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Detection of Pulse Code Modulated Signals with Adaptive Decision Circuits UARI Research Report No. 33

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"Task A"

University of Alabama Research Institute
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Dr. R. J. Polge is principal investigator and is especially responsible for "Part I" of the report, while Dr. J. C. Chang is especially responsible for "Part II."

CONTENTS OF PART I

LIST OF SYMBOLS FREQUENTLY USED

A	Peak amplitude of a single filtered pulse; Subscript for adaptive threshold
C	Capacity; subscript for constant threshold
D	Threshold level; symbol for threshold detector
E	Average probability of all possible errors
$E_{1C}(s_1^*)$	Average probability of error of the first type knowing the sampled value s_1^*
f(x)	Probability density of the random variable x
$f(s_1/s_1^*)$	Conditional probability density of s_1, knowing s_1^*
F	Filter; F(f) transfer function of F
G	Reduction in decibels of the average probability of error or increase of the signal to noise, ratio, as specified.
G_f, G_i	Power density spectrum at the output and input of F respectively, in general $G_i = G_i(f)$ and $G_f = G_f(f)$
i	Subscript for input; integer
I(x)	Integral, $I(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt$
K	Normalized width of the pulse (width KT)
K*	Normalized interval between sampled and detected value (interval K*T)
ℓ	$\ell = -1$ for error of the first type, $\ell = +1$ for error of the second type
n(t)	Filtered noise: $n(t = t_1) = n_1$, $n(t = t_1^*) = n_1^*$
$n_i(t)$	Input noise
$p_i(t)$	Input rectangular random pulses
$P(x > D)$	Probability that x is larger than D
P_s	Average power, $P_s = \frac{V^2 K}{2}$
Q	Fictitious signal to noise ratio, $Q = \frac{V^2 KT}{2\eta}$
R	Resistance; autocorrelation function

$s(t)$	Noisy signal at the input of the threshold detector $s(t = t_1) = s_1$, $s(t = t_1^*) = s_1^*$
$s_i(t)$	Input noisy signal, $s_i(t) = n_i(t) + p_i(t)$
t	Time; t_1^* sampling time; t_1 detection time.
T	Pseudo period
u	Ratio of the pulse width by the RC time constant of the filter, $u = Ky$
V	Height of input rectangular random pulse
y	$y = T/RC$, y determines the RC filter
1	Binary digit corresponds to one pulse; subscript which means $t = t_1$
0	Binary digit corresponds to no pulse; subscript which means $t = 0$
β	The RC noise can be obtained from the white noise using an RC filter and $\beta = T/RC$.
η	$\eta/2$ is the power density spectrum for white noise
$\rho(\tau)$	Normalized autocorrelation function of $n(t)$
ρ^*	Autocorrelation coefficient of n_1^* with n_1
σ_i^2, σ^2, σ_1^2	Variances of $n_i(t)$, $n(t)$ and n_1, respectively
*	Subscript which means sampling
/	Inside the brackets of f( ) or P( ), it means conditional
( )	Around 1 or * , it means experimental as opposed to theoretical value

DETECTION OF PULSE CODE MODULATED SIGNALS WITH ADAPTIVE DECISION CIRCUITS

Chapter I

INTRODUCTION

This paper studies the detection of coded pulse type signals in the presence of noise or jamming signals. A decision circuit compares the noisy signal to a threshold to determine whether or not a pulse was sent. The average probability of error is reduced if an adaptive threshold is used instead of a constant threshold; this improvement may be interesting in the reception of command and guidance signals, detection of coded radar signals and other applications, especially those involving transmissions where high data rates are important. The threshold is a function of the sequence of pulses and the predicted noise. The principle of adaptive threshold is explained in Chapter II. Chapter III and Chapter IV provide two examples of adaptive threshold.

In Chapter III, a pulse code modulated signal mixed with RC noise is detected with a threshold device. It is shown that the average probability of error in the detection is reduced when the noise is sampled before detection and the threshold level varied accordingly.

In Chapter IV, a pulse code modulated signal mixed with white normal noise is first filtered with a RC network and then detected with a threshold device. It is shown that the average probability of error in the detection is reduced when the filtered signal, which is a function of the noise and also the previous sequence of pulses, is sampled before detection and the threshold level varied accordingly. The gain in average probability of error, the optimum width of the pulse and the optimum RC of the filter are also determined.

In Chapters V and VI, an integrator is used during the interval of time where a pulse may be present. An adaptive integrator is investigated whereby the noise is sampled before integration, so that the "expected" value of the noise during the interval of integration can be computed by correlation technique; then a "corrected signal" equal to the unknown signal minus the "expected" value of the noise, is integrated.

In Chapter V, a pulse code modulated signal mixed with RC noise is processed through an integrator and then detected with a threshold device. It is shown that while a standard integrator already reduces the average probability of error in the detection, an adaptive integrator reduces the average probability of error considerably more.

In Chapter VI, the pulse code modulated signal mixed with RC noise is (1) detected by a linear detector, (2) integrated by a standard or adaptive integrator and (3) detected by a threshold device. The problam is much more complex because the linear detector is a non linear device, the probability distribution does not remain normal and the superposition principle does not apply. It is shown that the variance of the signal before threshold detection is considerably reduced when an adaptive integrator is used, which of course means a smaller average probability of error.

Chapters VII and VIII are the experimental verification of Chapter IV. Chapter VII explains the block diagram of the experimental set up. The pure signal, a train of rectangular pulses, is mixed with white noise, then filtered with an RC network, and finally is detected with a constant or with an adaptive threshold. The errors in the detection are determined by comparing the detected signal to the original pure signal in a coincidence circuit. The errors are counted to compare the constant and adaptive threshold. Chapter VIII discusses the measurements and give the experimental results. Special circuits are described in Appendix D.

Chapter IX is both the review and the conclusion of Part I. It compares the adaptive schemes discussed in Part I.

The Appendices A, B, and C concern the evaluation of the average probability of error of Chapter IV. Appendix A shows that the double integral which represents the average probability of error in the case of adaptive threshold can be condensed into one integral. Appendix B explains the computation of the average probability of error using a digital computer. Appendix C gives a technique to expand the average probability of error as a power series.

Appendix E compares the RC filter and the integrator for constant or adaptive threshold and for RC type or white noise. It completes and correlates the results of Chapters III, IV, and V.

Chapter II

DETECTION OF A PULSE CODE MODULATED SIGNAL

The block diagram of a PCM receiver which consists of a filter, F, followed by a threshold detector, TD, is shown in Figure II-1.

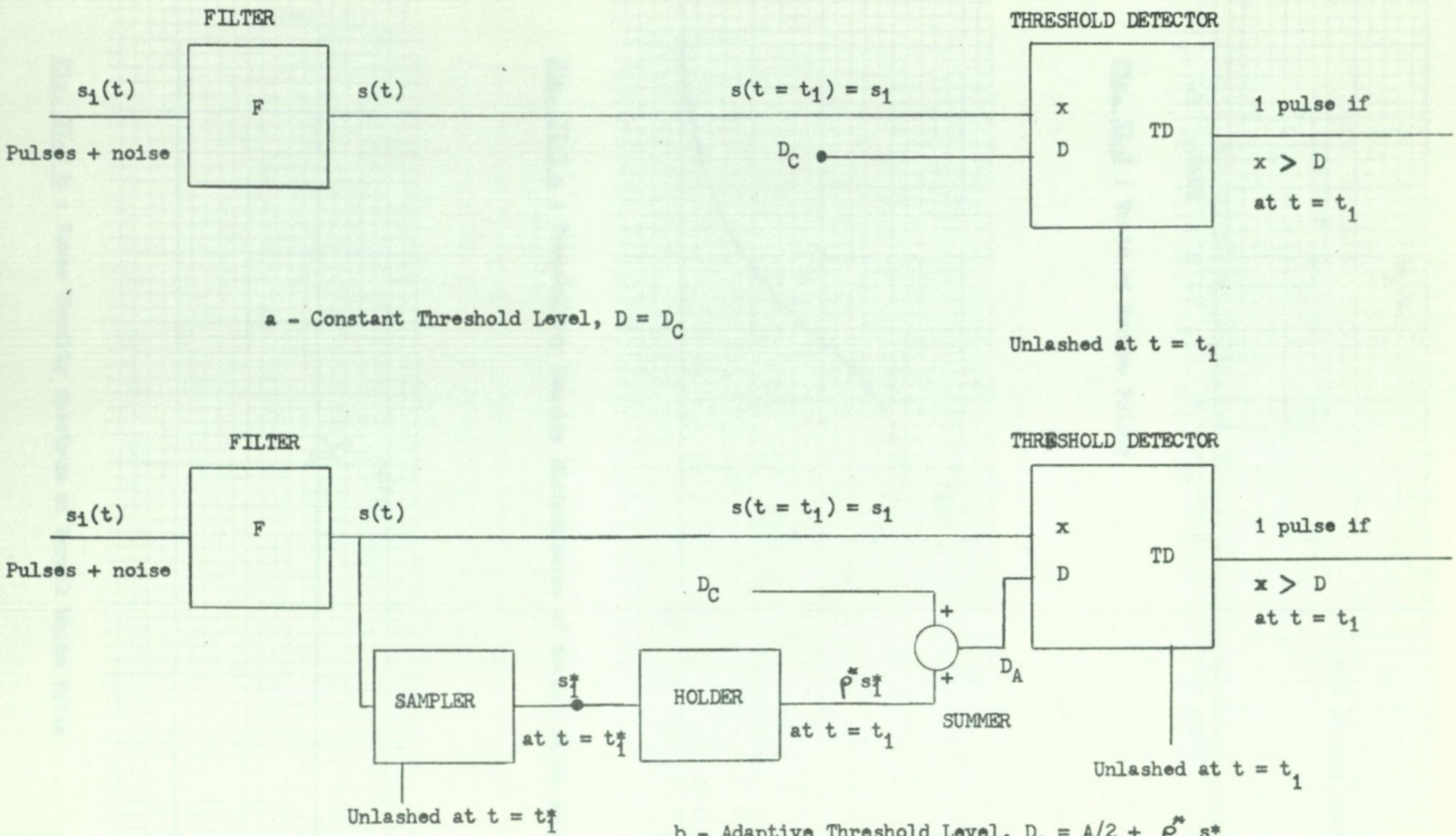
The input of the receiver, $s_i(t)$, is the sum of a binary coded signal represented by a train of random pulses (one for a pulse, zero for no pulse), and of a random noise, $n_i(t)$. The origin of time is chosen such that the pulses may be present only in the interval of time $mT < t < mT + KT$ where T is the pseudo period, KT is the width of the pulse, and m is an integer positive, negative or zero. The instantaneous amplitude of the train of rectangular random pulses, shown in Figure II-2, is defined by $p_i(t)$.

$$p_i(t) = \gamma_m V \qquad \text{for } mT < t < mT + KT$$
$$p_i(t) = 0 \qquad \text{elsewhere} \qquad \text{(II-1)}$$

where V is the amplitude of a pulse and γ_m is a random variable; $\gamma_m = 1$ if the mth pulse is present and $\gamma_m = o$ otherwise. The set of probabilities of γ_m does not depend on m for a stationary process. The capacity of the binary channel is maximum when the probabilities of zero and one are equal. This interesting case is assumed here

$$P(\gamma_m = 0) = P(\gamma_m = 1) = 1/2 \qquad \text{(II-2)}$$

The inputs of the threshold detector are marked X and D. The output of the detector is a pulse if and only if $X > D$ while the detector is unlashed. This occurs once every pseudo period at time $t = mT + KT$. D is called threshold level. Denote the output of F by $s(t)$ and consider the detection of the random pulse defined by $m = 1$; since the detector is unlashed only at time $t_1 = T + KT$, $s(t_1) = s_1$ is the only value of $s(t)$ which matters in the constant threshold detection; there $X = s_1$ and $D = D_C = A/2$, where D_C is the constant threshold level and A is the value of s_1 when the noise is null.



a - Constant Threshold Level, $D = D_C$

b - Adaptive Threshold Level, $D_A = A/2 + \rho^* s_1^*$

<u>Fig. II-1</u> : Block Diagram of a Pulse Code Modulation Receiver

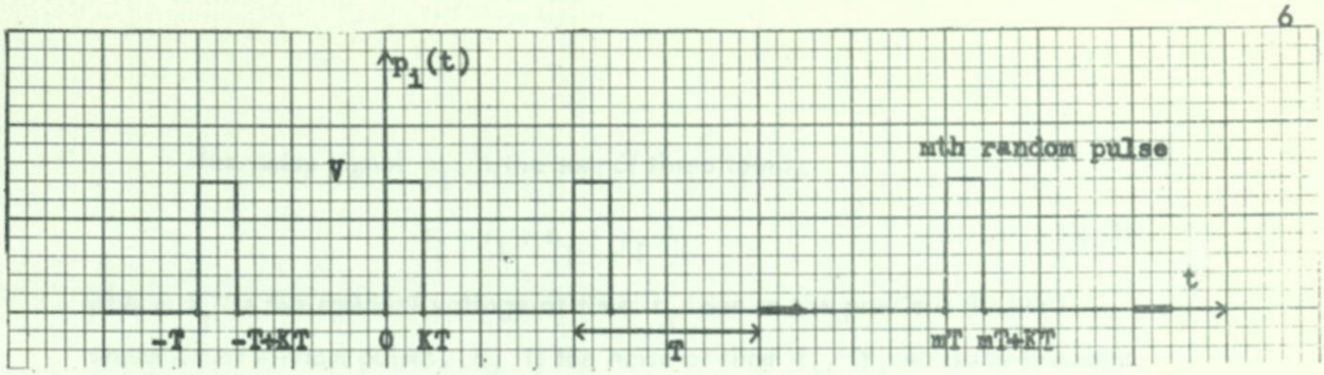


Fig. II-2 : Train of Random Pulses

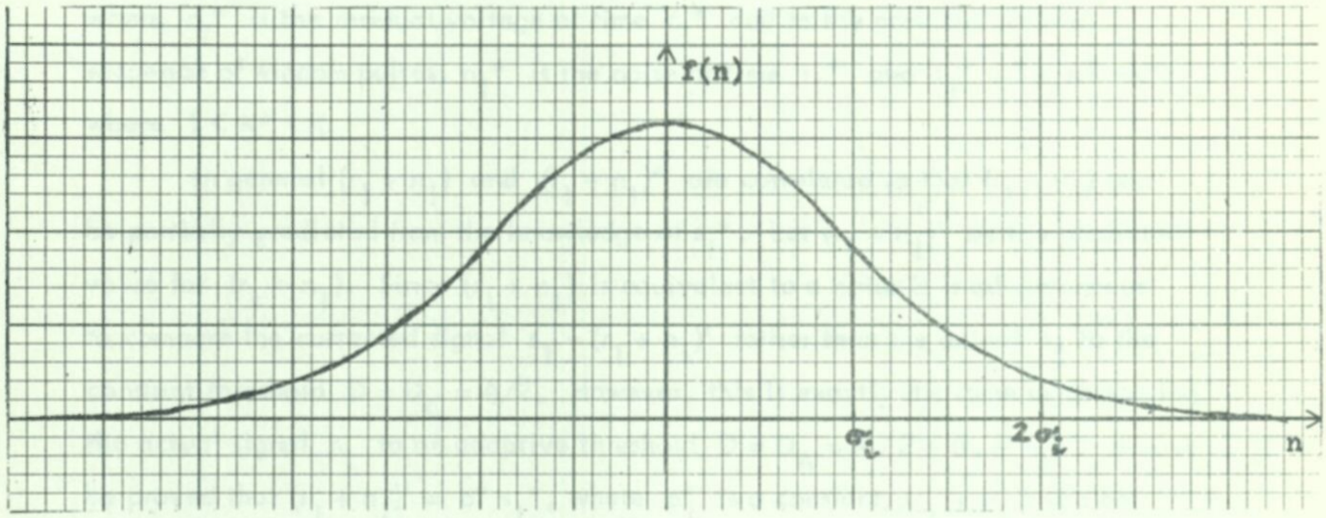


Fig. II-3 a : Probability Density Distribution of Normal White Noise

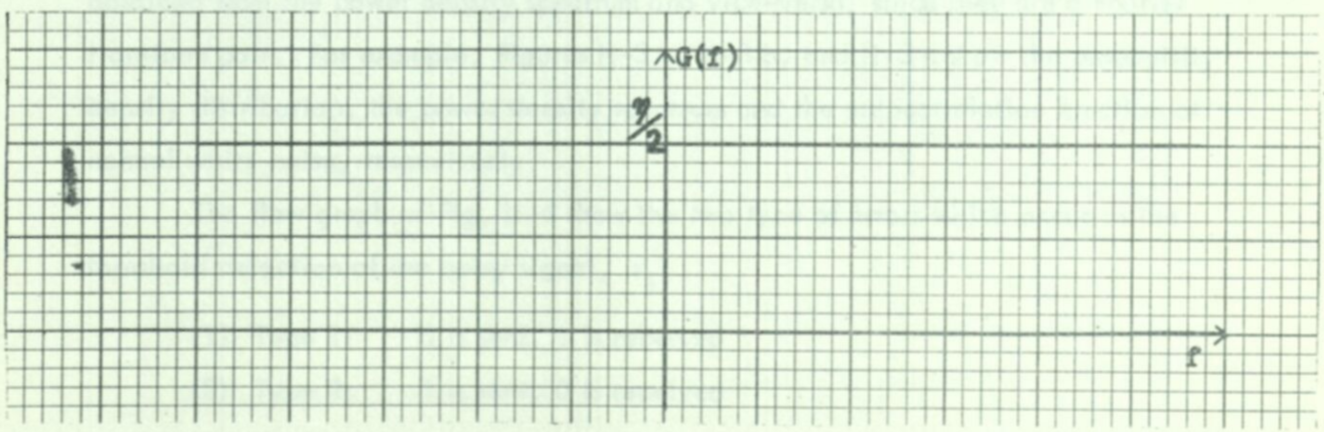


Fig. II-3 b : Power Density Spectrum of Normal White Noise

The noisy signal s_1 is the sum of three terms,

$$s_1 = r_1 + n_1 + \gamma_1 A \qquad \text{(II-3)}$$

where r_1 is the residual voltage at time t_1 due to the previous (known) sequence of random pulses, n_1 is the noise at time t_1, $\gamma_1 A$ is the unknown random pulse starting at time $t = T$. The detector must determine whether the pulse is present or not ($\gamma_1 = 1$ or 0)

The sampled signal, $s(t_1^*) = s_1^*$, at time t_1^* before the unknown random pulse is the sum of 2 terms, $s_1^* = r_1^* + n_1^*$ (II-4)
where r_1^* is the residual voltage at time t_1^* due to the previous (known) sequence of random pulses, n_1^* is the noise at time t_1^* and $t_1^* = T + KT - K^*T$ where $1 < K^* < K$

In general $(r_1 + n_1)$ and $(r_1 + n_1)^*$ are correlated so that $(r_1 + n_1)$ can be partially predicted from the knowledge of s_1^*. Let $(r_1 + n_1)'$ be the predicted value for $(r_1 + n_1)$. Since $(r_1 + n_1)$ is equivalent to a noise it is advantageous to compare the corrected signal $s_1 - (r_1 + n_1)'$ rather than the actual s_1 to the constant threshold $D = D_C = A/2$. However, this is exactly the same as comparing the actual signal s_1 to an adaptive threshold $D = D_A = A/2 + (r_1 + n_1)'$. It will be proved that $(r_1 + n_1)' = \rho^* s_1^*$, where ρ^* is a constant; $(r_1 + n_1)'$ can be obtained by using a sampler-holder; the sampled value s_1^* decreases to $\rho^* s_1^*$ from the time t_1^* to the time t_1.

The input random noise, $n_i(t)$, is defined by its probability density distribution, $f(n_i)$, and its autocorrelation function, $R_i(\tau)$. The autocorrelation is obtained from the power density spectrum and vice-versa, since they are a Fourier transform pair. For example, Figures II-3 a, II-3 b, and II-3 c show the probability density distribution, the power density spectrum and the autocorrelation function of normal white noise, respectively.

The flow graph of Fig. II-4 show the two type of errors which occur in the threshold detection of the noisy signal, s_1:

(1) Type 1: 0 is sent, 1 is received

(2) Type 2: 1 is sent, 0 is received

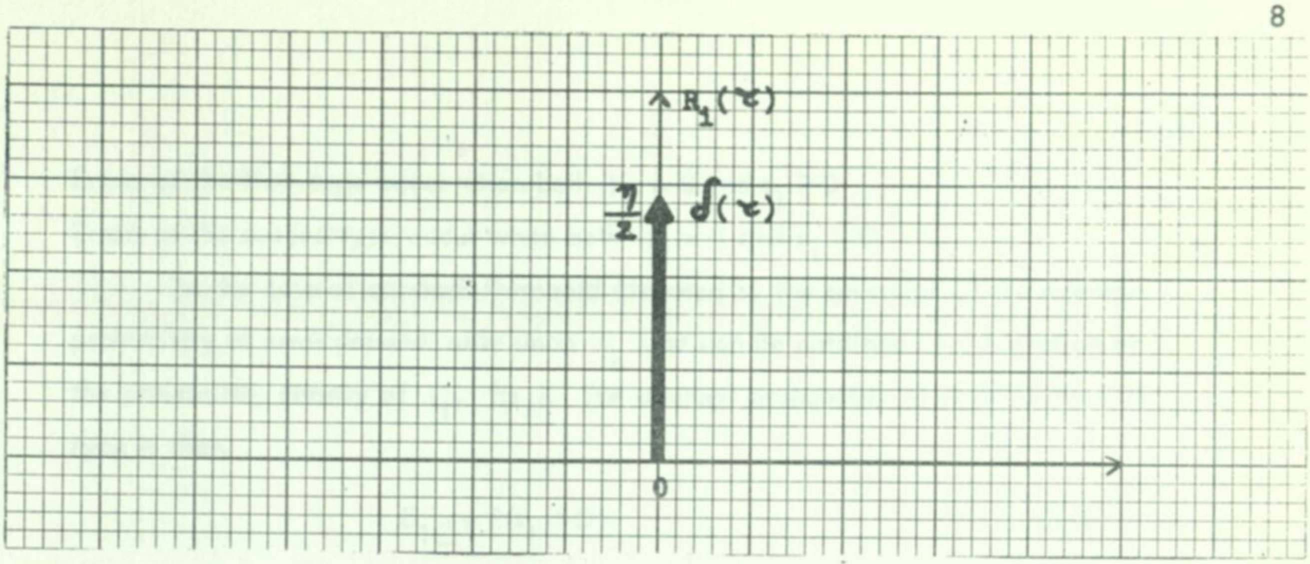


Fig. II-3 c : Autocorrelation Function of Normal White Noise

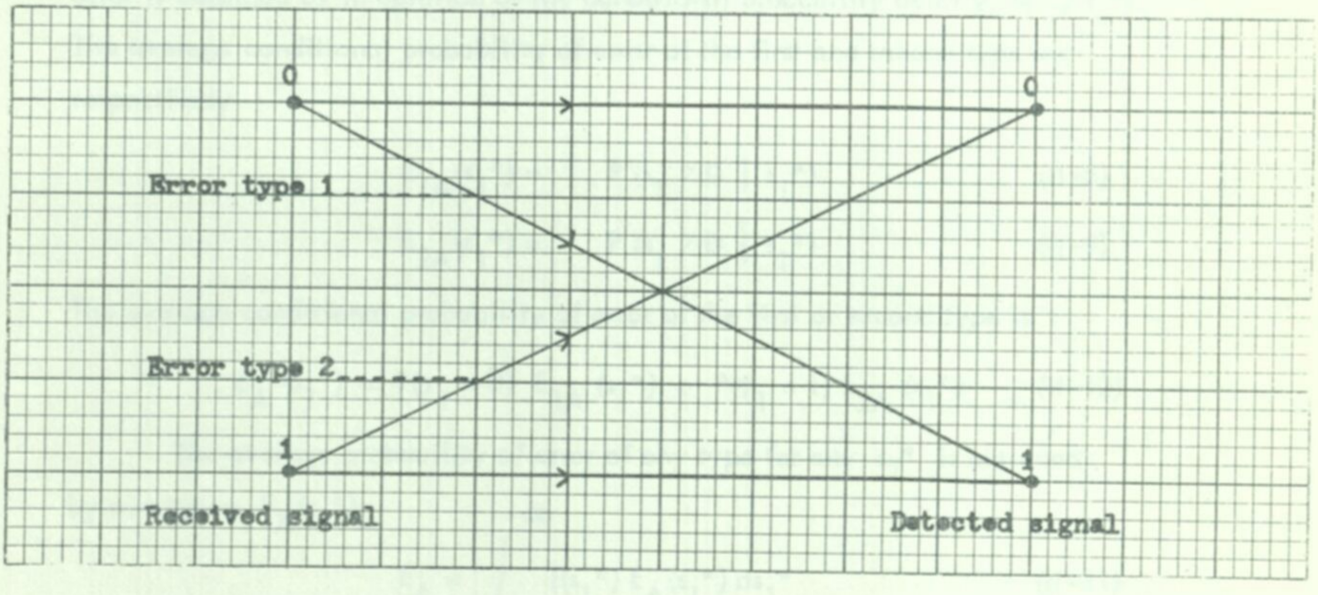


Fig. I -4 : Flow-graph showing the two Types of Error in the Detection

Errors of the first type occur when both $\gamma_1 = 0$ and $s_1 > D$

Errors of the second type occur when both $\gamma_1 = 1$ and $s_1 < D$

In the case of constant threshold (denoted by the subscript C) the average probability of error depends only upon s_1 and can be obtained by integration of the probability density of s_1, $f(s_1)$. The average probability of error of the first type is then

$$E_{1C} = P(s_1 > D) \qquad \text{(II-5)}$$

and the average probability of error of the second type is

$$E_{2C} = P(s_1 < D) \qquad \text{(II-6)}$$

The average probability of error of the first type or of the second type in the detection at constant threshold is:

$$E_C = P(\gamma_1 = 0)\, E_{1C} + P\,(\gamma_1 = 1)\;\, E_{2C}. \qquad \text{(II-7)}$$

In the case of adaptive threshold the average probability of error depends on both s_1 and $s_1{}^*$; for a given $s_1{}^*$, the average probability of error is obtained by integration of the conditional probability density, $f(s_1 / s_1{}^*)$. The average conditional probability of error of the first and second type are respectively

$$E_{1A}(s_1{}^*) = \quad P\,(s_1 > D \mid s_1{}^*) \qquad \text{(II-8)}$$

$$E_{2A}(s_1{}^*) = \quad P\,(s_1 > D \mid s_1{}^*) \qquad \text{(II-9)}$$

The average conditional probability of error of the first or second type is

$$E_A\,(s_1{}^*) = P(\gamma_1 = 0)\;\, E_{1A}\,(s_1{}^*) + P(\gamma_1 = 1)\, E_{2A}\,(s_1{}^*) \qquad \text{(II-10)}$$

The average probability of error of any type for any $s_1{}^*$ is obtained by averaging $E_A(s_1{}^*)$:

$$E_A = \int_{-\infty}^{+\infty} f(s_1{}^*)\, E_A(s_1{}^*)\, ds_1{}^* \qquad \text{(II-11)}$$

The technique outlined in this chapter is applied on specific examples in Chapter III and IV, V and VI. In Chapters III and IV, the signal s_1 is compared to an adaptive threshold, while in Chapters V and VI, a corrected signal s_1 is compared to a constant threshold.

Chapter III

III - DETECTION OF A PULSE CODE MODULATED SIGNAL IN PRESENCE OF RC NOISE

III-1 Statement of the Problem

The input signal consists of rectangular random pulses mixed with RC normal noise and the receiver is either a constant threshold detector or an adaptive threshold detector. The block diagram of the receiver is shown in Fig. II-1 where the transfer function of F is unity since no filter is used; therefore, $s_1(t) = s(t)$, $n_i(t) = n(t)$, etc.

Given the average power of the emitter and the characteristics of the noise, it is shown that the average probability of error in the detection of pulses is considerably reduced when an adaptive threshold is used.

The normalized autocorrelation function of RC noise is $\rho(\tau) = e^{\frac{-\beta}{T}|\tau|}$ where β is a coefficient proportional to the bandwidth of the RC noise and is called normalized bandwidth; more precisely, if a white noise is filtered through a RC network, the output is a RC noise with normalized autocorrelation function $\rho(\tau) = e^{\frac{-\beta}{T}|\tau|}$ and $\beta = \frac{T}{RC}$. The input signal is completely defined by the following independent variables: signal to noise ratio, S_i/N_i, normalized width of the pulse, K, the normalized bandwidth of the noise, β, and the set of probabilities for pulse and no pulse. It is convenient to assume that the input resistance of the receiver is one ohm and to introduce three independent variables: the amplitude of a pulse, V, the average power of the signal, P_s, and the variance of the noise, σ_i^2.

Assuming equal probabilities for pulse and no pulse, i.e. $P(\gamma = 0) = P(\gamma = 1) = 1/2$, the average power of the train of random pulses is

$$P_s = \frac{V^2 K}{2} \; ; \text{ hence, } \quad V = \frac{2 P_s}{K} \qquad \text{(III-1)}$$

The noise, n(t), has zero mean and variance σ_i^2; hence, σ_i^2 is the average power of the noise. Therefore, the input signal to noise ratio is

$$\frac{S_i}{N_i} = \frac{P_s}{\sigma_i^2} \qquad \text{(III-2)}$$

and the probability density is

$$f(n) = \frac{1}{\sqrt{2\pi}\,\sigma_i} e^{-n^2/2\sigma_i^2} \qquad \text{(III-3)}$$

The autocorrelation function of the RC normal noise, n(t), is

$$R(\tau) = \overline{n(t)\,n(t+\tau)} = \sigma_i^2\,\rho(\tau) = \sigma_i^2\,e^{-\frac{\beta}{T}|\tau|} \qquad \text{(III-4)}$$

The choice of the dependent variables V and σ_i is arbitrary; only the ratio $\frac{V}{\sigma_i}$ is important. $\frac{V}{\sigma_i}$ can be expressed in terms of signal to noise ratio:

$$\frac{V}{\sigma_i} = \frac{2}{K}\,\frac{S_i}{N_i} \qquad \text{(III-5)}$$

In order to find the average probability of error in the threshold detection of s(t) at time $t_1 = T + KT$, the probability density of $s_1 = s(t = T + KT)$ must be obtained, but first $p(t = T + KT) = p_1$ must be defined.

III-2 Average Probability of Error for Constant Threshold

Since the noise is symmetric and the rectangular pulses are independent, the best choice for the constant threshold, D_C, is V/2. The signal $s_i(t) = s(t)$ is compared to the threshold at the time $t_1 = T + KT$ and $s_i(t_1) = s(t_1) = s_1$.

Using formulas (II-5) and (II-6), the average probability of error of the first type E_{1C} (0 sent, 1 received) and the average probability of error of the second type E_{2C} (1 sent, 0 received) are respectively:

$$E_{1C} = P(s_1 > V/2,\ \gamma_1 = 0) = P(n_1 > V/2) \qquad \text{(III-6)}$$

$$E_{2C} = P(s_1 < V/2,\ \gamma_1 = 1) = P(n_1 < -V/2) \qquad \text{(III-7)}$$

The average probability of error for constant threshold is then:

$$E_C = 1/2\, E_{1C} + 1/2\, E_{2C} = \frac{1}{\sqrt{2\pi}\,\sigma_i} \int_{V/2}^{\infty} e^{-x^2/2\sigma_i^2}\, dx = \frac{1}{\sqrt{2\pi}} \int_{V/2\sigma_i}^{\infty} e^{-x^2/2}\, dx \qquad \text{(III-8)}$$

For short notation, let $I[u_I] = \int_{u_I}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}\, dx$; (III-9)

$$\text{then } E_C = I[V/2\sigma_i] \qquad \text{(III-10)}$$

III-3 Average Probability of Error for Constant or Adaptive Threshold

In the case of constant threshold there is no need for sampling before detection; however, the average probability of error in the detection is the same, with or without sampling, if the sample is not used to modify the threshold or the signal. It is, therefore, possible to establish a general formula for the average probability of error for a given threshold level D, constant or adaptive.

Since s_1 and s_1^* are not independent, the average probability of error in the detection of s_1 depends upon s_1^*, and the average conditional probability of error, $E(s_1^*)$, can be defined. The average probability of error E for any s_1^* can be obtained by averaging $E(s_1^*)$:

$$E = \int_{-\alpha}^{+\alpha} f(s_1^*)\, E(s_1^*)\, ds_1^* \qquad \text{(III-11)}$$

This computation of the average probability of error is always correct whether the threshold is constant or adaptive.

The received random pulses are rectangular and no memory type network is used before detection; therefore, the signal, s(t), is just equal to the noise, n(t), during the time interval between random pulses:

$$s(t) = n(t) \text{ for } nT + KT < t < (n+1)T$$

This relation is satisfied at the time $t_1{}^*$ of sampling

$$s(t = t_1{}^*) = s_1{}^* = n(t = t_1{}^*) = n_1{}^* \qquad \text{(III-12)}$$

$$t_1{}^* = T + KT - K^*T \qquad \text{where} \quad 1 > K^* > K$$

At the time t_1 of detection, and assuming additive noise

$$s(t_1) = s_1 = n_1 + \gamma_1 V \qquad \text{(III-13)}$$

$$t_1 = T + KT$$

When $s_1{}^*$ is known and used to modify the threshold, the average probability of error can be reduced. The relation between s_1 and $s_1{}^*$ is expressed in terms of conditional probability density $f(s_1 / s_1{}^*)$. The average conditional probability of error can be expressed in terms of the first moments, s_1, $s_1{}^*$, the variances, $\overline{(s_1 - \overline{s_1})^2}$, $\overline{(s_1{}^* - \overline{s_1{}^*})^2}$, and the autocorrelation coefficient, ρ^*. Here,

$$\overline{(s_1 - \overline{s_1})^2} = \overline{(s_1{}^* - \overline{s_1{}^*})^2} = \sigma_i^2 \qquad \text{(III-14)}$$

and
$$\rho^* = \frac{\overline{(s_1 - \overline{s_1})\,(s_1 - s_1{}^*)}}{\sqrt{\overline{(s_1 - \overline{s_1})^2}\;\overline{(s_1{}^* - \overline{s_1{}^*})^2}}} = \frac{\overline{(s_1 - \overline{s_1})\,(s_1{}^* - \overline{s_1{}^*})}}{\sigma_i^2} = \frac{\overline{n_1 n_1{}^*}}{\sigma_i^2} \qquad \text{(III-15)}$$

Therefore, ρ^* is simply the autocorrelation coefficient between the noise, $n_1{}^*$, at the time of sampling and the noise, n_1, at the time of the detection. The normalized autocorrelation of the noise is defined in paragraph III-1 as $\rho(\tau) = \dfrac{\overline{n(t)\;n(t+\tau)}}{\sigma_i^2} = e^{-\frac{\beta}{T}|\tau|}$

Therefore, $\rho^* = \dfrac{\overline{n_1{}^* n_1}}{\sigma_i^2}$ is obtained from $\rho(\tau)$ by letting

$t + \tau = t_1 = T + KT$ and $t = t_1{}^* = t_1 + KT - K^*T$ resulting in $\rho^* = e^{-\beta(t_1 - t_1{}^*)/T}$

$$= e^{-\beta K^*} \qquad \text{(III-16)}$$

$$f(s_1/s_1^*) = \frac{1}{\sqrt{2\pi}\ \sigma_i \sqrt{1-\rho^{*2}}}\ e^{\frac{\left[(s_1 - \overline{s_1}) - \rho^*(s_1^* - \overline{s_1^*})\right]^2}{2\sigma_i^2 (1-\rho^{*2})}} \qquad \text{(III-17)}$$

where the bar above a letter denotes averaging.

In terms of n_1 and n_1^*, $f(s_1/s_1^*)$ is

$$f(s_1/s_1^*) = \frac{1}{\sqrt{2\pi}\ \sigma_i \sqrt{1-\rho^{*2}}}\ e^{\frac{-(n_1 - \rho^* n_1^*)^2}{2\sigma_i^2(1-\rho^{*2})}} \qquad \text{(III-18)}$$

The average conditional probability of error of the first and second type, in the detection of s_1, knowing the sampled signal s_1^* before detection, are denoted $E_{1C}(s_1^*)$ and $E_{2C}(s_1^*)$ respectively. $E_{1C}(s_1^*)$ and $E_{2C}(s_1^*)$ are expressed by formulas similar to formulas (II-8) and (II-9):

$$E_{1C}(s_1^*) = P\left[(n_1 > V/2) \mid n_1^*\right] \qquad \text{(III-19)}$$

$$E_{2C}(s_1^*) = P\left[(n_1 < -V/2) \mid n_1^*\right] \qquad \text{(III-20)}$$

$$E_{1C}(s_1^*) = \int_{n_1 = V/2}^{\infty} f(s_1/s_1^*)\, ds_1$$

$$E_{1C}(s_1^*) = \frac{1}{\sqrt{2\pi}\ \sigma_i \sqrt{1-\rho^{*2}}} \int_{V/2}^{\infty} e^{\frac{-(n_1 - \rho^* n_1^*)^2}{2\sigma_i^2 (1-\rho^{*2})}}\ dn_1$$

After the change of variable $t = \dfrac{n_1 - \rho^* n_1^*}{\sigma_i \sqrt{1-\rho^{*2}}}$,

$$E_{1C}(s_1^*) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-t^2/2}\, dt \text{ , where } x = \frac{V/2 + \rho^* n_1^*}{\sigma_i \sqrt{1 - \rho^{*2}}}$$

which is simply $E_{1C}(s_1^*) = I\left[\dfrac{V/2 + \rho^* n_1^*}{\sigma_i \sqrt{1 - \rho^{*2}}}\right]$ (III-21)

with the notation of paragraph III-3.

$$E_{2C}(s_1^*) = \frac{1}{\sqrt{2\pi}\, \sigma_i \sqrt{1 - \rho^{*2}}} \int_{-\infty}^{n_1 = V/2} e^{-\frac{(n_1 - \rho^* n_1^*)^2}{2\sigma_i^2 (1 - \rho^{*2})}}\, dn_1$$

and after the change of variable $t = \dfrac{-(n_1 - \rho^* n_1^*)}{\sigma_i \sqrt{1 - \rho^{*2}}}$

$$E_{2C}(s_1^*) = I\left[\frac{V/2 + \rho^* n_1^*}{\sigma_i \sqrt{1 - \rho^{*2}}}\right] \quad \text{(III-22)}$$

The average conditional probability of error of the first or the second type in the detection of s_1 knowing s_1^* is defined as $E_C(s_1^*)$ which is obtained as the average of $E_{1C}(s_1^*)$ and $E_{2C}(s_1^*)$:

$$E_C(s_1^*) = 1/2\, I\left[\frac{V/2 - \rho^* n_1^*}{\sigma_i \sqrt{1 - \rho^{*2}}}\right] + 1/2\, I\left[\frac{V/2 + \rho^* n_1^*}{\sigma_i \sqrt{1 - \rho^{*2}}}\right] \quad \text{(III-23)}$$

The average probability of error of the first or the second type in the detection of s_1, for any s_1^*, is defined as E_C and is obtained through averaging of $E_C(s_1^*)$ with respect to s_1^*; since $s_1^* = n_1^*$ by formula (III-12), it results:

$$E_C = \int_{-\infty}^{+\infty} \frac{e^{-n_1^{*2}/2\sigma_i^2}}{\sqrt{2\pi}\ \sigma_i} \left\{ 1/2 \ \mathrm{I}\left[\frac{V/2 - \rho^* n_1^*}{\sigma_i \sqrt{1-\rho^{*2}}}\right] + 1/2 \ \mathrm{I}\left[\frac{V/2 + \rho^* n_1^*}{\sigma_i \sqrt{1-\rho^{*2}}}\right] \right\} dn_1^* \tag{III-24}$$

Fortunately, the double integral can be condensed into a single integral, because

$$\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\ \sigma} e^{-v^2/2\sigma^2} \ \mathrm{I}\left[a + \frac{bv}{\sigma}\right] dv = \mathrm{I}\left[\frac{a}{\sqrt{1+b^2}}\right] \tag{III-25}$$

as shown in Appendix A .

Hence, $$E_C = 1/2 \ \mathrm{I}\left[\frac{V}{2\sigma_i}\right] + 1/2 \ \mathrm{I}\left[\frac{V}{2\sigma_i}\right] = \mathrm{I}\left[\frac{V}{2\sigma_i}\right] \tag{III-26}$$

which is identical to the result obtained in paragraph III-2, without sampling. This shows that the average probability of error for constant threshold E_C can be obtained directly or by averaging $E_C(s_1^*)$, when s_1^* is not used to vary the threshold. In other words whether s_1^* is known or not is irrelevant if s_1^* is not used to vary the threshold.

The average probability of error for adaptive threshold, E_A , can only be obtained through averaging the average conditional probability of error $E_A(s_1^*)$ because the threshold varies with s_1^*. Since s_1 and s_1^* are related, s_1 can be partially predicted from s_1^* and the threshold varies accordingly. The adaptive threshold is

$$D_A = V/2 + g(s_1^*) \tag{III-27}$$

where $g(s_1^*)$ is a function of s_1^* chosen such as to minimize the average probability of error in the detection.

The average conditional probability of error in the detection with an adaptive threshold, D_A, of a signal, s_1, knowing the sampled value before detection s_1^*, is denoted by $E_A(s_1^*)$ which is obtained from the formula for $E_C(s_1^*)$ (formula III-23) simply by replacing $V/2$ by $V/2 + g(s_1^*)$.

$$E_A(s_1^*) = 1/2 \; I\left[\frac{V/2 + g(s_1^*) - \rho^* n_1^*}{\sigma_i \sqrt{1-\rho^{*2}}}\right] + 1/2 \; I\left[\frac{V/2 - g(s_1^*) + \rho^* n_1^*}{\sigma_i \sqrt{1-\rho^{*2}}}\right] \qquad \text{(III-28)}$$

The corrective function $g(s_1^*)$ is chosen such as to minimize the average probability of error, E_A. The average conditional probability of error, $E_A(s_1^*)$, is minimum for

$$g(s_1^*) = \rho^* s_1^* = \rho^* n_1^* \qquad \text{(III-29)}$$

this results from the comparison between the well known inequality

$$I(u_1) < 1/2 \; I(u_1 + \Delta) + 1/2 \; I(u_1 - \Delta) \qquad \text{(III-30)}$$

and formula (III-28).

E_A is the average of $E_A(s_1^*)$ with respect to s_1^*, $E_A = \overline{E_A(s_1^*)}$; when $g(s_1^*) = \rho^* n_1^*$ not only $E_A(s_1^*)$ is minimum for every s_1^* but also it is a constant, and there is no need for further averaging. After replacing $g(s_1^*)$ by $\rho^* n_1^*$ in formula (III-28),

$$E_A = E_A(s_1^*) = 1/2 \; I\left[\frac{V}{2\sigma_i \sqrt{1-\rho^{*2}}}\right] + 1/2 \; I\left[\frac{V}{2\sigma_i \sqrt{1-\rho^{*2}}}\right] \qquad \text{(III-31)}$$

$$E_A = I\left[\frac{V}{2\sigma_i \sqrt{1-\rho^{*2}}}\right] \qquad \text{(III-32)}$$

where ρ^* can be replaced, using the formula (III-16):

$$E_A = I\left[\frac{V}{2\sigma_i\sqrt{1-e^{-2\beta K^*}}}\right] \qquad \text{(III-33)}$$

The physical interpretation of the choice $g(s_1{}^*) = \rho^* n_1{}^*$ is straightforward. The adaptive threshold at time t_1 is obtained by combining formulas III-27 and III-29 and III-16:

$$D_A = V/2 + \rho^* n_1{}^* = V/2 + n_1{}^* \, e^{\frac{-\beta(t_1 - t_1{}^*)}{T}} \qquad \text{(III-34)}$$

The corrective term is the output of an RC network of time constant T/β discharged from the initial value $n_1{}^*$ at time $t_1{}^*$ to the value $n_1{}^* \, e^{\frac{-\beta(t_1 - t_1{}^*)}{T}}$ at time t_1, and it is obtained at the output of a sampler-holder which samples the filtered noise $s(t)$ at time $t = t_1{}^*$; during the sampling time the sampler-holder behaves as a zero-memory network but during the holding time the sampler-holder behaves as an RC network and the RC time constant of the sampler-holder must be equal to T/β, $RC = T/\beta$. The schematic of a sampler-holder is given in the experimental chapter, Ch. VII. The time of sampling and of detection have to be slightly modified in a practical model because the electronic components do not operate instantaneously.

III-4 Minimization of the Average Probability of Error

The minimum average probability of error for constant threshold for a given input signal to noise ratio can be obtained. The average probability of error is of the form $I(x)$ which is monotonic decreasing and is minimum when x is maximum.

