for a soft scatterer and

$$\left\{ \left(\cos\gamma \frac{\partial}{\partial r} + \frac{\sin\gamma}{r} \frac{\partial}{\partial \theta} \right) \sum_{n=m}^{N} h_n^{(2)}(kr) P_n^m(\cos\theta) A_{mn} = -j^m J_m(k_x \rho) \exp.(jk_z z) \right\}_{\substack{r = r_q \\ \theta = \theta_q \\ \gamma = \gamma_q}} \tag{13}$$

for a hard scatterer, where $q = 1, 2, ---- (N - m)$, and γ_q is the angle between the radial vector and the normal at point (r_q, θ_q). (See Fig. 2). The expansion coefficients A_{mn} can be obtained easily by a computer using Eq. (12) or (13). The series of Eq. (4) can be truncated in practice. Due to the asymptotic behaviour of the Bessel function, the series can be truncated for m where $J_m(k_x \rho_{max}) << 1$, or approximately $M = k_x \rho_{max} + 6$, where ρ_{max} is the maximum value of ρ of the scatterer. Therefore, in practice, it is only necessary to solve a finite number of systems of equations. Furthermore, one computer program is applicable to all these systems of equations. When all the A_{mn}'s of dominant contributions are found, the scattered wave and the scattering properties are readily determined by utilizing Eq. (9).

4. ELECTROMAGNETIC WAVES SCATTERED BY PERFECTLY CONDUCTING BODIES

In the previous section, it is seen that the three-dimensional accoustic problems can be reduced to the simple forms in the two-dimensional case if the scatterer has rotational symmetry. Similar results for the three-dimensional electromagnetic problems were achieved, though the formulation is more complicated. The present analysis is also started by resolving the tangential components of the incident wave into cylindrical modes by means of Eqs. (4) - (6). And the scattered fields are expressed in terms of the spherical solutions. The TE and TM incident waves are considered separately as follows:

A. TE incident plane wave.

Considering that the incident plane wave is polarized with the electric field perpendicular to the plane of incidence, then the incident wave function may be written as

$$E^i = \hat{y} \exp. [j(k_x x + k_z z)] \tag{14}$$

Observe that this is transverse electric to z. In terms of cylindrical modes, the tangential components of this incident field in the scatterer's coordinate system are given by

$$E_\phi^i = -\exp. (jk_z z) \sum_{m=0}^{\infty} \epsilon_m j^{m+1} J_m'(k_x \rho) \cos m\phi \tag{15}$$

$$E_v^i = -\exp. (jk_z z) \cos\alpha \sum_{m=1}^{\infty} \epsilon_m j^{m+1} (m/k_x \rho) J_m(k_x \rho) \sin m\phi \tag{16}$$

where Eqs. (3), (5) and (6) have been applied to Eq. (14). Since the scatterer is not uniform in the z-direction, the scattered field can not be a pure TE wave, but can be expressed by the superposition of TE and TM waves. A more convenient way to obtain an expression for the scattered field is the superposition of TE_r and TM_r waves in spherical coordinate system, where

TE_r and TM_r denote the transverse electric and the transverse magnetic with respect to the r-direction. Conventionally, the TE_r and TM_r outgoing waves are generated by constructing the magnetic and electric vector potentials respectively in the following forms:[7]

$$A^s = \hat{r} \sum_{\substack{m=1 \\ n=m}}^{\infty} a_{mn} G_{mn}(r,\theta) \sin m\phi \tag{17}$$

$$F^s = \hat{r} \sum_{\substack{m=0 \\ n=m}}^{\infty} b_{mn} Z G_{mn}(r,\theta) \cos m\phi \tag{18}$$

where $G_{mn}(r,\theta) = kr\, h_n^{(2)}(kr)\, P_n^m(\cos\theta)$. The a_{mn}'s and b_{mn}'s are constants to be determined by boundary conditions, and Z is the intrinsic impedance of the medium. The expansion coefficients a_{mn} and b_{mn} are of the same physical units due to the introduction of Z in Eq. (18) which shows convenience in the latter applications. Since the plane of incidence is the x-z plane, the choices of $\cos m\phi$ and $\sin m\phi$ are shown as in Eqs. (17) and (18).