The average probability of error for constant threshold given by formula (III-26) can be expressed in terms of the signal to noise ratio using the formula (III-5):

$$E_C = I\left(\frac{V}{2\sigma_i}\right) = I\left[\sqrt{\frac{S_i/N_i}{2K}}\right] \qquad \text{(III-35)}$$

Therefore, E_C decreases when K decreases; if the physically realizable minimum of K is .1, the minimum attainable average probability of error at constant threshold is

$$E_{Cmin} = I\left[\sqrt{5\,\frac{S_i}{N_i}}\right] \qquad \text{(III-36)}$$

where E_{Cmin} denotes the physically realizable minimum of E_C.

The average probability of error for adaptive threshold is given by formula (III-32) and can also be expressed in terms of input signal to noise ratio:

$$E_A = I\left[\frac{V}{2\sigma_i\sqrt{1-e^{-2\beta K^*}}}\right] = I\left[\sqrt{\frac{S_i/N_i}{2K(1-e^{-2\beta K^*})}}\right] \qquad \text{(III-37)}$$

The average probability of error is minimum when the product $2K(1-\rho^{*2}) = 2K(1-e^{-2\beta K^*})$ is minimum which requires that both K and K^* be minimum. Since the sampling must be made before the unknown signal, K^* must be less than K. Assuming again that $K = .1$ is the physical minimum and that the sampling can be made just before the unknown signal, E_A is minimum for $K = K^* = .1$ and is given by

$$E_{A\,min} = I\left[\frac{5\,S_i/N_i}{1-e^{-.2\beta}}\right] \qquad \text{(III-38)}$$

$E_{A\,min}$ is a function of β and increases when β decreases.

III-5 Comparison Between Adaptive and Constant Threshold

The received signal, a train of random rectangular pulses mixed with RC noise is defined by (1) the signal to noise ratio, S_i/N_i , (2) the normalized bandwidth, β , which determines the normalized autocorrelation function of the noise and (3) the relative width of the pulses, K. The probabilities of pulse and no pulse are assumed equal and the average power of the emitter (not the amplitude V of the pulses) is assumed constant.

The average probability of error at constant threshold is a function of S_i/N_i and K (formula III-35) and is minimum when K is minimum (say K = .1) $E_{C\ min}$ is given by formula III-36 and is function of S_i/N_i only.

The adaptive threshold detector is defined by the time of sampling, i.e. K*, and the law of prediction $D_A = V/2 + \rho^* n_1^*$. The average probability of error for adaptive threshold, E_A , is a function of S_i/N_i , β , K and K* (formula III-37). The average probability of error is minimum when K* = K and K is minimum (say K = .1). $E_{A\ min}$ is given by formula III-38 and is a function of S_i/N_i and β.

It is especially interesting to compare the minimum average probabilities of error for constant or adaptive threshold, $E_{C\ min}$ and $E_{A\ min}$ respectively. $(-\log_{10} E_{C\ min})$ and $(-\log_{10} E_{A\ min})$ are plotted instead of $E_{C\ min}$ and $E_{A\ min}$ which are very small numbers. The maximum of $-\log_{10} E$ is the minimum of E. Fig. III-1 shows $(-\log_{10} E_{C\ min})$ versus S_i/N_i for a useful range of S_i/N_i. Fig. III-2 shows $(-\log_{10} E_{A\ min})$ versus S_i/N_i for K= K* and different values of β . The average probability of error is smaller for an adaptive threshold than for a constant threshold; the smaller β is , the larger is the improvement. The reduction in the average probability of error can be expressed by a gain in decibels, $G = -20 \log_{10} (E_A/E_C)$. G is a function of S_i/N_i and β. Fig. III-3 shows G versus S_i/N_i for typical values of β.

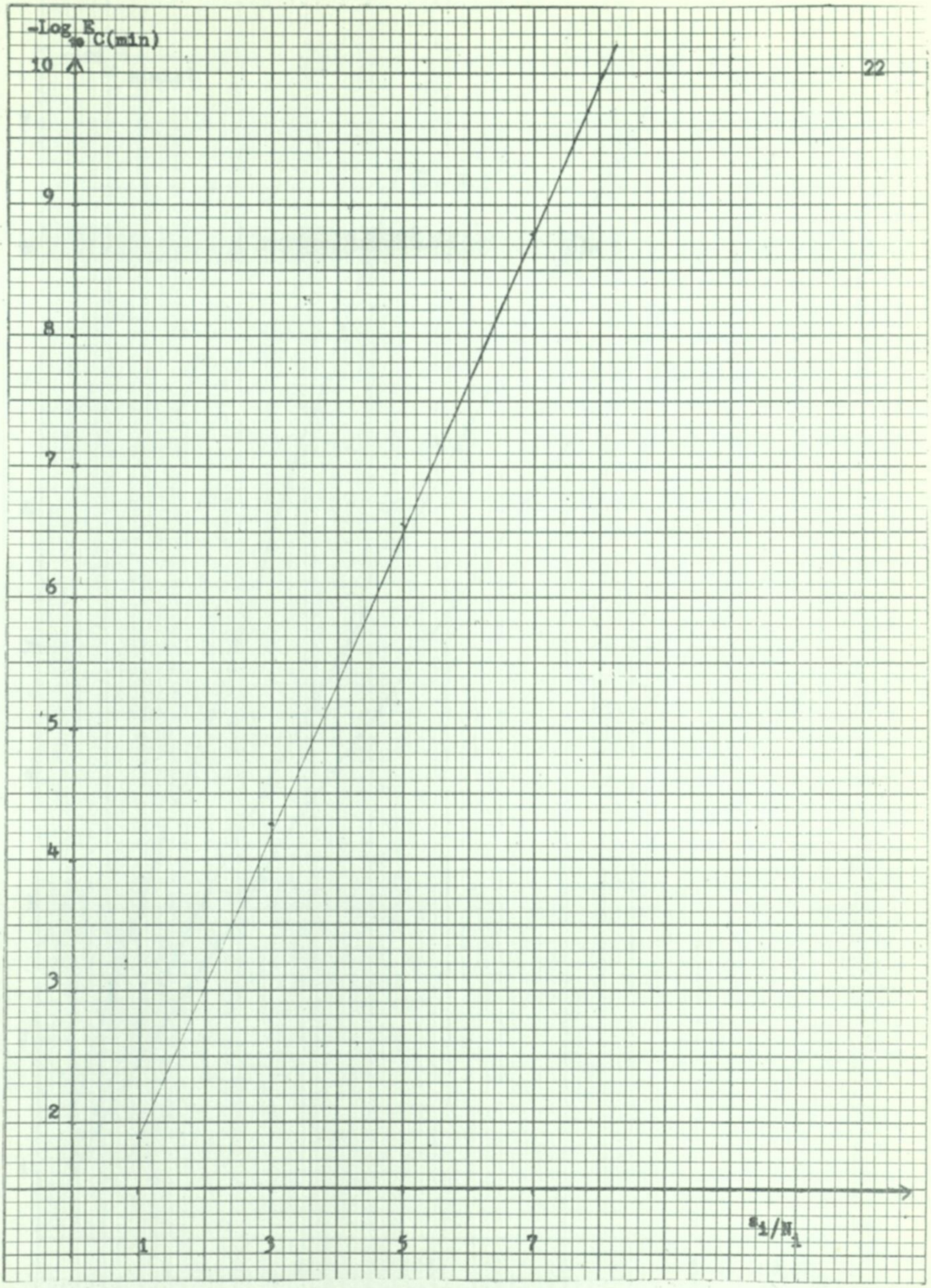


Fig. III-1 : Probability of Error versus Signal to Noise Ratio, Constant Threshold and RC Noise

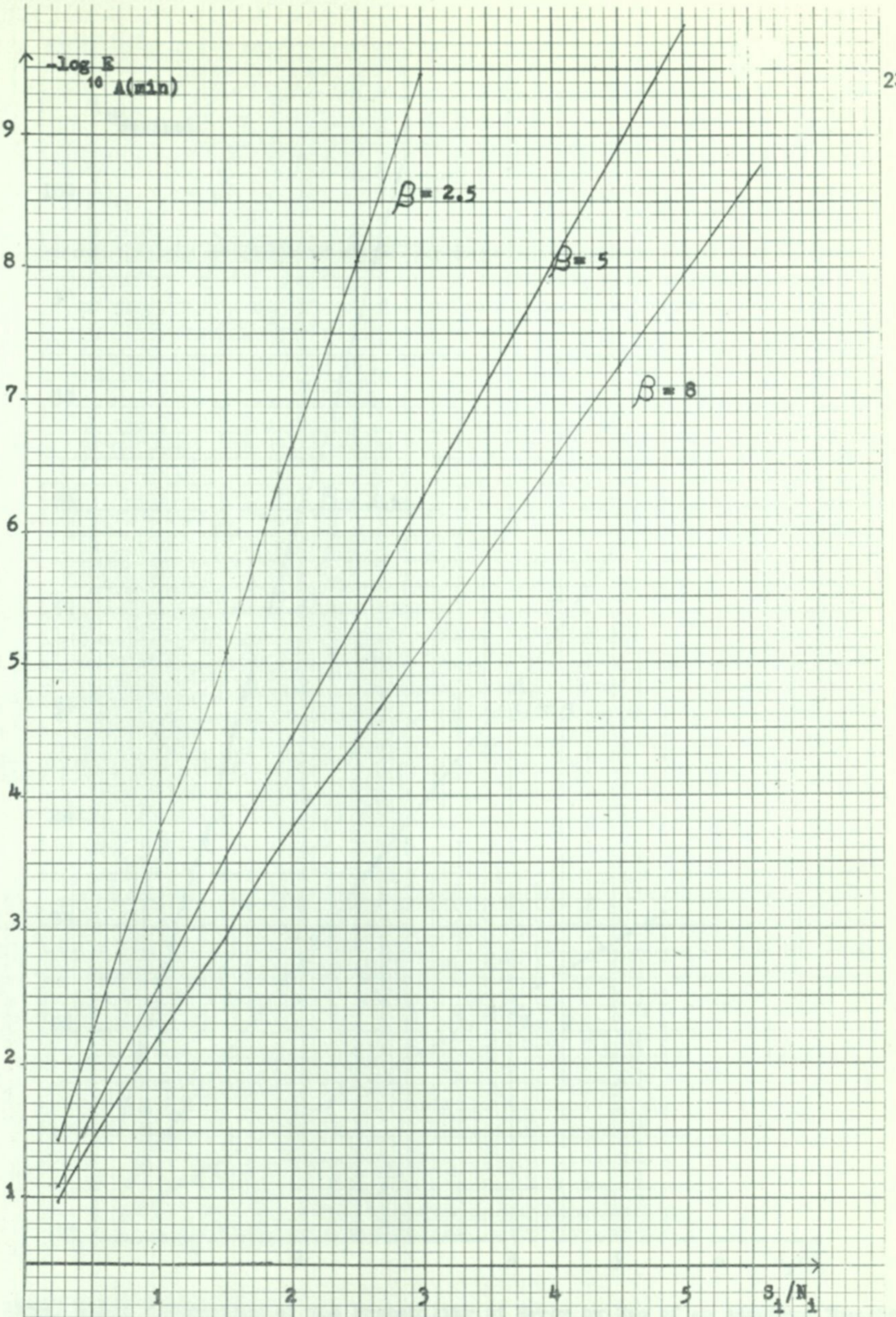


Fig. III-2 : Probability of Error versus Signal to Noise Ratio, Adaptive Threshold and RC Noise for β = 2.5, 5, 8

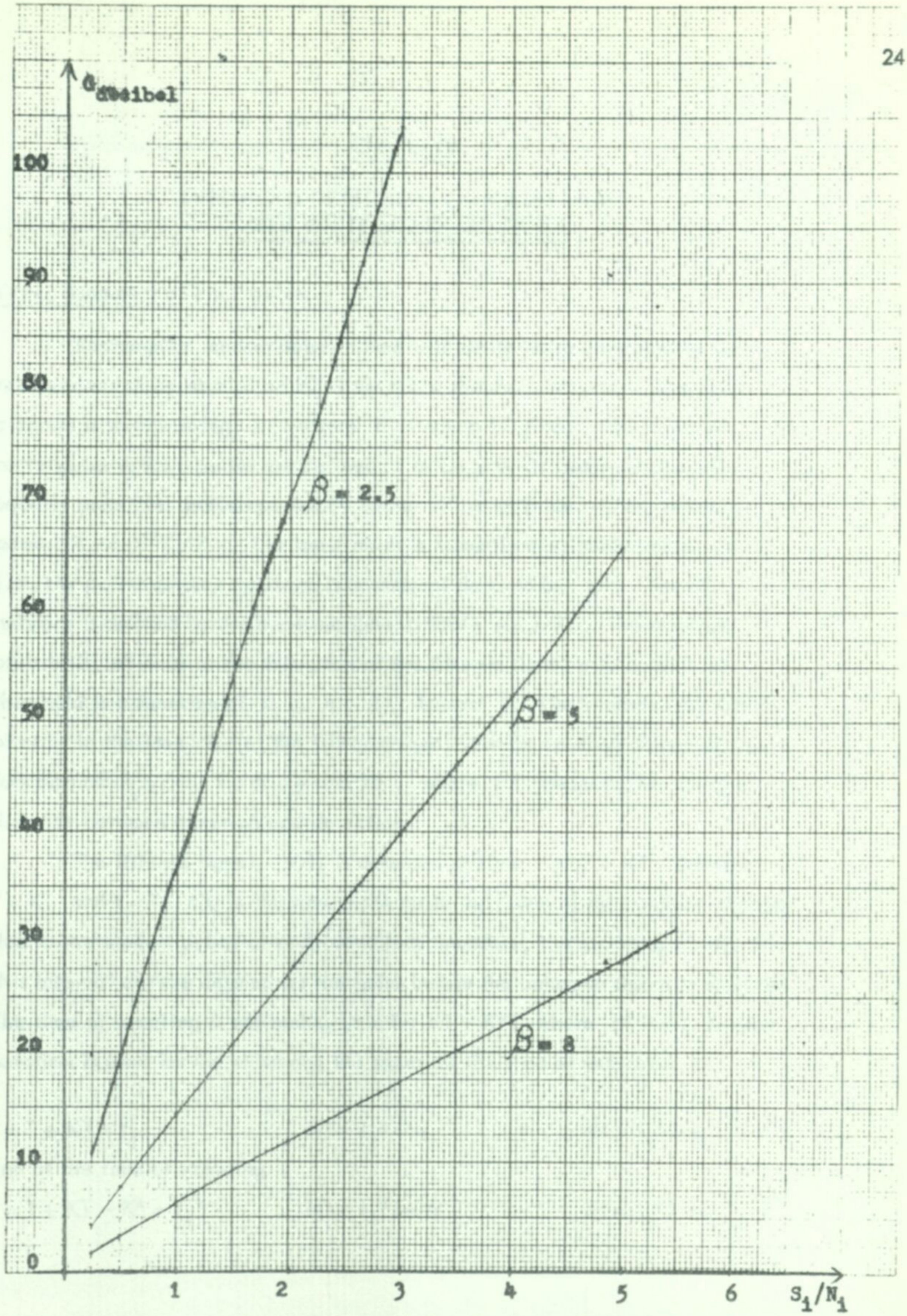


Fig. III-3 : Reduction of the Probability of Error by Use of an Adaptive Threshold.

RC Noise, $\beta = 2.5$, 5 and 8. $G = -20 \log_{10} \left(\frac{E_A}{E_C}\right)$

Chapter IV

THRESHOLD DETECTION OF PULSES MIXED WITH WHITE NOISE, RC FILTER

IV-1 Statement of the Problem

Rectangular random pulses mixed with white noise are detected by a receiver which consists of an RC filter and a threshold detector. The block diagram is shown in Fig. II-1 where F is now a RC filter. The received rectangular random pulses are defined as in Chapter II: pulse and no pulse equally probable, pseudo period T, width KT, amplitude V, and average power $P_s = V^2K/2$. The input noise is normally distributed and white, i.e. the power density spectrum is a constant denoted by $\eta/2$. The RC network is defined by the dimensionless variable $y = T/RC$. The average probability of error in the detection of the filtered signal with a constant threshold is a function of P_s, η, T, K and y. The average probability of error in the detection of the filtered signal with an adaptive threshold is a function of P_s, η, T, K, y and K*, where K* measures the relative distance between sampling and detection.

The filtered signal, s(t), is detected at time $t = T + KT$. Between $t = KT$ and $t = T$, s(t) is the sum of the noise n(t) and of the residual voltage due to the random pulses already detected; therefore, the sampling of s(t) in this interval of time determines the noise before the unknown signal is received. The time of sampling is defined by $t = T + KT - K^*T$ where $K^* > K$. A short notation is used for the noise and the signal at the time of detection

$t = T + KT = t_1$ $\qquad n(t = t_1) = n_1$ $\qquad s(t = t_1) = s_1$

and at the time of sampling

$t = T + KT - K^* = t_1^*$ $\qquad n(t = t_1^*) = n^*$ $\qquad s(t = t_1^*) = s_1^*$

The following problem is solved: the input signal consists of a train of rectangular random pulses mixed with white normal noise. Given the average power of the train of pulses, P_s, and the power density of the noise, η, and the frequency of transmission, $1/T$, find the optimum choice of RC filter and pulse width (i.e. y and K) to minimize the average probability of error in the threshold detection. Consider the case of constant and adaptive threshold and compare the average probabilities of error.

The average power of the white noise over all the range of frequency is infinite; hence, the average probability of error cannot be defined as a function of the input signal to noise ratio which is zero. Fortunately, P_s, η, and T appear only as a factor $Q = P_s T/\eta$; therefore, the problem reduces to: given Q, find the combination of K and y which minimizes the average probability of error where Q is similar to a signal to noise ratio and is called fictitious signal to noise ratio.

In order to find the average probability of error in the threshold detection of s(t) at time $t_1 = t + KT$, the probability density of $s_1 = s(t = T + KT)$ must be obtained, but first the effect of the filter must be determined.

IV-2 Effect of the RC Filter on the Noise

The effect of the RC filter on the power density spectrum, the autocorrelation and the probability density of the input noise, $n_i(t)$, are easily obtained. Since the network F is linear, the probability density remains normal; hence, the probability density of the filtered noise is completely defined by its first and second moments, m_1 and m_2. Only n_1 and n_1^*, the values of the noise at the time of detection and at the time of sampling, are important. Since the noise is stationary, the moments do not depend upon t; therefore,

$$m_1 = \overline{n(t)} = \overline{n_1} = \overline{n_1^*} \qquad \text{(IV-1)}$$

and

$$m_2 = \overline{n(t)^2} = \overline{n_1^{\,2}} = \overline{n_1^{*2}} \qquad \text{(IV-2)}$$

$G_i(f)$ and $G_F(f)$ are, respectively, the power density spectrum at the input and the power density spectrum at the output of a linear network defined by a transfer function, $F(f)$ (here f is the frequency). For a RC filter, $F(f) = \frac{1}{1 + j\,2\pi\, f\, RC}$. The power density spectrum at the input, G_i, is a constant since the noise is white, $G_i = \eta/2$. The power density spectrum, G_f, at the output of the filter, F, is simply the product of the power density spectrum at the input by the square of the modulus of the transfer function.

$$G_f = \left| F(f) \right|^2 G_i(f) = \frac{\eta/2}{1 + (2\pi\, f\, RC)^2} \qquad \text{(IV-3)}$$

The autocorrelation function is the Fourier cosine transform of the power density spectrum (Wiener's theorem). Therefore, the autocorrelation at the output of the filter is

$$R(\tau) = \eta/2 \int_{-\infty}^{+\infty} \frac{\cos\, \omega\tau}{1 + (2\pi\, RC\, f)^2}\, df = \frac{\eta}{4RC}\, e^{-|\tau|/RC} \qquad \text{(IV-4)}$$

The autocorrelation $R(\tau)$ is by definition

$$R(\tau) = \overline{n(t)\;\; n(t+\tau)} \qquad \text{(IV-5)}$$

The mean square value of the noise is obtained by letting $\tau = 0$ in formulas (IV-4) and (IV-5)

$$R(o) = \overline{n(t)^2} = \eta / 4RC \qquad \text{(IV-6)}$$

The normalized autocorrelation function of the noise, $\rho(\tau)$, is

$$\rho(\tau) = \frac{R(\tau)}{R(o)} = e^{-|\tau|/RC} \qquad \text{(IV-7)}$$

The response of the RC filter to a unit impulse is $\frac{1}{RC}\, e^{-t/RC}$. The output of a network is obtained by convolution of the input with the impulse response. Therefore,

$$n(t) = \int_{-\infty}^{+\infty} n_i(\tau) \frac{1}{RC} e^{\frac{-(t-\tau)}{RC}} d\tau$$

$$\overline{n(t)} = \int_{-\infty}^{+\infty} \overline{n_i(\tau)} \frac{1}{RC} e^{\frac{-(t-\tau)}{RC}} d\tau \qquad \text{(IV-8)}$$

The input noise has zero mean. Substituting $\overline{n_i(t)} = 0$ in formula IV-8 yields $\overline{n(t)} = 0$. (IV-9)

Finally, the first and second moments are obtained by combining formula IV-1 with formula IV-9 and formula IV-2 with formula IV-6.

$$m_1 = \overline{n_1} = \overline{n_1{}^*} = 0 \qquad \text{(IV-10)}$$

$$m_2 = \overline{n_1{}^2} = \overline{n_1{}^{*2}} = \eta / 4RC \qquad \text{(IV-11)}$$

The variance for n_1 and $n_1{}^*$ is

$$\sigma^2 = m_2 - m_1{}^2 = \eta / 4RC \qquad \text{(IV-12)}$$

A normal probability density remains normal after passing through a linear network; therefore, the probability density for n_1 or $n_1{}^*$ is of the form

$$f(n) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{\frac{-n^2}{2\sigma^2}} \qquad \text{(IV-13)}$$

IV-3 Effect of the RC Filter on the Random Sequence of Pulses

The RC filter changes the shape and the width of the pulses. The amplitude increases in a sequence of overlaping pulses because the initial charge of the condenser is increasing every time. A sequence of random pulses can be considered as the superposition of individual pulses starting at time (mT) where m is an integer; thus, it is sufficient to study a single pulse without initial charge of the capacitor.

Consider a rectangular pulse of amplitude V occuring at time $t = 0$; the instantaneous voltage, $e_i(t)$, is defined as $e_i(t) = V$ for $0 < t < KT$ and $e_i(t) = 0$ elsewhere. The pulse, $e_i(t)$, is applied to a RC network without initial charge and the output pulse is defined by $e(t)$. It is easily shown that:

$$e(t) = V(1 - e^{-t/RC}) \qquad \text{for } 0 \leq t \leq KT \qquad \text{(IV-14)}$$

and $$e(t) = V(1 - e^{-KT/RC})\, e^{-\frac{t - KT}{RC}} \qquad \text{for } KT \leq t \qquad \text{(IV-15)}$$

The maximum $e(t)$ occurs at time KT and is equal to

$$e(t = KT) = V(1 - e^{-KT/RC}) = A \qquad \text{(IV-16)}$$

If the input rectangular pulse originates at time $t = mT$ instead of $t = 0$, this corresponds to a translation of $e_i(t)$ by mT; $e_i(t)$ becomes $e_i(t - mT)$. Therefore, the output voltage is also translated by mT; $e(t)$ becomes $e(t - mT)$ which is denoted by $e_m(t)$ for a more convenient notation.

$$e_m(t) = 0 \qquad \text{for } t \leq mT \qquad \text{(IV-17)}$$

$$e_m(t) = V\,(1 - e^{-(t - mT)/RC}) \qquad \text{for } mT \leq t \leq mT + KT \qquad \text{(IV-18)}$$

$$e_m(t) = V\,(1 - e^{-KT/RC})\; e^{-[(t - mT) - KT]/RC} \qquad \text{for } mT + KT \leq t \qquad \text{(IV-19)}$$

The maximum of $e_m(t)$ occurs at time $mT + KT$ and is again equal to A.

The output of the filter to a random pulse $\gamma_m V$ is $\gamma_m e_m(t)$, which is defined by multiplying formulas IV-17, IV-18 and IV-19 by γ_m (γ_m was defined in Chapter II):

$$\gamma_m\, e_m(t) = 0 \qquad \text{for } t \leq mT \qquad \text{(IV-20)}$$

$$\gamma_m e_m(t) = \gamma_m V(1 - e^{-(t - mT)/RC}) \qquad \text{for } mT \leq t \leq mT + KT \qquad \text{(IV-21)}$$

$$\gamma_m e_m(t) = \gamma_m V(1 - e^{-KT/RC}) e^{-[(t - mT) - KT]/RC} \qquad \text{for } mT + KT \leq t \qquad \text{(IV-22)}$$

The maximum of $\gamma_m e_m(t)$ occurs at time $mT + KT$ and is equal to $\gamma_m A$.

A filter is said to have a n-step memory if the output becomes zero n pseudo periods after removal of the input. Assuming a two-step memory, the filtered random sequence of pulses, $p(t)$, during the interval of time $KT \leq t \leq T + KT$ is obtained by superposition of $\gamma_{-1} e_{-1}(t)$, $\gamma_o e_0(t)$ and $\gamma_1 e_1(t)$

$$p(t) = \gamma_{-1} e_{-1}(t) + \gamma_o e_0(t) + \gamma_1 e_1(t) \qquad \text{(IV-23)}$$

$e_m(t)$ is defined by one of the formulas IV-20, IV-21 and IV-22 for which t is in the proper range. The index m is equal to -1, 0 and 1.

To obtain the noise at the time of sampling, t_1^*, and at the time of detection, t_1, the variable t in formula (IV-23) is replaced by t_1^* and t_1, respectively:

$$p(t_1^*) = p_1^* = V(1 - e^{-KT/RC}) \left[\gamma_o e^{-\frac{T(1 - K^*)}{RC}} + \gamma_{-1} e^{-\frac{T(2 - K^*)}{RC}} \right] \qquad \text{(IV-24)}$$

and

$$p(t_1) = p_1 = V(1 - e^{-KT/RC}) \left[\gamma_1 + \gamma_o e^{-T/RC} + \gamma_{-1} e^{-2T/RC} \right] \qquad \text{(IV-25)}$$

It is convenient to express p_1^* and p_1 as functions of the normalized autocorrelation function of the noise, $\rho(\tau) = e^{-|\tau|/RC}$ (formula IV-7). Let

$$\rho(\tau = T) = \rho = e^{-\gamma} \text{ and } \rho(\tau = K^*T) = \rho^* = e^{-\gamma K^*}$$

Then

$$p_1^* = \frac{V(1-\rho^K)}{\rho^*}[\gamma_o\rho + \gamma_{-1}\rho^2] = \frac{A}{\rho^*}[\gamma_o\rho + \gamma_{-1}\rho^2] \qquad \text{(IV-26)}$$

and $$p_1 = V(1-\rho^K)[\gamma_1 + \gamma_o\rho + \gamma_{-1}\rho^2] = A[\gamma_1 + \gamma_o\rho + \gamma_{-1}\rho^2] \qquad \text{(IV-27)}$$

IV-4 Average Probability of Error

Assuming additive noise

$$s_1^* = n_1^* + \frac{A}{\rho^*}(\gamma_o\rho + \gamma_{-1}\rho^2) \qquad \text{(IV-28)}$$

$$s_1 = n_1 + A(\gamma_1 + \gamma_o\rho + \gamma_{-1}\rho^2) \qquad \text{(IV-29)}$$

Since s_1 and s_1^* are not independent, the average probability of error in the detection of s_1 is a function of s_1^*; this average conditional probability of error is denoted by $E(s_1^*)$ which is the average of the average conditional probabilities of error of the first and second type $E_1(s_1^*)$ and $E_2(s_1^*)$. The average probability of error in the detection of s_1 for any s_1^* is denoted by E which is obtained by averaging $E(s_1^*)$ with respect to s_1^*, that is, with respect to n_1^*, γ_{-1} and γ_o, successively. The threshold level of the detector is denoted by D. Two types of threshold level are investigated: constant threshold, D_C, and adaptive threshold, D_A. The different average probabilities of error defined above are a function of D; therefore, the subscript D is added to all the average probabilities of error defined previously, for example $E_D(s_1^*)$, $E_{1D}(s_1^*)$, E_D, etc. $E_{1D}(s_1^*)$ and $E_{2D}(s_1^*)$ are obtained by integration of the conditional probability density $f(s_1 | s_1^*)$ which must be determined.

The conditional probability density of two normally distributed random variables, x_1 and x_2, is expressed by a well known formula as a function of the variances σ_1 and σ_2, and the autocorrelation coefficient ρ_{12}.

$$f(x_1 \mid x_2) = \frac{1}{\sqrt{2\pi}\ \sigma_1\ \sqrt{1-\rho_{12}^2}}\ e^{-\frac{1}{2(1-\rho_{12}^2)}\left[\left(\frac{x_1-\overline{x_1}}{\sigma_1}\right)-\rho_{12}\left(\frac{x_2-\overline{x_2}}{\sigma_2}\right)\right]^2} \qquad \text{(IV-30)}$$

$$\text{where} \quad \rho_{12} = \frac{\overline{(x_1-\overline{x_1})(x_2-\overline{x_2})}}{\sqrt{\overline{(x_1-\overline{x_1})^2}\ \overline{(x_2-\overline{x_2})^2}}}$$

In this formula, let

$$s_1 = x_1 \qquad s_1{}^* = x_2$$

and note that

$$s_1 - \overline{s_1} = n_1 = n(t=t_1) \qquad s_1{}^* - \overline{s_1{}^*} = n_1{}^* = n(t=t_1{}^*)$$

$$\sigma_1{}^2 = \sigma_2{}^2 = \overline{n_1{}^2} = \overline{n_1{}^{*2}} = \sigma^2$$

and

$$\rho_{12} = \frac{\overline{n_1\ n_1{}^*}}{\sqrt{\overline{n_1{}^2}\ \overline{n_1{}^{*2}}}} = \rho(\tau = t_1 - t_1{}^*) = e^{\frac{-(t_1-t_1{}^*)}{RC}} = \rho^* \qquad \text{(IV-31)}$$

$$\text{Then}\quad f(s_1/s_1{}^*) = \frac{1}{\sqrt{2\pi}\ \sigma\ \sqrt{1-\rho^{*2}}}\ e^{\frac{-(n_1-\rho^* n_1{}^*)^2}{2\sigma^2(1-\rho^{*2})}} \qquad \text{(IV-32)}$$

$$\text{Also,}\qquad f(s_1 \mid s_1{}^*) = f(n_1 \mid n_1{}^*) = f(n_1 \mid s_1{}^*) \qquad \text{(IV-33)}$$

The average conditional probability of error, $E_D(s_1{}^*)$, in the detection of s_1 with a threshold D and knowing $s_1{}^*$ is evaluated next. Given $s_1{}^*$ <u>and</u> γ_o and γ_{-1}, the average conditional probability of error of the first type (0 sent, 1 received) is:

$$E_{1D}(s_1^*) = P[s_1 > D \mid s_1^*] \qquad \text{(IV-34)}$$

The inequality can be expressed as a function of n_1 by using formula IV-27 with $\gamma_1 = 0$.

$$E_{1D}(s_1^*) = P[n_1 > D - A(\gamma_o \rho + \gamma_{-1} \rho^2) \mid s_1^*] \qquad \text{(IV-35)}$$

The conditional probability density $f(n_1/s_1^*)$ is given by formula IV-33

$$E_{1D}(s_1^*) = \int_{n_1 = D - A(\gamma_o \rho + \gamma_{-1}\rho^2)}^{\infty} \frac{1}{\sqrt{2\pi}\ \sigma\sqrt{1-\rho^{*2}}}\, e^{-(n - \rho^* n_1^*)^2/2\sigma^2(1-\rho^{*2})}\, dn_1 \qquad \text{(IV-36)}$$

After the change $u = \dfrac{n_1 - \rho^* n_1^*}{\sigma\sqrt{1-\rho^{*2}}}$

$$E_{1D}(s_1^*) = \int_{u_{1D}}^{\infty} e^{-x^2/2}\, dx = I(u_{1D}) \qquad \text{(IV-37)}$$

$$u_{1D} = \frac{D - A(\gamma_o \rho + \gamma_{-1}\rho^2)}{\sigma\sqrt{1-\rho^{*2}}} \quad \frac{-\rho^*\ n_1^*}{\sigma\sqrt{1-\rho^{*2}}} \qquad \text{(IV-38)}$$

Given s_1^* <u>and</u> γ_o and γ_{-1}, the average conditional probability of error of the second type (1 sent, 0 received) is:

$$E_{2D}(s_1^*) = P[s_1 < D \mid s_1^*] \qquad \text{(IV-39)}$$

$$= P[n_1 < D - A(\gamma_o \rho + \gamma_1 \rho^2) \mid n_1^*] \qquad (IV-40)$$

$$E_{2D}(s_1^*) = \int_{-\infty}^{n = D - A - A(\gamma_o \rho + \gamma_{-1}\rho^2)} \frac{1}{\sqrt{2\pi}\ \sigma \sqrt{1 - \rho^{*2}}} e^{\frac{-(n_1 - \rho^* n_1^*)^2}{2\sigma^2 (1 - \rho^{*2})}} dn_1 \qquad (IV-41)$$

After the change $u = -\left(\frac{n_1 - \rho^* n_1^*}{\sigma \sqrt{1 - \rho^{*2}}} \right)$

$$E_{2D}(s_1^*,\ \gamma_o,\ \gamma_{-1}) = \int_{u_{2D}}^{\infty} e^{-x^2/2}\ du = I(u_{2D}) \qquad (IV42)$$

$$u_{2D} = \frac{A - D + A(\gamma_o \rho + \gamma_{-1} \rho^2)}{\sigma \sqrt{1 - \rho^{*2}}} + \frac{\rho^* n_1^*}{\sigma \sqrt{1 - \rho^{*2}}} \qquad (IV-43)$$

Given s_1^*, i.e. n_1^*, γ_o and γ_{-1}, the average conditional probability of error of the first or second type is:

$$E_D(s_1^*) = 1/2\ E_{1D}(s_1^*) + 1/2\ E_{2D}(s_1^*) \qquad (IV-44)$$

$$= 1/2\ I(u_{1D}) + 1/2\ I(u_{2D}) \qquad (IV-45)$$

The average conditional probability of error of the first or second type for given γ_o and γ_{-1} but for any n_1^* is obtained by averaging (IV-45) with respect to n_1^*.

$$E_D(\gamma_o,\ \gamma_{-1}) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \frac{e^{-n_1^{*2}/2\sigma^2}}{\sigma} \frac{1}{2} \left[I(u_{1D}) + I(u_{2D}) \right] dn_1^* \qquad (IV-46)$$

If u_{1D} and u_{2D} are functions of n_1^*, the double integral can be condensed in a simple integral as explained in Appendix A, resulting in:

$$E_D(\gamma_o, \gamma_{-1}) = 1/2 \; I\left[\frac{D - A(\gamma_o\rho + \gamma_{-1}\rho^2)}{\sigma}\right] + 1/2 \; I\left[\frac{A - D + A(\gamma_o\rho + \gamma_{-1}\rho^2)}{\sigma}\right] \qquad \text{(IV-47)}$$

The average probability of error of any type, for any n_1^*, and any γ_o, γ_{-1}, is obtained by averaging the average conditional probability of error, $E_D(\gamma_o, \gamma_{-1})$, with respect to γ_o and γ_{-1}.

$$E_D = 1/8 \sum_{\gamma_o = 0,1} \; \sum_{\gamma_{-1} = 0,1} \left[I\left[\frac{D - A(\gamma_o\rho + \gamma_{-1}\rho^2)}{\sigma}\right] + I\left[\frac{A - D + A(\gamma_o\rho + \gamma_{-1}\rho^2)}{\sigma}\right]\right] \text{(IV-48)}$$

IV-5 Choice of the Threshold for the Minimum Average Probability of Error

The threshold D is chosen to make the average probability of error minimum; two cases are considered: constant threshold and adaptive threshold.

Given s_1^* (i.e. n_1^*, γ_o and γ_{-1}), the average conditional probability of error of the first or second type is:

$$E_D(s_1^*) = 1/2 \; I(u_{1D}) + 1/2 \; I(u_{2D}) \qquad \text{(IV-49)}$$

Since $I(x)$ is monotonically decreasing, the sum $I(u_1) + I(u_2)$ where $u_1 + u_2 = C =$ Constant, is minimum when $u_1 = u_2 = C/2$.

Therefore, since $u_{1D} + u_{2D} = \dfrac{A}{\sigma\sqrt{1-\rho^2}}$, the average probability of error is minimum when $u_{1D} = u_{2D} = \dfrac{A}{2\sigma\sqrt{1-\rho^2}} = u_D$. Substituting formulas IV-38 and IV-43, yields

$$u_D = \frac{D - A(\gamma_o \rho + \gamma_{-1}\rho^2) - \rho^* n_1^*}{\sigma \sqrt{1 - \rho^{*2}}} = \frac{A - D + A(\gamma_o \rho + \gamma_{-1}\rho^2) + \rho^* n_1^*}{\sigma \sqrt{1 - \rho^{*2}}} = \frac{A}{2\sigma\sqrt{1 - \rho^{*2}}} \qquad \text{(IV-50)}$$

The equality is satisfied when

$$D = D_A = A/2 + A(\gamma_o \rho + \gamma_{-1}\rho^2) + \rho^* n_1^* \qquad \text{(IV-51)}$$

Using formulas (IV-26) and $s_1^* = p_1^* + n_1^*$

$$D_A = A/2 + \rho^* s_1^* \qquad \text{(IV-52)}$$

Assuming that the threshold level can be varied as a function of the sampling signal s_1^* , D_A is the optimum threshold (at time t_1) and is named adaptive threshold.

The physical interpretation for the choice of the corrective term $\rho^* s_1^*$ is similar to the one of paragraph III-3. The corrective term is the output of an "RC network" of time constant RC, discharged from the initial value s_1^* (here $s_1^* \neq n_1^*$) at time t_1^* to the value $n_1^* \, e^{-\frac{(t_1 - t_1^*)}{RC}}$ at time t_1 and it is obtained at the output of a sampler-holder which samples the filtered noise s(t) at time $t = t_1^*$; during the holding time the sampler-holder behaves as an RC network and the RC time constant of the sampler-holder must be equal to the RC time constant of the RC filter. Again, the experimental times of sampling and detection (Ch. VII) are slightly different from the theoretical values.

Replacing D by D_A in $E_D(s_1^*)$ (formula IV-44) and replacing subscript D_A by A for short notation yields

$$E_{D_A}(s_1^*) = E_A(s_1^*) = I\left[\frac{A}{2\sigma\sqrt{1-\rho^{*2}}}\right] \qquad \text{(IV-53)}$$

Since this expression does not depend on s_1^* , further averaging is not necessary. The average probability of error of the first or second type for any s_1^* is, in the case of adaptive threshold,

$$E_A = E_A(s_1^*) = I\left[\frac{A}{2\sigma\sqrt{1-\rho^{*2}}}\right] \qquad \text{(IV-54)}$$

If the threshold level cannot be varied, the best choice for D is $D = A/2$ when ρ is small; otherwise, D is slightly higher. The constant threshold is denoted by D_C and the subscript D_C is replaced by C in the average probabilities of error for constant threshold. Since $E_{D_C}(s_1^*) = E_C(s_1^*)$ is a function of s_1^* , it must be averaged with respect to n_1^* and then γ_o and γ_{-1} in order to obtain the average probability of error of the first or second type for any s_1^* at constant threshold, D_C. $E_{D_C} = E_C$ is obtained by substituting $D = D_C = A/2$ in formula (IV-48)

$$E_C = 1/8 \sum_{\gamma_o=0,1} \sum_{\gamma_{-1}=0,1} \left\{ I\left[\frac{A}{2\sigma}\left(1 + 2(\gamma_o\rho + \gamma_{-1}\rho^2)\right)\right] + I\left[\frac{A}{2\sigma}\left(1 - 2(\gamma_o\rho + \gamma_{-1}\rho^2)\right)\right]\right\} \qquad \text{(IV-55)}$$

or $$E_C = 1/8 \sum_{\gamma_o=0,1} \sum_{\gamma_{-1}=0,1} \sum_{\ell=1,-1} I\left[\frac{A}{2\sigma}\left(1 + 2\ell(\gamma_o\rho + \gamma_{-1}\rho^2)\right)\right] \qquad \text{(IV-56)}$$

In terms of the dimensionless variables $Q = \frac{P_s T}{\eta}$, y and K, the average probability of error for adaptive and constant threshold are respectively:

$$E_A = I\left[\frac{\sqrt{Q}\ \sqrt{2}\ (1 - e^{-Ky})}{\sqrt{Ky}\ \sqrt{1 - e^{-2K^*y}}}\right] \qquad \text{(IV-57)}$$

$$E_C = 1/8 \sum_{\gamma_o = 0,1}\ \sum_{\gamma_{-1} = 0,1}\ \sum_{\ell = 1,-1} I\left[\frac{\sqrt{Q}\ \sqrt{2}\ (1 - e^{-Ky})\ \left[1 + 2\ell(\gamma_o e^{-y} + \gamma_{-1} e^{-2y})\right]}{\sqrt{Ky}}\right] \qquad \text{(IV-58)}$$

The $\sqrt{2}$ comes from the choice of equal probability for pulse and no pulse.

IV-6 Choice of the Numerical Values for the Variables

The average probability of error for adaptive and constant threshold are expressed in formulas (IV-57) and (IV-58), respectively. Both depend upon the fictitious signal to noise ratio, Q, the relative width of the pulse, K, and the inverse of the time constant of the filter, $y = T/RC$.

The numerical values for the fictitious signal to noise ratio Q are chosen such that the average probability of error remains in the range of interest in telemetry or radar, say $10^{-3} < E < 10^{-7}$, resulting in the following values for Q, $Q = 15, 10, 25, 30, 35$. Since it is advantageous to use small values for K, let $K = .1, .3, .5$. Let $y_{min}(K)$ be the value of y which makes the average probability of error minimum for a given K. Although $y_{min}(K)$ varies widely with K, fortunately, $Ky_{min}(K)$ varies little. Therefore, it is convenient to take $u = Ky$ as variable instead of y. The values for u are in the range $.1 < u < 2.0$ since this insures a variation of y far below and far above $y_{min}(K)$, for any choice of K.

The average probability of error for adaptive threshold depends not only on Q, K and $u = Ky$, but also on K^*. It is clear that the average probability of error decreases with K^*, so that $K = K^*$ is the optimum choice. Therefore, the comparison between adaptive and constant threshold is made for $K^* = K$; that is, sampling just before the unknown signal.

The numerical values used for Q, K and $u = Ky$ are tabulated below.

$$Q_i = 15,\ 20,\ 25,\ 30,\ 35$$

$$K_j = .1,\ .3,\ .5$$

$$u_m = .1,\ .4,\ .6,\ .8,\ 1,\ 1.15.\ 1.25,\ 1.35,\ 1.45,\ 1.55,\ 1.65,\ 1.75,\ 1.85,\ 2,\ 2.5,\ 3,\ 5,\ 7,\ 10,\ 15,\ 25.$$

where the index i, j and m correspond to the place in the list, for example:

$$Q_1 = 15, \quad K_3 = .5, \quad u_5 = 1, \text{ etc.}$$

IV-7 Computation of the Average Probability of Error

The average probability of error for adaptive and constant threshold, given by formulas (IV-57) and (IV-58), are computed for all the numerical values of $Q = Q_i$, $K = K_j$ and $u = u_m$ tabulated in paragraph IV-6. Since all possible combinations of Q_i, K_j, and u_m, $(5 \times 3 \times 21)$, must be used, three successive variations are needed: m varies first, then j, then i.

Given a set of numerical values $Q_i = Q_i'$, $K_j = K_j'$, $u_m = u_m'$, $K^* = K_j'$, the average probabilities of error after replacing Ky by u in formulas (IV-57) and (IV-58), are:

(1) For adaptive threshold

$$E_A(Q_i', u_m') = I \quad \sqrt{Q_i'} \quad \sqrt{2} \quad \frac{1 - e^{-u_m'}}{\sqrt{u} \quad \sqrt{1 - e^{-2u_m'}}} \tag{IV-59}$$

(2) For constant threshold

$$E_C(Q_i', K_i', u_m') = \sum_{\gamma_0=0,1} \sum_{\gamma_{-1}=0,1} \sum_{\ell=1,-1} 1/8 \; I\left[\sqrt{Q_i'}\,\sqrt{2}\;\frac{1-e^{-u_m'}}{\sqrt{u_m'}}\left(1+2\ell(\gamma_0 e^{-u/K}+\gamma_{-1} e^{-2u/K})\right)\right] \qquad \text{(IV-60)}$$

where $I(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\,dt = .5 - \int_o^x \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\,dt$,

The average probability of error are linear combinations of integrals $I(x_i)$ where x_i assumes numerical values. The computer must evaluate the definite integrals $I(x)$ for any x; therefore, the computer computes and stores $I(x_i)$ for a large number of values $x = x_i$ and the intermediary $I(x)$ are obtained by interpolation. (See Appendix B).