With the assumed form of the magnetic and electric vector potential, the r, θ, and ϕ components of the scattered electric field can be derived easily as shown in text books.[7,8] Using the derived expressions for the r, θ, and ϕ components in Eq. (3), it can be shown that the ϕ and v components of the scattered electric field are given by

$$E_\phi^s = Z \sum_{m,n} \left[(-j/r \sin\theta) m \frac{\partial G_{mn}}{\partial \zeta} a_{mn} + \frac{1}{r} \frac{\partial G_{mn}}{\partial \theta} b_{mn} \right] \cos m\phi \tag{19}$$

$$E_v^s = Z \sum_{m,n} \left\{ (-j/r)[n(n+1) \sin(\theta - \alpha)(G_{mn}/\zeta) + \cos(\theta - \alpha) \frac{\partial^2 G_{mn}}{\partial \zeta \, \partial \theta} \right] .$$

$$\left. \cdot a_{mn} + [\cos(\theta - \alpha)/r \sin\theta]\, m\, G_{mn}\, b_{mn} \right\} \sin m\phi \tag{20}$$

where $\zeta = kr$, and the limits of the summations are the same as those in Eqs. (17) and (18). The boundary conditions require $E^i + E^s = 0$ and $E_v^i + E_v^s = 0$ at the surface of the conducting body, it follows that

$$\sum_{n=m}^{\infty} \left[(-j/r \sin\theta) m \frac{\partial G_{mn}}{\partial \zeta} a_{mn} + \frac{1}{r} \frac{\partial G_{mn}}{\partial \theta} b_{mn} \right] = \exp.(jk_z z)\, \epsilon_m\, j^{m+1} J_m{}'(k_x \rho) / Z \tag{21}$$

$$\sum_{n=m}^{\infty} (-j/r)[n(n+1) \sin(\theta - \alpha)(G_{mn}/\zeta) + \cos(\theta - \alpha) \frac{\partial^2 G_{mn}}{\partial \zeta \partial \theta}]\, a_{mn} + [\cos(\theta - \alpha)/r \sin\theta]\, m\, G_{mn}\, b_{mn} \tag{22}$$

$$= \exp.(jk_z z) \cos\alpha\, \epsilon_m\, j^{m+1}\, m\, J_m(k_x \rho)/k_x\, \rho Z$$

at the meridian contour of the rotational symmetric body. Observe that these two equations are valid for all m except that when $m = o$, the last equation does not exist. As in accoustics, the series expressions for the incident waves can be truncated for M where $J_M(k_x \rho_{max}) << 1$. Hence, there are only a finite number of Eqs. (21) and (22) necessary in practical applications. Again, in order to utilize the point-matching technique, the infinite summations of Eqs. (19) - (22) are replaced by finite summations with limits from $n = m$ to N, where N is an integer. Since Eqs. (21) and (22) must be satisfied simultaneously, 2(N-m) points are chosen along the meridian contour of the scatterer where these two equations hold. A system of 2(N-m) simultaneous algebraic equations with 2(N-m) unknowns for each m are formed:

$$\left\{ \sum_{n=m}^{N} \left[(-j/r \sin\theta) m \frac{\partial G_{mn}}{\partial \zeta} a_{mn} + \frac{1}{r} \frac{\partial G_{mn}}{\partial \theta} b_{mn} \right] \right.$$

$$= \exp.(jk_z z)\,\epsilon_m\, j^{m+1} J_m'(k_x\rho)/Z \Big\}_{\substack{r = r_q \\ \theta = \theta_q}}$$

$$\sum_{n=m}^{\infty} \left\{ (-j/r)[n(n+1)\sin(\theta - \alpha)\, G_{mn}/\zeta + \cos(\theta - \alpha)\frac{\partial^2 G_{mn}}{\partial\zeta\,\partial\theta}\right]a_{mn} \tag{23}$$

$$+ [\cos(\theta - \alpha)/r \sin\theta]\, m\, G_{mn}\, b_{mn} \Big\}_{r = r_q,\ \theta = \theta_q,\ \alpha = \alpha_q}$$

$$= [\exp.(jk_z z)\cos\alpha\, \epsilon_m\, j^{m+1}\, m\, J_m(k_x\rho)/k_x\rho Z]_{\substack{r = r_q \\ \theta = \theta_q \\ \alpha = \alpha_q}}$$

where (r_q, θ_q) is a point at the meridian contour of the body, α_q is the angle α evaluated at the point (r_q, θ_q), $q = 1, 2, 3, \ldots\ldots 2(N - m)$ for $m \neq 0$; $q = 1, 2, \ldots\ldots N$, and $a_{on} = 0$ for $m = 0$. Similarly, these systems of equations represented by Eq. (23) can be solved numerically for the expansion coefficients a_{mn} and b_{mn} by a computer without difficulty. The program must run $(M + 1)$ times.

B. TM incident plane wave.

The analysis of the scattering of a TM plane wave is quite similar to that of a TE plane wave. The incident wave is polarized with the electric field parallel to the plane of incidence. Let the normalized electric field vectoral function be given by

$$E^i = (-\hat{x}\cos\theta_o + \hat{z}\sin\theta_o)\exp.[j(k_x x + k_z z)] \tag{24}$$

Using Eqs. (3) - (6), one obtains the tangential components in the scatterer's coordinate system of the incident field such as

$$E_\phi^i = -\cos\theta_o \exp.(jk_z z) \sum_{m=1}^{\infty} \epsilon_m\, j^{m+1} (m/k_x\rho)\, J_m(k_x\rho)\sin m\phi \tag{25}$$

$$E_v^{\,i} = \exp.\,(jk_z z) \sum_{m=0}^{\infty} [\,j \cos\theta_o \cos\alpha\, J_m{}'\,(k_x\rho) - \sin\theta_o \sin\alpha\, J_m\,(k_x\rho)\,] \cdot j^m \epsilon_m \cos m\phi \tag{26}$$

Again, it is convenient to express the scattered fields by the superposition of outgoing TE_r and TM_r waves in spherical coordinate system. As in the TE case, these outgoing TE_r and TM_r waves can be derived from the magnetic and electric vector potentials which are given by

$$A^s = \hat{r} \sum_{\substack{m=0 \\ n=m}}^{\infty} a_{mn}\, G_{mn} \cos m\phi \tag{27}$$

$$F^s = \hat{r} \sum_{\substack{m=1 \\ n=m}}^{\infty} b_{mn}\, Z\, G_{mn} \sin m\phi \tag{28}$$

where the symbols are as previously given. Note that due to the difference in polarization of the incident waves, the choices of cos $m\phi$ and sin $m\phi$ in Eqs. (27) and (28) are different from those of Eqs. (17) and (18). Following the steps as stated in the TE case, from Eqs. (27) and (28), and utilizing Eq. (3), one can derive the ϕ and v components of the scattered electric fields which are given by

$$E_\phi^{\,s} = Z \sum_{m,n} [\,j/r \sin\theta)\, m \frac{\partial G_{mn}}{\partial \zeta}\, a_{mn} + \frac{1}{r} \frac{\partial G_{mn}}{\partial \theta}\, b_{mn}\,] \sin m\phi \tag{29}$$

$$E_v^{\,s} = Z \sum_{m,n} \{\, (1/jr)[\sin(\theta-\alpha)\, n(n+1)\, G_{mn}/\zeta + \cos(\theta-\alpha) \frac{\partial^2 G_{mn}}{\partial \zeta\, \partial \theta}\,] a_{mn} - [\cos(\theta-\alpha)/r \sin\theta\,]\, m\, G_{mn}\, b_{mn}\,\} \cos m\phi \tag{30}$$