Since $I(x > 6.35)$ is quite negligible, $I(x)$ is computed only for discrete values of $x = x_i = i(.005)$ where i assumes every integer value in the range $1 \leq i \leq 1272$.

To simplify the notation, let

$$I(x_i) = \int_{i(.005)}^\infty \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\,dt = I(i) \qquad \text{(IV-61)}$$

An iteration formula can be used:

$$I(o) = \int_o^\infty \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\,dt = .50000000 \qquad \text{(IV-62)}$$

$$I(i+1) = I(i) - \int_{i(.005)}^{(i+1)(.005)} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\,dt \qquad \text{(IV-63)}$$

The small definite integral $\int_{i(.005)}^{(i+1)(.005)} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\,dt$ is evaluated as a sum, using Simpson technique as explained in Appendix B.

If $0 < x < 6.35$, then $I(i+1) \leq I(x) \leq I(i)$ where i is the integer obtained by truncation of $\frac{x}{.005}$, $i = Tr\left(\frac{x}{.005}\right)$. Since $I(i+1)$ and $I(i)$ are very close, $I(x)$ can be obtained by linear interpolation.

$$\text{If } x > 6.35, \text{ then } I(x) \approx 0 \tag{IV-64}$$

$$\text{If } x < 0\ , \text{ then } I(x) = .5 + I(|x|) \tag{IV-65}$$

$$\text{If } x > 0\ , \text{ then } I(x) = I(|x|) \tag{IV-66}$$

and

$$I(|x|) = I(i) - [I(i) - I(i+1)]\left[\frac{|x|}{.005} - (i-1)\right] \tag{IV-67}$$

where $i = Tr.\left[\frac{|x|}{.005} + .00001\right] + 1$ and the symbol Tr. means truncation.

IV-8 Interpretation of the Computed Results

The average probability of error for adaptive threshold, E_A, is a function of $Q = \frac{P_s T}{\eta}$ and $u = Ky$; the average probability of error for constant threshold is a function of $Q = \frac{P_s T}{\eta}$, K and $u = Ky$. E_A and E_C were computed in paragraph IV-7 for the various Q_i and u_m tabulated in paragraph IV-6 and are now plotted versus Q. Since the average probabilities of error are very small numbers, it is convenient to plot $-\log_{10} E_A$ and $-\log_{10} E_C$ instead of E_A and E_C. The minimum of E_A and E_C correspond to the maximum of $-\log_{10} E_A$ and $-\log_{10} E_C$, respectively.

E_A depends upon Q and Ky. Fig. IV-1 shows $-\log_{10} E_A$ versus Q for Ky = .1 and Ky = .5(K* = K). For constant Ky, $-\log_{10} E_A$ increases about linearly with Q; for example, if Ky = .1, $-\log_{10} E_A \approx .226\,Q + .89$: for a given value of Q, $-\log_{10} E_A$ decreases when Ky increases. Therefore, the average probability of error

(1) decreases exponentially when Q increases at constant Ky.

(2) decreases when Ky decreases at constant Q.

Assuming that Ky = .1 is the physical minimum for Ky, E_A is minimum for Ky = .1. The value of the minimum decreases when Q increases and denoted

$$E_A(Ky = .1\ ,\ Q) = E_{A\ min}\,(Q)$$

E_C depends upon Q, K and y = u/K. The influence of the choice of the RC filter on the average probability of error for constant threshold is demonstrated in Fig. IV-2, which shows the variation of $-\log_{10} E_C$ versus y = T/RC for constant Q and constant K. Typical values are used for Q_i and K_i: $Q_i = 15$, $Q_i = 35$ and $K_i = .1$, $K_i = .3$, $K_i = .5$. Each curve $-\log_{10} E_C$ versus y for given Q_i and K_i presents a maximum for a specific value of $y = y_{min}(K_i)$; the corresponding minimum average probability of error is denoted $E_{C\ min}(Q_i\ ,\ K_i)$. The minimum average probability of error, $E_{C\ min}(Q_i, K_i)$, depends upon both Q_i and K_i, for given K the minimum decreases when Q increases; whereas, for given Q, the minimum decreases when K decreases.

The optimum value of y, $y_{min}(K_i)$, where the minimum average probability of error for given Q_i and K_i occurs, depends only upon K_i (not Q_i). Both $y_{min}(K_i)$ and K_i vary inversely and while $y_{min}(K_i)$ varies widely, the product $K_i\, y_{min}(K_i)$ varies little. Fig. IV-3 shows the variation of $y_{min}(K)$ as a function of K; therefore, this curve determines the optimum RC filter for each choice of the pulse width. If K = .1 is the physical minimum of K, $E_{C\ min}(Q, K = .1) = E_{C\ min}(Q)$ is the smaller minimum of E_C versus y for given Q and any realizable K, the corresponding value of y is $y = y_{min}(K = .1) = 12.5$.

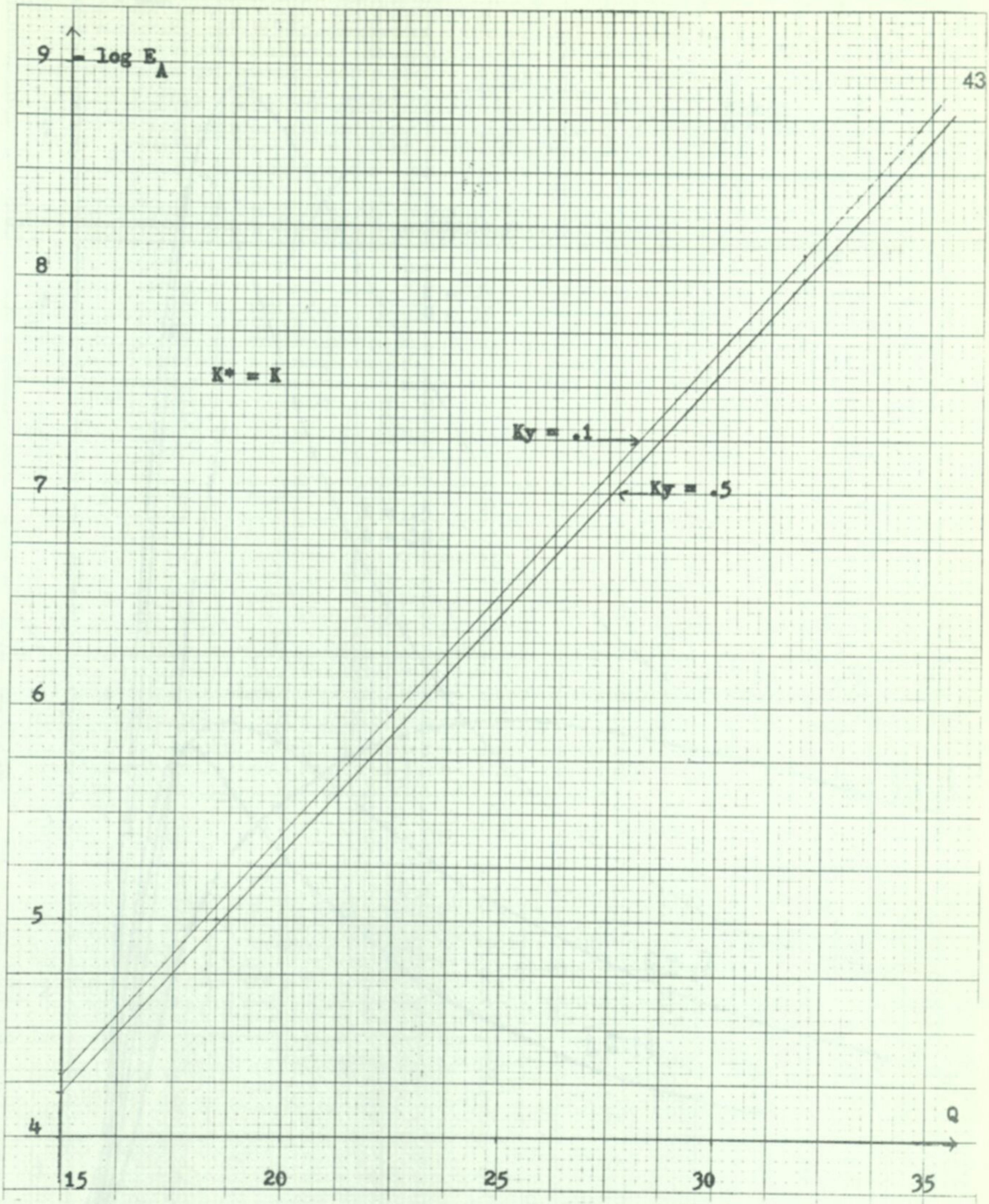


Fig. IV-1 : Probability of Error for Adaptive Threshold versus $Q = \frac{V^2 KT}{2\eta}$, using an Analog Computer (White Noise and RC Filter)

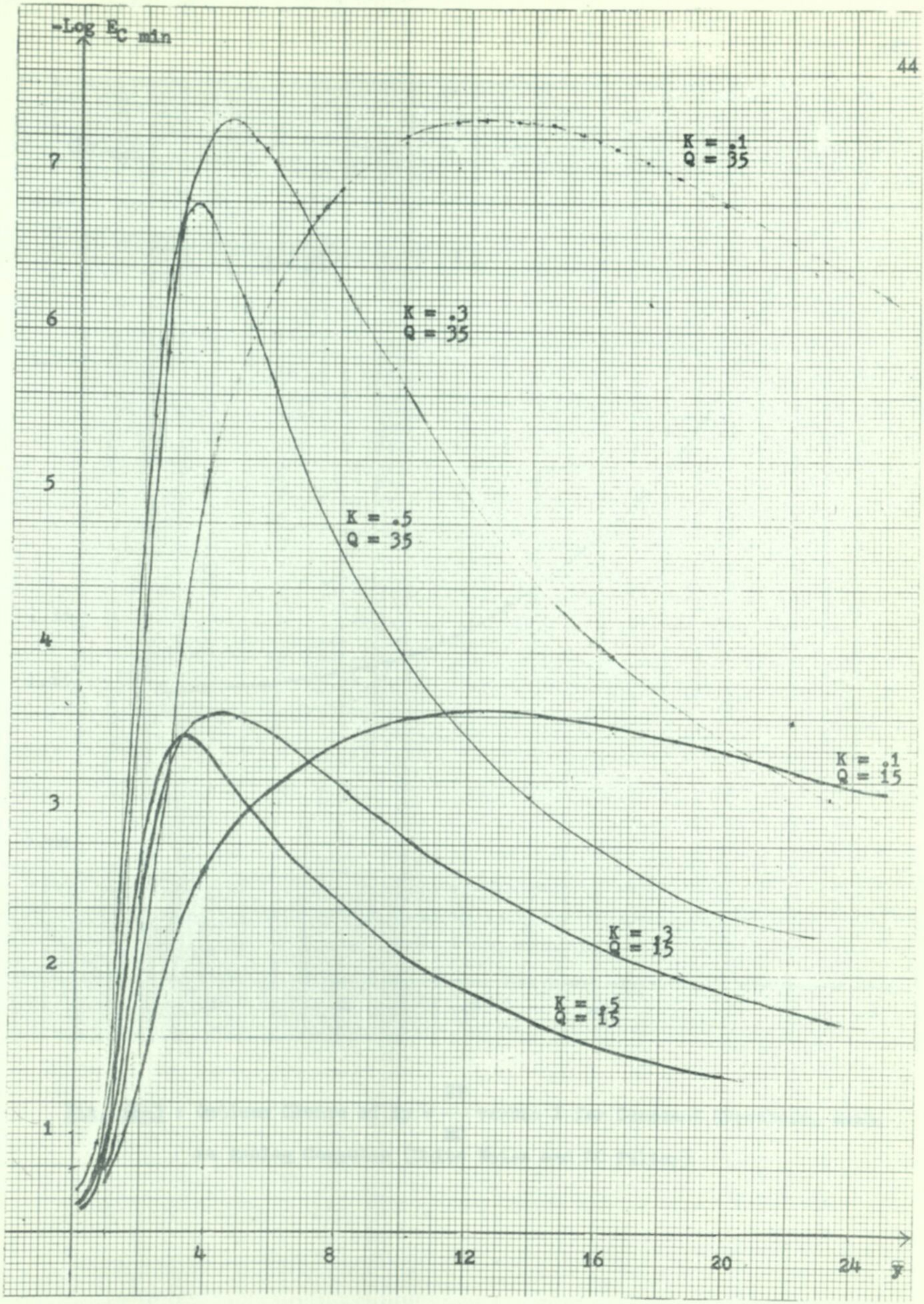


Fig. IV-2 : Probability of Error for Constant Threshold versus y = T/RC

(White Noise and RC Filter)

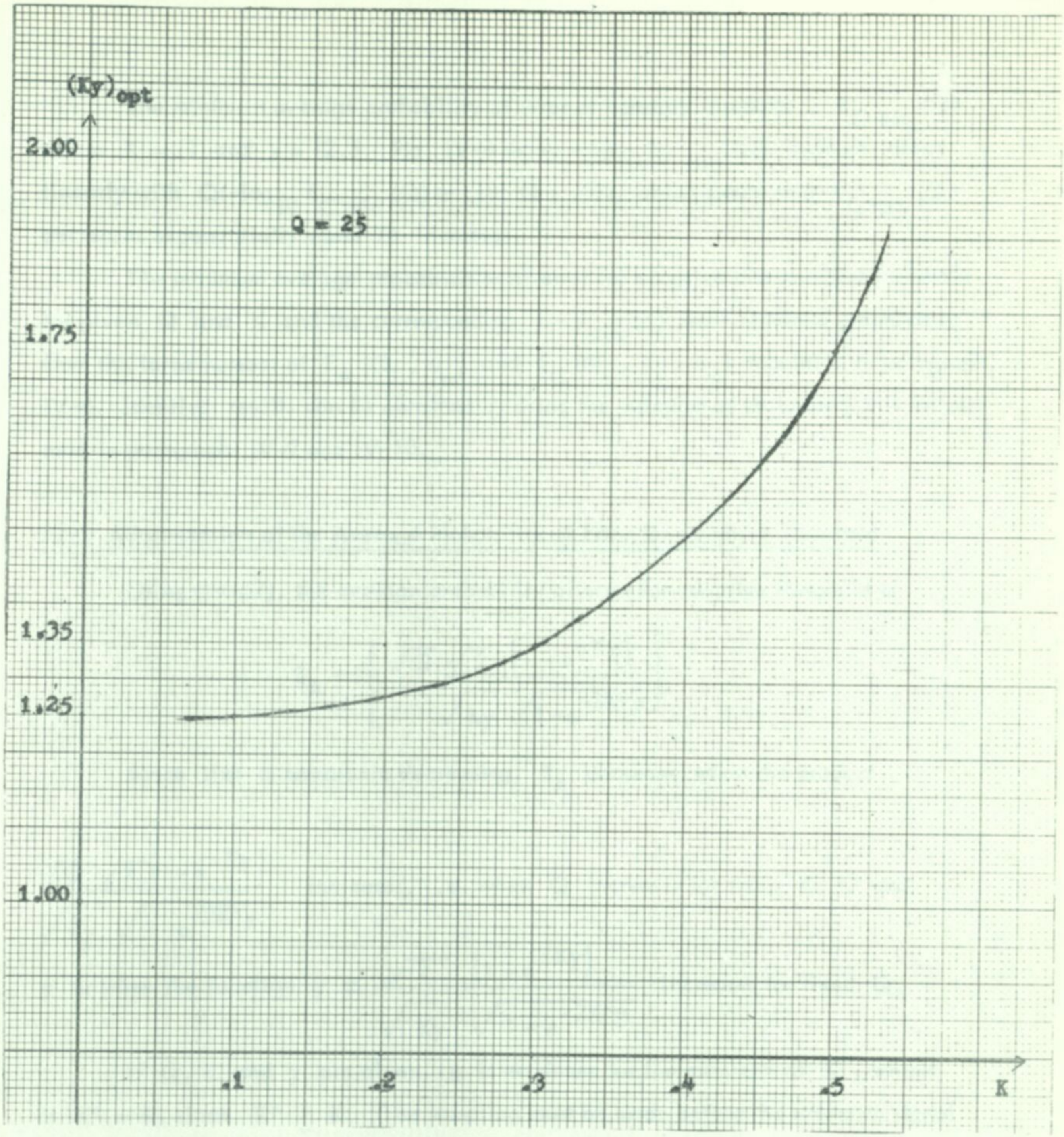


Fig. IV-3 : Optimum Choice of $Ky = \frac{KT}{RC}$ versus K for Constant Threshold, using an Analog Computer (White Noise and RC Filter)

The minimum average probability of error at constant threshold is $E_{C\,min}(Q)$ where $K = .1$ and $y = 12.5$. Fig. IV-4 shows $-\log_{10} E_{C\,min}(Q)$ increases about linearly with $Q(-\log_{10} E_{C\,min}(Q) \approx .185\,Q + .87)$, which means that $E_{C\,min}(Q)$ decreases exponentially when Q increases.

The minimum average probabilities of error for adaptive and constant threshold for given Q and K and the corresponding $y = y_{min}(K)$ were obtained graphically from the large number of $E_A(Q_i, u_m)$ and $E_C(Q_i, K_j, u_m)$ tabulated in paragraph IV-6. Identical results can be obtained by using the calculus of variations; this follows next in paragraphs IV-9 and IV-10.

IV-9 Minimization of the Average Probability of Error for Adaptive Threshold

Using (IV-57), the average probability of error for adaptive threshold is

$$E_A = I\left[\frac{\sqrt{Q}\,\sqrt{2}\,\left(1 - e^{-Ky}\right)}{\sqrt{Ky}\,\sqrt{1 - e^{-2K^*y}}}\right]$$

Since $I(x)$ is monotonic decreasing, E_A decreases when the product

$\dfrac{\sqrt{Q}\,\sqrt{2}\left(1 - e^{-Ky}\right)}{\sqrt{Ky}\,\sqrt{1 - e^{-2K^*y}}}$ increases, i.e. when Q increases for fixed K, K^* and y or when the function $g = \dfrac{\sqrt{2}\,\left(1 - e^{-Ky}\right)}{\sqrt{Ky}\,\sqrt{1 - e^{-2K^*y}}}$ increases for fixed Q.

If Q, K and y are given, g is maximum for the minimum of K^*, which is $K^* = K$, since $K^* \geq K$. (The sampling must be made before the unknown signal starts.)

When $K = K^*$, the expression for g is simplified

$$g = \frac{\sqrt{2}\,\left(1 - e^{-Ky}\right)}{\sqrt{Ky}\,\sqrt{1 - e^{-2Ky}}} = \sqrt{\frac{2\left(1 - e^{-Ky}\right)}{Ky\left(1 + e^{-Ky}\right)}}$$

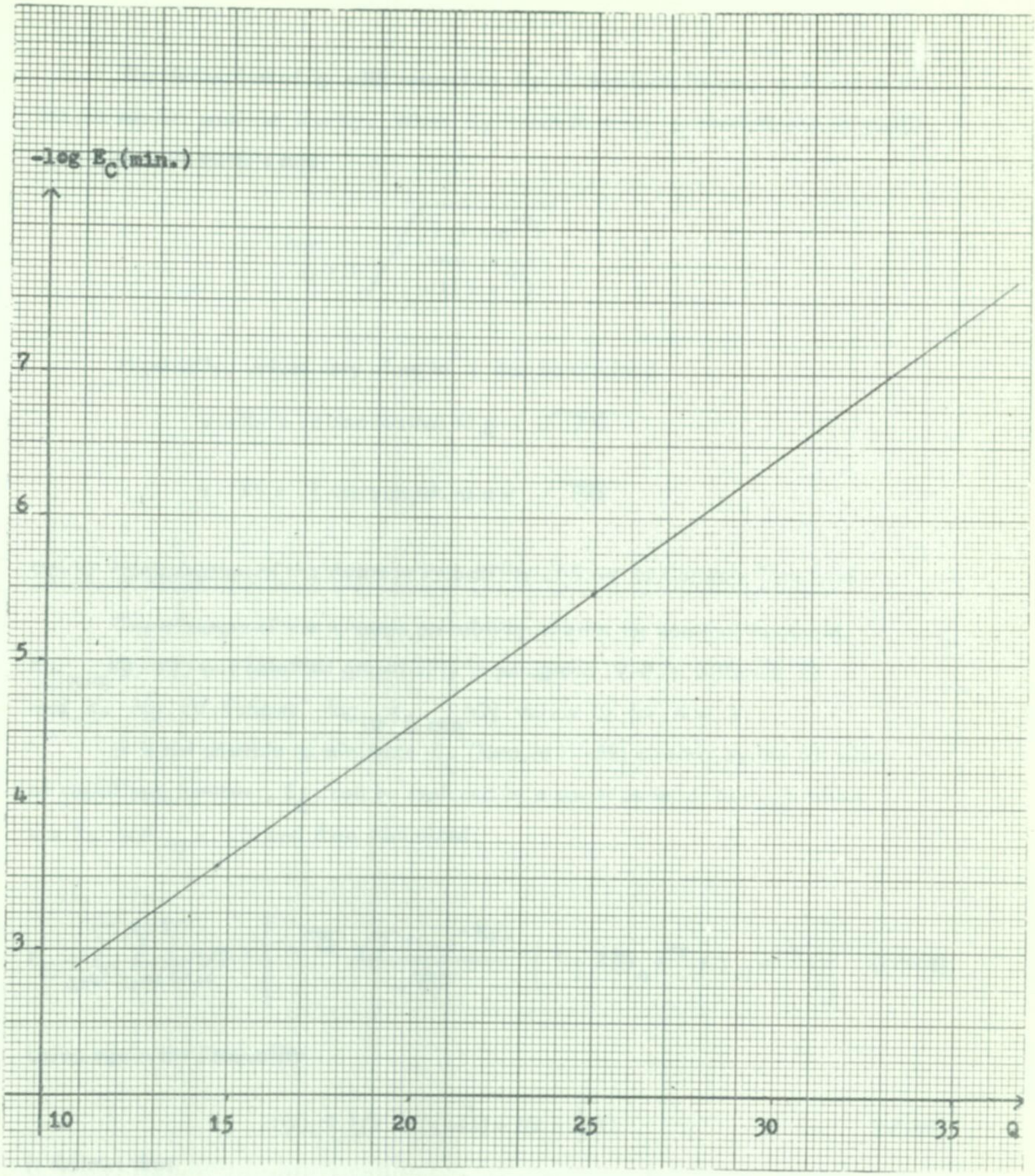


Fig. IV-4 : Minimum Probability of Error for Constant Threshold versus $Q = \frac{V^2KT}{2\eta}$, using an Analog Computer (White Noise and RC Filter)

The derivative of g with respect to Ky is negative; hence, g is minimum for the smaller possible $u = Ky$. When Ky tends to zero, g tends to one by application of L'Hospital's rule:

$$\lim_{u \to 0} \sqrt{\frac{2(1-e^{-u})}{u(1+e^{-u})}} = 1$$

Physically, values very close to 1 can be attained

$u = Ky = .3$ corresponds to $g = \sqrt{.992}$

$u = Ky = .1$ corresponds to $g = \sqrt{.998}$

IV-10 Minimization of the Average Probability of Error for Constant Threshold

The minimum of the average probability of error for constant threshold, $E_{C\,min}(Q, K)$, was obtained graphically in paragraph IV-8 for different Q and K. Fig. IV-4 shows $-\log_{10} E_{C\,min}(Q)$ versus Q for $K = .1$.

It is not possible to minimize E_C (formula IV-58) directly by the calculus of variations; however, E_C can be replaced by a power series of $(u - u_0)$ where $u = Ky$ and $u_0 = 1.6$ and then minimized.

$$E_C = \sum_{\gamma_0=0,1} \sum_{\gamma_{-1}=0,1} \sum_{\ell=1,-1} 1/8 \; I\left[\sqrt{Q}\,\frac{\sqrt{2}\,(1-e^{-Ky})}{\sqrt{Ky}}\left(1+2\ell(\gamma_0 e^{-y}+\gamma_{-1} e^{-2y})\right)\right] \quad \text{(IV-68)}$$

Using again the inequality

$$I(x) < 1/2\,[I(x+\Delta) + I(x-\Delta)] \quad \text{(IV-69)}$$

it follows that

$$E_C > I[\alpha] \quad \text{(IV-70)}$$

where $\alpha = \sqrt{Q}\,\dfrac{\sqrt{2}\,(1-e^{-Ky})}{\sqrt{Ky}}$ (IV-71)

The minimum of $I(\alpha)$ is easily shown to occur at $Ky = 1.26$.

Because of the factor $1 + 2\ell(\gamma_o e^{-y} + \gamma_{-1} e^{-2y})$, the minimum of the average probability of error for constant threshold occurs for $Ky > 1.26$. When K is small, the minimum occurs very close to $Ky = 1.26$ because $y = \frac{1.26}{K}$ is large and e^{-y} and e^{-2y} are negligible. When K is large, the minimum occurs for Ky well above 1.26, because $y = \frac{1.26}{K}$ is small and e^{-y} and e^{-2y} cannot be neglected.

The problem is to minimize the sum of 8 transcendental integrals, the limits of which are complicated functions.

The solution is to develop the integrals in a series. Taking advantage of the fact that the maximum occurs for a rather limited range of $u = Ky$, the integrals are developed into a power series of $z = (u - u_0)$ about the point $u_0 = 1.55$.

Assuming one step memory, the average probability of error for constant threshold is

$$E_C = \sum_{\gamma_o = 0,1} \sum_{\ell = -1,1} 1/4 \; I\left[\sqrt{Q}\ \sqrt{2}\ \frac{(1 - e^{-Ky})}{\sqrt{Ky}} (1 + 2\ell\gamma_o e^{-y})\right] \qquad \text{(IV-72)}$$

$$E_C = \sum_{\gamma_o = 0,1} \sum_{\ell = -1,1} 1/4 \; P(\gamma_o, \ell, z) \qquad \text{(IV-73)}$$

where

$$P(\gamma_o, \ell, z) = \sum_{m=0}^{4} a_i^{(m)} z^m \qquad \text{(IV-74)}$$

$i = 1, 2, 3, 4$, represents the four combinations of γ_o and ℓ

$$z = (u - u_0) \qquad \text{(IV-75)}$$

and $$\alpha_i^{(m)} = \left[\frac{d^m}{du^m}\ I\ [\phi(u, K, i)]\right] \qquad \text{(IV-76)}$$

in which $$I(\phi) = \int_{\phi}^{\infty} \frac{e^{-t^2/2}}{\sqrt{2\pi}}\, dt \qquad \text{(IV-77)}$$

and $\phi(u, K, j) = \sqrt{Q}\ \sqrt{2}\ \frac{(1 - e^{-u})}{\sqrt{u}} (1 + 2\ell \gamma_o e^{-u/K})$ (IV-78)

The procedure to obtain the coefficients $a_i^{(m)}$ is described in Appendix C for one step memory; for a two-step memory the calculations are even more complex.

Since the value of u for which E_C is minimum is not too far from $u_0 = 1.6$, the series converges rapidly with four terms giving a good approximation. When E_C is expressed as a fourth degree polynomial, the minimum of E_C is the real root of a third degree polynomial. For a given fictitious signal to noise ratio Q and width of pulse K, the value of u which minimizes E_C, $u_{min}(K)$, and the minimum of E_C, $E_{C\ min}$, can be obtained. Fig. IV-5 shows $u_{min}(K)$ versus K and Fig. IV-6 shows $-\log_{10} E_{C\ min}(K = .1)$ versus Q, resulting from the series approximation. The agreement with the exact results of paragraph IV-8 is surprisingly good considering that the approximation

(1) is for one step memory,

(2) contains only four terms

IV-11 Comparison Between Adaptive and Constant Threshold

The average probability of error in the detection of rectangular pulses mixed with white noise, using an RC filter followed by a threshold device, has been obtained for the case of a constant threshold and of an adaptive threshold.

In the adaptive scheme, the sum of the noise and residual voltage due to the previous pulses is sampled just before the signal is received and the threshold level modified accordingly.

It is shown that the minimum average probability of error for adaptive threshold depends only upon the product $u = Ky$ and decreases with Ky. Hence, given the average power of the emitter, the average probability of error is the same for different widths of pulses if Ky is maintained constant. The plots are made for $Ky = .1$ and $.5$

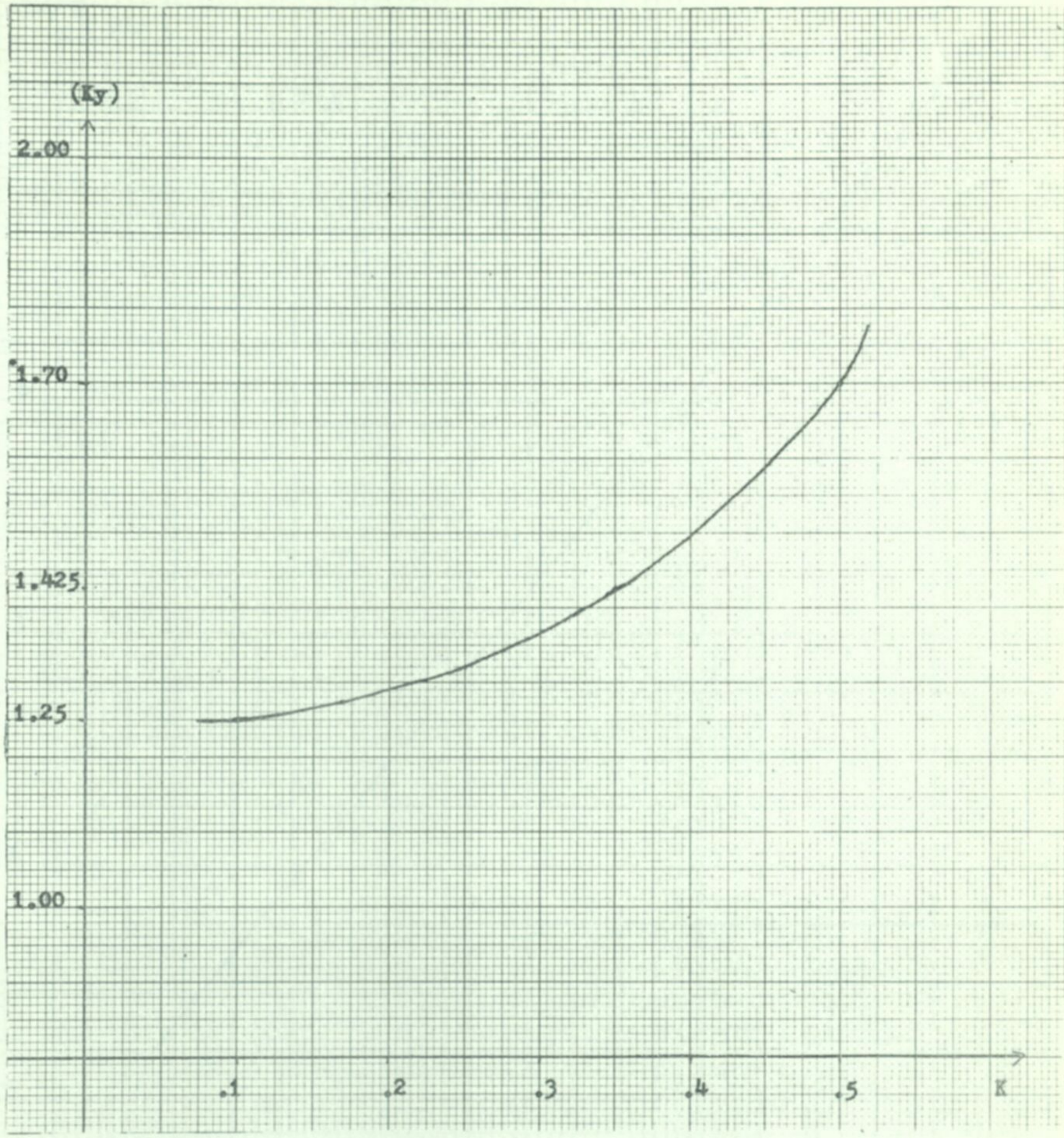


Fig. IV-5 : Optimum Choice of Ky $= \frac{KT}{RC}$ versus K for Constant Threshold, using an Analog Computer (White Noise and RC Filter)

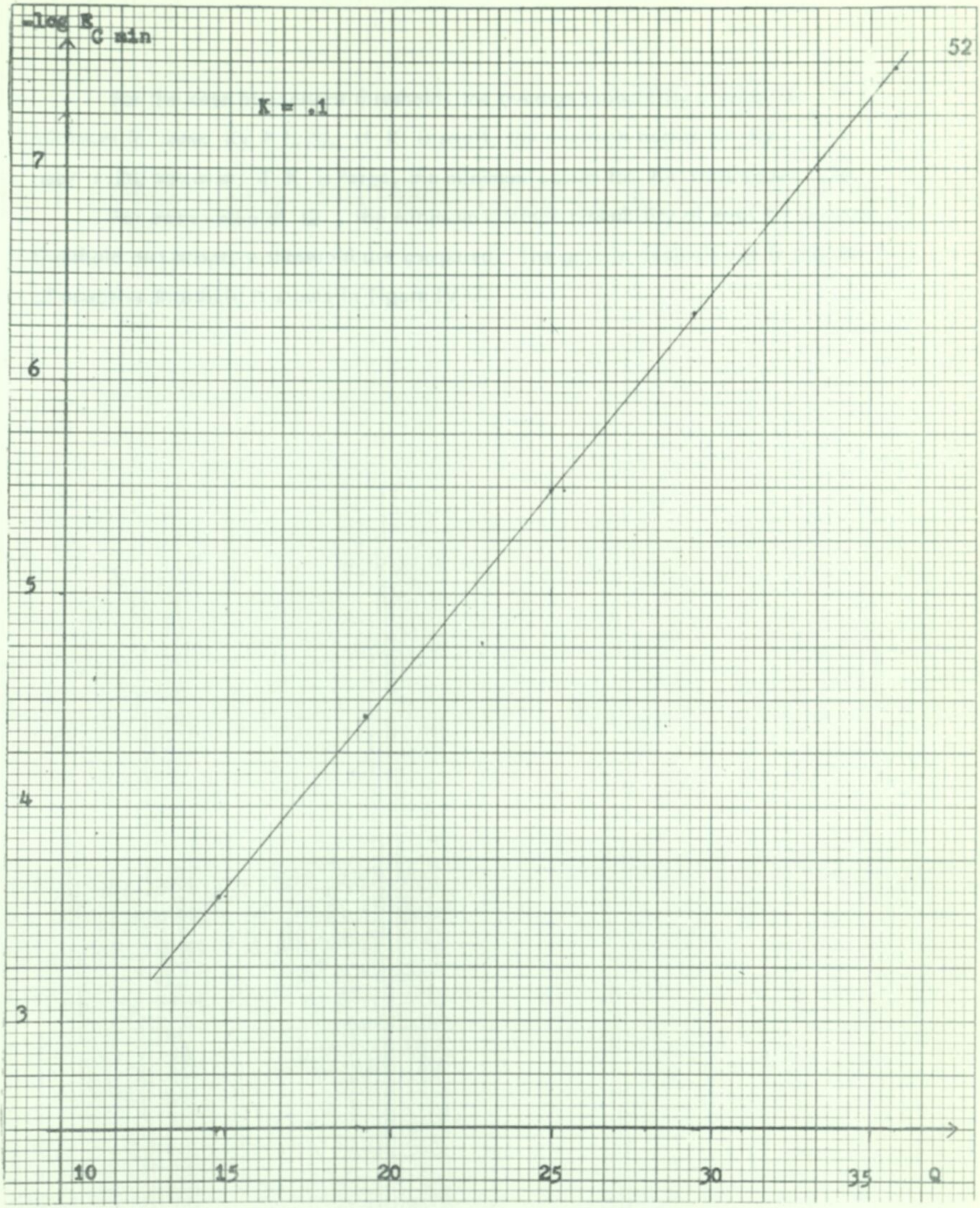


Fig. IV-6 : Minimum Probability of Error for Constant Threshold versus $Q = \frac{V^2 KT}{2\eta}$, using Series Expansion (White Noise and RC Filter)

For the case of constant threshold, the average probability of error is minimum for some $K\gamma > 1.26$. If K is small, the minimum occurs for $K\gamma$ near 1.26; however, if K is large, the minimum occurs for $K\gamma$ well above 1.26.

For each value of Q, the minimum average probability of error for constant threshold and adaptive threshold are compared. The gain in decibels due to the use of adaptive threshold is shown in Figure IV-7.

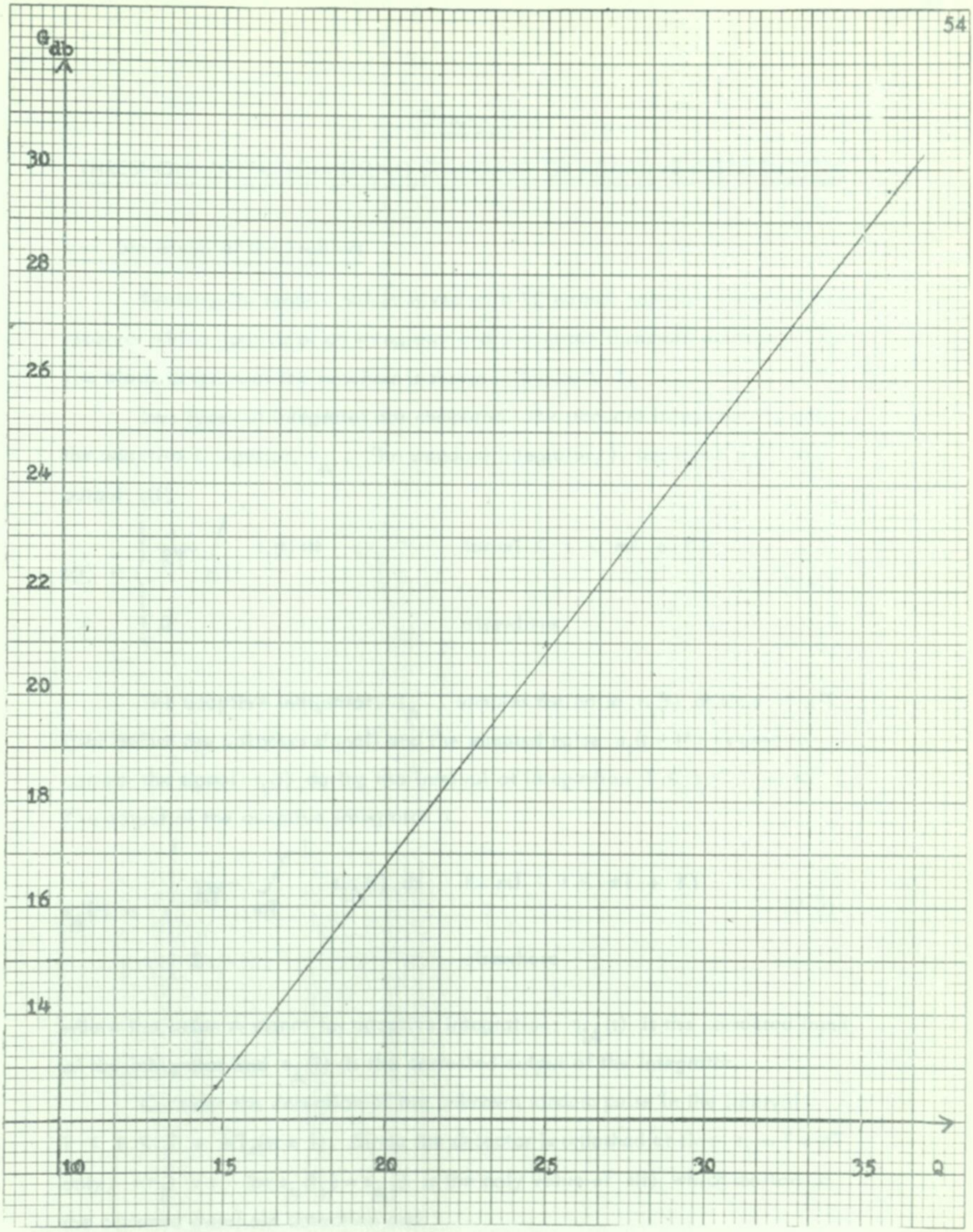


Fig. IV-7 : Reduction of the Probability of Error by Use of an Adaptive Threshold.

(White Noise and RC Filter). $G = -20 \log_{10} (E_A/E_C)$

Chapter V

THRESHOLD DETECTION USING STANDARD OR ADAPTIVE INTEGRATION

V-1 Statement of the Problem

Rectangular random pulses mixed with RC normal noise are detected by a receiver which consists of an integrator and a constant threshold detector, TD. The block diagram of the receiver is shown in Figure V-1.

Two types of integrator are compared: the standard integrator, I, and the adaptive integrator, I_A. The standard integrator, I, has input $s_i(t)$ and output $s(t)$:

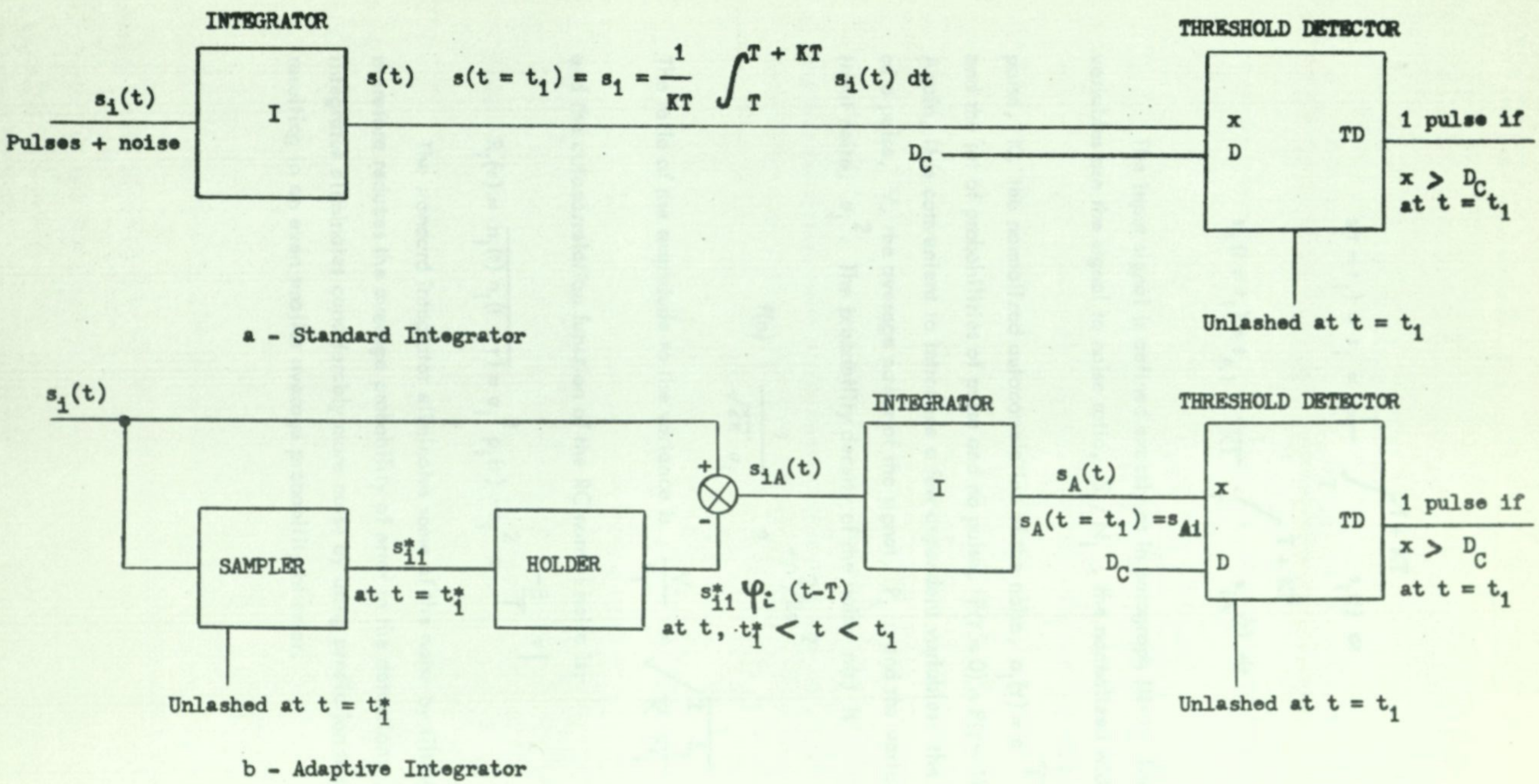
$$s(t) = \begin{cases} \dfrac{1}{KT} \displaystyle\int_{mT}^{t} s_i(t)\, dt & \text{for } mT < t < mT + KT \\ 0 & \text{elsewhere} \end{cases} \qquad (V-1)$$

The adaptive integrator, I_A, samples the noise $n_i(t)$ at time $t^* = T$ (just before the unknown signal) and the sampled noise $n_i(t = t^*)$ is used to correct the signal $s_i(t)$ during the interval of integration $T < t < T + KT$. The output of the adaptive integrator is:

$$s_A(t) = \begin{cases} \dfrac{1}{KT} \displaystyle\int_{mT}^{t} s_{iA}(t)\, dt & \text{for } mT < t < mT + KT \\ 0 & \text{elsewhere} \end{cases} \qquad (V-2)$$

where the index A stands for adaptive integration: $s_{iA}(t)$ is the corrected input of the integrator and $s_A(t)$ is the corrected output of the integrator.