The boundary conditions are the same as before, i.e., $E_\phi^i + E_\phi^s = 0$ and $E_v^i + E_v^s = 0$ at the surface of the conducting body. The point-matching technique is used to evaluate the expansion coefficients A_{mn} and b_{mn}. The coefficient evaluation procedure is the same as in Section 4A, a system of 2(N-m) [N for m = 0] inhomogeneous algebraic equations with 2(N-m) [N for m = 0] unknowns are formed for each m. They are

$$\left\{ \sum_{n=m}^{N} \left[(z/r \sin\theta)\, m \frac{\partial G_{mn}}{\partial \zeta} a_{mn} + \frac{1}{r} \frac{\partial G_{mn}}{\partial \theta} b_{mn} \right] = \cos\theta_o \exp.(jk_z z)\, m\, \epsilon_m (j^{m+1}/Z k_x \rho) J_m(k_x \rho) J_m(k_x \rho) \right\}_{\substack{r=r_q \\ \theta=\theta_q}}$$

$$\sum_{n=m}^{N} \left\{ (j/r)[\sin(\theta-\alpha)\, n(n+1)\, G_{mn}/\zeta + \cos(\theta-\alpha) \frac{\partial^2 G_{mn}}{\partial\zeta\,\partial\theta}] a_{mn} - [\cos(\theta-\alpha)/r\sin\theta]\, m\, G_{mn}\, b_{mn} \right\}_{\substack{r=r_q \\ \theta=\theta_q \\ \alpha=\alpha_q}} \tag{31}$$

$$= \left\{ \exp.(jk_z z)\, j^m \epsilon_m\, [j \cos\theta_o \cos\alpha\, J_m'(k_x\rho) - \sin\theta_o \sin\alpha\, J_m(k_x\rho)] \right\}_{\substack{r=r_q \\ \theta=\theta_q \\ \alpha=\alpha_q}}$$

where (r_q, θ_q) is a point at the meridian contour of the body, α_q, is the angle α evaluated at the point (r_q, θ_q), q = 1, 2, ... 2(N - m) for $m \neq 0$; while q = 1, 2, ... N and $b_{on} = 0$ for m = 0. Similarily, Eq. (31) can be solved for a_{mn}'s and b_{mn}'s by a computer and in practical applications, only a finite number of m is considered.

5. NOSE ON SCATTERING

In the case of nose on scattering, i.e., $\theta_o = 0$, the problems are simpler. Under this condition, the right hand sides of Eqs. (4) - (6) are reduced to 1, $\cos\phi$, and $\sin\phi$ respectively. Eqs. (12), (13), (23) and (31) are reduced to only one system of equations, i.e., $m = 0$ for Eqs. (12) and (13), $m = 1$ for Eqs. (23) and (31). Hence, in each case, it is necessary to evaluate only one system of equations. Note that in the electromagnetic scattering problem, two cases are the same except with a phase of 90 degrees difference in space. Both cases are reducable to those formulated by Schultz et al.[9]

6. THE SCATTERING CROSS-SECTIONS.

In accoustics, the normalized total scattered power can be defined as

$$P_{at} = \int_{\Omega} |\psi^s|^2 r^2 \, d\Omega \tag{32}$$

where Ω is the solid angle. Using the orthogonal relationships, of the spherical harmonics, the integrations of Eq. (32) are easy to perform and the total scattered power is given by

$$P_{at} = 4\pi \sum_{\substack{m=0\\n=m}}^{N,M} \epsilon_m \, |h_n^{(2)}(kr)|^2 \frac{r^2}{2n+1} \frac{(n+m)!}{(n-m)!} |A_{mn}|^2 \tag{33}$$

At large distances, the factor $|h_n^{(2)}(kr)|$ can be replaced by $1/kr$. Therefore, the total scattered power at large distances from the scatterer is inversely proportional to the square of the frequency.

In the scattering of electromagnetic waves, the scattered fields can be determined by Eqs. (17), (18) and (27), (28) for the TE and the TM cases, respectively. The expansion coefficients a_{mn} and b_{mn} are computed by the point matching method. For far field considerations, it is convenient to express the scattered field in spherical components. The ϕ components for both cases are given by Eqs. (19) and (29), respectively. The θ component for the TE case is given by

$$E_\theta^{\,s} = \sum_{\substack{n=m\\m=1}}^{N,M} \left[(Z/jr)\, a_{mn} \frac{\partial^2 G_{mn}}{\partial \zeta \, \partial \theta} + (1/r \sin\theta)\, m\, b_{mn} G_{mn} \right] \sin m\phi \tag{34}$$

while

$$E_\theta^{\,s} = \sum_{\substack{m=0\\n=m}}^{N,M} \left[(Z/jr)\, A_{mn} \frac{\partial^2 G_{mn}}{\partial \zeta \, \partial \theta} - (1/r \sin\theta)\, b_{mn}\, m\, G_{mn} \right] \cos m\phi \tag{35}$$

for the TM case. The r component is omitted since it is negligible when compared with the θ and ϕ components in the far field region. The scattering cross-section in a particular direction (θ_p, ϕ_p) can be defined by

$$A_s(\theta_p, \phi_p) = 4\pi R^2 \; |E^s(R, \theta_p, \phi_p)/E^i|^2$$

where $|E^s(R, \theta_p, \phi_p)|^2 = |E_\theta{}^s(R, \theta_p, \phi_p)|^2 + |E_\phi{}^s(R, \theta_p, \phi_p)|^2$

and R is the distance between the observation point and the origin.

The total scattered power is given by

$$P_t = \frac{R^2}{Z} \int_\Omega \left(|E_\theta|^2 + |E_\phi|^2 \right) d\Omega \tag{36}$$

Substituting Eqs. (19) and (32), or (29) and (33) into Eq. (34), and noting that the orthogonal relationships of the trigonometric functions and

$$\int_0^\pi \left[\frac{dP_n{}^m}{d\theta} \; \frac{dP_\ell{}^m}{d\theta} + (m/\sin\theta)^2 \, P_n{}^m \, P_\ell{}^m \right] \sin\theta \, d\theta$$

$$= \frac{2}{2n+1} \; \frac{(n+m)!}{(n-m)!} \; n(n+1) \; \delta_{n\ell}$$

yields.

$$P_t = (2\pi Z) \sum_{\substack{m=0 \\ n=m}}^{N,M} (1/\epsilon_m) \left[2n(n+1)/(2n+1) \right] \left[(n+m)!/(n-m)! \right] \left[|b_{mn}|^2 + |a_{mn}|^2 \right] \tag{37}$$

where the asymptotic values of the spherical Bessel functions

$$\lim_{r \to \infty} h_n^{(2)}(kr) = j^{n+1} \; \exp.(-jkr)/kr$$

and $$\lim_{r \to \infty} \frac{d}{dr} \left[r h_n^{(2)}(kr) \right] = j^n \; \exp.(-jkr)$$

have been used. Eq. (35) is valid for both the TE and the TM cases, of course, the values of a_{mn} and b_{mn} are different. The total scattering cross section is defined as the ratio of the total scattered power to the incident power density. Thus,

$$\sigma_t = (2\pi Z^2) \sum_{\substack{m=0 \\ n=m}}^{N,M} (1/\epsilon_m)[2n(n+1)(n+m)!/(2n+1)(n-m)!][|a_{mn}|^2 + |b_{mn}|^2] \qquad (38)$$

In this formulation, the expansion coefficients a_{mn} and b_{mn} are proportional to 1/Zk, hence, the total scattering cross section is , in fact, expressed in forms of $1/k^2$.

7. EXAMPLE.

To demonstrate the accuracy of the point-matching method for this application, the scattering of a plane wave by a conducting sphere will be considered. The good agreement between the approximate solutions and the exact answers shows that the boundary conditions are approximately satisfied when applying the point-matching technique.[6,7]