Consider the detection of the unknown random pulse in the interval $T < t < T + KT, (m = 1)$. Since the detector is unlashed at time $t_1 = T + KT$ only, $s(t_1) = s_1$ (or $s_A(t_1) = s_{A1}$) is the only value of $s(t)$ which matters in the constant threshold detection and



<u>Fig. V-1</u> : Block Diagram of a Receiver with Integrator and Threshold Detector

$$s(t = t_1) = s_1 = \frac{1}{KT} \int_{T}^{T+KT} s_i(t)\ dt$$

$$s_A(t = t_1) = s_{A1} = \frac{1}{KT} \int_{T}^{T+KT} s_{iA}(t)\ dt$$

The input signal is defined exactly as in paragraph III-1. The independent variables are the signal to noise ratio, S_i/N_i , the normalized width of the pulse, K, the normalized autocorrelation of the noise, $\rho_i(\tau) = e^{-\frac{\beta}{T}|\tau|}$, and the set of probabilities of pulse and no pulse, $P(\gamma = 0) = P(\gamma = 1) = 1/2$. Again, it is convenient to introduce a few dependent variables: the amplitude of a pulse, V, the average power of the signal, P_s , and the variance of the input noise, σ_i^2. The probability density of the noise n(t) is

$$f(n) \quad \frac{1}{\sqrt{2\pi}\ \sigma_i} \quad e^{-n^2/2\sigma_i^2}$$

The ratio of the amplitude to the variance is $\frac{V}{\sigma_i} = \sqrt{\frac{2}{K}\frac{S_i}{N_i}}$ (V-3)

and the autocorrelation function of the RC normal noise is:

$$R_i(\tau) = \overline{n_i(t)\ n_i(t+\tau)} = \sigma_i^2\ \rho_i(\tau) = \sigma_i^2\ e^{-\frac{\beta}{T}|\tau|} \qquad \text{(V-4)}$$

The standard integrator eliminates some of the noise by filtering and therefore reduces the average probability of error in the detection. The adaptive integrator eliminates considerably more noise by using prediction besides filtering, resulting in an even smaller average probability of error.

V-2 Effect of the Standard Integrator I

Given a random pulse, $\gamma_1 V$ at time $T + KT$, the output of the network I is $p_1 = \frac{1}{KT} \gamma_1 \int_T^{T+KT} V\,dt = \gamma_1 V$. Similarly given a random noise, $n_i(t)$, the output n_1 of the network I is $n_1 = \frac{1}{KT} \int_T^{T+KT} n_i(t)\,dt$. Since I is linear, the noise remains normal; n_1 is normal and completely determined by its mean zero and its variance, σ_1^2.

Assuming additive noise, the input to the network I is $s_i(t) = n_i(t) + p_i(t)$ and the output is $s(t) = n(t) + p(t)$. The signal, $s(t)$, is detected at time $t = T + KT$; therefore, the statistic properties of $s(t = T + KT) = s_1 = n_1 + \gamma_1 V$, (and hence of n_1) must be obtained. Since integration is a linear process, the probability density distribution remains normal and the probability density of s_1, $f(s_1)$ is completely determined by its first moment, $\overline{s_1}$, and its variance, $\overline{s_1^2} - \overline{s_1}^2 = \sigma_1^2$.

$$s_1 = \frac{1}{KT} \int_T^{T+KT} s_1(t)\,dt = \frac{1}{KT} \int_T^{T+KT} (n_i(t) + \gamma_1 V)\,dt \qquad \text{(V-5)}$$

$$\overline{s_1} = \frac{1}{KT} \int_T^{T+KT} \overline{n_i(t)}\,dt + \frac{1}{KT} \int_T^{T+KT} \gamma_1 V\,dt = \gamma_1 V \qquad \text{(V-6)}$$

since $\overline{n_i(t)} = 0$

The signal s_1 is translated so that the new variable x_1 has zero mean:

$$x_1 = s_1 - \overline{s_1} \qquad \text{(V-7)}$$

$\overline{s_1} = \gamma_1 V$, hence

$$x_1 = s_1 - \overline{s_1} = \frac{1}{KT} \int_T^{T+KT} n_i(t)\,dt \qquad \text{(V-8)}$$

Note that x_1 is independent of γ_1, i.e. whether or not a pulse is received.

The variance of s_1, σ_1^2, is equal to $\overline{x_1^2}$

$$\sigma_1^2 = \overline{s_1^2} - \overline{s_1}^2 = \overline{(s_1 - \overline{s_1})^2} = \overline{x_1^2} \qquad \text{(V-9)}$$

$$x_1^2 = \frac{1}{K^2T^2} \int_T^{T+KT} \int_T^{T+KT} n_i(t)\; n_i(t')\; dt'\, dt \qquad \text{(V-10)}$$

$$\text{and } \sigma_1^2 = \overline{x_1^2} = \frac{1}{K^2T^2} \int_T^{T+KT} \int_T^{T+KT} \overline{n_i(t)\, n_i(t')}\; dt\, dt' \qquad \text{(V-11)}$$

Note that $\overline{n_i(t)\, n_i(t')} = R_i(t' - t)$ the autocorrelation of the input noise.

With the changes of variable $t = u + T$ and $t' = u' + T$, σ_1^2 becomes

$$\sigma_1^2 = \frac{1}{K^2T^2} \int_o^{KT} \int_o^{KT} R_i(u - u')\, du\, du'$$

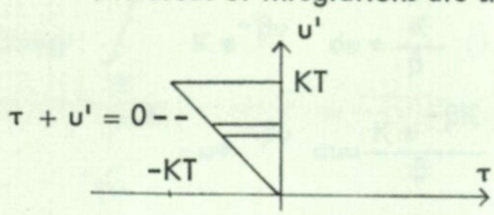
Let $u - u' = \tau$

$$\sigma_1^2 = \frac{1}{K^2T^2} \int_o^{KT} du' \int_{-u'}^{KT-u'} R_i(\tau)\, d\tau \qquad \text{(V-12)}$$

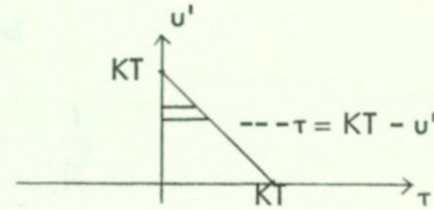
after breaking the first integral in two integrals

$$\sigma_1^2 = \frac{1}{K^2T^2} \left[\int_o^{KT} du' \int_{-u'}^{o} R_i(\tau)\, d\tau + \int_o^{KT} du' \int_o^{KT-u'} R_i(\tau)\, d\tau \right] \qquad \text{(V-13)}$$

The areas of integrations are shown below



and



Changing the order of integration yields

$$\sigma_1^{\,2} = \frac{1}{K^2 T^2}\left[\int_{-KT}^{o} R_i(\tau)\,d\tau \int_{-\tau}^{KT} du' + \int_{o}^{KT} R_i(\tau)\,d\tau \int_{o}^{KT-\tau} du'\right] \qquad \text{(V-14)}$$

where the limits are determined by the areas of integration.

The limits can be obtained mathematically, without a graph;

(1) if $d\tau$ is integrated first, using $0 < u < KT$ and $u - u' = \tau$, it follows $-u' < \tau < -u' + K$ (or $-u' < \tau < 0$ and $0 < \tau < -u' + KT$)

(2) if du' is integrated first, using $-u' < \tau < 0$ and $0 < u' < KT$, it follows $-\tau < u' < KT$; similarly, using $0 < \tau < -u' + KT$ and $o < u' < KT$, it follows $0 < u' < KT - \tau$.

The first integral of formula V-14 is easily integrated, resulting in

$$\sigma_1^{\,2} = \frac{2}{K^2 T^2}\int_{o}^{KT} (KT - \tau)\, R_i(\tau)\, d\tau \qquad \text{(V-15)}$$

It is convenient to use the dimensionless variable $u = \tau/T$

$$\text{Then } \sigma_1^{\,2} = \frac{2}{K^2}\int_{o}^{K} (K - u)\, R_i(u)\, du \qquad \text{(V-16)}$$

$$R_i(\tau) = \sigma_i^{\,2}\, e^{-\frac{\beta}{T}|\tau|} \text{ ; hence, } R_i(u) = \sigma_i^{\,2}\, e^{-\beta u} \text{ for } u > 0$$

$$\text{and } \sigma_1^{\,2} = \sigma_i^{\,2}\,\frac{2}{K^2}\int_{o}^{K} (K - u)\, e^{-\beta u}\, du \qquad \text{(V-17)}$$

Using $\int_{o}^{K} K e^{-\beta u} \, du = \frac{K}{\beta} (1 - e^{-\beta K})$ and

$$\int_{o}^{K} - ue^{-\beta u} \, du = \frac{K e^{-\beta K}}{\beta} + \frac{e^{-\beta K}}{\beta^2} - \frac{1}{\beta^2}$$

$$\sigma_1^{\,2} = \sigma_i^{\,2} \, \frac{2}{K^2 \beta^2} \, [K\beta + e^{-K\beta} - 1] \qquad \text{(V-18)}$$

Let $\mu = 2[\frac{K\beta + e^{-\beta K} - 1}{K^2 \beta^2}]$, which is always smaller than one; therefore, the integration reduces the variance of the noise by a factor μ; $\sigma_i^{\,2}$ becomes $\sigma_1^{\,2} = \mu \, \sigma_i^{\,2}$.

V-3 Effect of the Adaptive Integrator, I_A

The noise does not vary instantaneously and the noise, $n_i(t)$, during the time interval of integration $(T < t < T + KT)$ is related to the noise, $n_i(t=T) = n_i^*$, just before integration. The expected or predicted value of the noise at time t $(T < t < T + KT)$ is $n_i^* \; \rho_i(t - T)$ where $\rho_i(\tau)$ is the normalized autocorrelation of the noise. The signal, $s_i(t)$, at time t can be corrected by subtracting the expected value of the noise, $n_i^* \; \rho_i(t - \tau)$. The corrected signal, $s_{iA}(t)$, is integrated and the corrected output of the adaptive integrator, s_{1A}, is

$$s_{1A} = \frac{1}{KT} \int_{T}^{T+KT} [\, s_i(t) - \rho_i(t - \tau) \, n_i^* \,] \; dt \qquad \text{(V-19)}$$

$$\overline{s_{1A}} = \frac{1}{KT} \int_{T}^{T+KT} \overline{s_i(t)} \, dt - \frac{\overline{n_i^*}}{KT} \int_{T}^{T+KT} \rho_i(t - \tau) \, dt \qquad \text{(V-20)}$$

$$\overline{s_{1A}} = \gamma_1 V$$

The variance of s_{1A} is obtained as in paragraph V-2; letting

$$x_{1A} = s_{1A} - \overline{s_{1A}}, \text{ then } \sigma_{1A}^2 = \overline{x_{1A}^2}$$

$$x_{1A} = \frac{1}{KT} \int_T^{T+KT} (n_i(t) - \rho_i(t-\tau)\, n_i^*)\, dt \qquad \text{(V-21)}$$

$$x_{1A} = \frac{1}{KT} \int_T^{T+KT} (n_i(t) - e^{-\frac{\beta}{T}(t-\tau)}\, n_i^*)\, dt$$

and

$$\sigma_{1A}^2 = \overline{x_{1A}^2} = \frac{1}{K^2T^2} \int_T^{T+KT}\int_T^{T+KT} \overline{(n_i(t) - e^{-\frac{\beta}{T}(t-\tau)}\, n_i^*)(n_i(t') - e^{-\frac{\beta}{T}(t'-\tau)}\, n_i^*)}\, dt\, dt' \qquad \text{(V-22)}$$

The first double integral is broken into four double integrals; the first double integral was evaluted in the case of constant threshold and the three remaining integrals are equal; thus, two out of the three integrals cancel each other:

$$\overline{n_A^2} = J_1 - J_2 - J_3 + J_4 \qquad \text{(V-23)}$$

$$J_1 = \frac{1}{K^2T^2} \int_T^{T+KT}\int_T^{T+KT} n(t)\, n(t')\, dt\, dt' = \sigma_i^2 \left[\frac{2}{K^2\beta^2} (K\beta + e^{-K\beta} - 1)\right] \qquad \text{(V-24)}$$

$$J_2 = J_3 = J_4 = \frac{1}{K^2T^2} \int_T^{T+KT}\int_T^{T+KT} \overline{n^{*2}}\, e^{-\frac{\beta}{T}(t-T)}\, e^{-\frac{\beta}{T}(t'-T)}\, dt\, dt' \qquad \text{(V-25)}$$

$$\overline{n^{*2}} = \sigma_i^2 \text{ and } \frac{1}{KT} \int_T^{T+KT} e^{-\frac{\beta}{T}(t-T)}\, dt = \frac{1 - e^{-\beta K}}{\beta K} \qquad \text{(V-26)}$$

$$\sigma_{1A}^{2} = \overline{n_A^{2}} = \sigma_i^{2}\left\{ \frac{2}{K^2\beta^2}\left(K\beta + e^{-K\beta} - 1\right) - \frac{(1-e^{-K\beta})^2}{K^2\beta^2} \right\} \qquad \text{(V-27)}$$

$$\sigma_{1A}^{2} = \sigma_i^{2}\,\frac{[2K\beta - 3 - e^{-2K\beta} + 4e^{-K\beta}]}{K^2\beta^2} \qquad \text{(V-28)}$$

V-4 Comparison Between Standard and Adaptive Integration

The RC input noise, n_i, is defined by its autocorrelation,

$R_i(\tau) = \sigma_i^2 \, e^{\frac{-\beta\tau}{T}}$, i. e. by its variance, σ_i^2, and β. The variance after standard adaptive integration at time t_1 are respectively σ_1^2 and σ_{1A}^2 which are given by formulas V-18 and V-28.

When a rectangular pulse of amplitude V is received, the amplitude at the input of the threshold detector at time t_1 is V, whether an integrator is used or not. The signal-to-noise ratio at the input of the threshold detector at time t_1 increases when an integrator is used because the power of the signal remains constant while the power of the noise decreases. The increase in the signal to noise ratio due to the integrator is best expressed as a gain in decibels.

Let (S_i/N_i), (S/N), $(S/N)_A$ be the initial signal to noise ratio, the signal to noise ratio after a standard integrator and the signal to noise ratio after an adaptive integrator, respectively. Then

$$G = +20 \log_{10}\left[\frac{(S/N)}{(S_i N_i)}\right] = +20 \log_{10}\left(\sigma_i^2/\sigma_1^2\right) \qquad \text{(V-29)}$$

$$G_A = +20 \log_{10}\left[\frac{(S/N)_A}{(S_i/N_i)}\right] = +20 \log_{10}\left(\sigma_i^2/\sigma_{1A}^2\right) \qquad \text{(V-30)}$$

where G and G_A are the gain in signal to noise ratio in decibels, for standard and adaptive integrator, respectively. G and G_A do not depend on the input signal to noise ratio and they decrease when $K\beta$ increases. The variation of G and G_A versus $K\beta$ is plotted in Fig. V-2 showing that G_A is considerably larger than G.

Since integration is a linear process, the noise remains normal; therefore, the average probability of error in threshold detection with a constant level, $V/2$, of random pulses of amplitude V mixed with noise of variance, σ, is $I(V/2\sigma)$ where

$$I(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \, dt.$$

The average probability of error without integrator, with standard integrator and with adaptive integrator are respectively

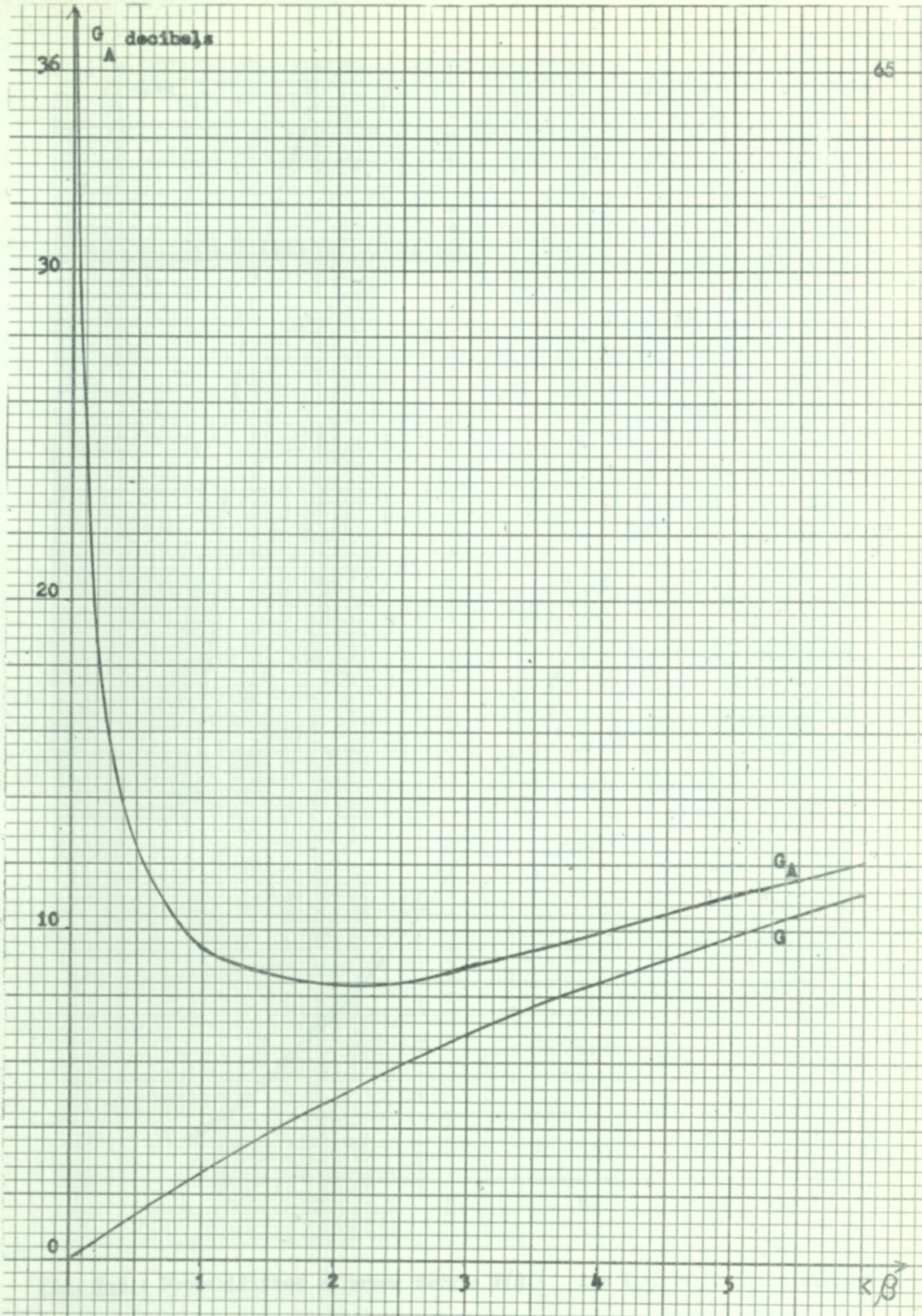
$$E_o = I(V/2\sigma_i) \qquad (V-31)$$

$$E_I = I(V/2\sigma_I) \qquad (V-32)$$

$$E_{IA} = I(V/2\sigma_{IA}) \qquad (V-33)$$

Since the average probabilities of error, E_o, E_I, and E_{IA}, are very small numbers, it is convenient to plot instead $-\log_{10} E_o$, $-\log_{10} E_I$ and $-\log_{10} E_{IA}$. Fig. V-3 and Fig. V-4 show the logarithms of the probabilities of error versus $V/2\sigma_i$ for given $K\beta$. Fig. V-3 is for $K\beta = .2$ and Fig. V-4 is for $K\beta = 2$.

The average probability of error is a very non linear function of the signal-to-noise ratio; therefore, the advantage of an adaptive integrator before threshold detector is even more apparent in terms of reduction of the average probability of error than in terms of increase of the signal to noise ratio.



<u>Fig. V-2</u> : Increase of the Signal-to-Noise Ratio by Standard Integration (G) and by Adaptive Integration (G_A).

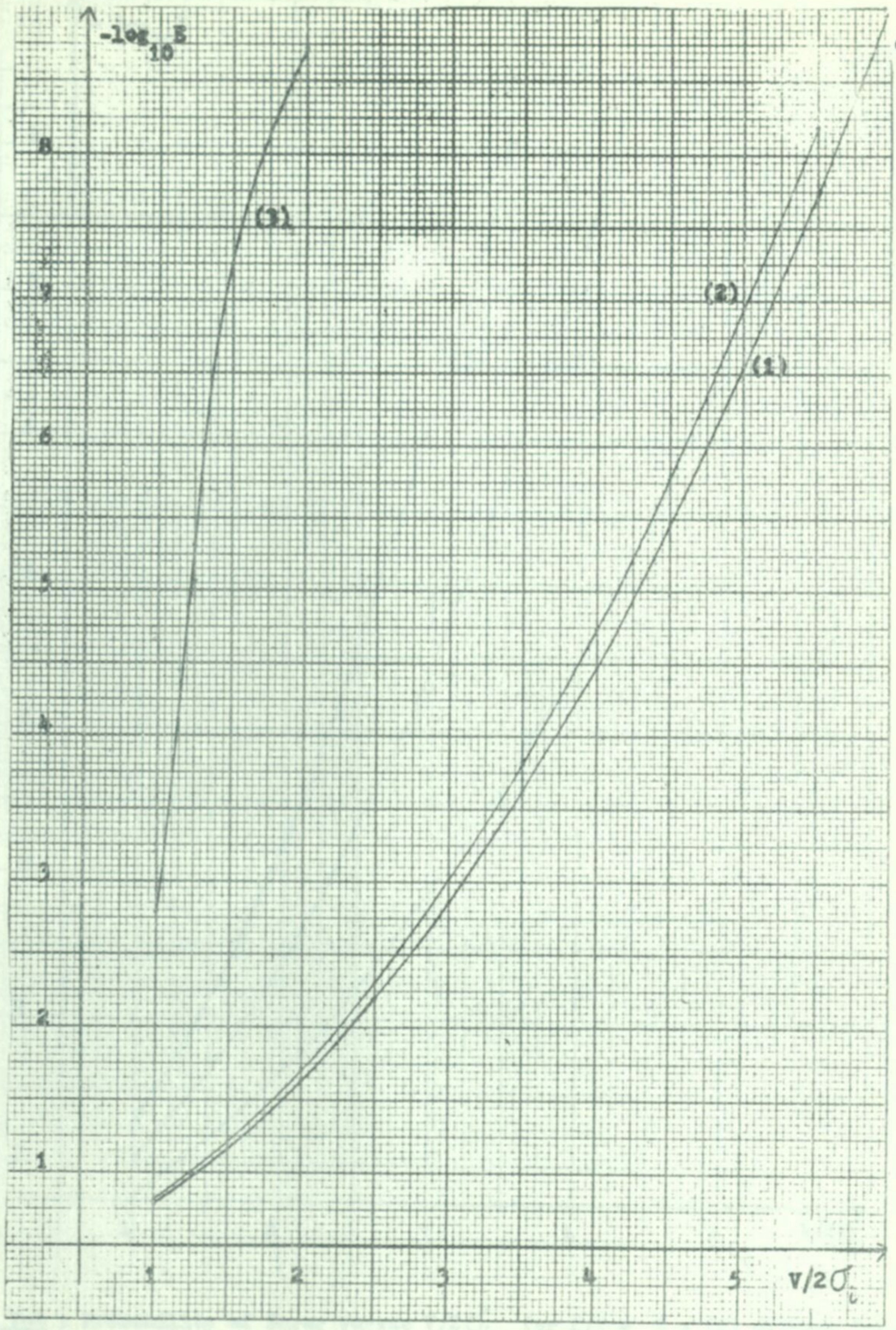


Fig. V-3 : Probabilities of error versus $V/2\sigma_i$ for $K\beta = .2$: (1) without Integrator ($-\log_{10}E_o$), (2) with Standard Integrator ($-\log_{10}E_I$), (3) with Adaptive Integrator ($-\log_{10}E_{IA}$)

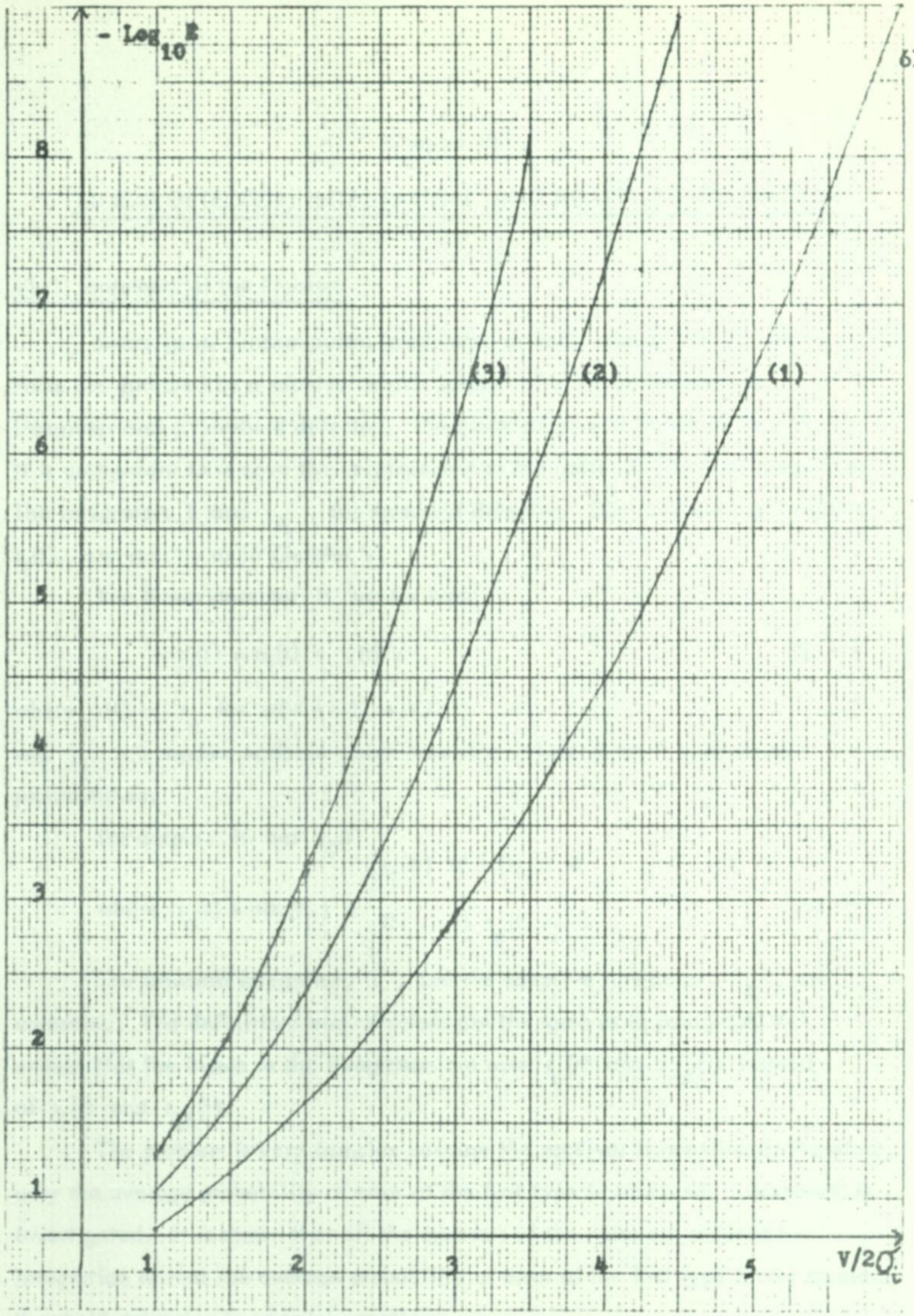


Fig. V-4 : Probability of Error versus $V/2\sigma_i$ for $K\beta = 2$: (1) without Integrator ($-\log_{10}E_0$), (2) with Standard Integrator ($-\log_{10}E_I$), (3) with Adaptive Integrator ($-\log_{10}E_{IA}$)

Chapter VI

LINEAR DETECTOR, INTEGRATION, AND ADAPTIVE INTEGRATION

VI-1 Statement of the Problem

Rectangular random pulses (modulated or not), mixed with RC normal noise are detected by a receiver which consists of a linear detector, an integrator and a threshold detector. The block diagram is shown in Fig. VI-1. It is convenient to remove the dc component of the noise $(\overline{n_o})$ before integration. The integrators I and I_A, the threshold detector, TD, and the input signal, $s_i(t)$, are exactly as in Chapter V.

The linear detector H has for input

$$s_i(t) = n_i(t) + \gamma(t)\,V \tag{VI-1}$$

where $\gamma(t) = \gamma_n$ for $nT < t < nT + KT$

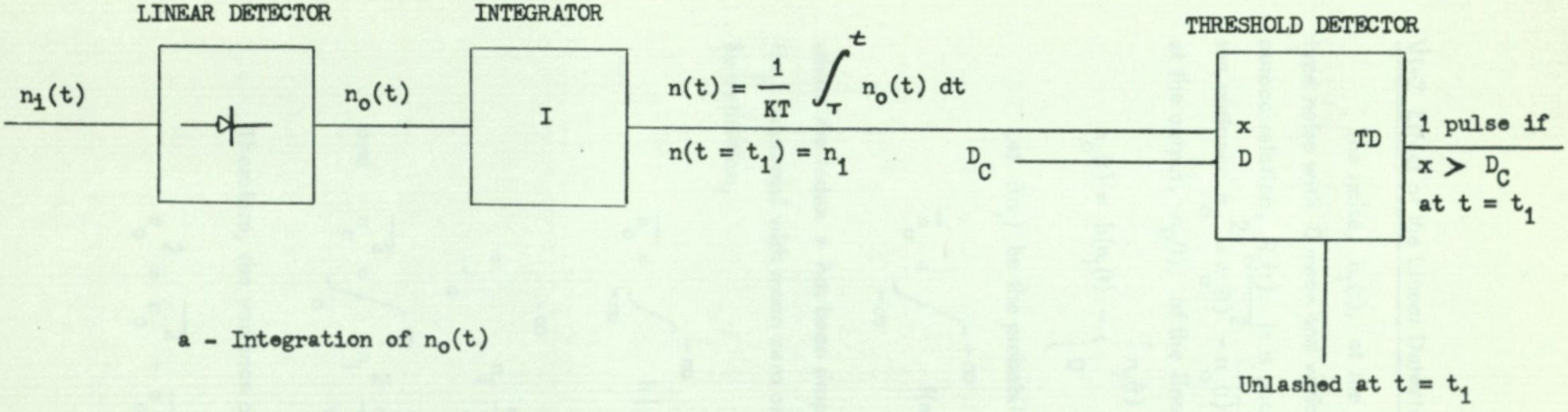
and γ_n is a random variable which takes the values 0 or 1 with equal probabilities.

For output, H has $s_0(t)$

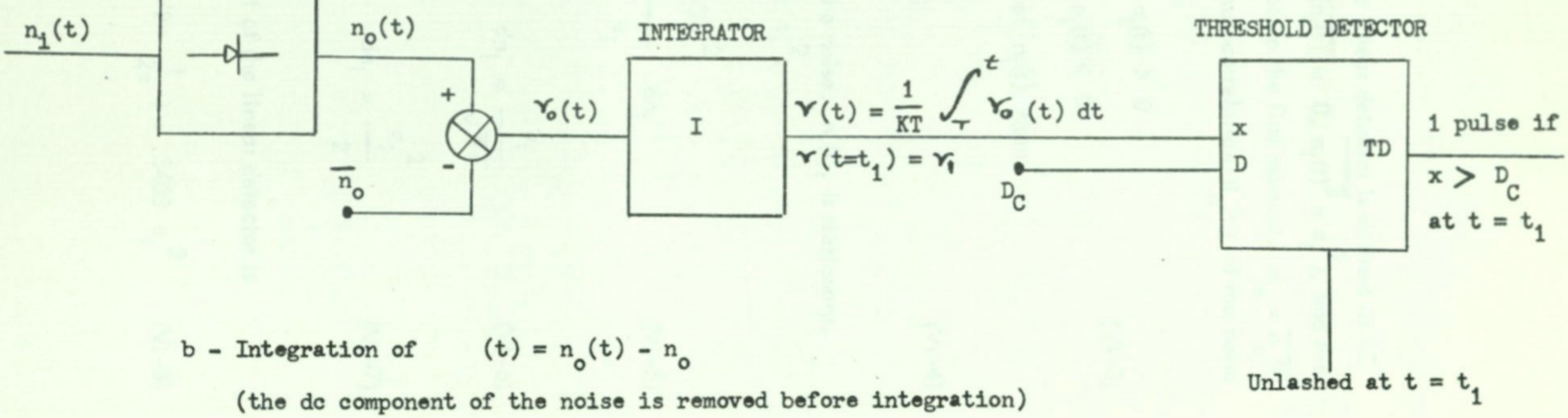
$$\text{where}\quad s_o(t) = h(s_i(t)) = \begin{cases} s_i(t) & \text{if } s_i(t) > 0 \\ 0 & \text{if } s_o(t) \leq 0 \end{cases} \tag{VI-2}$$

The standard integrator, I, and the adaptive integrator, I_A, are compared. The definitions and notations are the same as in paragraph V-1 except that the inputs to the integrators are now $s_o(t)$ and $s_{oA}(t)$ instead of $s_i(t)$ and $s_{iA}(t)$.

The problem is very complex because the receiver is non-linear; therefore, only the average probability of error of the first type (0 received, 1 detected) is investigated. It is shown that all the moments of the noise are reduced by integration so that the average probability of error of the first type in the constant threshold detection is certainly smaller.



a - Integration of $n_o(t)$



b - Integration of $(t) = n_o(t) - \overline{n_o}$

(the dc component of the noise is removed before integration)

FIG. VI-1 : Block Diagram of Receiver with Linear Detector, Standard Integrator, and Threshold Detector.

VI-2 Effect of the Linear Detector

The noise, $n_i(t)$, at the input of the linear detector is defined as RC type noise with 0 mean and variance, σ_i^2 $(\overline{n_i(t)} = 0,\ \overline{n_i(t)^2} = \sigma_i^2)$, and the autocorrelation, $R_i(\tau)$. It is necessary to obtain the first moment, $m_o = \overline{n_o(t)}$, the variance $\sigma_o^2 = \overline{n_o(t)^2} - \overline{n_o(t)}^2$ and the autocorrelation $R_o(\tau)$ of the noise at the output, $n_0(t)$, of the linear detector.

$$n_o(t) = h(n_i(t)) = \begin{cases} n_i(t) & \text{for } n_i(t) > 0 \\ 0 & \text{for } n_i(t) < 0 \end{cases} \qquad \text{(VI-3)}$$

Let $f(n_i)$ be the probability density of $n_i(t)$, then

$$\overline{n}_o = \int_{-\infty}^{+\infty} f(n_i)\, h(n_i)\, dn_i \qquad \text{(VI-4)}$$

where the index t has been dropped since the noise, $n_i(t)$, is stationary. $f(n_i)$ is normal with mean zero and variance σ_i^2.
By definition,

$$\overline{n}_o = \int_{-\infty}^{+\infty} h[n_i]\, \frac{e^{-n_i^2/2\sigma_i^2}}{\sqrt{2\pi}\ \sigma_i}\, dn_i \qquad \text{(VI-5)}$$

$$= \int_{0}^{\infty} n_i\, \frac{e^{-n_i^2/2\sigma_i^2}}{\sqrt{2\pi}\ \sigma_i}\, dn_i = \frac{\sigma_i}{\sqrt{2\pi}} \qquad \text{(VI-6)}$$

and $$\overline{n_o^2} = \int_{0}^{\infty} n_i^2\, \frac{e^{-n_i^2/2\sigma_i^2}}{\sqrt{2\pi}\ \sigma_i}\, dn_i = \frac{\sigma_i^2}{2} \qquad \text{(VI-7)}$$

Therefore, the variance at the output of the linear detector is

$$\sigma_o^2 = \overline{n_o^2} - \overline{n}_o^2 = \sigma_i^2(1/2 - \tfrac{1}{2\pi}) = .3408\ \sigma_i^2 \qquad \text{(VI-8)}$$

The input noise is of type RC; therefore, it is defined by an exponential autocorrelation function:

$$R_i(\tau) = \sigma_i^2 \, e^{\frac{-\beta\tau}{T}} = \sigma_i^2 \, \rho_i(\tau) \qquad \text{(VI-9)}$$

where $\rho_i(\tau)$ is the normalized autocorrelation function.

The autocorrelation at the output of the linear detector, $R_o(\tau)$, is by definition:

$$R_o(\tau = t_1 - t_2) = \overline{n_o(t_1)\, n_o(t_2)} \qquad \text{(VI-10)}$$

For a more convenient notation, let $x_1 = n_i(t_1)$ and $x_2 = n_i(t_2)$. Then $f(x_1, x_2)$ is the joint probability density of $n_i(t_1)$ and $n_i(t_2)$ and

$$R_o(\tau = t_1 - t_2) = \iint_{-\infty}^{+\infty} f(x_1, x_2)\, h(x_1)\, h(x_2)\, dx_1\, dx_2 \qquad \text{(VI-11)}$$

$f(x_1, x_2)$ is normal and can be expressed as a function of the autocorrelation coefficient, ρ_i, of x_1 and x_2:

$$f(x_1, x_2) = \frac{1}{2\pi} \, e^{-\frac{1}{2(1-\rho_i^2)\,\sigma_i^2}\left(x_1^2 + x_2^2 - 2\rho_i x_1 x_2\right)} \qquad \text{(VI-12)}$$

$$\text{where } \rho_i = \frac{\overline{(x_1 - \overline{x_1})\,(x_2 - \overline{x_2})}}{\sqrt{\overline{(x_1 - \overline{x_1})^2}\;\overline{(x_2 - \overline{x_2})^2}}} = e^{-\frac{\beta}{T}|\tau|} \qquad \text{(VI-13)}$$

Since the joint probability density is normal, Price method [16] can be used. The function $h(x)$ is differentiated until δ -functions are obtained; in this case, one must obtain the second derivative of $h(x)$

$$\frac{d^2h}{dx_i} = h^{(2)}(x_i) = \delta\,(x_i) \qquad \text{(VI-14)}$$

where δ is the well known δ-function and x_i is x_1 or x_2,

Using Price theorem,

$$\frac{\partial^2 R_o(\tau)}{\partial \rho_i^2} = \iint_{-\infty}^{+\infty} h^{(2)}(x)\, h^{(2)}(x)\, e^{-\frac{1}{2(1-\rho_i^2)\sigma_i^2}(x_1^2 + x_2^2 - 2\rho_i x_1 x_2)} dx_1\, dx_2 \qquad \text{(VI-15)}$$

which is immediately integrable since $h^{(2)}(x_1)$ and $h^{(2)}(x_2)$ are δ-functions

Price [16] shows that

$$R_o(\tau) = \frac{\sigma_i^2}{2\pi}\left[\sqrt{1-\rho_i^2} + \rho_i \cos^{-1}(-\rho_i)\right] \qquad \text{(VI-16)}$$

$$R_o(\tau) = \frac{\sigma_i^2}{2\pi}\left[\sqrt{1-e^{\frac{-2\beta|\tau|}{T}}} + e^{\frac{-\beta|\tau|}{T}} \cos^{-1}\left(-e^{\frac{-\beta|\tau|}{T}}\right)\right] \qquad \text{(VI-17)}$$

where $\rho_i = e^{\frac{-\beta|\tau|}{T}}$ and $\tau = t_1 - t_2$, (VI-18)

$R_o(0)$ is obtained by substituting $\tau = 0$ in formula (VI-17)

$$R_o(0) = \overline{n_o^2} = \frac{\sigma_i^2}{2} \qquad \text{(VI-19)}$$

It is convenient to express the autocorrelation at the output of the half wave linear device in series form, using the result of Davenport [3].

$$R_o(\tau) = \frac{\sigma_i^2}{2\pi}\left(1 + \frac{\pi}{2}\rho_i + \frac{1}{2}\rho_i^2 + \frac{1}{24}\rho_i^4 + \frac{1}{80}\rho_i^6 + \ldots\right) \qquad \text{(VI-20)}$$

and approximately $R_o(\tau) \approx \sigma_i^2\left(\frac{1}{2\pi} + .25\rho_i + .07958\rho_i^2\right)$ (VI-21) where $\rho_i = e^{\frac{-\beta|\tau|}{T}}$ and $\tau = t_1 - t_2$

The approximate value of $R_o(0)$ is obtained by substituting $\tau = 0$, i. e. $\rho_i = 1$, in formula (VI-21);

$$R_o(0) \approx \sigma_i^2 \,(1/2\pi \;+\; .25 \;+\; .07958)$$
$$R_o(0) \approx \sigma_i^2 \,(.48873) \qquad \text{(VI-22)}$$

The approximate value of the variance is

$$\sigma_o^2 = \overline{n_o^2} - \overline{n_o}^2 = R(0) - \frac{\sigma_i^2}{2\pi}$$

$$\sigma_o^2 \approx \sigma_i^2 \,(.48873 - 1/2\pi) = \sigma_i^2 (.32958) \approx \sigma_i^2 (.33) \qquad \text{(VI-23)}$$

The approximation of formula (VI-17) by formula (VI-21) is best for small values of ρ_i, i. e. large value of τ, but it is still very good at the limit when $\rho_i = 1$, ($\tau = 0$) as shown by comparing the exact variance (formula VI-8) and the approximate variance (formula VI-23).

VI-3 Effect of a Standard Integrator on the Variance

The noise, $n_o(t)$, at the output of the linear detector is integrated during the time from $t = mT$ to $t = KT$ where m is an integer. The output of the integrator is defined by

$$n(t) = \frac{1}{KT}\int_{mT}^{t} n_o(t)\ dt \qquad \text{for } mT < t < mT + KT$$

Consider the detection of the first random pulse; the threshold detector is unlashed only at time $t = t_1 = T + KT$; therefore, the only important value of the output of the integrator is for $t = t_1$ and

$$n(t_1) = n_1 = \frac{1}{KT} \int_T^{T+KT} n_o(t)\, dt \qquad \text{(VI-24)}$$

The first moment of n_1 is easily obtained

$$\overline{n_1} = \frac{1}{KT} \int_T^{T+KT} \overline{n_o}(t)\, dt = \overline{n_o} \qquad \text{(VI-25)}$$

The second moment $\overline{n_1^{\,2}}$ is

$$\overline{n_1^{\,2}} = \frac{1}{K^2 T^2} \int_T^{T+KT} \int_T^{T+KT} \overline{n_o(t)\, n_o(t')}\, dt\, dt' \qquad \text{(VI-26)}$$

The variance of n_1 (output of the integrator at time t_1) is

$$\sigma_1^{\,2} = \overline{n_1^{\,2}} - \overline{n_1}^{\,2} = \frac{1}{K^2 T^2} \int_T^{T+KT} \int_T^{T+KT} \overline{n_o(t)\, n_o(t')}\, dt\, dt' - \overline{n_o}^{\,2} \qquad \text{(VI-27)}$$

Since $\overline{n_o}^{\,2} = \frac{1}{K^2 T^2} \int_T^{T+KT} \int_T^{T+KT} \overline{n_o}^{\,2}\, dt\, dt'$, the two terms can be condensed into one double integral resulting in

$$\sigma_1^{\,2} = \frac{1}{K^2 T^2} \int_T^{T+KT} \int_T^{T+KT} [\overline{n_o(t)\, n_o(t')} - \overline{n_o}^{\,2}]\, dt\, dt' \qquad \text{(VI-28)}$$

The integral can be expressed in terms of

$\nu_o(t) = n_o(t) - \overline{n_o}$, where $\nu_o(t)$ is the noise minus its dc component and is called dc filtered noise. The mean of $\nu_o(t)$ is zero by definition

$$\overline{n_o(t)\, n_o(t')} - \overline{n_o}^{\,2} = \overline{\nu_o(t)\, \nu_o(t')} \qquad \text{(VI-29)}$$

and $$\sigma_1^2 = \frac{1}{K^2T^2} \int_T^{T+KT} \int_T^{T+KT} \overline{\nu_o(t)\ \nu_o(t')}\ dt\ dt' \tag{VI-30}$$

Physically, this means that the variance at the output of the integrator at time t_1 is the same whether $\nu_o(t)$ or $n_o(t)$ is integrated; that is, whether the dc component of the input to the integrator is removed or not. The proof is quite simple. Let

$$\nu_1 = \frac{1}{KT} \int_T^{T+KT} \nu_o(t)\ dt \tag{VI-31}$$

The variance of $\nu_1 (\sigma_{\nu_1}^2)$ is obtained immediately

$$\overline{\nu_1} = \frac{1}{KT} \int_T^{T+KT} \overline{\nu_o}(t)\ dt = 0 \tag{VI-32}$$

$$\sigma_{\nu_1}^2 = \overline{\nu_1^2} = \frac{1}{K^2T^2} \int_T^{T+KT} \int_T^{T+KT} \overline{\nu_o(t)\ \nu_o(t')}\ dt\ dt' \tag{VI-33}$$

Hence, $\sigma_{\nu_1}^2 = \sigma_1^2$, which was to be shown.

Since integrating $n_0(t)$ or $\nu_o(t)$ is equivalent, $\nu_o(t)$ is chosen because the mathematics is simplified. In other words, the block diagram VI-1 b is preferred to the block diagram VI-1 a.

The autocorrelation, $\Psi_o(\tau)$, of $\nu_o(t)$ is simply expressed in terms of the autocorrelation, $R_o(\tau)$, of $n_o(\tau)$:

$$\Psi_o(\tau) = \overline{\nu_o(t)\ \nu_o(t+\tau)} = \overline{(n_o(t) - \overline{n_o})\ (n_o(t+\tau) - \overline{n_o})} \tag{VI-34}$$

$$\Psi_o(\tau) = R_o(\tau) - \overline{n_o}^2 \tag{VI-35}$$

The approximate value of $\Psi_o(\tau)$ is obtained by substituting formulas VI-6 and VI-21 in formula VI-35, resulting in

$$\Psi_o(\tau) = \sigma_i^2 \left(.25 \; e^{\frac{-\beta|\tau|}{T}} + .07958 \; e^{\frac{-2\beta|\tau|}{T}}\right) \qquad \text{(VI-36)}$$

σ_i^2 is given in terms of σ_o^2 by formula VI-23; therefore,

$$\Psi_o(\tau) = \sigma_o^2 \left[.7587 \; e^{\frac{-\beta|\tau|}{T}} + .2415 \; e^{\frac{-2\beta|\tau|}{T}} \right] \qquad \text{(VI-37)}$$

The normalized autocorrelation of $\nu_o(t)$, $\phi_o(\tau)$, is defined as usual:

$$\phi_o(\tau) = \frac{\Psi_o(\tau)}{\Psi_o(o)} = \frac{\Psi_o(\tau)}{\sigma_o^2} \qquad \text{(VI-38)}$$

$$\phi_o(\tau) = \left[.7587 \; e^{\frac{-\beta|\tau|}{T}} + .2415 \; e^{\frac{-2\beta|\tau|}{T}} \right] \qquad \text{(VI-39)}$$

The variance, σ_1^2 (formula VI-30), is now expressed in terms of $\phi_o(\tau)$.

$$\sigma_1^2 = \sigma_o^2 \frac{1}{K^2T^2} \int_T^{T+KT} \int_T^{T+KT} \phi_o(t - t') \, dt \, dt' \qquad \text{(VI-40)}$$

which reduces to a simple integral as shown in paragraph V-3.

$$\sigma_1^2 = \sigma_o^2 \frac{2}{K^2T^2} \int_T^{KT} (KT - \tau) \, \phi_o(t) \, d\tau \qquad \text{(VI-41)}$$

and after (1) replacing $\phi_o(\tau)$ by its value of formula (VI-28) and

(2) using the dimensionless variable $u = \tau/T$,

$$\sigma_1^2 = \sigma_o^2 \frac{2}{K^2} \int_o^K (K - u) \left(.75873 \, e^{-\beta u} + .24152 \, e^{-2\beta u}\right) du \qquad \text{(VI-42)}$$

It was shown in paragraph V-2 that

$$\frac{2}{K^2}\int_o^K (K-u)\, e^{-\beta u}\, du = \frac{2}{K^2\beta^2}\,(K\beta + e^{-K\beta} - 1) \qquad \text{(VI-43)}$$

Changing β into 2β,

$$\frac{2}{K^2}\int_o^K (K-u)\, e^{-2\beta u}\, du = \frac{1}{2K^2\beta^2}\,(2K\beta + e^{-2K\beta} - 1) \qquad \text{(VI-44)}$$

$\sigma_1^{\,2}$ is obtained by substituting formulas (VI-37) and (VI-38) in (VI-36)

$$\sigma_1^{\,2} = \sigma_o^{\,2}\left[.75873\,\frac{2}{K^2\beta^2}\,(K\beta + e^{-\beta K} - 1) + .24152\,\frac{1}{2K^2\beta^2}\,(2K\beta + e^{-2K\beta} - 1)\right] \qquad \text{(VI-45)}$$

After grouping the terms,

$$\sigma_1^{\,2} = \sigma_o^{\,2}\left[\frac{1.75897}{K\beta} + 1.51746\,\frac{e^{-\beta K}}{K^2\beta^2} + .12076\,\frac{e^{-2\beta K}}{K^2\beta^2} - \frac{1.63822}{K^2\beta^2}\right] \qquad \text{(VI-46)}$$

$$\text{or}\quad \sigma_1^{\,2} = \sigma_i^{\,2}\left[\frac{.57958}{K\beta} + \frac{.5\,e^{-\beta K}}{K^2\beta^2} + .03979\,\frac{e^{-2K\beta}}{K^2\beta^2} - \frac{.53979}{K^2\beta^2}\right] \qquad \text{(VI-47)}$$

VI-4 Effect of an Adaptive Integrator on the Variance

The input and output of the linear detector are $n_i(t)$ and $n_o(t)$, respectively. The dc component of $n_o(t)$ is removed and the dc filtered noise is called $\nu_o(t)$, exactly as in paragraph VI-3.

Consider again the detection of the first random pulse, i. e. the pulse which may be present in the interval $T < t < T + KT$. The dc filtered noise does not vary instantaneously; consequently, the dc filtered noise, $\nu_o(t)$, during the time interval of integration $(T < t < T + KT)$ is related to the dc filtered noise just before integration, $\nu_o(t = T) = \nu_o{}^*$. If the dc filtered noise is sampled at time $t^* = T$,

ν_o^* is known and the expected value of the dc filtered noise is $\nu_o^* \phi_o(t - T)$ where $\phi_o(\tau)$ is the normalized autocorrelation function of $\nu_o(t)$ and given by formula VI-39.

The noise, $\nu_o(t)$, at time $t(T < t < T + KT)$ can be partially eliminated by subtracting the expected value of the noise. The corrected noise is called $\nu_{oA}(t)$:

$$\nu_{oA}(t) = \nu_o(t) - \phi_o(t - T)\,\nu_o^* \qquad \text{(VI-48)}$$

The input of the integrator, $\nu_{oA}(t)$, and the output, $\nu_A(t)$, are shown in the block diagram VI-2.

$$\nu_A(t) = \frac{1}{KT}\int_T^t \nu_{oA}(t)\,dt \qquad \text{(VI-49)}$$

Again, since the threshold detector is unlashed only at time $t = t_1 = T + KT$, the only important value of $\nu_A(t)$ is

$$\nu_A(t_1) = \nu_{1A} = \frac{1}{KT}\int_T^{T+KT} \nu_{oA}(t)\,dt \qquad \text{(VI-50)}$$

The first moment of ν_{1A} is desired

$$\overline{\nu_{1A}} = \frac{1}{KT}\int_T^{T+KT} \overline{\nu_{oA}}(t)\,dt \qquad \text{(VI-51)}$$

$\overline{\nu_{oA}}(t)$ is obtained using formula (VI-48):

$$\overline{\nu_{oA}}(t) = \overline{\nu_o(t)} - \phi_o(t - T)\,\overline{\nu_o^*} \qquad \text{(VI-52)}$$

(for a given value of t, $\phi_o(t - T)$ is just a coefficient)

Since $\overline{\nu_o(t)} = \overline{\nu_o^*} = 0$, $\overline{\nu_{oA}(t)} = 0$ it follows that $\overline{\nu_{1A}} = 0$ (VI-53)

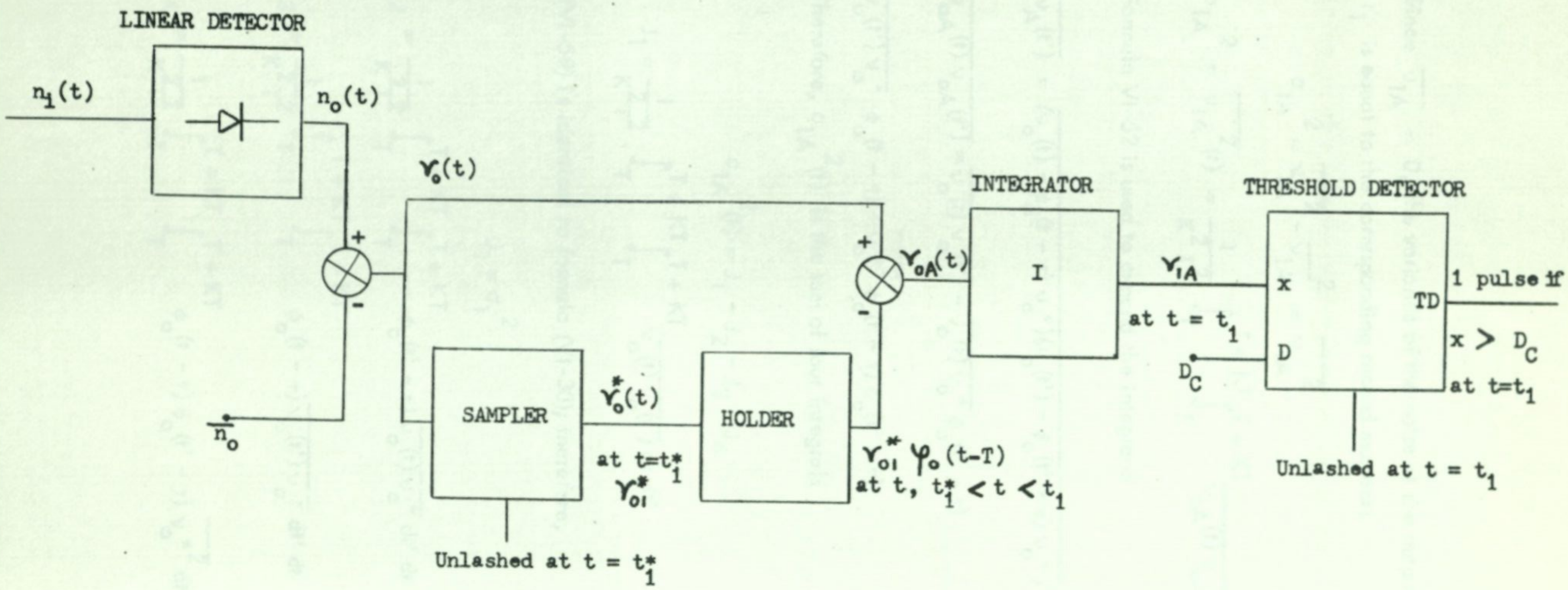


Fig. VI-2 : Block Diagram of a Receiver with Linear Detector, Adaptive Integrator, and Threshold Detector.

Since $\overline{\nu_{1A}} = 0$, the variance of the noise at the output of the integrator at time t_1 is equal to the corresponding second moment:

$$\sigma_{1A}^2 = \overline{\nu_{1A}^2} - \overline{\nu_{1A}}^2 = \overline{\nu_{1A}^2} \qquad \text{(VI-54)}$$

$$\sigma_{1A}^2 = \overline{\nu_{1A}^2}(t) = \frac{1}{K^2 T^2} \int_T^{T+KT} \int_T^{T+KT} \overline{\nu_{oA}(t)\, \nu_{oA}(t')}\, dt\, dt' \qquad \text{(VI-55)}$$

Formula VI-52 is used to expand the integrand

$$\overline{\nu_{oA}(t)\;\nu_A(t')} = \overline{(\nu_o(t) - \phi_o(t-\tau)\,\nu_o^*)(\nu_o(t') - \phi_o(t'-\tau)\,\nu_o^*)} \qquad \text{(VI-56)}$$

$$\overline{\nu_{oA}(t)\,\nu_{oA}(t')} = \overline{\nu_o(t)\,\nu_o(t')} - \overline{\nu_o(t)\,\nu_o^*}\,\phi_o(t'-\tau) - \overline{\nu_o(t')\,\nu_o^*}\,\phi_o(t-\tau) + \overline{\nu_o^{*2}}\,\phi_o(t-\tau)\,\phi_o(t'-\tau) \qquad \text{(VI-57)}$$

Therefore, $\sigma_{1A}^2(t)$ is the sum of four integrals

$$\sigma_{1A}^2(t) = J_1 - J_2 - J_3 + J_4 \qquad \text{(VI-58)}$$

where $$J_1 = \frac{1}{K^2 T^2} \int_T^{T+KT} \int_T^{T+KT} \overline{\nu_o(t)\,\nu_o(t')}\, dt\, dt' \qquad \text{(VI-59)}$$

Formula (VI-59) is identical to formula (VI-30); therefore,

$$J_1 = \sigma_1^2 \qquad \text{(VI-60)}$$

$$J_2 = \frac{1}{K^2 T^2} \int_T^{T+KT} \int_T^{T+KT} \phi_o(t'-\tau)\,\overline{\nu_o(t)\,\nu_o^*}\, dt'\, dt \qquad \text{(VI-61)}$$

$$J_3 = \frac{1}{K^2 T^2} \int_T^{T+KT} \int_T^{T+KT} \phi_o(t-\tau)\,\overline{\nu_o(t')\,\nu_o^*}\, dt'\, dt \qquad \text{(VI-62)}$$

$$J_4 = \frac{1}{K^2 T^2} \int_T^{T+KT} \int_T^{T+KT} \phi_o(t-\tau)\,\phi_o(t'-\tau)\,\overline{\nu_o^{*2}}\, dt'\, dt \qquad \text{(VI-63)}$$

From the definition of autocorrelation,

$$\overline{\nu_o(t)\,\nu_o^*} = \sigma_o^{\,2}\,\phi_o(t - T) \qquad \text{(VI-64)}$$

$$\overline{\nu_o(t')\,\nu_o^*} = \sigma_o^{\,2}\,\phi_o(t' - T) \qquad \text{(VI-65)}$$

$$\overline{\nu_o^{*2}} = \sigma_o^{\,2} \qquad \text{(VI-66)}$$

After substituting formulas (VI-64) and (VI-65) and (VI-66) in J_2, J_3, and J_4 respectively,

$$J_2 = J_3 = J_4 = \frac{1}{K^2T^2}\int_T^{T+KT}\int_T^{T+KT} \sigma_o^{\,2}\,\phi_o(t' - \tau)\,\phi_o(t - \tau)\,dt'\,dt \qquad \text{(VI-67)}$$

Let $J = \frac{1}{KT}\,\sigma_o \int_T^{T+KT} \phi_o(t - \tau)\,dt$. (VI-68)

Then $J_1 = J_2 = J_3 = (J)^2$ (VI-69)

After substituting (VI-60) and (VI-69) into (VI-58),

$$\overline{\sigma_{1A}^{\,2}}(t) = \sigma_1^{\,2} - (J)^2 \qquad \text{(VI-70)}$$

$\phi_o(t - T)$ is replaced by formula (VI-67) and after the change of variable, $u = \frac{t - T}{T}$,

$$J = \frac{\sigma_o}{K}\int_o^K \left[.7587\ e^{-\beta u} + .2415\ e^{-2\beta u}\right] du \qquad \text{(VI-71)}$$

After integrating and grouping the terms,

$$J = \frac{\sigma_o}{K\beta}\left[.8795 - .7587\ e^{-K\beta} - .1208\ e^{-2K\beta}\right] \qquad \text{(VI-72)}$$

$\sigma_{1A}^{2}(t)$, given by formula (VI-70), can be expressed in terms of the initial variance before detection, σ_i^2 , by using formulas (VI-47), (VI-23) and (VI-72).

$$J = \frac{\sigma_i}{K\beta}\left(.5048 - .4355 \ e^{-K\beta} - .0693 \ e^{-2K\beta}\right)$$

$$\sigma_{1A}^{2}(t) = \sigma_i^{2}\left\{\frac{.5796}{K\beta} + \frac{.5\ e^{-K\beta}}{K^2\beta^2} + \frac{.0398\ e^{-2K\beta}}{K^2\beta^2} - \frac{.5398}{K^2\beta^2} - \left[\frac{1}{K\beta}(.5048 - .4355\ e^{-K\beta} - .0693\ e^{-2K\beta})\right]^2\right\} \qquad \text{(VI-73)}$$

VI-5 Higher Moments are also Reduced by Integration

Before integration, the noise is stationary, after integration, the noise is non stationary. Only the moments at the time of detection, $t_1 = T$, are important. The nth moments at $t = t_1$ for standard and adaptive integrator are respectively,

$$\overline{\nu_1^{\,n}} = \frac{1}{(KT)^n}\int_T^{T+KT}\int_T^{T+KT}\cdots\int_T^{T+KT}\overline{\nu_o(t_1)\,\nu_o(t_2)\,\cdots\,\nu_o(t_n)}\,dt_1\,dt_2\cdots dt_n \qquad \text{(VI-74)}$$

and

$$\overline{\nu_{1A}^{\,n}} = \frac{1}{(KT)^n}\int_T^{T+KT}\int_T^{T+KT}\cdots\int_T^{T+KT}\overline{\nu_{oA}(t_1)\,\nu_{oA}(t_2)\,\cdots\,\nu_{oA}(t_n)}\,dt_1\,dt_2\cdots dt_n \qquad \text{(VI-75)}$$

It was shown in paragraph VI-3 and VI-4 that the variance of the noise at time t_1 decreases when an integrator is used. The decrease is greater for an adaptive than for a standard integrator; this result can be expressed by an inequality:

$$\overline{\nu_o^{\,2}(t_1)} > \overline{\nu_1^{\,2}} > \overline{\nu_{1A}^{\,n}} \qquad \text{(VI-76)}$$

The inequality (VI-76) can be extended for higher moments:

$$\overline{\nu_o^{\,n}(t_1)} > \overline{\nu_1^{\,n}} > \overline{\nu_{1A}^{\,n}} \qquad \text{(VI-77)}$$

using basic properties of autocorrelation functions and multiple integrals.

The inequality $\overline{\nu_o^n(t)} > \overline{\nu_1^n}$ is shown first. The generalized autocorrelation function

$$R_n(t_1, t_2, \ldots, t_n) = \overline{\nu_o(t_1)\,\nu_o(t_2)\ldots\nu_o(t_n)} \qquad \text{(VI-78)}$$

is maximum at $t_1 = t_2 = t_n$ (the origin of time is arbitrary); that is

$$\overline{\nu_o(t_1)\,\nu_o(t_2)\ldots\nu_o(t_n)} < \overline{\nu_o^n(t_1)} \qquad \text{(VI-79)}$$

Substituting the inequality (VI-79) in formula (VI-74) it results

$$\nu_1^n \leq \frac{1}{(KT)^n} \int_T^{T+KT} \int_T^{T+KT} \ldots \int_T^{T+KT} \overline{\nu_o^n(t_1)}\, dt_1 \ldots dt_n \qquad \text{(VI-80)}$$

and after integration,

$$\overline{\nu_1^n} \leq \overline{\nu_o^n(t_1)} \qquad \text{(VI-81)}$$

which was to be proved.

It remains to be shown that $\overline{\nu_{1A}^n}^n \leq \overline{\nu_{1A}^n}^n$. Let $n = 3$ to simplify the mathematics. By definition,

$$\nu_{oA}(t) = \nu_o(t) - \phi_o(t - T)\,\nu_o^* \qquad \text{for} \quad T < t < T + KT \qquad \text{(VI-82)}$$

Therefore,

$$\overline{\nu_{oA}(t_1)\,\nu_{oA}(t_2)\,\nu_{oA}(t_3)} = \overline{\nu_o(t_1)\,\nu_o(t_2)\,\nu_o(t_3)}$$

$$\overline{-\nu_o^*\,\nu_o(t_2)\,\nu_o(t_3)\phi_o(t_1 - T)} - \overline{\nu_o^*\,\nu_o(t_3)\,\nu_o(t_1)\,\phi_o(t_2 - T)}$$

$$\overline{-\nu_o^*\,\nu_o(t_1)\,\nu_o(t_2)\,\phi_o(t_3 - T)} + \overline{\nu_o^{*2}\,\nu_o(t_3)\,\phi_o(t_1 - T)\,\phi_o(t_2 - T)} \qquad \text{(VI-83)}$$

$$\overline{+\nu_o^{*2}\,\nu_o(t_1)\,\phi_o(t_2 - T)\,\phi_o(t_3 - T)} + \overline{\nu_o^{*2}\,\nu_o(t_2)\,\phi_o(t_3 - T)\,\phi_o(t_1 - T)}$$

$$\overline{- \nu_o{}^{*3} \phi_o(t_1 - T)\, \phi_o(t_2 - T)\, \phi_o(t_3 - T)}$$

After combining the integrals,

$$\overline{\nu_{1A}^3} = \frac{1}{K^3T^3} \int_T^{T+KT} \int_T^{T+KT} \int_T^{T+KT} \overline{\nu_{oA}(t_1)\, \nu_{oA}(t_2)\, \nu_{oA}(t_3)}\; dt_1\, dt_2\, dt_3 \qquad \text{(VI-84)}$$

$$= \overline{\nu_1^3} - 3 \frac{J}{\sigma_o} \frac{1}{K^2T^2} \int_T^{T+KT} \int_T^{T+KT} \overline{[\, \nu_o{}^* \nu_o(t_2)\,]\, [\nu_o(t_3) - \nu_o{}^* \phi_o(t_3 - T)\,]}\; dt_2\, dt_3 - (J)^3 \qquad \text{(VI-85)}$$

where J is an integral given by formula (VI-72) and

$$\overline{\nu_{1A}^3} < \overline{\nu_1^3} \qquad \text{(VI-86)}$$

which was to be shown. By extension,

$$\overline{\nu_{1A}^n} < \overline{\nu_1^n} \qquad \text{(VI-87)}$$

and finally

$$\overline{\nu_{1A}^n} < \overline{\nu_1^n} < \overline{\nu_o^n} \qquad \text{(VI-88)}$$

VI-6 Reduction in Average Probability of Error and Variance due to an Adaptive Integrator

The receiver consists of a linear detector and an integrator. When the input of the receiver consists only of normal type RC noise (0 received), three cases are compared: no integrator, standard integrator and adaptive integrator. The nth moments are denoted respectively by $\overline{\nu_o^n}(t)$, $\overline{\nu_1^n}$ and $\overline{\nu_{1A}^n}$. Let $f(\nu_o)$, $f(\nu_1)$, $f(\nu_{1A})$ be the probability density at the input of the threshold detector for the three cases. The corresponding average probabilities of error of the first type are

$$E_o = \int_\Delta^\infty f(\nu_o)\, d\nu_o \;, \quad E_1 = \int_\Delta^\infty f(\nu_1)\, d\nu_1 \;,$$

$E_{1A} = \int_{\infty}^{\Delta} f(\nu_{1A})\, d\nu_{1A}$, respectively where D is the threshold level.

Since the linear rectifier is a non linear network, the probability densities $f(\nu_o)$, $f(\nu_1)$, $f(\nu_{1A})$ are not normal and quite difficult to obtain. However, the inequality $E_o > E_1 > E_{1A}$ follows directly from the inequality $\overline{\nu_o^{\,n}(t)} > \overline{\nu_1^{\,n}} > \overline{\nu_{1A}^{\,n}}$ for any n; in other words, integration, and especially adaptive integration, reduces the average probability of error.

Although the average probability of error of the first type is not defined completely by the variance and the mean, the reduction of the variance gives an idea of the reduction in average probability of error. The input variance of the noise, σ_i^2 , serves as a reference; the variance after detection is $\sigma_o^2 = .33\ \sigma_i^2$; the variance after standard integration is σ_1^2 given by formula VI-47 and the variance after adaptive integration is σ_{1A}^2 given by formula VI-73. σ_1^2 and σ_{1A}^2 are function of the product $K\beta$, that is, the width of the pulse and the autocorrelation functions of the RC type input noise. Fig. VI-3 shows the normalized variances, $\frac{\sigma_o^2}{\sigma_i^2}$, $\frac{\sigma_1^2}{\sigma_i^2}$, and $\frac{\sigma_{1A}^2}{\sigma_i^2}$, versus $K\beta$. The variance of the noise is considerably reduced by the use of adaptive integration especially for small values of $K\beta$.

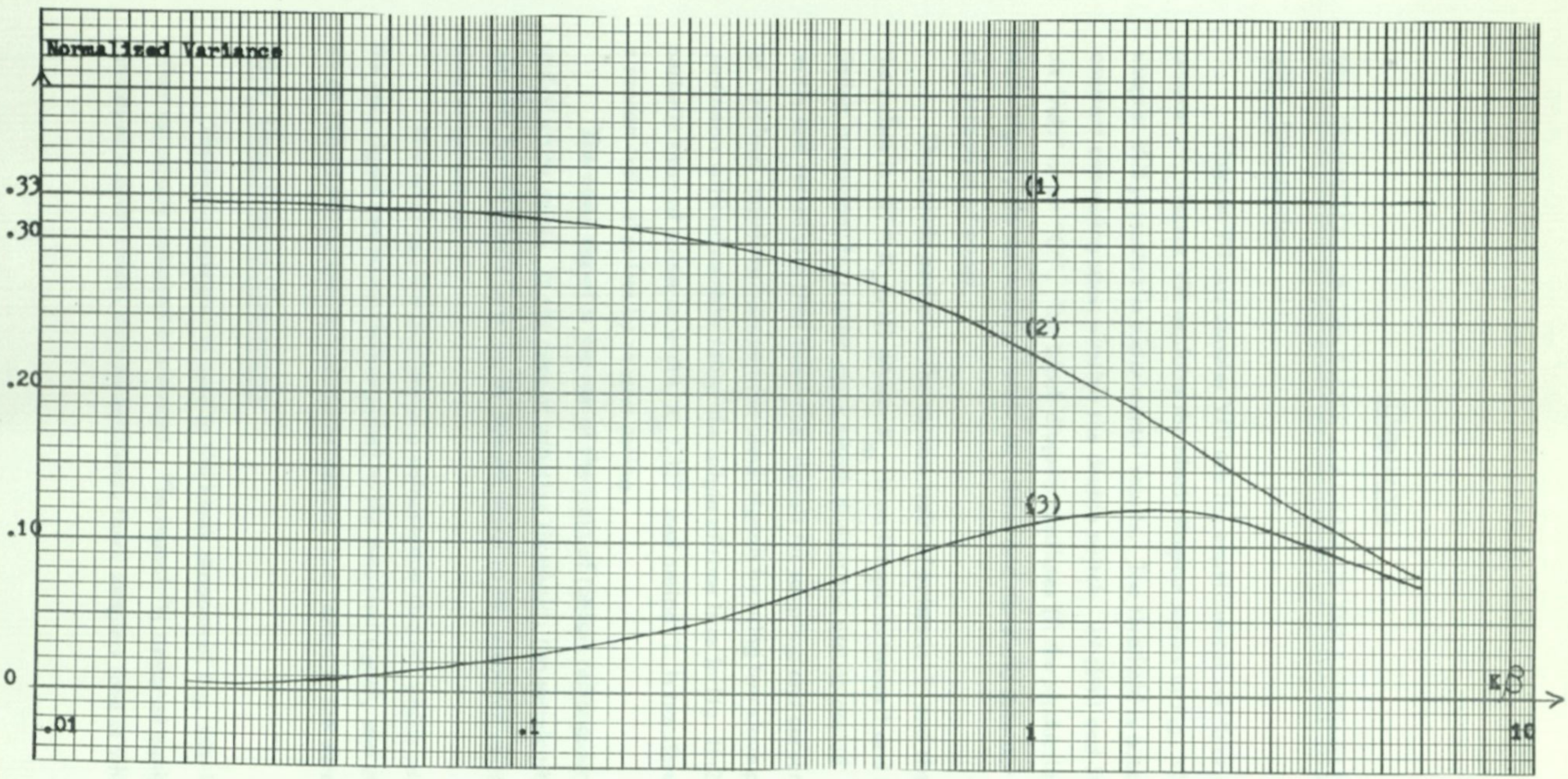


Fig. VI-3 : Normalized Variance versus $K\beta$: (1) Linear Detection, (σ_o^2/σ_i^2), (2) Linear Detection and integration (σ_1^2/σ_i^2) and (3) Linear Detection and Adaptive Integration $(\sigma_{1A}^2/\sigma_i^2)$.

Chapter VII

VII- OPERATION OF THE BLOCK DIAGRAM OF THE EXPERIMENTAL SET UP

VII-I Introduction

Chapter VII and chapter VIII are the experimental verification of chapter IV; chapter VII explains the block diagram shown in Fig. VII-1 while chapter VIII presents the experimental results. The threshold is adaptive or constant depending upon whether the double switch SW (SW1 and SW2) is on "Adp" or on "Ct". The block diagram simulates (I) the production of the noisy signal, (2) the receiver with RC filter and threshold detector and (3) the detection of errors.

The pseudo period of the random train of pulses is T and the width of the pulses is $KT = K_m(.1)T$ where K_m is an integer between 1 and 5. Precise timing is provided by a pilot clock which divides the pseudo period T into tenths and serves as a time reference; signals for sampling and detecting can be obtained anywhere between the clock signals by using two slave pulse generators with adjustable delay; the origin of time is as in Fig. II-2. The periodic train of pulses of width $K_m(.1)T$ and period T is transformed into a random train of pulses by random gating .

The noisy signal is the sum of the train of random rectangular pulses and of white noise. In the case of adaptive threshold, the sampler-holder corrects the threshold level by an amount equal to the expected noise at the time of detection.

The error counter consists of a coincidence circuit, which compares the true signal (noiseless random pulse) to the detected signal, and of a counter which counts the number of non-coincidences, i.e. errors in the detection.

VII-2 Block Diagram Components

The basic components used in the block diagram of Fig. VII-1 are either especially designed or standard. The especially designed equipment consists of a pilot clock, six preamplifiers, three samplers (one of which is

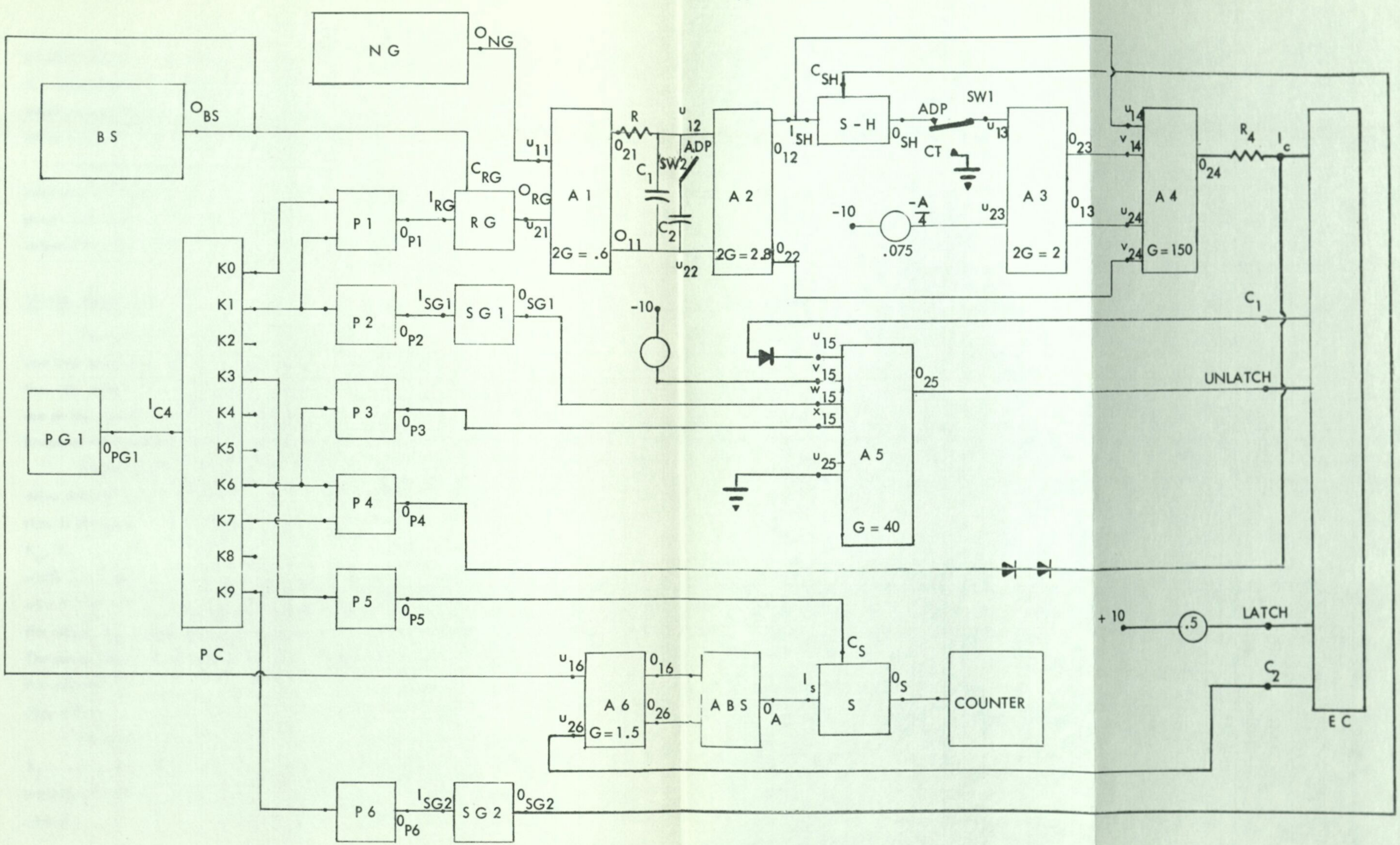


Fig. VII-1 Block Diagram of the Experimental Set Up.

a sampler holder), one binary source, seven summer amplifiers (one of which is a component of the sampler holder) and one absolute value network. The standard equipment consists of an electronic comparator, three generators, a white noise generator and an electronic counter.

Throughout this chapter, the components are described in terms of the mathematical operations performed. The circuits for specially designed components are given in Appendix D; for the standard components the reader is referred to manufacturer's literature.

VII-2a Pilot Clock

The pilot clock behaves as a ten step rotary switch which progresses by one step every time a pulse is applied at the input. The rotary switch connects the nth output to a positive dc voltage (+8 v) and the nine other outputs are at the ground. When a pulse is applied at the input, the (n + 1)th output becomes +8 v while the nth output becomes zero.

A pilot generator provides a periodic train of pulses of period $T/10$ to drive the pilot clock; the shape of the input pulse is arbitrary. The origin of time is chosen as in Fig. II-2 and the ten outputs of the pilot clock are labeled $K_0, K_1 \ldots\ldots K_9$. The output K_0 is a periodic train of rectangular pulses of width $T/10$ and period T; the pulses appear in the intervals of time $mT < t < mT + .1\,T$, where m is an integer. The output K_1 is identical to the output K_0 except that the train of rectangular pulses is translated by $.1\,T$. The pulses appear in the intervals of time $mT + .1\,T < t < mT + .2\,T$. Similarly the outputs $K_2 \ldots\ldots K_9$ are obtained successively by translation as shown in Fig. VII-2.

The pilot clock divides each period T into tenths; the outputs K_0, $K_1 \ldots\ldots K_9$ of the pilot clock are successively positive for one tenth of a period and can be used to synchronize the receiver and the error detecting circuit.

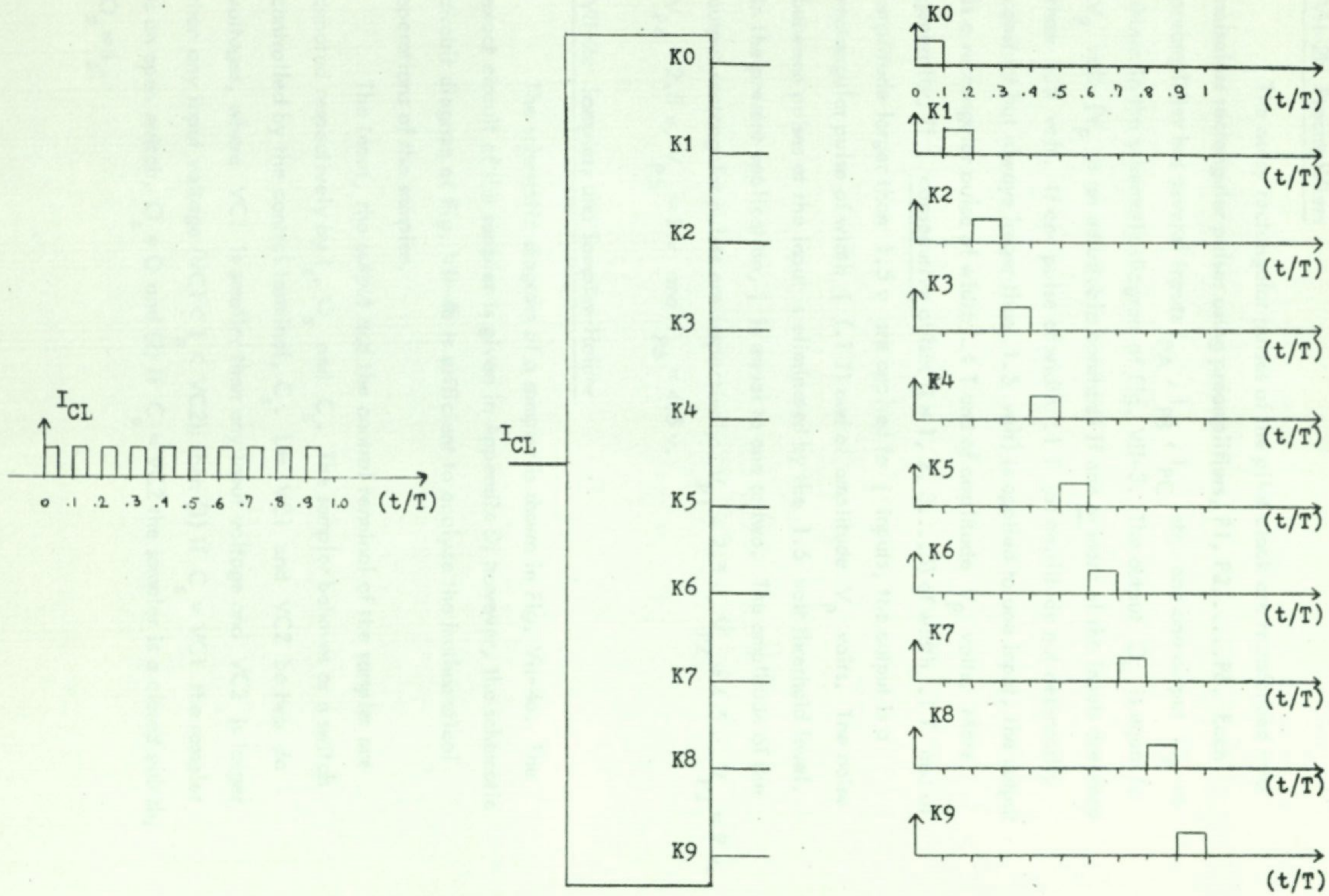


Fig. VII-2 : Schematic Diagram of the Pilot Clock

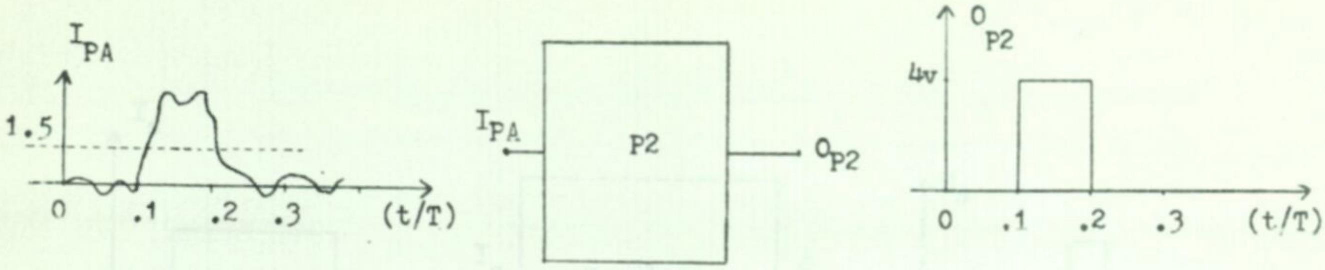
VII-2b Preamplifiers

The noisy rectangular pulses of the pilot clock are transformed into noiseless rectangular pulses using preamplifiers, P1, P2,.....P6. Each preamplifier has several inputs I_{PA}, I_{PB}, I_{PC} etc. and one output O_P as shown in the schematic diagram of Fig. VII-3. The output O_P is equal to V_P volts (V_P is an adjustable constant) if one or more of the inputs are more than 1.5 volt. If one pulse of width .1 T (of amplitude not necessarily constant but always larger than 1.5 volt) is applied to one input, the output is a rectangular pulse of width .1 T and of amplitude V_P volts. More generally, if j consecutive pulses (j = 1, 2, 3....5) of width .1 T and of amplitude larger than 1.5 v are applied to j inputs, the output is a rectangular pulse of width j (.1 T) and of amplitude V_P volts. The noise between pulses at the input is eliminated by the 1.5 volt threshold level. In the present application, j is equal to one or two. The amplitude of the output rectangular pulses are respectively: $V_{P1} = 2.5$ v, $V_{P2} = 4.5$ v, $V_{P3} = 9$ v, $V_{P4} = 2.5$ v, $V_{P5} = 8$ v, and $V_{P6} = 4.5$ v.

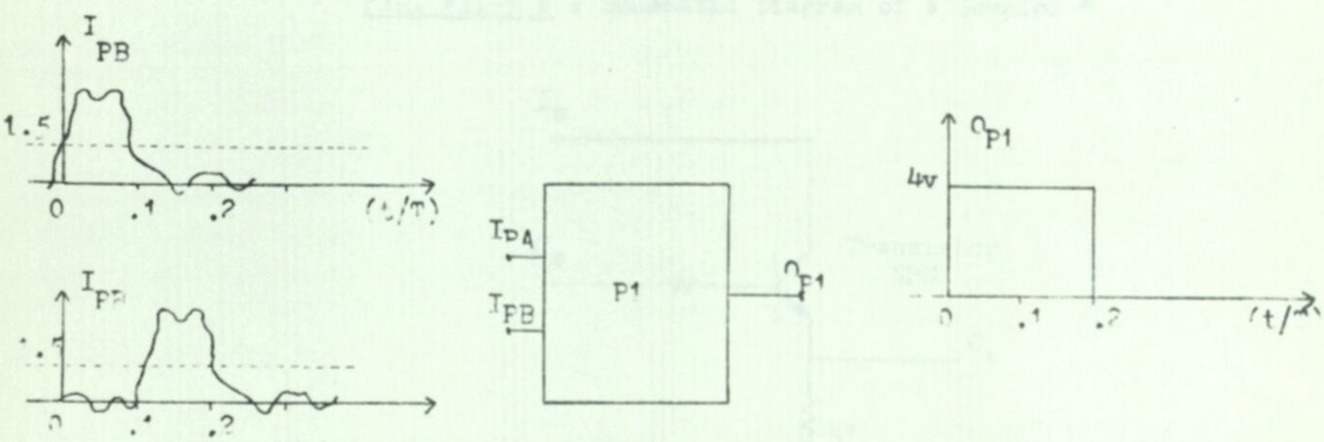
VII-2c Samplers and Sampler-Holder

The schematic diagram of a sampler is shown in Fig. VII-4a. The exact circuit of the sampler is given in Appendix D; however, the schematic circuit diagram of Fig. VII-4b is sufficient to analyze the mathematical operations of the sampler.

The input, the output and the control terminal of the sampler are denoted respectively by I_s, O_s and C_s. The sampler behaves as a switch controlled by the control terminal, C_s. Let VC1 and VC2 be two dc voltages, where VC1 is smaller than any input voltage and VC2 is larger than any input voltage (VC1 < I_s < VC2); then (1) if C_s = VC1 the sampler is an open switch, $O_s = O$ and (2) if C_s = VC2 the sampler is a closed switch, $O_s = I_s$.