Consider the plane wave $E^i = -x \exp.(jkz)$ scattered by a perfectly conducting sphere of radius a. Since $\theta_o = 0$ and $\alpha = \theta$, Eq. (31) is reduced to a quite simple form. In Table I, the expansion coefficients a_n and b_n, calculated by the point-matching method, are compared with those of rigorous solutions for $ka = 1$. The electromagnetic field satisifes the boundary conditions exactly at three points for the three-point approximation, while the field satisfies the boundary conditions at four points for the four-point approximation. The chosen points in these calculations are $r = a$, $\theta = 0$, $90°$, and $180°$ for the three-point approximation; $r = a$, $\theta = 0$, $60°$, $120°$, and $180°$ for the four-point approximation. Note that the points of $\theta = 0°$ and $\theta = 180°$ give the same algebraic equation. If the point $r = a$, $\theta = 0°$ is chosen, the solutions satisfy the boundary conditions at the point $r = a$, $\theta = 180°$ automatically. Therefore, the three-point approximation has a system of four equations and the four-point approximation has a system of six equations only. Of course, if neither $\theta = 0°$ nor $\theta = 180°$ is chosen, the situation is the same as discussed previously. One should note that degenerate equations may arise in other cases.

The exact solutions in Table I are obtained by assuming that the magnetic and electric vector potentials for the scattered field are given by

$$A^s = \hat{r} \sum_{n=1}^{\infty} a_n G_{1n} \cos\phi \tag{39}$$

$$F^s = \hat{r} \sum_{n=1}^{\infty} b_n Z G_{1n} \sin\phi \tag{40}$$

TABLE I

Comparison of the Expansion Coefficients of a Sphere Computed by the Point-Matching Method and the Exact Solutions at ka = 1

	Three-Point Approximation	Four-Point Approximation	Exact
a_1	$-0.001719 + j0.001104$	$-0.001821 + j0.0011694$	$-0.001809 + j0.001162$
b_1Z	$-0.29068 - j0.06336$	$-0.31501 - j0.068659$	$-0.31211 - j0.068027$
a_2	$-(0.19378 + j6.377) \times 10^{-5}$	$-(0.19678 + j6.4759) \times 10^{-5}$	$-(0.20391 + j6.7103) \times 10^{-5}$
b_2Z	$(0.03656 - j2.1246) \times 10^{-2}$	$(0.02401 - j1.3956) \times 10^{-2}$	$(0.02467 - j1.434) \times 10^{-2}$
a_3		$(11.526 - j0.008712) \times 10^{-7}$	$(11.695 - j0.00884) \times 10^{-7}$
b_3Z		$(4.1795 + j0.002262) \times 10^{-4}$	$(3.157 + j0.00171) \times 10^{-4}$

respectively. The scattered field obtained from Eqs. (39) and (40) plus the incident field satisfies the boundary conditions exactly at $r = a$. The good agreements of the four-point approximation with the exact solutions reveals that the boundary conditions are satisfied quite well by the point-matching technique.

8. DISCUSSION.

It was shown that the scattering by rotational symmetric bodies in accoustics is quite similar to the two-dimensional problems as those discussed by Yee[5] and Mullin et al.[4] For each cylindrical mode of the incident wave, a corresponding scattered field can be obtained by the point-matching method, similar to the solution of the two-dimensional problems except the Hankel functions are replaced by the spherical Hankel functions. One can easily convince himself that the validity of the point-matching method for each cylindrical mode is the same as those in the references [4] - [6]. That is, the method works well and gives acceptable numerical results to bodies of smooth meridian contour which are not gross perturbations from the circular,[4] and at low frequencies, the boundary conditions are satisfied around the meridian contour of the body as shown in Figs. 5 and 6 of references [5]. Obviously, this method is not applicable to needles or dishes. Of course, all these statements for the applicability of the point-matching method are valid when applied to the superposition of the cylindrical modes, i.e. to the scattering by rotational symmetric bodies.

Returning to the electromagnetic problems now, for each cylindrical mode of the incident wave, the same situation arises except that two tangential components of the electric field satisfy the boundary condition simultaneously. The applicability of the point-matching method for each component is the same as in accoustics. It can then be concluded that the same statements for the applicability of the method are valid for scattering of electromagnetic waves by perfectly conducting bodies of rotational symmetry.