(a)Preamplifier with one Input



(b) Preamplifier with several Inputs

Fig. VII-3 : Schematic Diagram of a Preamplifier

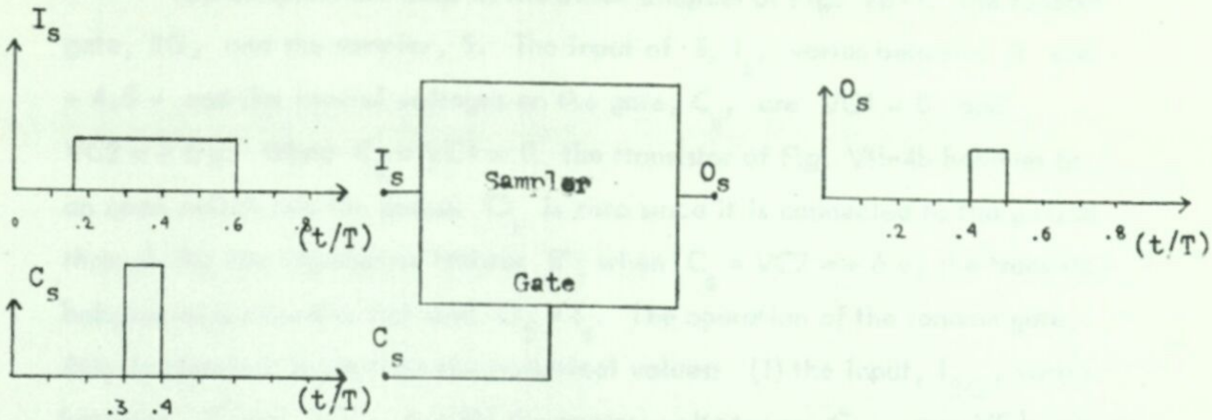


Fig. VII-4 a : Schematic Diagram of a Sampler *

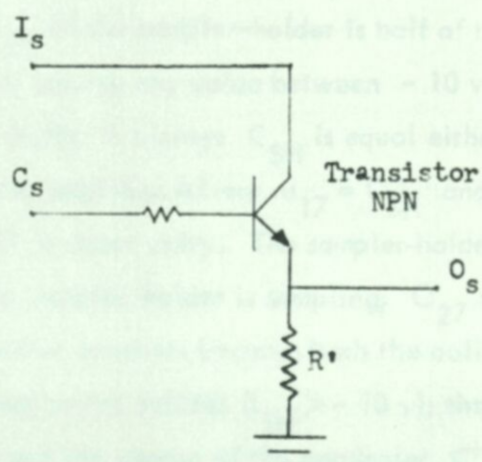


Fig. VII-4 b : Schematic Circuit of a Sampler

* Fig. VII-4 a corresponds to the operation of the sampler S (see coincidence circuit) for an error of the first type (0 sent, 1 detected)

Two samplers are used in the block diagram of Fig. VII-1: the random gate, RG, and the sampler, S. The input of S, I_s, varies between 0 and + 4.5 v and the control voltages on the gate, C_s, are VC1 = 0 and VC2 = + 6 v. When C_s = VC1 = 0 the transistor of Fig. VII-4b behaves as an open switch and the output O_s is zero since it is connected to the ground through the low impedance resistor R'; when C_s = VC2 = + 6 v, the transistor behaves as a closed switch and $O_S = I_s$. The operation of the random gate, RG, is identical except for the numerical values: (1) the input, I_{RG}, varies between 0 and 2.5 v and (2) the control voltages on C_{RG} are VC1 = 0 and VC2 = + 4.5 v.

The schematic diagram of a sampler-holder is shown in Fig. VII-5a. The sampler-holder, SH, consists of a sampler, a memory device (capacitor C') and an amplifier as shown in the schematic circuit diagram of Fig. VII-5b. The input I_{SH} of the sampler-holder is half of the output of A_2, i.e. $I_{SH} = \frac{s(t)}{2}$, and can assume any value between - 10 v and + 10 v. The control of the sampler-holder is binary: C_{SH} is equal either to + 10 v or to - 10 v. The inputs of the amplifier A7 are $u_{17} = C_{SH}$ and u_{27} = - 10 v and the one-side-gain of A7 is about unity. The sampler-holder has two states: (1) When C_{SH} = + 10 v, the sampler-holder is sampling; O_{27} = - 10 v by formula VII-4 and the transistor conducts because both the collector and the base are positive with respect to the emitter (I_{SH} > - 10 v); the transistor behaves as a closed switch and the charge of the condenser C' by the summer amplifier A2 connected to I_{SH} can be considered instantaneous because the output impedance of a summer amplifier is very small; therefore, the voltage O_{SH} follows closely the voltage I_{SH}, $O_{SH} = I_{SH} = \frac{s(t)}{2}$ (2) When C_{SH} = - 10 v, the sampler-holder is holding; O_{27} = 0 since identical voltages are applied on u_{17} and u_{27}; therefore, the transistor does not conduct because the emitter is negative with respect to the base. The condenser C' discharges itself on the resistance R' (the output impedance of A7 is negligible) and O_{SH} decays from its initial value to zero with a time constant R'C'.

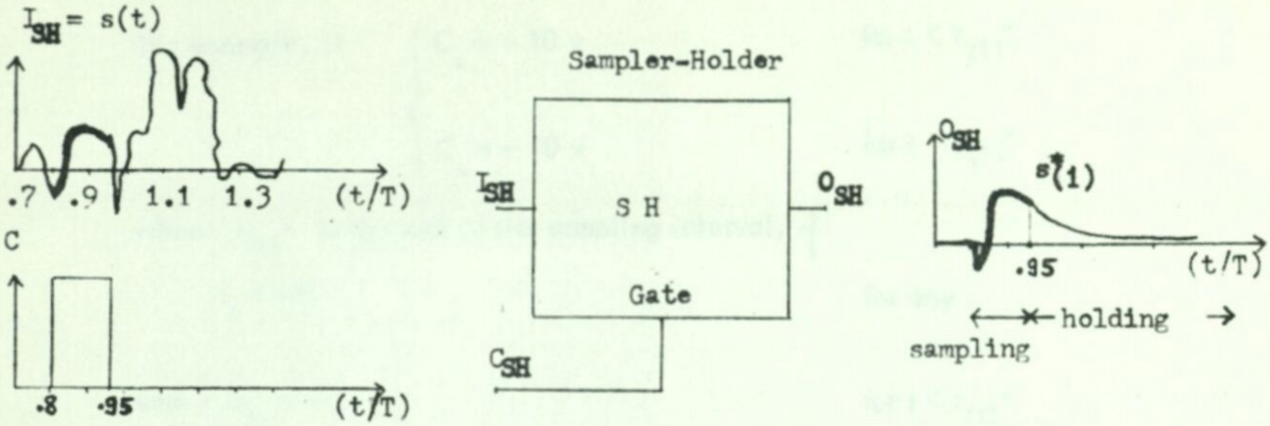


Fig. VII-5 a : Schematic Diagram of a Sampler-Holder*

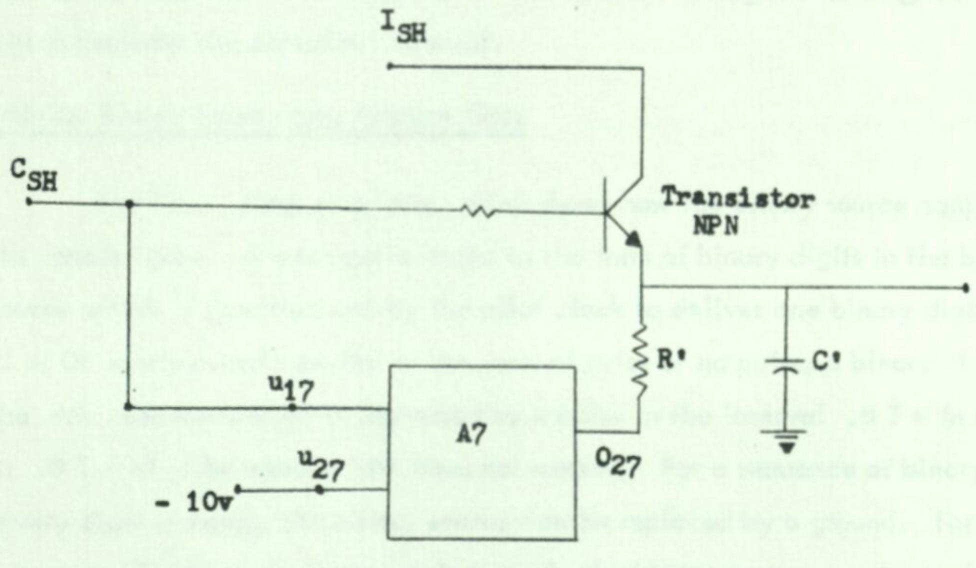


Fig. VII-5 b : Schematic Circuit of a Sampler-Holder

*** In Fig. VII-5 a, the filtered signal s(t) is sampled during the interval .8 T to $t^*_{(1)}$ = .95 T and s(t = $t^*_{(1)}$) = $s^*_{(1)}$**

For example, if
$$\begin{cases} C_s = +10\text{ v} & \text{for } t < t_{(1)}^* \\ C_s = -10\text{ v} & \text{for } t > t_{(1)}^* \end{cases}$$

where $t_{(1)}^*$ is the end of the sampling interval, †

$$I_s = s(t) \qquad \text{for any } t$$

and
$$\begin{cases} O_H = s(t) & \text{for } t < t_{(1)}^* \\ O_H = s(t = t')\, e^{-(t - t_{(1)}^*)/R'C'} & \text{for } t > t_{(1)}^* \end{cases}$$

In conclusion, the sampler-holder behaves as a zero-memory network during the sampling interval and as a RC network during the holding time, which includes the detection interval.

VII-2d Binary Source and Random Gate

The block diagram of Fig. VII-6 shows how the binary source controls the random gate. A message is stored in the form of binary digits in the binary source which is synchronized by the pilot clock to deliver one binary digit (1 or 0) every pseudo-period in the form of pulse or no pulse; a binary 1 in the nth pseudo-period is expressed by a pulse in the interval $.8\,T + (n-1)\,T$ to $.8\,T + nT$ (the exact width does not matter). For a sequence of binary 0 (every digit is zero), the binary source can be replaced by a ground. For a sequence of binary 1 (every digit is one), the binary source can be replaced by + 4.5 volts. The messages, i.e. the sequence of digits, can be completely random; however, recurring messages such as 111111....., 000000....., 101010....., 1001001....., are very useful to check the detection system; for example, to show that the average probability of error for adaptive threshold is independent from the sequential order of the digits.

† t_1, t_1^*, ρ^* denote theoretical values; $t_{(1)}$, $t_{(1)}^*$, $\rho^{(*)}$ denote the corresponding experimental values.

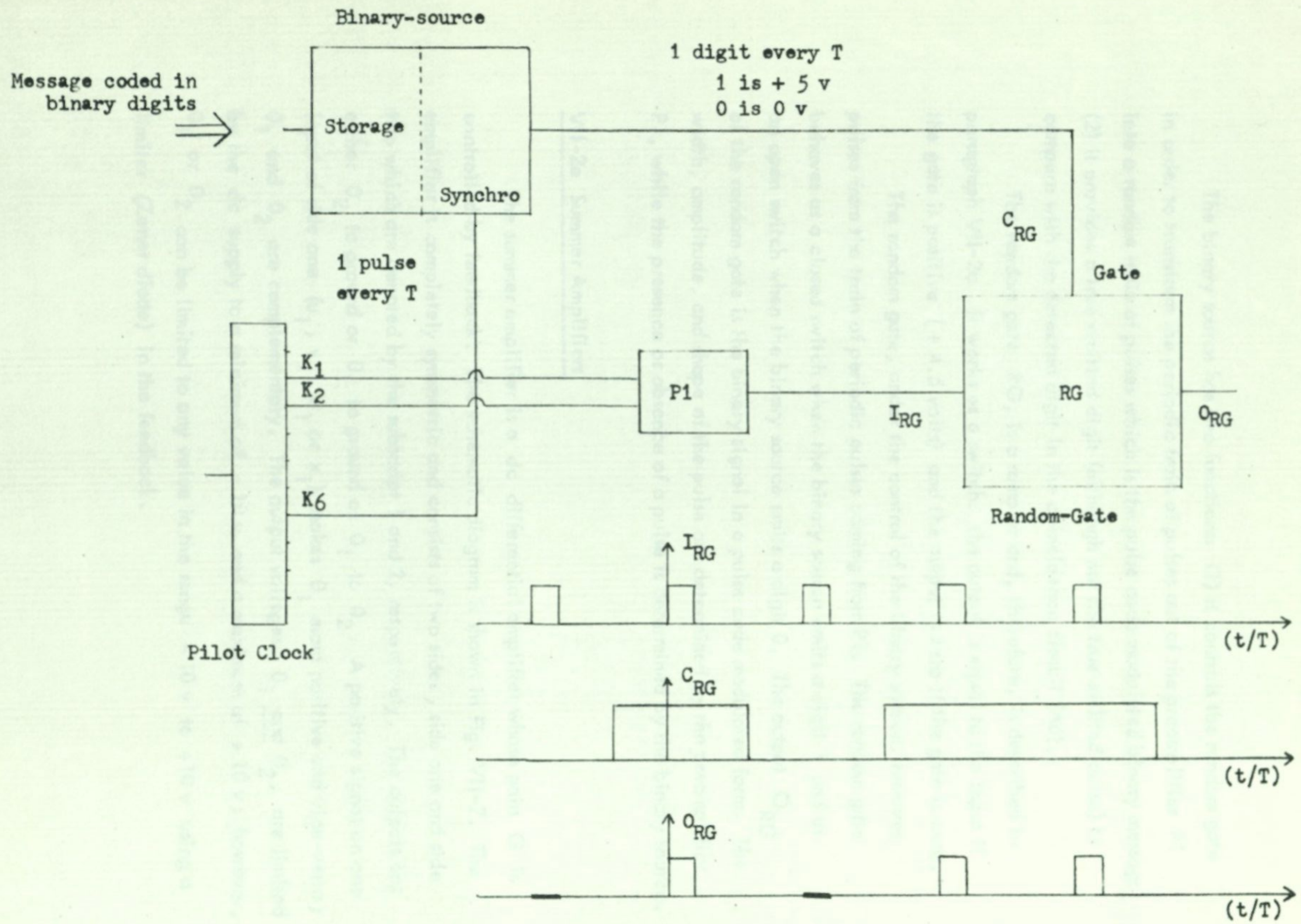


Fig. VII-6 : Schematic Diagram of the Binary-Source and the Random-Gate

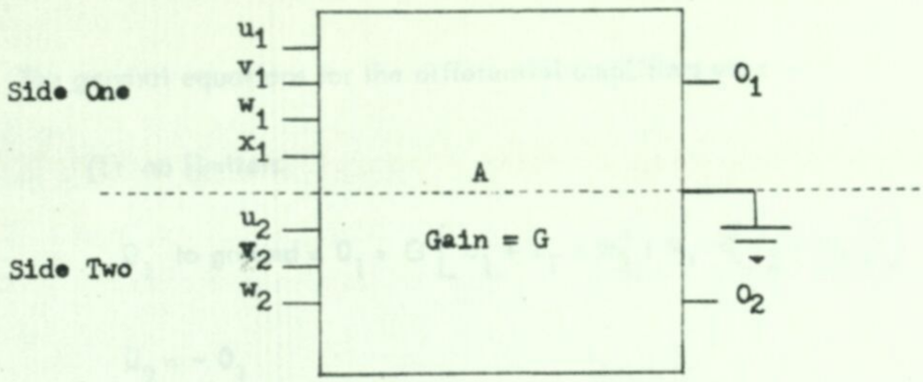
The binary source has two functions: (1) it controls the random gate in order to transform the periodic train of pulses out of the preamplifier P1 into a random train of pulses which is the pulse code modulated binary message; (2) it provides a true emitted digit (although not the true emitted pulse) to compare with the detected digit in the coincidence circuit (A6).

The random gate, RG, is a sampler and, therefore, is described in paragraph VII-2c. It works as a switch: the output is equal to the input if the gate is positive (+ 4.5 volts) and the output is zero if the gate is zero.

The random gate, under the control of the binary source, removes pulses from the train of periodic pulses coming from P1. The random gate behaves as a closed switch when the binary source emits a digit 1 and as an open switch when the binary source emits a digit 0. The output O_{RG} of the random gate is the binary signal in a pulse code modulated form. The width, amplitude, and shape of the pulse are determined by the preamplifier P1, while the presence or absence of a pulse is determined by the binary source.

VII-2e Summer Amplifiers

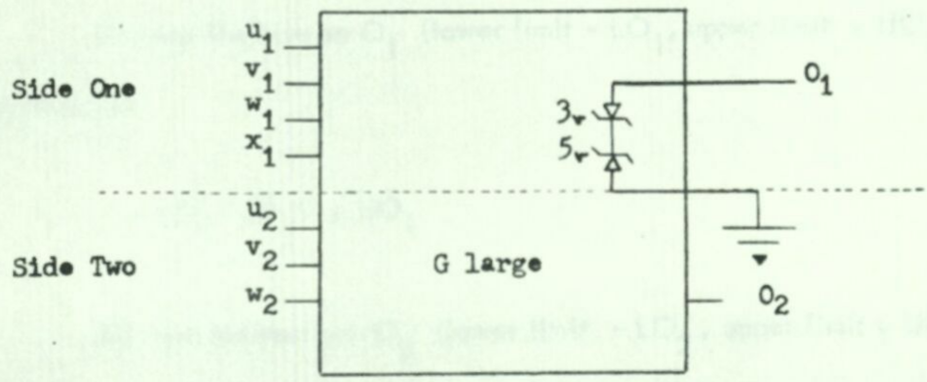
The summer amplifier is a dc differential amplifier whose gain G is controlled by feedback. The schematic diagram is shown in Fig. VII-7. The amplifier is completely symmetric and consists of two sides, side one and side two which are denoted by the subscript 1 and 2, respectively. The outputs are either 0_2 to ground or 0_1 to ground or 0_1 to 0_2. A positive signal on one input of side one (u_1, v_1, w_1 or x_1) makes 0_1 more positive and vice-versa; 0_1 and 0_2 are complementary. The output voltages, 0_1 and 0_2, are limited by the dc supply to a minimum of - 10 v and a maximum of + 10 v; however, 0_1 or 0_2 can be limited to any value in the range - 10 v to +10 v using a limiter (Zener diode) in the feedback.



$$-10 < O_1 = G\,[u_1 + v_1 + w_1 - (u_2 + v_2)] < +12$$

$$-10 < O_2 = -O_1 < +12$$

(a) Linear Amplifier



(b) High Gain Amplifier with Two Limiters on O_1

If $u_1 + v_1 + w_1 - (u_2 + w_2) > 0$ $\quad O_1 = 5$ v

If $u_1 + v_1 + w_1 - (u_2 + w_2) < 0$ $\quad O_1 = -3$ v

Fig. VII-7 : Schematic Diagram of a Summer Amplifier

The general equations for the differential amplifiers are :

(1) no limiters

$$0_1 \text{ to ground} = 0_1 = G\left[u_1 + v_1 + w_1 + x_1 - \left(u_2 + v_2\right)\right] \qquad \text{(VII-1)}$$

$$0_2 = -0_1 \qquad \text{(VII-2)}$$

$$0_1 \text{ to } 0_2 = 2\ 0_1 \qquad \text{(VII-3)}$$

$$-10\text{ v} < \left(0_1 \text{ or } 0_2\right) < +10\text{ v} \qquad \text{(VII-4)}$$

where G is the gain of the amplifier

(2) two limiters on O_1 (lower limit $-LO_1$, upper limit $+UO_1$)

(VII-4) becomes

$$-LO_1 < O_1 < +UO_1 \qquad \text{(VII-5)}$$

(3) two limiters on O_2 (lower limit $-LO_2$, upper limit $+UO_2$)

(VII-4) becomes

$$-LO_2 < O_2 < +UO_2 \qquad \text{(VII-6)}$$

The voltage gain, G, is limited to 30 for linear operation but may be made much larger to perform logic operations. In the block diagram of Fig. VII-1 the differential amplifiers are denoted by A1, A2, A3, A4, A5, A6. A1, A2, A3 and A6 are linear low gain amplifiers ; A4 and A5 are high gain amplifiers with voltage limiters on O_2. When the output of the amplifier is between O1 (or O2) and the ground, the gain is G which is called one side gain; when the output of the amplifier is between Ol and O2, the gain of the amplifier is 2G which is called double side gain.

The equations for the output of a high gain amplifier with two limiters on O_2 are obtained after replacing G by ∞ in formulas (VII-1), (VII-2), (VII-3) and (VII-6):

(1) If $u_1 + v_1 + x_1 \left(u_2 + v_2\right) > O$,

then $$O_2 = - LO_2 \qquad \text{(VII-7)}$$

(2) If $u_1 + v_1 + w_1 + x_1 - \left(u_2 + v_2\right) < O$,

then $$O_2 = + UO_2 \qquad \text{(VII-8)}$$

(3) $O_1 = - O_2$

Since several amplifiers are used in the block diagram of Fig. VII-1, a second subscript is necessary to denote the number of the amplifier. For example, u_{15} means: input u, side 1, amplifier 5.

VII-2f Absolute Value Network and Coincidence Circuit

The absolute value circuit, ABS, which is shown in Fig. VII-8a, has two inputs, I_{AB1} and I_{AB2}, and one output, O_{AB}. Since the inputs of ABS are the outputs of the differential amplifier, AG, they are equal and opposite:

If $I_{AB1} = - I_{AB2}$

$O_A = I_{AB1} = I_{AB2}$

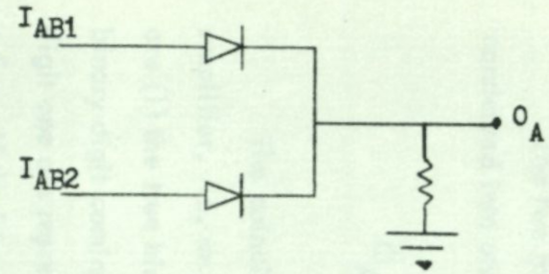


Fig. VII-8 a : Absolute Value Circuit, ABS

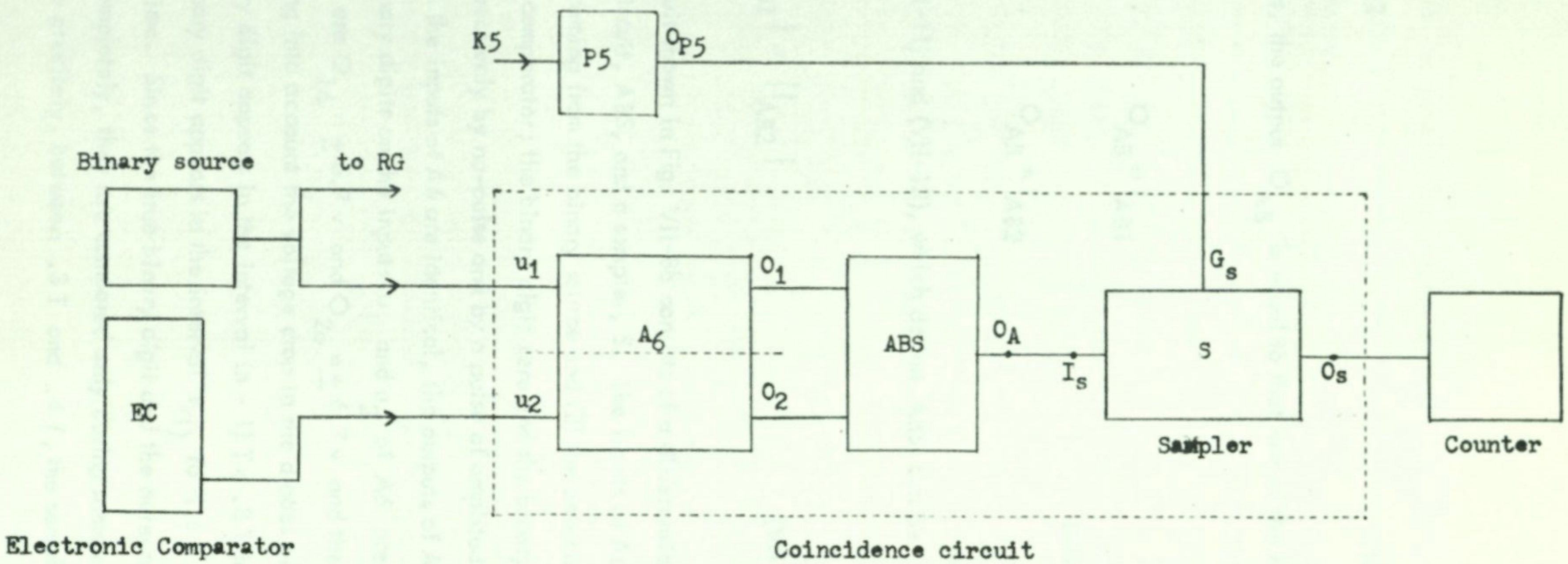


Fig. VII-8 b : Schematic Diagram of the Coincidence Circuit

$$I_{AB1} = -I_{AB2} \qquad \text{(VII-10)}$$

Because of the diodes, the output O_{AB} is equal to that one of the two inputs which is positive :

$$\text{If } I_{AB1} > 0, \quad O_{AB} = I_{AB1}$$

$$\text{If } I_{AB2} > 0, \quad O_{AB} = I_{AB2} \qquad \text{(VII-11)}$$

The two formulas (VII-11) and (VII-12), which define ABS can be condensed into one :

$$O_{AB} = |I_{AB1}| = |I_{AB2}| \qquad \text{(VII-12)}$$

The coincidence circuit shown in Fig. VII-8b consists of a differential amplifier, A6, an absolute circuit, ABS, and a sampler, S. The inputs of A6 are (1) the true binary digit coming from the binary source and (2) the detected binary digit coming from the comparator; the binary digit zero and the binary digit one are represented respectively by no-pulse and by a pulse of amplitude + 5 v. If the binary digits on the inputs of A6 are identical, the outputs of A6 and ABS are zero; if the binary digits on the inputs u_1 and u_2 of A6 are different, the outputs of A6 are $O_{16} = \pm 6.7$ v and $O_{26} = \pm 6.7$ v and the output of ABS is + 6 v (taking into account the voltage drop in the diodes of the rectifier). The true binary digit appears in the interval (n - 1) T + .8 T to (n) T + .8 T; the detected binary digit appears in the interval $t_{(1)}$ to 1.6 T, where $t_{(1)}$ is the detection time. Since the true binary digit and the detected binary digit do not overlap completely, they are compared only during some of the interval of overlap, more precisely, between .3 T and .4 T; the sampler

S is used for this purpose. The sampler S is controlled by the preamplifier P5 and behaves as a closed switch in the interval of time $nT + .3T < t < nT + 4T$ and as an open switch elsewhere.

During the interval of time $nT + .3T < t < nT + .4T$, the output of the sampler S is either zero or + 6 v: it is zero if the nth emitted digit and the nth detected digit are identical and it is 6 v if they are different. Outside of this interval of time, the output of the sampler is zero.

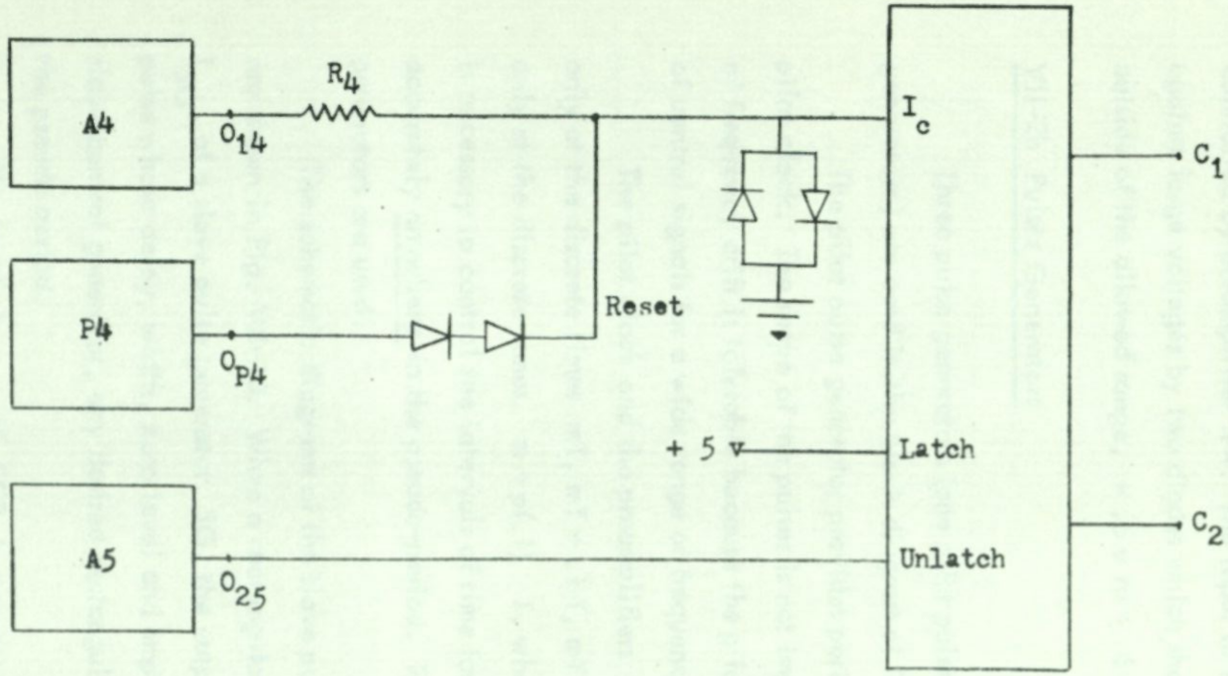
For every error in the detection, a pulse (amplitude 6 v, width .1 T) appears at the output of O_s ; this pulse activates the counter, CNT, which counts the errors in the detection of binary digits mixed with white noise; if the frequency of the binary digits and the duration of the experiment are measured, the average probability of error in the detection can be determined.

VII-2g Electronic Comparator

The electronic comparator unit, E. C., is made by Electronic Associate Inc. The block diagram of the electronic comparator is shown in Fig. VII-9. The input is I_C and the outputs are C_1 and C_2 which are complementary; that is, if one is 5 volts, the other is 0 volt and vice-versa. The comparator possesses two logic controls: latch and unlatch. When + 5 volts are applied on the latch terminal, the input of the comparator is virtually disconnected and the outputs C_1 and C_2 remain in their previous state independently of the input. The unlatch terminal overrides the latch terminal; therefore, when - 2 volts are applied on the unlatch terminal, the input voltage regains control of the comparator. When the comparator is unlatched the correspondence between input and output is as follows:

$$I_C \geq .5 \qquad \begin{cases} C_1 = +5\text{v} \\ C_2 = 0\text{v} \end{cases} \qquad \text{(VII-13)}$$

$$I_C \leq .5 \qquad \begin{cases} C_1 = 0\text{v} \\ C_2 = +5\text{v} \end{cases} \qquad \text{(VII-14)}$$



(1) If $O_{25} > 0$ C_1 and C_2 keep previous values independently of I_c

(2) If $O_{25} < 0$ (Detection and reset)

$I_c > 0$	$C_1 = 5$	$C_2 = 0$
$I_c < 0$	$C_1 = 0$	$C_2 = 5$

Fig. VII-9 : Schematic Diagram of the Electronic Comparator

In the block diagram of Fig. VII-1, the comparator is latched all the time except (1) at the time of detection where the input I_C is controlled by the summer amplifier A4 and (2) at the time of reset where the input I_C is controlled by preamplifier P4. The input of the comparator (I_C) is protected against large voltages by two diodes which short circuit the input for voltages outside of the allowed range, - .6 v to + .6 v.

VII-2h Pulses Generators

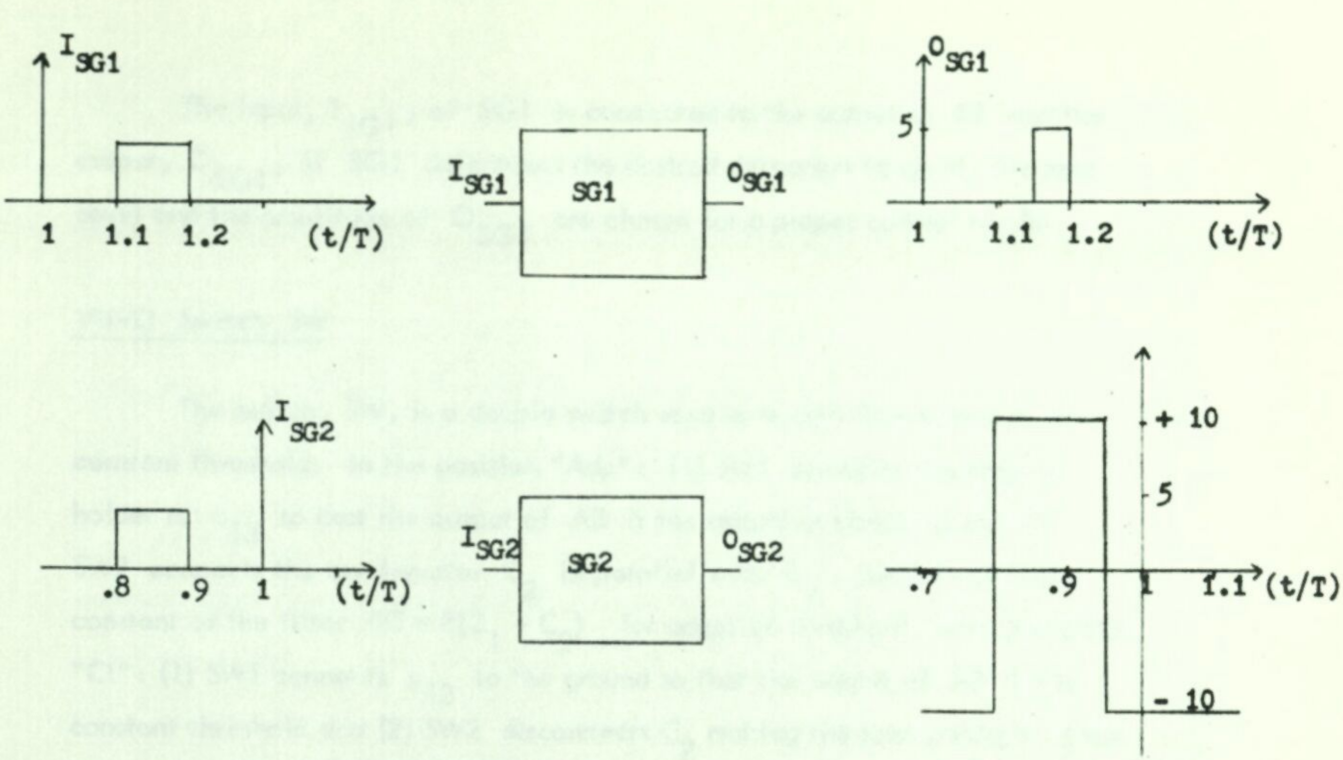
Three pulse generators (one pilot pulse generator and two slave pulse generators) are used in the block diagram of Fig. VII-1.

The pilot pulse generator provides periodic pulses at the input of the pilot clock. The shape of the pulses is not important and a certain amount of frequency drift is tolerable because the pilot clock insures a proper sequence of control signals for a wide range of frequency.

The pilot clock and the preamplifiers P1 to P6 provide control signals only at the discrete times mT, mT + .1 T, mT + .2 T, etc., or more generally, only at the discrete times m + p(.1) T. where m and p are integers. It is necessary to control the intervals of time for sampling and for detection accurately anywhere in the pseudo-period. For this purpose, slave pulse generators are used.

The schematic diagrams of the slave pulse generators SG1 and SG2 are shown in Fig. VII-10. When a rectangular pulse is applied to the input, I_{SG}, of a slave pulse generator, SG, the output, O_{SG}, is a delayed rectangular pulse whose delay, width, zero level and amplitude are adjustable. Using a slave control generator, any desired rectangular pulse can be obtained during the pseudo period.

The input, I_{SG2}, of SG2 is connected to the output of P6 and the output, O_{SG2}, of SG2 determines the desired sampling interval; the zero level and the amplitude of O_{SG2} are chosen for a proper control of the sampler-holder, SH.



The four adjustments for SG1 and SG2 are :

	SG1	SG2
Delay	.05 T	0
Width	.05 T	.15 T
Zero setting	0	- 10 v
Amplitude	5 v	20 v

<u>Fig. VII-10</u> : Schematic diagram of the slave pulse generators, SG1 and SG2

The input, I_{SG1}, of SG1 is connected to the output of P2 and the output, O_{SG1}, of SG1 determines the desired detection interval; the zero level and the amplitude of O_{SG1} are chosen for a proper control of A5.

VII-2i Switch SW

The switch, SW, is a double switch used to switch from adaptive to constant threshold. In the position "Adp": (1) SW1 connects the sampler holder to u_{13} so that the output of A3 is the adaptive threshold and (2) SW2 connects the condensator C_2 in parallel with C_1, making the time constant of the filter $RC = R(C_1 + C_2)$ for adaptive threshold. In the position "Ct": (1) SW1 connects u_{13} to the ground so that the output of A3 is the constant threshold and (2) SW2 disconnects C_2 making the time constant of the filter $RC = RC_1$ for constant threshold.

VII-2j White Noise Generator

The random noise generator is manufactured by General Radio (type 1390-B). It is used to deliver white noise of normal distribution on a bandwidth 0 to 22 Kcs, which is a large bandwidth in comparison of the inverse of the width of a pulse. The maximum open circuit output is more than two volts and the spectral voltage density with 2 volt output is about 2.4 milivolts for one-cycle band. The output impedance is less than 900 ohms which is small compared to the input impedance of the amplifier A1.

VII-2k Electronic Counter

The electronic counter is manufactured by Hewlett-Packard Company (type 5245L) which is a high frequency general purpose electronic counter that measures frequencies from 0 to 50 Mcs, periods from 1 second to 10 seconds and period average from 10 to 100,000 periods. The electronic counter is used to count the pulses coming from the coincidence circuit (one pulse for each error in the detection) for a duration of several hours.

VII-3 Choice of the Sampling Time, The Detection Time and the Holding Network

The experimental model of Fig. VII-1 differs slightly from the mathematical model of chapter IV because the response of the electronic components, although fast, is not instantaneous.

The experimental sampling time finishes at time $t = t^*_{(1)}$ and $t^*_{(1)} = .95$ T instead of at the theoretical value, $t^*_1 = T$, because the sampling must be terminated before the start of the unknown pulse and a safety margin is necessary.

The experimental time of detection is at the end of the pulse; however, it cannot be defined exactly (as the theoretical time was) because the detection is not instantaneous. For example, if the width of the pulse is .2 T, the detection takes place at a time $t_{(1)}$ in the interval 1.15 T to 1.2 T instead of the theoretical value $t_1 = 1.2$ T.

The optimum adaptive threshold, D_A, at the time, $t_{(1)}$, of detection is obtained by using formulas (IV-52) and (IV-31) and replacing t_1 by $t_{(1)}$, t^*_1 by $t^*_{(1)}$ and ρ^* by $\rho^{(*)}$:

$$D_A = A/2 + s^*_{(1)} e^{-(t_{(1)} - t^*_{(1)})/RC} = A/2 + \rho^{(*)} s^*_{(1)} \dagger \quad \text{(VII-15)}$$

$\rho^{(*)}$ is not precisely defined because $t_{(1)}$ is anywhere between 1.15 T and 1.2 T.

$$e^{-.25\ T/RC} < \rho^{(*)} = e^{-(t_{(1)} - t^*_{(1)})/RC} < e^{-.2\ T/RC} \quad \text{(VII-16)}$$

As an approximation, let $\rho^{(*)} = e^{-.22\ T/RC}$ (VII-17)

The optimum adaptive threshold, $D_A(t)$, at time t anywhere during the interval of detection (1.15 T < t < 1.2 T) is obtained by generalizing formula (VII-15):

$$D_A(t) = A/2 + s^*_{(1)} e^{-(t - t^*_{(1)})/RC} \quad \text{for } 1.15\ T < t < 1.2\ T \quad \text{(VII-18)}$$

† The peak amplitude A of a filtered single pulse is smaller for adaptive than for constant threshold (RC larger).

On the other hand, the output of a sampler-holder at time t, after sampling until time $t^*_{(1)}$, is given by paragraph VII-2 e:

$$O_H(t) = s^*_{(1)}/2 \; e^{-(t - t^*_{(1)})/R'C'} \qquad \text{for } t > t^*_{(1)} \qquad \text{(VII-19)}$$

and at the time $t_{(1)}$ of detection:

$$O_H(t = t_{(1)}) = s^*_{(1)}/2 \; e^{-(t_{(1)} - t^*_{(1)})/R'C'} = \rho^{(*)} \, s^*_{(1)} \qquad \text{(VII-20)}$$

The output of the summer amplifier A3 (gain 2) at time t during the interval of detection is

$$O_{13}(t) = A/2 + s^*_{(1)} \; e^{-(t - t^*_{(1)})/R'C'} \qquad \text{for } 1.15\,T < t < 1.2\,T \qquad \text{(VII-21)}$$

From comparison between formulas (VII-18) and (VII-21), it follows that the output $O_{13}(t)$ of the amplifier A3 can be made equal to the adaptive threshold $D_A(t)$, as desired, during the detection interval by taking R'C' = RC:

$$O_{13}(t_{(1)}) = D_A(t = t_{(1)}) = D_A \qquad \text{for } R'C' = RC \text{ and } t = t_{(1)} \qquad \text{(VII-22)}$$

Since the output of P4 overrides the output of A4, EC is sensitive to A4 only for $1.15\,T < t < 1.2\,T$ when, both, EC is unlatched and $O_{P4} = 0$.

Therefore, the decision circuit, which consists of A4 and EC, compares the output of A3 to the output of A2 (i.e. D(t) to s(t)) only during the short interval $1.15\,T < t < 1.2\,T$. For all practical purposes, this short interval is an instant of time, $t_{(1)}$, where $1.15\,T < t_{(1)} < 1.2\,T$. Thus the comparator compares $D_A = D_A(t = t_{(1)})$ to $s_{(1)}$ as desired. In conclusion, the experimental block diagram of Fig. VII-1 simulates the theoretical block diagram of Fig. II-1 except for minor differences due to the noninstantaneous operation of the electronic components.

VII-4 Analysis of the Experimental Block Diagram

VII-4a Outline of the Analysis

The block diagram of Fig. VII-1 can be divided into four subdivisions: (1) the clock system, (2) the generator of noisy signal and the decision circuit, (3) the unlash control and (4) the coincidence circuit and error counter. In each of the four subdivisions, the operation of each component versus time is analyzed for about one pseudo-period using (a) the description of the components of paragraph VII-2 and (b) the experimental sampling time and the experimental detection time ($t_{(1)}^*$ and $t_{(1)}$ respectively) defined in paragraph VII-3.

The following types of detection are considered successively:

(1) The detection of one pulse (width KT = .2 T) mixed with white noise using an adaptive threshold detector (see VII-4 b);

(2) The detection of no-pulse, i.e. white noise alone using an adaptive threshold detector (see VII-4 c);

(3) The detection of one pulse (width KT = .2 T) mixed with white noise, using a constant threshold detector (see VII-4 d);

(4) The detection of no-pulse, i.e. white noise alone, using a constant threshold detector (see VII-4 e).

The analysis for a pulse width other than .2 T would be very similar.

VII-4b Detection of One Pulse Mixed with White Noise Using an Adaptive Threshold Detector

One pulse is emitted by the random gate during the interval of time $T < t < 1.2\ T$ and the switch, SW, of the block diagram of Fig. VII-1 is on the position Adaptive Threshold (Adp). The detector either detects a pulse, which is a correct detection, or detects a zero, which is an error of type 2 (see Fig. II-4).

The block diagram of Fig. VII-1 is decomposed into four elementary block diagrams: (1) the clock system; (2) the generator of noisy signal and the decision circuit; (3) the unlash control and (4) the coincidence circuit. These are analyzed successively in paragraphs VII-4b-1, VII-4b-2, VII-4b-3 and VII-4b-4.

VII-4b-1 The Clock System

The clock system (which consists of a periodic pulse generator, a pilot clock, a group of preamplifiers and two slave generators) is used for control and synchronization.

The functions of the preamplifiers, P1 .... P6, and of the two slave generators, SG1 and SG2, are shown in Fig. VII-11 for the one pseudo-period. The detection of one pulse in the interval of time T to 1.2 T starts by the reset of the electronic comparator in the previous period, at time t = .6 T.

The clock system controls (1a) the reset of the comparator and (1b) the gating of the sampler-holder; it synchronizes (2a) the generator of a noisy signal, (2b) the unlatch control and (2c) the coincidence circuit.

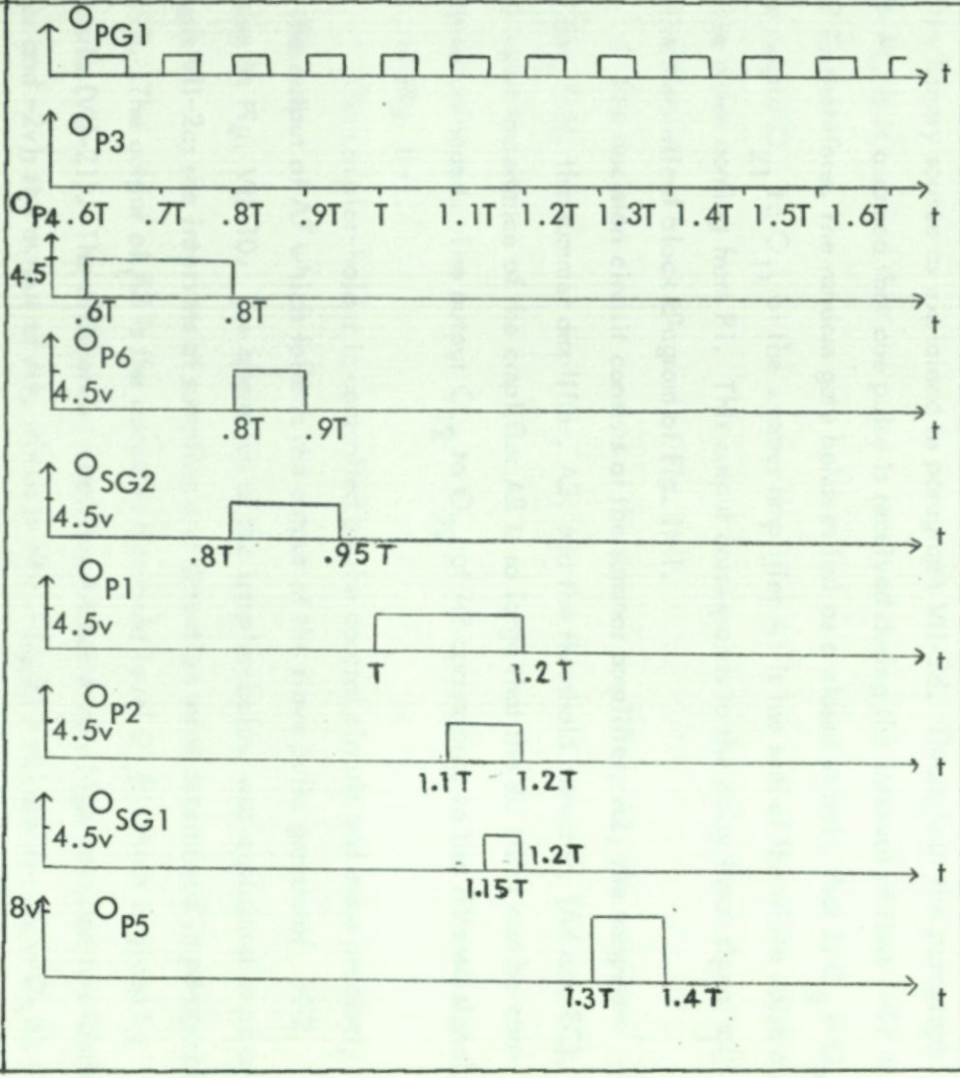
(1a) The clock system controls the reset of the comparator by the preamplifier P4 which makes the input of the comparator positive while the comparator is unlatched ($.6\,T < t < .7\,T$); P4 is positive during the time $.6\,T < t < .7\,T$ which insures that the comparator is in the position $I_c > 0$ ($C_1 = 5\,v$, $C_c = 0\,v$) when it is latched at time $t = .7\,T$.

(1b) The clock system controls the sampler-holder by the slave generator, SG2, which applies +10 v to the gate during the sampling interval $.8\,T < t < (.95\,T = t_{(1)}{}^*)$ and which applies -10 v to the gate during the holding interval $t_{(1)}{}^* < t < 1.8T$). The operation of the slave generator and of the sampler-holder are analyzed in detail in paragraph VII-2b and VII-2c, respectively.

(2a) The train of random pulses representing the binary signal is obtained by gating the periodic train of pulses produced by P1. (See paragraph VII-2d.)

(2b) The preamplifiers P2 and P3 synchronize the amplifier A5 which controls the unlash of the comparator. (See paragraph VII-2g.)

2(c) The preamplifier P5 controls the gate of the sampler, S, which synchronizes the coincidence circuit so that the true signal coming from the binary



Components	Type	Output Versus Time	Function in the Block Diagram
PG1	Pilot Pulse Generator		Drives the clock
P3	Preamplifier		Unlatches the comparator during the reset
P4	Preamplifier		Provides a positive signal on the input of the comparator during the reset
P6	Preamplifier		Synchronizes the slave pulse generator SG2
SG2	Slave Pulse Generator		Determines the sampling interval
P1	Preamplifier		This periodic train of pulses becomes the p.c.m. signal after gating by the random gate, RG
P2	Preamplifier		Synchronizes the slave generator SG1
SG1	Slave Pulse Generator		Unlatches the comparator during the detection interval
P5	Preamplifier		Determines the coincidence interval for comparison of the true and of the detected digit.

Table VII-11: Clock System, Outputs of the Components Versus Time source and the detected signal coming from the comparator are compared only during the interval of time .3 T < t < .4 T which is common to the two partially overlapping signals (see paragraph VII-21).

VII-4b-2 The Generator of Noisy Signal and the Decision Circuit

The generator of noisy signal consists of a noise generator, NG, a random gate, RG, and a summer amplifier, A1. The random gate transforms the train of periodic pulses at the output of P1 into a train of random pulses under the control of the binary source as explained in paragraph VII-2d. Throughout the paragraph VII-4b, it is assumed that one pulse is received during the interval of time T < t < 1.2 T; therefore, the random gate behaves just as a closed switch, that is $O_R = O_{P1}$. The output O_{21} to O_{11} of the summer amplifier A1 is the sum of the white noise and of the pulse coming from P1. This output corresponds to the noisy input signal $s_i(t)$ of the theoretical block diagram of Fig. II-1.

The decision circuit consists of the summer amplifier, A2, the sampler-holder, S-H, the summer amplifier, A3, and the threshold detector, (A4 and EC). The input impedance of the amplifier A2 is so large that the RC filter can be considered unloaded. The output O_{12} to O_{22} of A2 corresponds to the filtered signal, s(t), of Fig. II-1.

The sampler-holder is controlled by the control circuit and more precisely by the output of A7 which follows the output of the slave pulse generator, SG2, shown in Fig. VII-10. The operation of the sampler-holder was explained in paragraph VII-2c; the intervals of sampling and detection were determined in paragraph VII-3. The output of A3 is the adaptive threshold level $D_A(t)$ which is given by formula (VII-21). The differential amplifier A4 has a very high gain and two limiters (+2v and -2v); the output of A4, which is +2 v if $D_A(t) > s(t)$ and is -2 v if $D_A(t) < s(t)$, is applied to the input of the electronic comparator, I_c, through a small resistance, R_4 (R_4 is used for protection of both A4 and E_c).

The comparator E_c is latched by +5 v on the latch terminal except when the output of the amplifier A5 of the control circuit provides -2v on the unlatch terminal which occurs (1) during the reset interval (.6 T < t < .7 T) and (2) during the detection interval (1.15 T < t < 1.2 T).

The operation of the electronic comparator is explained in paragraph VII-2g. During the reset interval (.6 T < t < .7 T) the comparator switches to the testing state $C_1 = 5$ v, $C_2 = 0$ v,(or remains there) because both (1) P4 makes I_c positive and (2) A5 unlatches the comparator. During the detection interval ($1.15\ T < t_{(1)} < 1.2\ T$) the comparator is sensitive to the output of A4 because A5 unlatches the comparator; two situations arise: (1) if $s(t = t_{(1)}) = s_{(1)} > D_A = A/2 + \rho^{(*)}\ n^*_{(1)}$, then $O_{24}\ (t = t_{(1)}) < 0$ and the comparator switches to the detecting state $C_2 = 5$ v which means a pulse has been detected; (2) if $s_{(1)} < D_A$, then $O_{24}\ (t = t_{(1)}) < 0$ and the comparator remains on $C_2 = 0$ v; no pulse has been detected (this is an error of the second type. The variations of the output voltage of the components of the noisy signal generator and of the decision circuit versus time are shown in Fig. VII-12.

VII-4b-3 The Latch and Unlatch Control

The latching and unlatching of the comparator is controlled by the amplifier A5. The operation of the comparator was explained in paragraph VII-2g; the comparator is unlatched when the output of A5, O_{25}, is negative and is latched otherwise.

A5 is a high gain amplifier with two limiters which limit the output O_{25} between -2v and +10 v. Using paragraph VII-2e, O_{25} is equal to -2 v when the sum of the inputs to A5 (denoted by Σ) is positive:

If $\Sigma = u_{15} + v_{15} + w_{15} + 2x_{15} < 0$, $O_2 = +10$ v; EC is latched
If $\Sigma > 0$, $O_2 = -2$ v; EC is unlatched.

The control of latch and unlatch for one pseudo-period is shown in Fig. VII-13. During the detection interval (1.15 T < t < 1.2 T), $v_{15} = O_{SG1} = +5$ v; at time t = 1.15 T, the comparator EC is unlatched because both $v_{15} = +5$ v and

Components	Outputs	.6T	.8T	.95T	T	1.15T	< $t_{(1)}$ < 1.2T	1.6T
NG	$n_i(t)$	$n_i(t)$	....	....	....	....	....	....
BS	O_{BS}	+ 5 v	....	....	....	....	....	....
RG	$O_R = p_i(t)$	0	0	0	0	1.65 v	1.65 v	0
A_1	$0_{21} - 0_{11} = s_i(t)$ $= n_i(t) + p_i(t)$	$n_i(t)$	$n_i(t)$	$n_i(t)$	$n_i(t)$	$p_i(t)+n_i(t)$	$p_i(t)+n_i(t)$	$n_i(t)$
A_2	$0_{12} - 0_{22} = s(t)$	s(t)	s(t)	s(t)	s(t)		$s_{(1)}$	s(t)
SH	O_{SH}	$\pm \epsilon$	$\pm \epsilon$	s(t) $s_{(1)}^*$	$(s_{(1)}^* e^{-(t-t_{(1)}^*)/RC})$		$s_{(1)}^* \rho^*$	$s_{(1)}^* e^{-(t-t_{(1)}^*)/RC}$
A_3	$0_{23} - 0_{13} = D_A(t)$	$D_A(t)$	$D_A(t)$	$D_A(t)$	$D_A(t)$		$D_A(t_{(1)}) = D_A$	$D_A(t)$
If $s_{(1)} < D_A$								
A_4	0_{14}	$\pm$ 3 v	$\pm$ 3 v	$\pm$ 3 v	$\pm$ 3 v	$\pm$ 3 v	+ 3 v	$\pm$ 3 v
E_c	C1 C2	0 or 5 v 5 or 0 v	5 v 0	5 v 0	5 v 0	5 v 0	5 v 0	5 v 0
If $s_{(1)} < D_A$								
A_4	0_{14}	$\pm$ 3 v	$\pm$ 3 v	$\pm$ 3 v	$\pm$ 3 v	$\pm$ 3 v	- 3 v	$\pm$ 3 v
E_c	C1 C2	0 or 5 v 5 v or 0	5 v 0	5 v 0	5 v 0	5 v 0	0 5 v	0 5 v

Table VII-12: Operation of the Noisy Signal Generator and of the Decision Circuit (One Sent)

Terminals \ Time	– .6T	.6T – .7T	.7T – .8T	.8T < $t_{(1)}$	$t_{(1)}$ < .9T	.9T – 1.6T
v_{15}	- 7 v	- 7 v	- 7 v	- 7 v		- 7 v
$w_{15} = 0_{SG1}$	0	0	0	4.5v		0
$x_{15} = 0_{P3}$	0	4.5	0	0		0
No pulse detected at $t_{(1)}$: $u_{15} \approx C$	0 or 4.5 v	4.5 v	4.5 v	4.5 v		4.5 v
No pulse detected at $t_{(1)}$: Σ	-	+	-	+		-
No pulse detected at $t_{(1)}$: 0_{25}	+ 10 v	- 3 v	+ 10 v	- 3 v		+ 10 v
No pulse detected at $t_{(1)}$: EC	Latch	Unlatch	Latch	Unlatch		Latch
One pulse detected at $t_{(1)}$: $u_{15} \approx C$	0 or 4.5 v	4.5 v	4.5 v	4.5 v	0	0
One pulse detected at $t_{(1)}$: Σ	-	+	-	+	-	-
One pulse detected at $t_{(1)}$: 0_{25}	+ 10 v	- 3 v	+ 10 v	- 3 v	+ 10 v	+ 10 v
One pulse detected at $t_{(1)}$: EC	Latch	Unlatch	Latch	Unlatch	Latch	Latch

Table VII-13: Control of Latch and Unlatch on the Comparator by A5

$u_{15} = C_1 = +5\,v$ (the comparator was reset at $t = .6\,T$). The comparator remains unlatched until $T = 1.2\,T$ if no pulse is detected; however, if a pulse is detected at time $t_{(1)}$, $u_{15} = C_1$ becomes zero and the comparator is latched since v_{15} alone cannot unlatch it. The purpose of u_{15} is to prevent the detection of several pulses in one pseudo-period.

During the reset ($.6\,T < t < .7\,T$) the comparator is unlatched by a $+9\,v$ on x_{15}, whether ($u_{15} = C_1$) is zero or five volts.

VII-4b-4 Coincidence Circuit

The operation of the coincidence circuit was examined in detail in paragraph VII-2f. Table VII-14 shows the operation of the coincidence circuit and of the counter when one pulse is sent for the two cases: (1) no pulse detected (error of the second type) and (2) one pulse detected.

VII-4b-5 Operation of the Block Diagram When One Pulse is Sent

The main steps in the detection of one pulse mixed with white noise using a RC filter and an adaptive threshold are shown in Table VII-15 which condenses the main results of Tables VII-11, VII-12, VII-13 and VII-14 into one table for easy reference.

VII-4c Detection of no Pulse Mixed with White Noise Using an Adaptive Threshold Detector

No pulse is emitted by the random gate during the interval of time $T < t < 1.2\,T$; the detector either detects no pulse, which is a correct detection, or detects a pulse, which is an error of type 1. The switch, SW, of the block diagram of Fig. VII-1 is on the position "Adaptive" (Adp).

The clock system operates exactly as in paragraph VII-4b-1. The output of the random gate is zero; therefore, the output of A1 is just the white noise and

	Components	Outputs	0 – .6T	.6T – 1.15T	1.15T < $t_{(1)}$	$t_{(1)}$ < 1.2T	1.2T – 1.3T	1.3T – 1.4T	1.4T – 1.6T
	BS	$0_{BS} = u_{16}$	0 or 5 v	5 v	5 v		5 v	5 v	5 v
	P5	$0_{P5} = C_S$	0	0	0		0	6 v	0
One detected	EC	$C_2 = v_{16}$	0	0	0	5 v	5 v	5 v	5 v
	A_6	0_{16}	0 or 6.7 v	6.7 v	6.7 v	0	0	0	0
		0_{26}	0 or −6.7 v	−6.7 v	−6.7 v	0	0	0	0
	ABS	$O_A = \|0_{16}\|$	0 or 6 v	6 v	6 v	0	0	0	0
	S	O_s	0	0	0	0	0	0	0
	Counter	0	0	0	0	0	0	0	0
Zero detected (Error)	EC	$C_2 = v_{16}$	0	0	0		0	0	0
	A_6	0_{16}	0 or 6.7 v	6.7 v	6.7 v		6.7 v	6.7 v	6.7 v
		0_{26}	0 or −6.7 v	−6.7 v	−6.7 v		−6.7 v	−6.7 v	−6.7 v
	ABS	$O_A = \|0_{16}\|$	0 or 6 v	6 v	6 v		6 v	6 v	6 v
	S	0	0	0	0		0	6 v	0
	Counter	0	0	0	0		0	1	0

Table VII-14: Operation of the Coincidence Circuit and the Counter (One Pulse Sent)

		0 – .6T	.6T – .7T	.7T – .8T	.8T – $t_{(1)}^*$	$t_{(1)}^* = .95T$	.95T – T	T – 1.15T	$1.15T < t_{(1)} < 1.2T$	1.2T – 1.3T	1.3T – 1.4T	1.4T – 1.6T
Main Operation			Reset		Sampling		Holding		Decision		Error detect.	
Filtered Signal $0_{12} - 0_{22} = s(t)$						$s_{(1)}^*$			$s_{(1)}$			
Sampler-holder, O_{SH}					$s(t)$	$s_{(1)}^*$			$\rho(*)\, s_{(1)}^*$			
Threshold Level $0_{23} - 0_{13} = D_A(t)$									$A/2 + \rho(*) s_{(1)}^*$			
(1) One detected	Unlatch of EC by 0_{25}		Unlatch						Unlatch			
	Output of EC, C_2	0 or 5 v	0	0	0		0	0	5 v	5 v	5 v	5 v
	Coincidence Circuit, O_s		0	0	0		0	0	0	0	0	0
	Counter										0	
(2) Zero detected (error)	Unlatch of EC by 0_{25}		Unlatch						Unlatch	Unlatch		
	Output of EC, C_2	0 or 5 v	0	0	0		0	0	0	0	0	0
	Coincidence Circuit, O_s	0	0	0	0		0	0	0	0	5 v	0
	Counter										1	

Table VII-15: Condensed Results of the Detection of One Pulse Mixed with Noise, Using an Adaptive Threshold the output of A2 is the sum of the filtered noise and of the small residual voltage due to the previously detected pulses. The decision circuit and the reset operate exactly as in paragraphs VII-4b-2 and VII-4b-3.

The operation of the coincidence circuit and of the counter is explained in paragraph VII-2f for any possible case and is shown in Table VII-16 for the case of no pulse sent. Since no pulse is sent, u_{16} is zero for $.6\ T < t < 1.6\ T$ and in particular during the coincidence check ($1.3\ T < t < 1.4\ T$).

VII-4d Detection of One Pulse Mixed with White Noise, Using a Constant Threshold Detector

One pulse is emitted by the random gate during the interval of time $T < t < 1.2\ T$; the detector either detects a pulse, which is a correct detection, or detects a zero, which is an error of type 2.

The switch SW of the block diagram of Fig. VII-1 is on the position "constant threshold" (Ct); therefore, by SW1 the output of the sampler holder is not used and by SW2 the time constant of the RC network is reduced to $RC = RC_1$.

The operations of the clock circuit, the generator of noisy signal, the control circuit, the coincidence circuit and the counter are identical for constant or for adaptive threshold and are described in paragraphs VII-4b.

In the case of constant threshold, the input u_{13} of A3 is equal to zero; therefore, (1) the amplifier, A4, compares $D_c(t) = A/2$ to $s(t)$ and (2) the comparator, EC, detects the output of A4 at time $t_{(1)}$. Together, A4 and EC compare $D_c = A/2$ to $s(t = t_{(1)}) = s_{(1)}$.

In conclusion, the detections of one pulse mixed with noise using (1) an adaptive threshold and (2) a constant threshold, are identical except for two points (see Table VII-17):

a) The sampler-holder is not used in the case of constant threshold detection,

b) In the case of constant threshold, the decision circuit compares $D_c = A/2$ to $s_{(1)}$, while in the case of adaptive threshold it compares $D_A = A/2 + \rho^{(*)} s_{(1)}^{*}$ to $s_{(1)}$.

	Components	Outputs		.6T	1.15T < $t_{(1)}$		< 1.2T	1.3T	1.4T	1.6T
	BS	$O_{BS} = u_{16}$	0 or 5 v	0	0		0	0	0
	P5	$O_{P5} = C_S$	0	0	0		0	6 v	0
One detected (Error)	EC	$C_2 = v_{16}$	0	0	0	5 v	5 v	5 v	5 v
	A_6	0_{16}	0 or -6.7v	0	0	- 6.7v	-6.7v	-6.7v	-6.7v
		0_{26}	0 or +6.7v	0	0	+ 6.7v	+6.7v	+6.7v	+6.7v
	ABS	$O_A = \lvert 0_{16} \rvert$	0 or + 6.7v	0	0	+ 6v	+ 6v	+ 6v	+ 6v
	S	O_s	0	0	0	0	0	+ 6v	0
	Counter	0	0	0	0	0	0	1	0
Zero detected	EC	$C2 = v_{16}$			0		0	0	0
	A_6	0_{16}	0 or - 6.7v	0	0		0	0	0
		0_{26}	0 or + 6.7v	0	0		0	0	0
	ABS	$O_A = \lvert 0_{16} \rvert$	0 or + 6 v	0	0		0	0	0
	S	0	0	0	0		0	0	0
	Counter	0	0	0	0		0	0	0

Table VII-16: Operation of the Coincidence Circuit and the Counter

(No Pulse Sent)

sw	Adaptive Threshold	Constant Threshold
	"Adp"	"Ct"
Time constant of filter	$RC = R(C_1 + C_2)$	$RC = RC_1$
u_{13} at $t_{(1)}$	$\rho^{(*)} s_{(1)}^*$	0
Output of A_3 at $t_{(1)}$ O_{13} to O_{23}	$D_A = A/2 + \rho^{(*)} s_{(1)}^*$	$D_c = A/2$
Output of EC at $t_{(1)}$	(1) If $D_A < s_{(1)}$ $C_1 = +5v$ $C_2 = 0$ Error of the second type (2) If $D_A > s_{(1)}$ $C_1 = 0$ $C_2 = +5v$ Correct detection	(1) If $D_c < s_{(1)}$ $C_1 = +5v$ $C_2 = 0$ Error of the second type (2) If $D_c > s_{(1)}$ $C_1 = 0$ $C_2 = +5v$ Correct detection

Table VII-17 Comparison Between Adaptive and Constant Threshold During the Interval of Detection

VII-4e Detection of no Pulse Mixed with White Noise, Using a RC Filter and a Constant Threshold Detector.

The switch, SW, of Fig. VII-1 is on the position "Constant Threshold" (Ct); therefore, $u_{13} = 0$ and the sampler-holder is not used. By hypothesis, the output of the binary source is zero during the interval .6 T to 1.6 T. It follows: (1) the random gate is an open switch, the output of A1 is just the white noise and the output of A2 is the sum of the filtered noise and of the residual voltage due to the previously detected pulses; (2) the threshold detector compares $s_{(1)}$ to $D_c = A/2$ as in paragraph VII-4d; (3) the coincidence circuit compares a zero coming from the binary source to the detected signal coming from the comparator (C_2) during the interval $1.3\ T < t < 1.4\ T$. If a pulse is detected, which corresponds to an error of the first type, (1) $C_2 = 5$ v for $t_{(1)} < t < 1.6\ T$, and (2) $O_5 = +4.5$ v for $1.3\ T < t < 1.4\ T$ which activates the counter.

VII-5 Conclusion of Chapter VII

The experimental block diagram of Fig. VII-1 consists of functional blocks: clock system, noisy signal generator, receiver, and error detector. The operation of the block diagram is explained in three steps by considering (1) the basic components, (2) the functional blocks and (3) the complete diagram. It was shown that except for minor differences, Fig. VII-1 simulates the theoretical block diagram of Fig. II-2 as desired.

In chapter VIII, numerical values are assigned to the pseudo-period and the RC filter, and the block diagram is tested.

CHAPTER VIII

EXPERIMENTAL RESULTS

VIII-1 Introduction

Chapter VIII is the continuation of Chapter VII; the purpose of Chapter VIII is twofold: (1) numerical values are assigned to the following: the pseudo-period, the time constant of the filter and of the sampler-holder, the amplitude V, the gain of the amplifiers, the threshold and the fictitious signal to noise ratio; (2) the experimental average probabilities of error for constant and adaptive threshold are determined and compared with the theoretical results of Chapter IV.

VIII-2 Choice of the Pseudo-Period, T, and of the Normalized Width of the Pulse, K.

Although most telemetry systems operate at rather high frequency, they can be simulated at lower frequency using time scaling. The pseudo-period, T, can be determined by three considerations: (a) if T is short the design of the components is more difficult, (b) if T is long the experiment is very lengthy and (c) the bandwidth of the noise generator must be small for a truly normal distribution but still large compared to 1/T in order to simulate white noise; as a compromise, T is taken equal to 500 μs.

It was shown in Chapter IV that the average probability of error for constant or adaptive threshold is minimum for K minimum; however, K would be at least .1 or .2 for a practical system. In Chapter VIII as in Chapter VII, K is taken equal to .2. The pulse width is then: KT = 100 μs.

VIII-3 Choice of the Time Constant of the Filter (RC) and of the Sampler-Holder (R'C')

The time constant of the RC filter, which is determined by the product $Ky = \frac{KT}{RC}$, is chosen to minimize the average probability of error. For constant threshold and K = .2, the optimum value of Ky is Ky = 1.25 by Fig. IV-3. For adaptive threshold the average probability of error is minimum for the minimum Ky (see Paragraph IV-9), for realizability Ky is taken equal to .312.

The values of the time constant for constant and adaptive thresholds follow:

(1) Constant Threshold $RC = RC_1$, $Ky = 1.25$, $y = \frac{T}{RC} = 6.25$

$$RC_1 = \frac{KT}{Ky} = \frac{.2(500)}{1.25} = 80\ \mu s \qquad \text{(VIII-1)}$$

$$R = 1600\,\omega \quad , \quad C_1 = .05\ \mu f \qquad \text{(VIII-2)}$$

(2) Adaptive Threshold $RC = R(C_1 + C_2)$, $Ky = .312$, $y = \frac{T}{RC} = 1.56$

$$R(C_1 + C_2) = \frac{.2(500)}{.312} = 320\ \mu s \qquad \text{(VIII-3)}$$

$$R = 1600\,\omega, \ C_1 + C_2 = .2\,\mu f\ , \ C_2 = .15\,\mu f \qquad \text{(VIII-4)}$$

The variation of capacity from $C_1 = .05\mu f$ to $C_1 + C_2 = .2\mu f$ is controlled by the pole SW2 of the switch SW which connects $C_2 = .15\mu f$ in parallel with C_1.

The time constant R'C' of the sampler-holder is equal to the time constant of the RC network by paragraph IV-5; therefore, R'C' = 320μs: R' = 1600ω C' = .20 μf.

VIII-4 Choice of the Amplitude V of the Pulse and of the Gain of A1 and A2

Since the amplitude V of the rectangular pulse must be small compared to the maximum instantaneous amplitude of the noise but must be large for accuracy, V is taken equal to 1.65 volts.

In Chapter IV, $s_i(t) = n_i(t) + p_i(t)$ and $s(t)$ is $s_i(t)$ after RC filtering; this corresponds to a gain one for both A1 and A2. The threshold detection which compares the filtered signal to the threshold level is unchanged if both the filtered signal and the threshold level are multiplied by the same factor; this means that the gains of the amplifier A1 and A2 are arbitrary.

The gains of A1 and A2 are chosen small for linear operation of amplifiers, even when the noise has a large amplitude. The double gain of A1 is equal to one. The double gain of A2 is about 3.4 and is adjusted for a peak amplitude A = 4 volts when the noise is null. The output of A2 (0_{12} to 0_{22}) is called the filtered noise and denoted by s(t); because of the symmetry the voltage between 0_{12} and the ground is $s(t)/2$.

VIII-5 Choice of the Constant and of the Adaptive Threshold Levels

The constant threshold level, D_C, which is the output of the amplifier A3 when the switch SW is on "Ct", was defined in Chapter VII as equal to A/2 where A is the peak amplitude of a single filtered pulse without noise. Since the gain of A2 is chosen to make A=4v, it follows that $D_C = 2$ v.

The adaptive threshold level, $D_A(t)$, which is the output of the amplifier A3 when the switch SW is on "Adp", is defined by formula (VII-15) at the time $t_{(1)}$ of detection:

$$D_A = A/2 + s_{(1)}^* \, e^{-(t_{(1)} - t_{(1)}^*)/RC} \qquad \text{(VIII-5)}$$

A is proportional to $(1 - e^{-Ky})$, therefore, for adaptive threshold

$A = 4 \dfrac{(1 - e^{-.312})}{(1 - e^{-.25})} = 1.495$v. Substituting 1.495 v for A, $t_{(1)}^* = .95\,T$, $t_{(1)} \approx 1.17\,T$ and $y\,T/RC = 1.56$ gives

$$D_A = .748 + s_{(1)}^* \, e^{-(.22)(1.56)} = .748 + .709\, s_{(1)}^* \qquad \text{(VIII-6)}$$

Since the sampler-holder samples only one for the two sides of the amplifier A3, the output of the sampler is $\dfrac{s_{(1)}^*}{2} \, e^{\frac{-(t - t_{(1)}^*)}{RC}}$ for $t > t_{(1)}^*$;

therefore, (1) the gain of the amplifier A3 must be exactly two and (2) the input u_{23} of A3 must be equal to $A/4 = .374$v.

VIII-6 Adjustment of the Fictitious Signal to Noise Ratio Q

The numerical values of the fictitious signal to noise ratio Q are chosen exactly as in Paragraph IV-6 so that the theoretical and experimental results can be compared: Q = 15 and 25. Since,

$$Q = \frac{P_s T}{\eta} = \frac{V^2 KT}{2\eta}$$

by Paragraph IV-1 and that V, T and K are held constant, Q is determined by adjusting the white noise generator. When,

$$Q = 15, \ \eta = \frac{V^2 KT}{2Q} = \frac{(1.65)^2 (.2) 500 \ 10^{-6}}{30} = 9.1 \ 10^{-6} \text{ volts}^2/\text{ cycle}$$

When $Q = 25$, $\eta = 5.44 \ 10^{-6}$ volts^2/cycle

The peak amplitude of the output of A2 is $A = G_T V(1 - e^{-Ky})$ where G_T is the overall gain of the amplifiers A1 and A2. Substituting the numerical values for A, V, K and y, yields $G_T = \frac{4}{(1.65)(.714)} = 3.40$

The mean square power of the noise at the output of A2 is $\sigma_e^2 = G_T^2 \sigma^2$, where σ^2 is the variance of the noise when $G_T = 1$ and is given by formula IV-12

$$\sigma_e^2 = G_T^2 \sigma^2 = G_T^2 \frac{\eta}{4RC} = .361 \ 10^5 \eta \text{ volts}^2$$

Instead of measuring the power density, η, of the noise, it is more practical to measure the rms value of the noise σ_e at the output of A2.

$Q = 15$, $\eta = 9.1 \ 10^{-6}$ volts^2/cycle , $\sigma_e^2 = .328$ volts^2 , $\sigma_e = .573$ volt

$Q = 25$, $\eta = 5.44 \ 10^{-6}$ volts^2/cycle , $\sigma_e^2 = .196$ volts^2 , $\sigma_e = .443$ volt

VIII-7 Error for Noise with Limited Bandwidth

Since the average probabilities of error are rather small, the normal curve must be correct up to 6σ at least; therefore, noise with limited bandwidth and true normal curve was preferred to noise with large bandwidth and truncated normal curve to approximate the white normal noise. The variance for a limited bandwidth is smaller than for an infinite bandwidth; however, the error is small when the bandwidth is large with respect to 1/RC.

The power density spectrum at the input of A1 is G_i and the power density spectrum at the input of A2 is G_f. The bandwidth of the noise is 0 to f_c, then:

$$G_i \begin{cases} = \eta/2 & \text{for } 0 < f < 22 \text{ Kc} \\ = 0 & \text{otherwise} \end{cases}$$

$$G_f \begin{cases} = \dfrac{G_T^{\,2} \quad \eta/2}{1 + (2\,fRC)} & \text{for } 0 < f < 22 \text{ Kc} \\ = 0 & \text{otherwise} \end{cases}$$

The variance at the output of A2 is

$$\sigma_e^{\,2} = 2\,G_T^{\,2} \int_0^{f_c} G_f \; df$$

$$= \quad G_T^{\,2} \frac{\eta}{2\pi RC} \tan^{-1} (2\pi\, RC\, f_c)$$

When $f_c = \infty$, $\tan^{-1}(2\pi RC\, f_c) = \tan^{-1}(\infty) = \pi/2$; When $f_c = 22$ Kc,

$\tan^{-1}(2\pi\, RC\, f_c) = \tan^{-1}(11.2) = 1.49$. The error on the variance due to the limited bandwidth is in per cent

$$\frac{(\pi/2 - 1.49)}{\pi/2}\ 100 = 5\%$$

However, the bandwidth is not limited sharply at 22 Kc, so that the error is even less.

VIII-8 Photographs of the Signals and of the Threshold

The operation of the experimental set up is observed on the oscilloscope and is recorded using a polaroid camera. The oscilloscope is triggered by the pilot clock at time $t = .6T$. The random rectangular pulse is received (or not received) in the interval of time $T < t < 1.2T$. The photographs are made for a fictitious signal to noise ratio $Q = 15$.

The three cases: (1) pulse without noise, (2) noise without pulse and (3) pulse plus noise are considered successively: (1) Pulse without noise: in this case $s_i(t)$ is a rectangular pulse and $s(t)$ is the filtered pulse. Photo VIII-1 shows the input signal $s_i(t) = p_i(t)$ (output of A1) versus time; Photo VIII-1b shows the filtered signal $s(t) = p(t)$ (output of A2) versus time for constant threshold; Photo VIII-1c shows the filtered signal $s(t) = p(t)$ (output of A2) versus time for adaptive threshold. (2) Noise without pulse: Photo VIII-2 is for constant threshold (RC = 80 μs). Photo VIII-2a shows the noise before filtering, $s_i(t) = n_i(t)$, versus time. Photo VIII-2b shows the noise after filtering, $s(t) = n(t)$, versus time. Photo VIII-3 is for adaptive threshold (RC = 320 μs). Photo VIII-3a shows the noise before filtering versus time and would be identical to Photo VIII-2a if the noise was random. Photo VIII-3b shows the noise after filtering, $s(t) = n(t)$, versus time. (3) Pulse plus noise: Photo VIII-4 is for constant threshold. Photo VIII-4a shows the rectangular noisy pulse, $s_i(t) = n_i(t) + p_i(t)$, before filtering. Photo VIII-4b shows the noisy pulse after filtering, $s(t) = n(t) + p(t)$. Photo VIII-5 is for adaptive threshold. Photo VIII-5a shows the rectangular pulse, $s_i(t) = n_i(t) + p_i(t)$, before filtering and is statistically identical to photo VIII-4a. Photo VIII-5b shows the noisy pulse after filtering, $s(t) = n(t) + p(t)$.

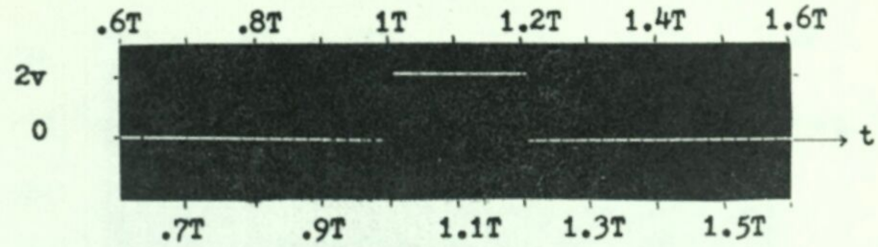


Photo VIII-1a: Input random rectangular pulse, $p_i(t)$, versus time
Scales: vertical, 1 sq = 2 v; horizontal, 1 sq = 50 μs

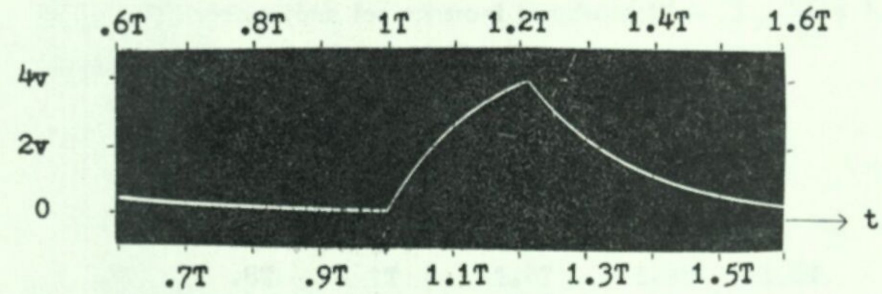


Photo VIII-1b: RC filtered random pulse, p(t), for constant threshold (K = .2, KT/RC = 1.25)
Scales: vertical, 1 sq = 2 v; horizontal, 1 sq = 50 μs

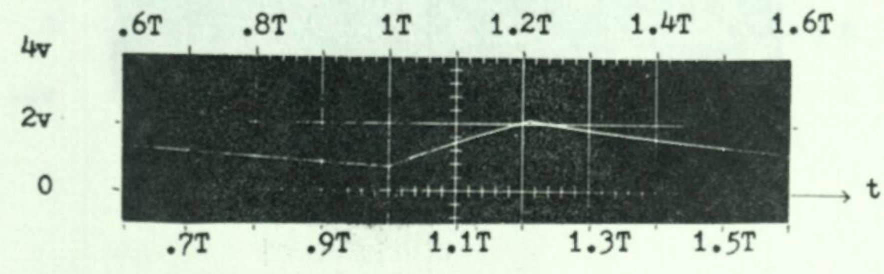


Photo VIII-1c: RC filtered random pulse, p(t), for adaptive threshold (K = .2. KT/RC = .312)
Scales: vertical, 1 sq = 2 v; horizontal, 1 sq = 50 μs

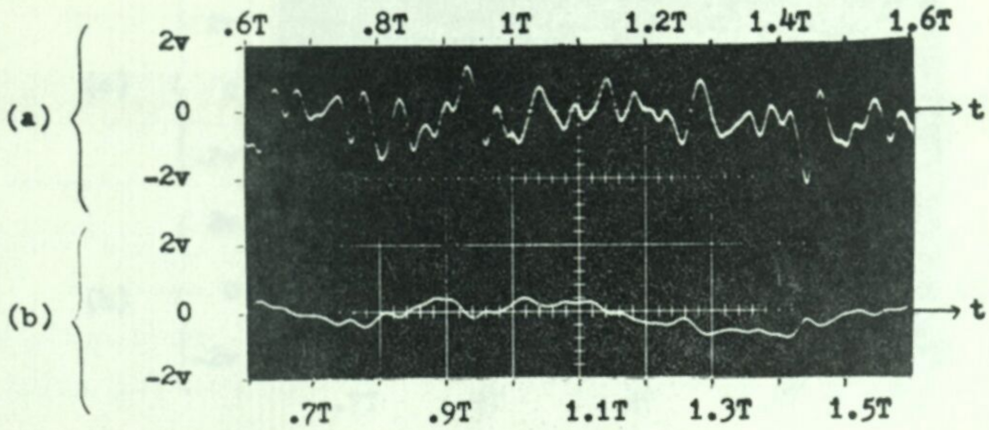


Photo VIII-2: (a) Input random noise, $n_i(t)$, versus time, (b) RC filtered random noise, n(t), versus time for constant threshold ($K = .2$, $\frac{KT}{RC} = 1.25$)
Scales: vertical, 1 sq = 2v; horizontal, 1 sq = 50μs

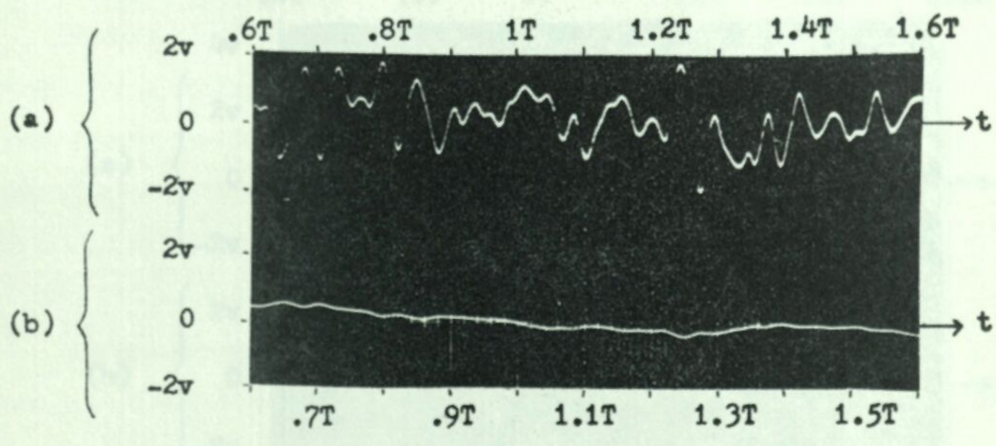


Photo VIII-3: (a) Input random noise, $n_i(t)$, versus time, (b) RC filtered random noise, n(t), versus time for adaptive threshold ($K = .2$, $\frac{KT}{RC} = .312$)
Scales: vertical, 1 sq = 2 v; horizontal, 1 sq = 50μs

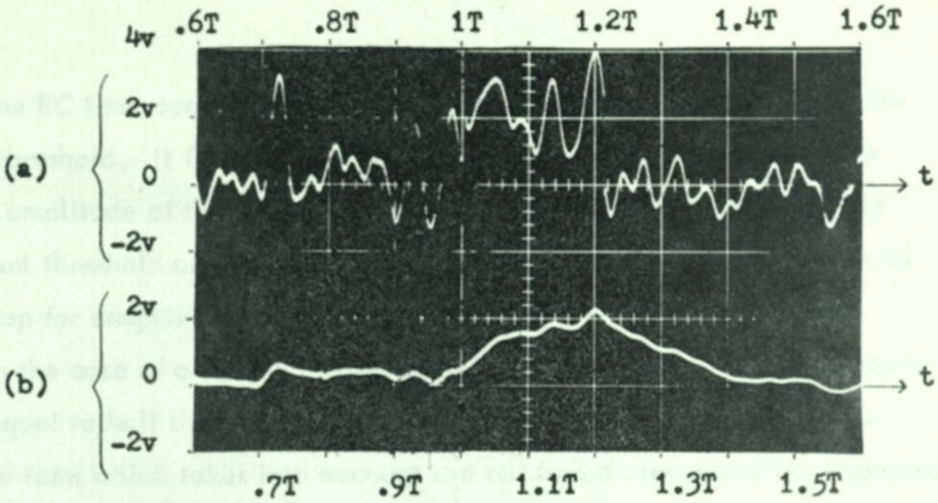


Photo VIII-4: (a) Pulse plus noise before filtering, $s_i(t)$, (b) pulse plus noise after filtering, s(t), for constant threshold (K = .2, $\frac{KT}{RC} = 1.25$)

Scales: vertical, 1 sq = 2v; horizontal, 1 sq = 50 μs

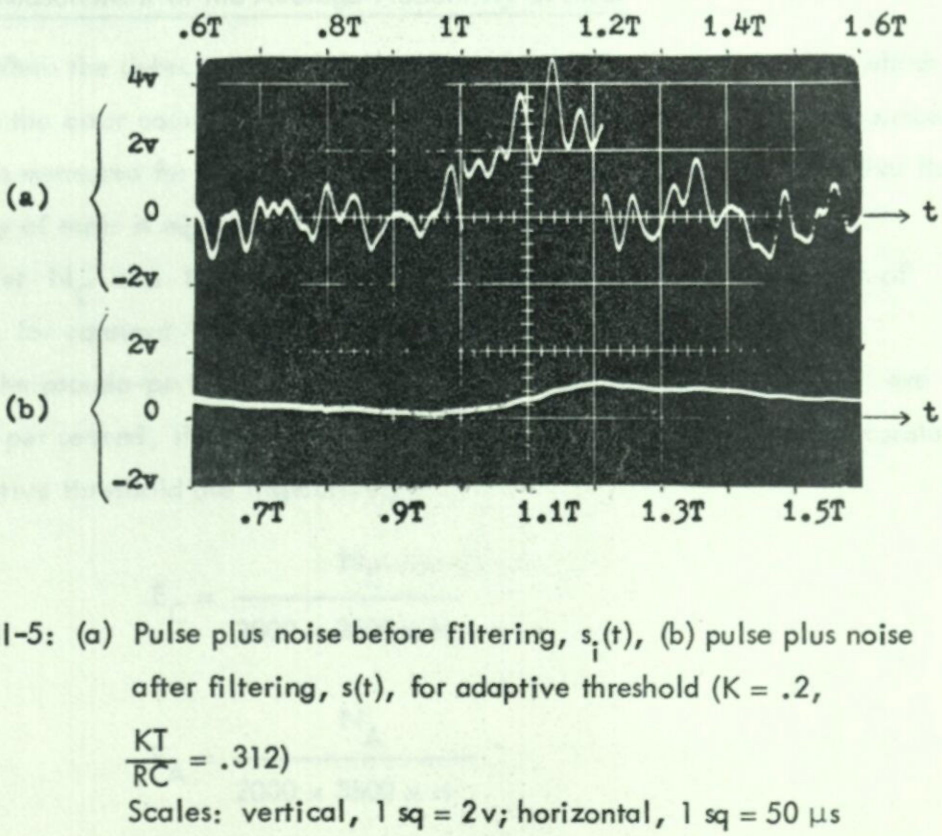


Photo VIII-5: (a) Pulse plus noise before filtering, $s_i(t)$, (b) pulse plus noise after filtering, s(t), for adaptive threshold (K = .2, $\frac{KT}{RC} = .312$)

Scales: vertical, 1 sq = 2v; horizontal, 1 sq = 50 μs

The RC time constant of the filter is much larger for adaptive than for constant threshold. It follows: (1) both the rms value of the noise and the maximum amplitude of the pulse after filtering are smaller for adaptive than for constant threshold and (2) the pulses do not overlap for constant threshold and overlap for adaptive threshold.