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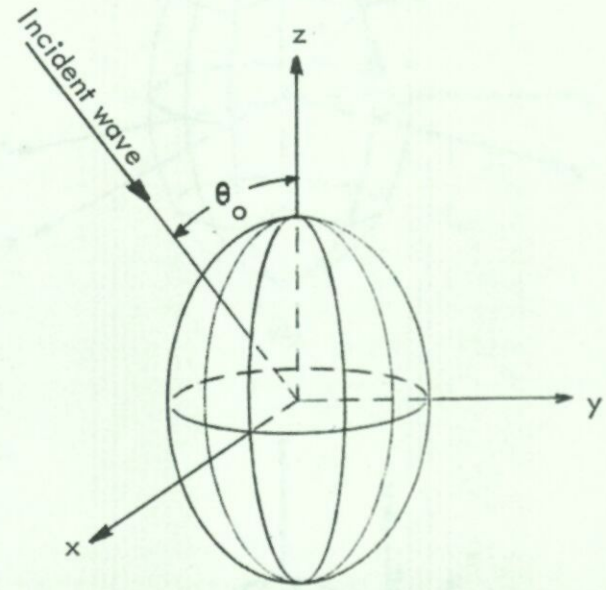

Fig. 1 - The scatterer and the incident wave in the cartesian coordinate system.

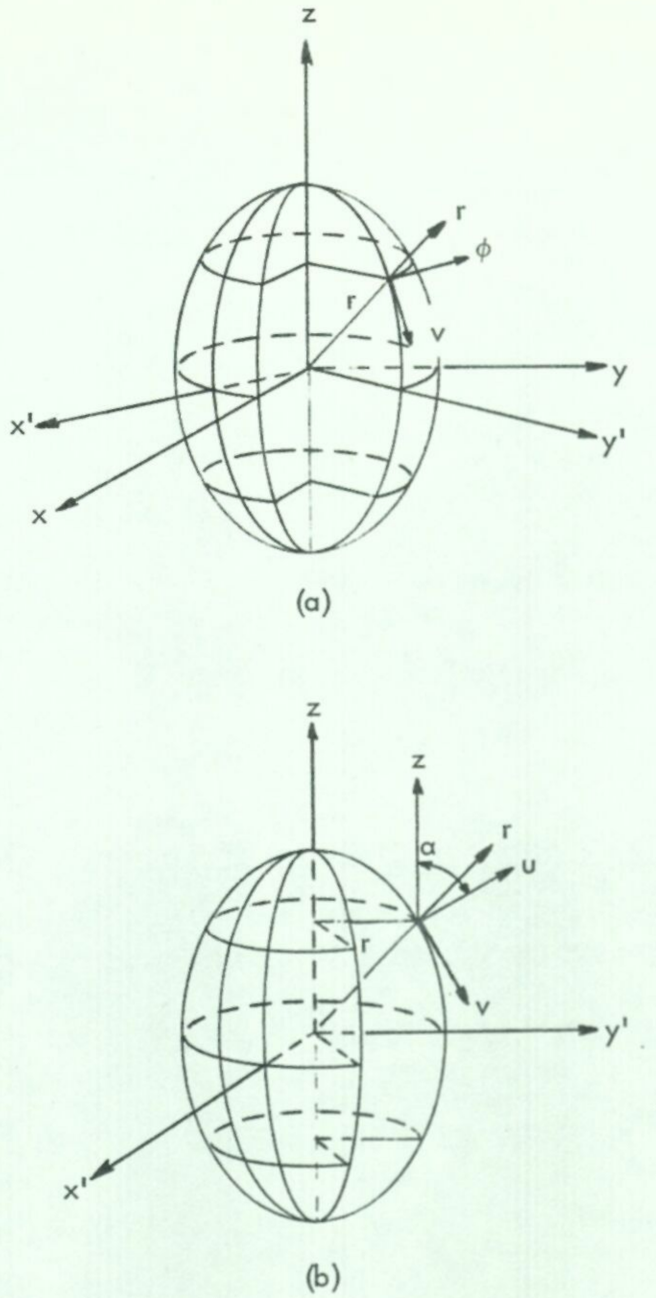

Fig. 2 – (a) The unit vectors v and ϕ of the scatterer's coordinate system and the cartesian coordinate system.

(b) The unit vectors u and v of the scatterer's coordinate coordinate system and the cartesian coordinate system [A rotational view of (a)].

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