In the case of adaptive threshold the threshold is the sum of a constant voltage equal to half the peak amplitude of a single filtered pulse and of a corrective term which takes into account the residual dc level and the expected value of the noise. The corrective term is obtained at the output of the sampler-holder which performs as shown in Photo VIII-6. Photo VIII-6a shows the filtered signal plus noise s(t) versus time. Photo VIII-6b shows the output of the sampler-holder versus time, i.e. $0_{SH}(t)$ of Fig. VII-1. $0_{SH}(t)$ follows s(t) during the sampling period $.8T < t < .95T$, and then decays exponentially during the holding period $.95T < t$.

VIII-9 Measurement of the Average Probability of Error

When the detected signal differs from the received signal a pulse which activates the error counter appears on the output of the sampler, S. The number of error is measured for a time sufficiently long (several hours) to assume that the frequency of error is equivalent to the average probability of error.

Let N_C and N_A be the number of errors occuring for a duration of H hours, for constant and adaptive threshold respectively.

The pseudo-period is $T = 500\,\mu s$ which means that 2000 "0" or "1" are received per second, the experimental average probabilities of errors for constant and adaptive threshold are respectively:

$$E_C = \frac{N_C}{2000 \times 3600 \times H}$$

and

$$E_A = \frac{N_A}{2000 \times 3600 \times H}$$

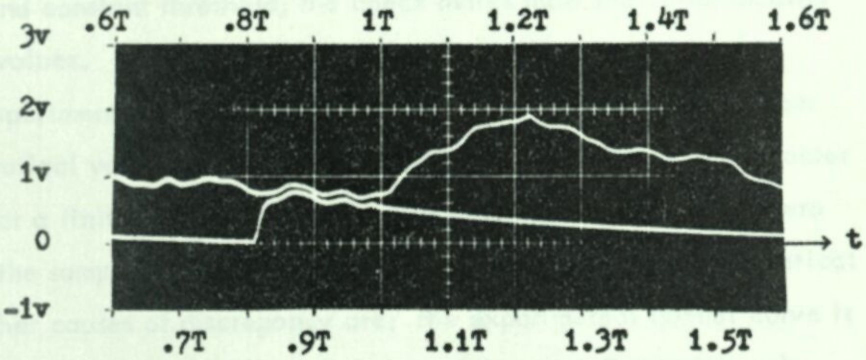


Photo VIII-6 Operation of the sampler-holder for a pulse mixed with noise: the upper curve is the filtered signal, the lower curve is the output of the sampler.

Scales: vertical, 1 sq = 1 v; horizontal, 1 sq = 50 μs

Since the error of the first and the second types are equal, the errors are practically the same whether only "1" or only "0" are received.

VIII-10 Comparison Between Experimental and Theoretical Results

The experimental and the theoretical average probabilities of error are compared in Fig. VIII-7. The continuous line shows the theoretical values (minus the logarithm of the average probability of error) versus Q, for adaptive and constant threshold; the check marks show the corresponding experimental values.

The experimental average probabilities of error are somewhat larger than the theoretical values. The main reasons are: (1) the threshold detector is unlatched for a finite interval of time rather than for the theoretical zero duration, (2) the sampling terminates at .95T instead of T for the theoretical optimum. Other causes of discrepancy are: the experimental normal curve is not exact for large voltages, slight unbalance of the amplifier which makes the error of the first and second type unequal, small drift, small internal noise of the amplifier. The experiment is especially difficult for large signal to noise ratio when the average probability of error becomes extremely small. More work will be done to make the measurements more accurate.

In conclusion the experimental results confirm the theoretical results. The experimental average probabilities of error, which are somewhat larger than the theoretical average probabilities of error (which is justified), are still much smaller for adaptive than for constant threshold as was true in the theoretical case.

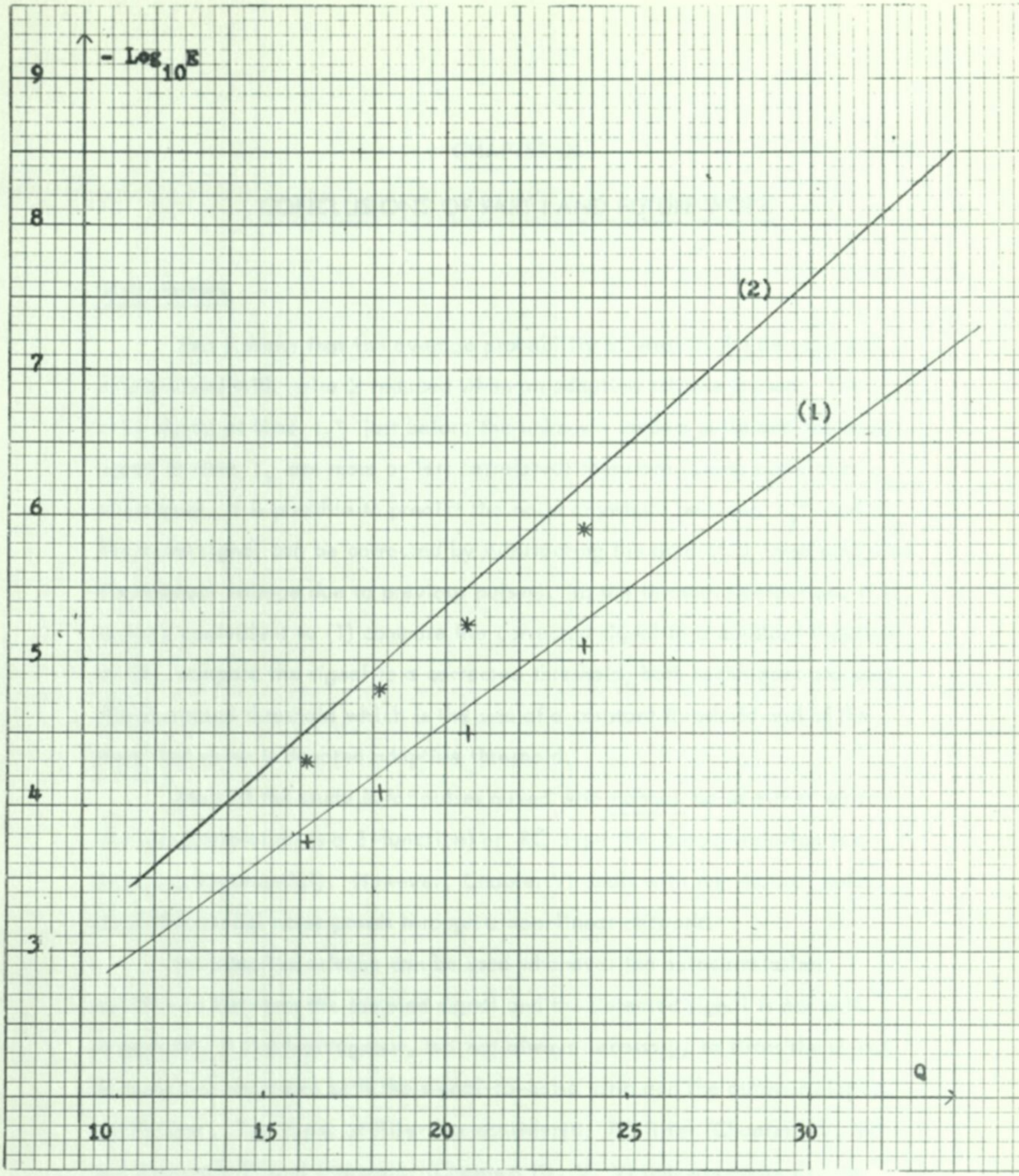


Fig. VIII-7 : Comparison Between Experimental and Theoretical Results.
Theoretical : (1) Constant Threshold, (2) Adaptive Threshold
Experimental : + Constant Threshold, * Adaptive Threshold

CHAPTER IX

CONCLUSIONS OF PART I AND FUTURE WORK

IX-1 Review

Part I shows that the average probability of error in the detection of pulses mixed with noise is reduced when an adaptive scheme is used. The noisy signal at time $t_1{}^*$ just before the unknown random pulse and the noisy signal at time t_1 are more or less correlated; therefore, using correlation techniques, it is possible to predict the noise anywhere between $t_1{}^*$ and t_1. The noisy signal can be corrected by substracting the predicted noise (and also the residual voltage due to previous pulses). In the threshold detection, it is exactly equivalent to (1) compare the corrected signal to a constant threshold or (2) compare the signal to a corrected threshold; both techniques are used. The corrected threshold which is the sum of a constant threshold and of the predicted noise is denoted adaptive threshold.

Most of the study is for pulses mixed with normal noise either white (the autocorrelation is a δ-function) or RC type (the autocorrelation is a decaying exponential). However, Chapter VI deals with non-normal noise, since the normal noise passes through a non-linear network.

An adaptive scheme can be used whenever the autocorrelation coefficient between the sampled signal and the detected signal is not null. In general, it is advantageous, but not always necessary, to filter the signal because this increases both the signal-to-noise ratio and the autocorrelation coefficient. Two types of filter are considered: RC filter (Chapters IV, VII and Appendix E) and integrator (Chapters III, V, VI and Appendix E). In Chapter IV, an RC network is used to filter the pulses mixed with white noise. The analysis, which is performed for constant threshold and for adaptive threshold, determines both the width of the pulse and the RC network which minimize the average probability of error for a given fictitious signal-to-noise ratio. The minimum average probability of error corresponds to the maximum signal-to-noise ratio only when the process is normal. Instead of (1) expressing the effect of the adaptive threshold as an increase of the signal-to-noise ratio and (2) maximizing the signal-to-noise ratio, the optimization is performed directly on the average probability of error since this is a more general technique. In Chapters V and VI, an integrator is used. The output of the integrator is the integral of the input only during the interval where a pulse might be present and it is zero otherwise. The integrator is an active network which increases the signal to noise ratio more than a passive network. When the autocorrelation function of the noise is not a δ-function (i.e. except for white noise), the noisy signal can be corrected using an adaptive integrator where the predicted value of the noise is subtracted continuously from the noisy signal before the integration takes place. The adaptive integrator can be used after a linear network (Chapters III and V) or after a non-linear network (Chapter VI). An adaptive threshold cannot be used behind the integrator because the output voltage before detection is always zero and is not correlated to the noise. The standard integrator and the adaptive integrator are compared with the RC filter in Appendix E.

IX-2 Conclusions

In Chapters III, IV, V, VI and Appendix E, a train of rectangular pulses mixed with noise is detected; the noise is either white and normal, or RC type and normal.

For white normal noise, the following cases are considered: (1) RC filter and constant threshold; (2) RC filter and adaptive threshold and (3) integrator and constant threshold. The optimum designs (minimum average probability of error) are (a) for constant threshold: $K = .1$, $RC = .08\ T$, (b) for adaptive threshold: $K = .1$, $RC = .32\ T$, $D_A = D_C + \rho^* s_1^*$, (c) for the integrator $s(t) = \frac{1}{KT} \int_T^{T+KT} s_i(t)\, dt$. The logarithm of the average probabilities of error versus the fictitious signal to noise ratio are shown in Fig. E-1, for the optimum choice of K, RC, D_A and the integrator. The average probabilities of error for adaptive threshold and integrator are equal and are smaller than those for constant threshold; the reduction in average probability of error, expressed in decibels, increases linearly with Q from 12 db for Q = 15 to 28 db for Q = 35.

For RC type normal noise, the following cases are considered: (1) no filter and constant threshold, (2) no filter and adaptive threshold, (3) integrator and constant threshold, (4) adaptive integrator and constant threshold, (5) RC filter and constant threshold, (6) RC filter and adaptive threshold. The effect of the adaptive threshold is, as before, a reduction of the variance by $\sqrt{1 - \rho^{*2}}$. The adaptive integrator subtracts continuously the predicted value of the noise before the integration takes place. The three cases: RC noise and no filter, RC noise and standard integrator, RC noise and adaptive integrator are compared in Fig. V-2. The adaptive integration is especially advantageous for large autocorrelation of the RC noise ($K\beta$ small). For example: if $K\beta = .5$, the signal to noise ratio for adaptive integrator is 12.4 db better than without integrator and 11 db better than with standard integrator; if $K\beta = 2$, the signal to noise ratio for adaptive integrator is 8.4 db better than without integrator and 3.6 db better than with standard integrator. At the limit when β becomes ∞, the RC noise becomes white noise, the autocorrelation becomes an impulse and nothing can be gained by an adaptive integrator. Instead of an integrator, an RC filter can be used with either a constant threshold or an adaptive threshold. For example, if $K\beta = 2$, the optimum RC filter is defined by $Ky = 1.4$ for constant threshold and by $Ky = 0.8$ for adaptive threshold. The increase in signal to noise ratio in decibels for $K\beta = 2$ are: (1) for RC filter and constant threshold 2.88 db (Appendix E), (2) for RC filter and adaptive threshold 5.6 db (Appendix E), (3) for standard integrator and constant threshold 4.9db (Fig. V-2), (4) for adaptive integrator and constant threshold 8.4 db (Fig. V-2). The reduction of the average probability of error by use of an adaptive threshold without filtering is shown by Fig. III-3; the reduction of the average probability of error by use of a standard or an adaptive integrator and constant threshold is shown by Fig. V-4; the reduction of the average probability of error by use of the optimum RC filter and constant or adaptive threshold is shown by Fig. E-3. The six systems investigated are listed in order of decreasing average probability of error: (a) no filter and constant threshold, (b) RC filter and constant threshold, (c) no filter and adaptive threshold, (d) standard integrator and constant threshold, (e) RC filter and adaptive threshold (f) adaptive integrator and constant threshold. The advantage of an adaptive integrator increases with the signal to noise ratio; for example, if $K\beta = 2$, the average probability of error decreases by 32 db for $V/2\sigma_i = 2$ and by 90 db for $V/2\sigma_i = 3.5$.

In Chapter VI, the use of an integrator on a nonlinear system is investigated. The signal to noise ratio is considerably increased by an adaptive integrator; the effect on the average probability of error is not straightforward since the probability distribution is not normal.

Chapters VII and VIII are experimental check of Chapter IV. Precise timing is obtained by using a dekatron counter and the errors are detected by comparing the true signal to the detected signal.

IX-3 Future Work

Research efforts to date were mainly concerned with the minimization of the average probability of error in telemetry using adaptive network and linear circuits. The future work will be oriented towards (1) application of adaptive detection on radar, and (2) use of non-linear networks in telemetry and radar.

In telemetry, the unknown signal is located in a known interval of time so that sampling can be used to determine the noise before detection and the threshold varied accordingly. This technique is not applicable in radar detection because the signal can appear everywhere and any sample would be a mixture of signal and noise. In order to transform the problem of telemetry in a problem of

APPENDIX A

AVERAGING THE CONDITIONAL PROBABILITY OF ERROR IN THE THRESHOLD DETECTION OF PULSES

A train of random rectangular pulses mixed with normal noise is filtered through a network F. The output, $s(t)$, of the filter at time $t = T + KT$ ($S(t = T + KT) = S_1$) is detected with a threshold detector of threshold level D. If the time constant of the filter F is large, the pulses overlap. Thus, the amplitude of s_1 depends upon whether the previous signal contained a pulse or not; in other words, s_1 depends upon $s(KT) = s_0$. More generally, the amplitude of the unknown signal, s_1, is often related to the amplitude of a known previous signal, $s_1{}^*$; therefore, the average probability of error in the threshold detection is a function of $s_1{}^*$ and it is an average conditional probability of error denoted by $E(s_1{}^*)$ and

$$E(s_1 / s_1{}^*) = \int_{g(s_1{}^*)}^{\infty} e^{-x^2/2}\, dx = I(g(s_1{}^*)) \qquad \text{where}$$

$g(s_1{}^*)$ is a function of $s_1{}^*$. The average probability of error in the detection of s_1 for any previous $s_1{}^*$ is obtained by averaging $E(s_1{}^*)$ with respect to $s_1{}^*$

$$E(s_1) = \int_{-\infty}^{+\infty} f(s_1{}^*)\; E(s_1{}^*)\; ds_1$$

where $f(s_1{}^*)$ is the probability density of $s_1{}^*$. $E(s_1)$ is a double integral, fortunately when $g(s_1{}^*)$ is linear, the double integral can be condensed in a simple integral using the following formula:

$$\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\,\sigma}\; e^{-v^2/2\sigma^2}\; I\left[a + b\,\frac{v}{\sigma}\right] dv = I\left[\frac{a}{\sqrt{1 + b^2}}\right]$$

which is proved next.

$$\text{Let } J = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-v^2/2\sigma^2} I\left[a + \frac{bv}{\sigma}\right] dv = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-v^2/2\sigma^2} \int_{a+\frac{bv}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt\, dv$$

$$J = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-v^2/2\sigma^2} \left(.5 - \int_{0}^{a+\frac{bv}{\sigma}} e^{-t^2/2}\, dt \right) dv$$

$$J = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-v^2/2\sigma^2} \left(.5 - \int_{0}^{\frac{bv}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt - \int_{\frac{bv}{\sigma}}^{a+\frac{bv}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt \right) dv$$

$$J = .5 - J_1 - J_2$$

$$\text{where } J_1 = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-v^2/2\sigma^2} \int_{0}^{\frac{bv}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt\, dv$$

$$\text{and } J_2 = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-v^2/2\sigma^2} \int_{\frac{bv}{\sigma}}^{a+\frac{bv}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt\, dv$$

$J_1 = 0$ because $\int_{0}^{\frac{bv}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt$ is an odd function of v.

J_2 is simplified after the change of variable

$$y = t - bv/\sigma$$

$$J_2 = \int_{-\infty}^{+\infty} \int_{0}^{a} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-v^2/2\sigma^2} e^{-\frac{\left(y + \frac{bv}{\sigma}\right)^2}{2}}\, dy\, dv$$

$$J_2 = \int_{-\infty}^{+\infty} \int_{0}^{a} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\left(y^2 + (b^2+1)\frac{v^2}{\sigma^2} + 2\frac{byv}{\sigma}\right)}{2}}\, dy\, dv$$

Reversing the order of integration of J_2 yields:

$$J_2 = \int_0^a \frac{1}{\sqrt{2\pi}} e^{\frac{-y^2}{2(b^2+1)}} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-1/2\left(\frac{\sqrt{b^2+1}}{\sigma} v + \frac{by}{\sqrt{b^2+1}}\right)^2} dv \; dy$$

The inside integral is evaluated immediately after the change of variable:

$$w = \frac{\sqrt{b^2+1}}{\sigma} v + \frac{by}{\sqrt{b^2+1}}$$

$$\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-1/2\left(\frac{\sqrt{b^2+1}}{\sigma} v + \frac{by}{\sqrt{b^2+1}}\right)^2} dv = \frac{1}{\sqrt{b^2+1}} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}} dw = \frac{1}{\sqrt{b^2+1}}$$

J_2 has been reduced to a single integral:

$$\int_0^a \frac{1}{\sqrt{2\pi}\,\sqrt{b^2+1}} e^{-\frac{y^2}{2(b^2+1)}} dy$$

Letting $t = y / \sqrt{b^2+1}$, then

$$J_2 = \int_0^{\frac{a}{\sqrt{b^2+1}}} e^{-\frac{t^2}{2}} dt$$

Finally,

$$J = .5 - \int_{o}^{\frac{a}{\sqrt{b^2+1}}} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{t^2}{2}}\, dt = \int_{\frac{a}{\sqrt{b^2+1}}}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{t^2}{2}}\, dt = I\left[\frac{a}{\sqrt{b^2+1}}\right]$$

which was to be proved.

APPENDIX B

COMPUTATION OF THE AVERAGE PROBABILITY OF ERROR OF DETECTION

B-1 Computer Program

The average probability of error in a threshold detection is a linear combination of integrals of the type $I\left[g_i\left(\frac{S_{io}}{N_{io}}, K, y\right)\right]$. For example, the average probability of error for constant threshold E_C is given in paragraph IV-8 for a two step memory:

$$E_C = \sum_{\gamma_{-1}=0,1} \sum_{\gamma_0=0,1} \sum_{\ell=-1,1} \frac{1}{8} I\left[\sqrt{\frac{S_{io}}{N_{io}}}\ \sqrt{8}\ \frac{(1-e^{-Ky})}{\sqrt{Ky}}\left(1+2\ell(\gamma_o e^{-y}+\gamma_{-1} e^{-2y})\right)\right]$$

$g_i\left(\frac{S_{io}}{N_{io}}, y, K\right)$ is written as a product of two functions

$$g_i\left(\frac{S_{io}}{N_{io}}, y, K\right) = h\left(\frac{S_{io}}{N_{io}}, y, K\right) W_i(y)$$

where $h\left(\frac{S_{io}}{N_{io}}, y, K\right) = \sqrt{\frac{S_{io}}{N_{io}}}\ \sqrt{8}\ \frac{1-e^{-Ky}}{Ky}$

and $W_i(y) = 1 + 2\ell(\gamma_o e^{-y} + \gamma_{-1} e^{-2y})$

where ℓ assumes the values 1 and -1, while γ_o and γ_{-1} assume the values 0 and 1. There are 8 possible values for $W_i(y)$, one for each combination of ℓ, γ_o and γ_{-1}.

It is necessary to obtain E_C for a large number of combinations of S_{io} / N_{io}, y, K, because the optimum combination of y and K for a given S_{io} / N_{io} is desired. A Univac 1107 computer is used for all numerical computations.

Given values for S_{io} / N_{io}, u, K, ℓ, γ_o and γ_{-1}, a numerical value is obtained for $g_i(S_{io} / N_{io}, y, K)$.

$I\left[g_i\ (S_{io}/N_{io}\ ,\ y\ ,\ K)\right]$ is then computed as explained in the next section. For a given combination of S_{io}/N_{io} , y and K, there are eight values for $I\left[g_i\ (S_{io}/N_{io}\ ,\ y\ ,\ K)\right]$ and E_C is the average for those eight values.

Tables for significant values of S_{io}/N_{io} , K and y are read in the computer. For given S_{io}/N_{io} , y , and K , the computer varies ℓ , γ_o and γ_{-1} , successively; the value obtained for E_C is typed together with the corresponding S_{io}/N_{io} , K , and y. Then the computer varies y , K and S_{io}/N_{io} successively, resulting in a large number of E_C for each S_{io}/N_{io} , the smaller of which corresponds to the best choice for y and K.

It is necessary to compute many values of E_C in the neighborhood of the minimum of E_C , in order to locate the minimum accurately. Since the minimum of E_C occurs in a small range of $u = Ky$, K is chosen first and then y is varied in the range $1.15\, y_0 < y < .85\, y_0$ where $y_0 = \dfrac{1.1 + K\,1.6}{K}$.

B-2 Computation of I(x)

The integral $I(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}}\, e^{-t^2/2}\, dt$ is expressed in the form

$I(x) = .5 - J(x)$ where $J(x) = \int_o^x \frac{1}{\sqrt{2\pi}}\, e^{-t^2/2}\, dt$. The integral

$J(x) = \int_o^x \frac{1}{\sqrt{2\pi}}\, e^{-t^2/2}\, dt$ is evaluated by Simpson's technique, which is one of the most widely used and simplest methods of numerical integration. This technique is developed and described in detail in most standard calculus and numerical analysis references. Therefore, we will only state the rule and explain how it was incorporated into our computer program.

Simpson's Rule: If f is continuous function in the interval $[a , b]$, if n is an even integer, and if $r_n = [x_o , x_1 , \ldots , x_n]$ is the regular partition of [a , b] into n subintervals then

$$\int_a^b f(x)\,dx = \frac{b-a}{3n}\Big[f(x_o) + 4f(x_1) + 2f(x_2)$$

$$+ 4f(x_3) + 2f(x_4) + \ldots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_2)\Big]$$

For the purposes needed in this work, it is necessary to have the value of the integral

$$\int_a^b \frac{1}{\sqrt{2\pi}}\, e^{-t^2/2}\, dt$$

over each interval [a , b] such that $b - a = .005$ and $0 \le a \le b \le 5.0$. Referring to the formula stated in Simpson's rule, in our case $n = 2$ since our interval [a , b] is quite small and this gives required accuracy. The $f(x_i)$'s needed for evaluating integral are obtained by evaluating $f(x_i) = \frac{1}{\sqrt{2\pi}} e^{-x_i/2}$ for $x_o = a;\ x_1 = \frac{b-a}{2}$; $x_2 = b$. Therefore, our formula simplifies as follows:

$$J(x) = \frac{1}{\sqrt{2\pi}} \frac{b-a}{3 \cdot n} \Big(f(x_o) + 4 * f(x_1) + f(x_2)\Big)$$

$$= \frac{.005}{\sqrt{2\pi} * 6} \Big(f(x_o) + 4 * f(x_1) + f(x_2)\Big)$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{1200}\Big(f(x_o) + 4 * f(x_1) + f(x_2)\Big)\right]$$

where $f(x_i)$ and x_i $(i = 0 , 1 , 2)$ are as defined above.

The average probabilities of errors in the detection of pulses is very small for a usable system. Therefore, the integrals $I(x)$ are desired for a very large range of x, say from $0 < |x| < 6.25$. For large value of x, $I(x)$ is very small, for example: when $x = 5$, $I(x) = .28665157 \times 10^{-6}$ and when $x = 6$, $I(x) = .986588 \times 10^{-9}$.

In order to obtain $I(x)$ with three significant digits, it would be necessary to compute $J(x)$ with twelve significant digits, which is impractical. Therefore, $I(x)$ is corrected by reading exact values, $I_T(x)$, into the computer from a table for $x = x_i = i(.005)$ where $(i - 1)$ is a multiple of one hundred for $1 < i < 701$ and $(i-1)$ is a multiple of 25 for $701 < i < 1272$. The incremental values of $J(x)$ are computed with eight digits. Since the greatest number of intervals between two read in values is one hundred, the maximum loss in accuracy would not exceed three digits. This leaves two extra digits to account for errors inherent in the Simpson formula.

Let $J^*(x)$ be the computed value of $J(x)$ for the discrete value of $x = x_i = i(.005)$ where i assumes every integer value between 1 and 1272.

Then $I(x_i) = I_T(x_i)$ if $I_T(x_i)$ exists; otherwise $J(x_i) = J^*(x_i)$ and $I(x_i) = I(x_{i-1}) - J(x_i)$. Finally, 1272 values are stored for $I(x_i)$.

$$\text{If } x > 6.35 \text{ , then } I(x) \approx 0$$

$$\text{If } x < 0 \text{ , then } I(x) = .5 + I(|x|)$$

$$I(|x|) = I(i) - \Big[I(i) - I(i+1) \Big] \Big[\frac{x}{.005} - (i - 1) \Big]$$

where $i = \mathrm{Tr}\left(\frac{|x|}{.005} + .00001 \right) + 1$

The integral $\int_{x_1}^{x_2} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt$ can be obtained as a function of $I(|x_1|)$ and $I(|x_2|)$.

$$\int_{x_1}^{x_2} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt = (\text{sign } x_2)\, I(|x_2|) - (\text{sign } x_1)\, I(|x_1|) \ .$$

APPENDIX C

EXPANSION OF $\frac{1}{\sqrt{2\pi}} \int_{g(u)}^{\infty} e^{-t^2/2}\, dt$ IN POWER OF $(u - u_o)$

The average probability of error, E, in the threshold detection of pulses mixed with normal noise is a linear combination of integrals:

$$E = \sum_{i=1}^{n} C_i \, I\,[g_i(u)] \tag{C-1}$$

where C_i's are constants, $I(x) = \int_x^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt$, and $g_i(u)$ is a function of u.

For example, if the amplitude of the detected signal is a function of the two previous random pulses, there are 8 combinations for γ_o and γ_{-1} and eight different functions $f_i(u)$.

Given the $f_i(u)$ and the value for u, E can be computed; from the curve E versus u, the minimum of E and the value of u which makes E minimum can be read. Since the minimum of E cannot be obtained directly by the calculus of variations, it is very convenient to develop $I(f_i(u))$ into a series of u, so that $E = \sum_{i=1}^{n} C_i \, I[f_i(u)]$ is a polynomial in u. If the series converges rapidly for all the $I[f_i(u)]$, the minimum of the polynomial is easily obtained. An example for this technique follows.

The probability of error for a constant threshold A/2 and a two step memory was obtained in paragraph IV-8:

$$E_C = \sum_{\gamma-1=0,1} \; \sum_{\gamma_o=0,1} \; \sum_{\ell=-1,1} \frac{1}{8} \, I\left[\sqrt{\frac{S_{io}}{N_{io}}} \, \frac{\sqrt{8}\,(1 - e^{-ky})}{\sqrt{ky}} \left(1 + 2\ell\,(\gamma_o e^{-y} + \gamma_{-1} e^{-2y})\right)\right] \tag{C-2}$$

where all the variables have been defined.

Formula (C-2) can be developed as a power series, but to simplify the algebra, one step memory is assumed:

$$E_C = \sum_{\gamma_o=0,1} \sum_{\ell=-1,1} \frac{1}{4}\, I\left[\sqrt{\frac{S_{io}}{N_{io}}}\, \frac{\sqrt{8}\,(1-e^{-Ky})}{\sqrt{Ky}} \left(1 + 2\ell\, \gamma_o\, e^{-y}\right)\right] \qquad \text{(C-3)}$$

$I\left[\frac{S_{io}}{N_{io}} \sqrt{8}\, \frac{1-e^{-Ky}}{\sqrt{Ky}}\right]$ is minimum for $Ky = 1.25$. The minimum of E_C occurs for a larger value of Ky because of the factor

$$W = 1 + 2\ell\, \gamma_o\, e^{-y}$$

When K is small (for $Ky = 1.25$), y is large so that the factor W is close to one; when K is large for $Ky = 1.25$, y is small and W is not neglitible. Therefore, the minimum of E_C occurs for a larger value of Ky. However, the value of $u = Ky$ which makes E_C minimum remains in the small range $1.25 < u < 2$, so that it becomes practical to develop $I[g_i(u)]$ (and hence E_C) in power of u about the point $u = u_0 = 1.6$:

$$I\left[g_i(u)\right] = I\left[g_i(u_o)\right] + \sum_{m=1}^{\infty} \left[\frac{d^m}{du^m}\, I[g_i(u)]\right]_{u=u_o} \frac{(u-u_o)^m}{m!} \qquad \text{(C-4)}$$

It is necessary to obtain the first four derivatives of $I[g_i(u)]$. Let

$$J[g(u)] = \int_o^{g(u)} \frac{1}{\sqrt{2\pi}}\, e^{-t^2/2}\, dt,$$

then

$$I[g(u)] = .5 - J[g(u)]$$

Hence, the derivatives of $I[g(u)]$ are the derivatives of $J[g(u)]$ except for a change of sign.

$$\frac{d}{du}\, I[g(u)] = -\frac{d}{du}\, J[g(u)] = -g'\, e^{-g^2/2}$$

$$\frac{d^2}{du^2}\, I\,[g(u)] = -\,[g'' - (g')^2\, g]\, e^{-g^2/2}$$

$$\frac{d^3}{du^3}\, I\,[g(u)] = -\,[g''' - 3g\,g'\,g'' - (g')^3 + (g)^2\,(g')^3]\, e^{-g^2/2}$$

$$\frac{d^4}{du^4}\, I\,[g(u)] = -\,[g'''' - 4g\,g'\,g''' - 4(g')^2\, g'' - 3g\,(g'')^2 + 6(g)^2\,(g')^2\, g'' + 3g\,(g')^4 - g^3\, g'^4]\, e^{-g^2/2}$$

where $g = g(u)$, $g' = \frac{d\,g(u)}{du}$, $g'' = \frac{d^2\, g(u)}{d^2 u}$, etc.

Since $g_i(u)$ is not a simple function, it is convenient to write $g_i(u)$ as a product of three factors

$$g_i(u) = x(u)\; x(u)\; w_i(u)$$

where $x(u) = \sqrt{\frac{S_{io}}{N_{io}}}\;\; \sqrt{8}\;\; (1 - e^{-u})$, $z(u) = u^{-1/2}$, and $w_i(u) = 1$ or $1 + 2e^{-u/k}$ or $1 - 2e^{-u/k}$. The derivatives of $g_i(u)$ can be expressed in terms of $x(u)$, $y(u)$, $w_i(u)$, and their derivatives x', y', w', x'', etc.

$$g' = x'zw + z'wx + w'xz$$

$$g'' = x''zw + xz''w + xzw'' + 2x'z'w + 2z'w'x + 2w'x'z$$

Before writing the higher derivatives of $g(u)$, we define an operator "Perm [ ]" which means permutation of x, z, w and summation. This notation divides by three the length of the formulas.

Using the operator "Perm [ ]" yields

$$g' = \text{Perm}\,[x'zw]$$

$$g'' = \text{Perm}\,[x''zw + 2x'z'w]$$

$$g''' = \text{Perm}\,[x'''zw + 3(x''z'w + x''zw')] + 6x'z'w'$$

$$g'''' = \text{Perm}\,[x''''zw + 4(x'''z'w + x'''zw') + 6x''z''w + 12x''z'w']$$

Finally, x, z, w and their derivatives are expressed as functions of the variable u and the parameters S_{io}/N_{io} and K.

$$x = \sqrt{\frac{S_{io}}{N_{io}}}\,\sqrt{8}\,(1 - e^{-u}), \quad x' = \sqrt{\frac{S_{io}}{N_{io}}}\,\sqrt{8}\,e^{-u}, \quad x'' = -x', \quad x''' = +x'$$

$$z = u^{-1/2}, \quad z' = -\frac{1}{2}\frac{z}{u}, \quad z'' = -\frac{3}{2}\frac{z'}{u}, \quad z''' = -\frac{5}{2}\frac{z''}{u}, \quad z'''' = -\frac{7}{2}\frac{z'''}{u}$$

$$w_i = 1 + 2a_i\,e^{-u/k} \qquad \text{where } a_i = 0,\ 0,\ +1,\ -1$$

$$w_i' = -\frac{2a_i}{K}\,e^{-u/k}, \quad w_i'' = -\frac{1}{K}\,w_i', \quad w_i''' = -\frac{1}{K}\,w_i'', \quad w_i'''' = -\frac{1}{K}\,w_i'''$$

For any given value of S_{io}/N_{io} and K, the coefficients of the power series can be obtained. The procedure is as follows:

(1) In x, z, w, and their derivatives, replace K and S_{io}/N_{io} by their assigned values and u by u_o.

(2) The numerical values obtained for x, z, and w, and their derivatives are substituted in the formula giving g, g', g'', g''', g''''.

(3) The numerical values obtained for g, g', g'', g''', g'''' in step (2) are substituted in the formulas giving $\frac{d^m}{du^m} I\,[g_i(u)]$.

Once the power series for the four different $I\,[g_i(u)]$ are obtained, the power series for E_C result immediately by formula (C-3). Since E_C is a polynomial of degree 4, the derivative of E_C is a third degree polynomial. To minimize E, all that is necessary is to find the roots of a third degree polynomial.

Figure IV-5 shows $u_{min}(K)$ versus K ($u = Ky$) and Figure IV-6 shows $-\log_{10} E_{C\,min}$ ($K = .1$) versus Q, obtained from the series approximation. These curves are in good agreement with the curves of Figure IV-3 and Figure IV-4, respectively, which result from a direct computation.

APPENDIX D

ESPECIAL CIRCUITS USED IN THE EXPERIMENT

D-1 Pilot Clock Circuit

The circuit of the pilot clock, shown in Fig. D1, consists of a decade counter tube (decatron EZ10B, Elesta Switzerland) and of its driving circuit. The decatron is a cold cathode gas tube and the arc moves from one cathode to the next every time a negative pulse is applied to the guide. The driving circuit transforms the periodic square wave coming from the pilot generator into periodic pulses of more than 100 v amplitude at the output of the transformer. The pilot clock operates correctly for frequencies up to 200 Kcs.

D-2 Preamplifier

The circuit of a preamplifier is shown in Fig. D-2. The preamplifier is used in two different ways: (1) A noisy pulse is applied on I_{PA} and the output, O_P, is a rectangular pulse of width .1 T, and (2) Two contigous noisy pulses are applied, one on I_{PA} and one on I_{PB} and the output, O_P, is a rectangular pulse of width .2 T. The amplitude of the rectangular pulse is determined by the Zener diode Z. The transistor Tr1 eliminates the noise on top of the pulse, while the transistor Tr2 eliminates the noise between pulses.

D-3 Sampler-Holder Circuit

The circuit of the sampler-holder is shown in Fig. D-3. It is exactly the same as the schematic circuit of Fig. VII-5 b. However, the sampler-holder samples not 0_{12} but $0'_{12}$, which is the output of an emitter follower; this makes the charge of the capacitor more instantaneous. It would be quite easy to include $0'_{12}$ in the feedback loop of the amplifier A2, but this in not necessary.

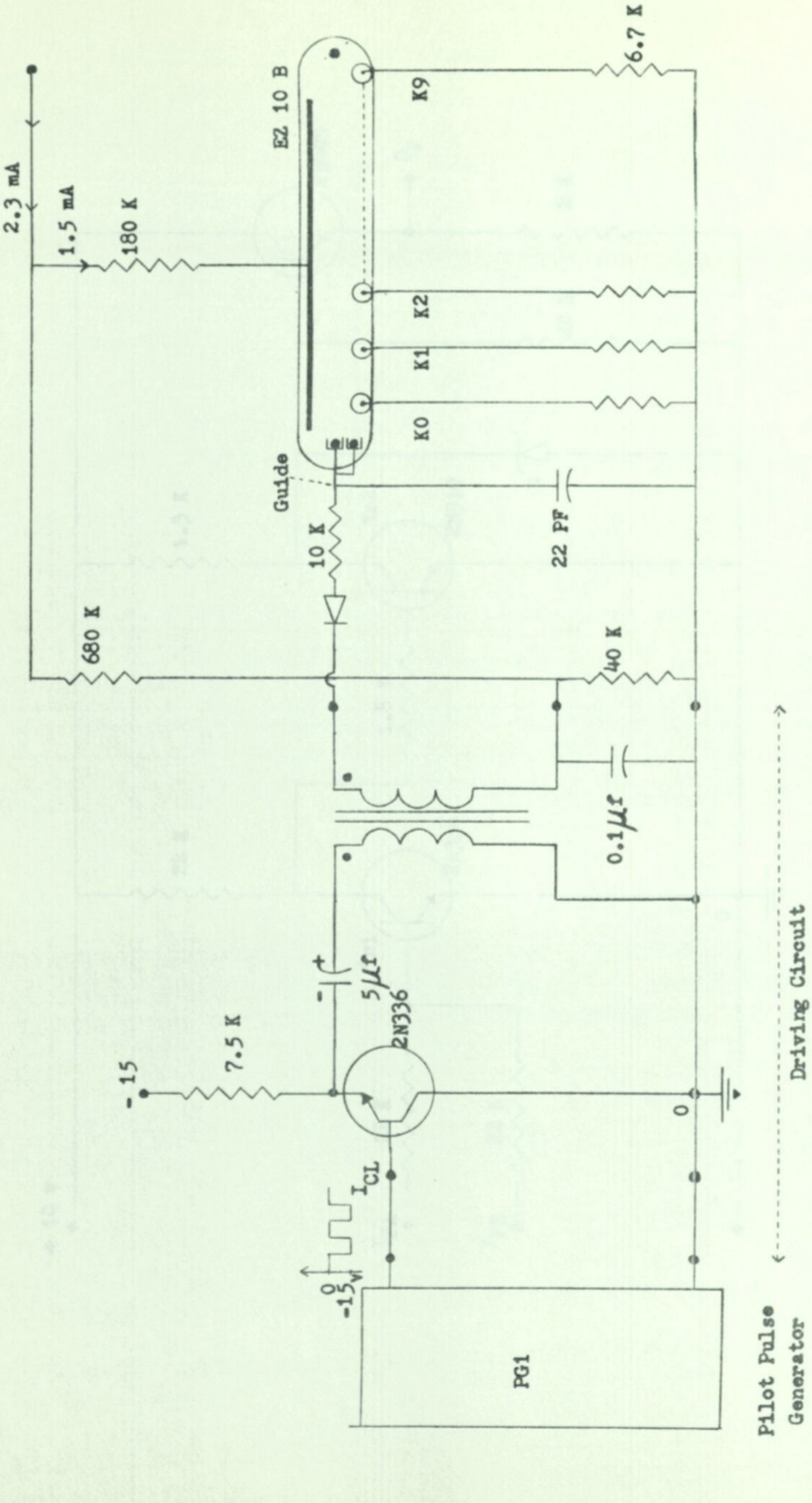


Fig. D1 : Pilot Clock Circuit

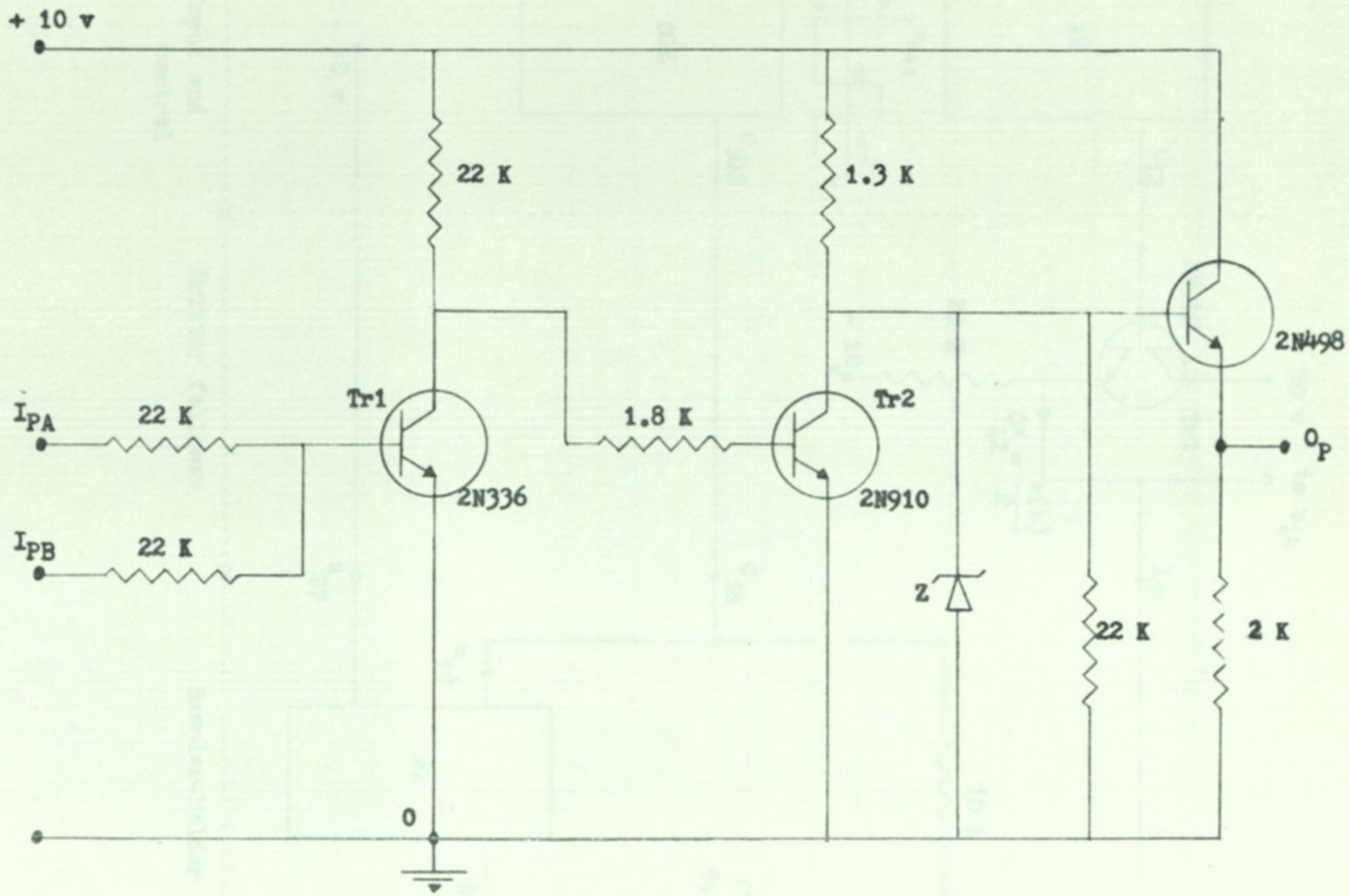


Fig. D 2 : Preamplifier Circuit

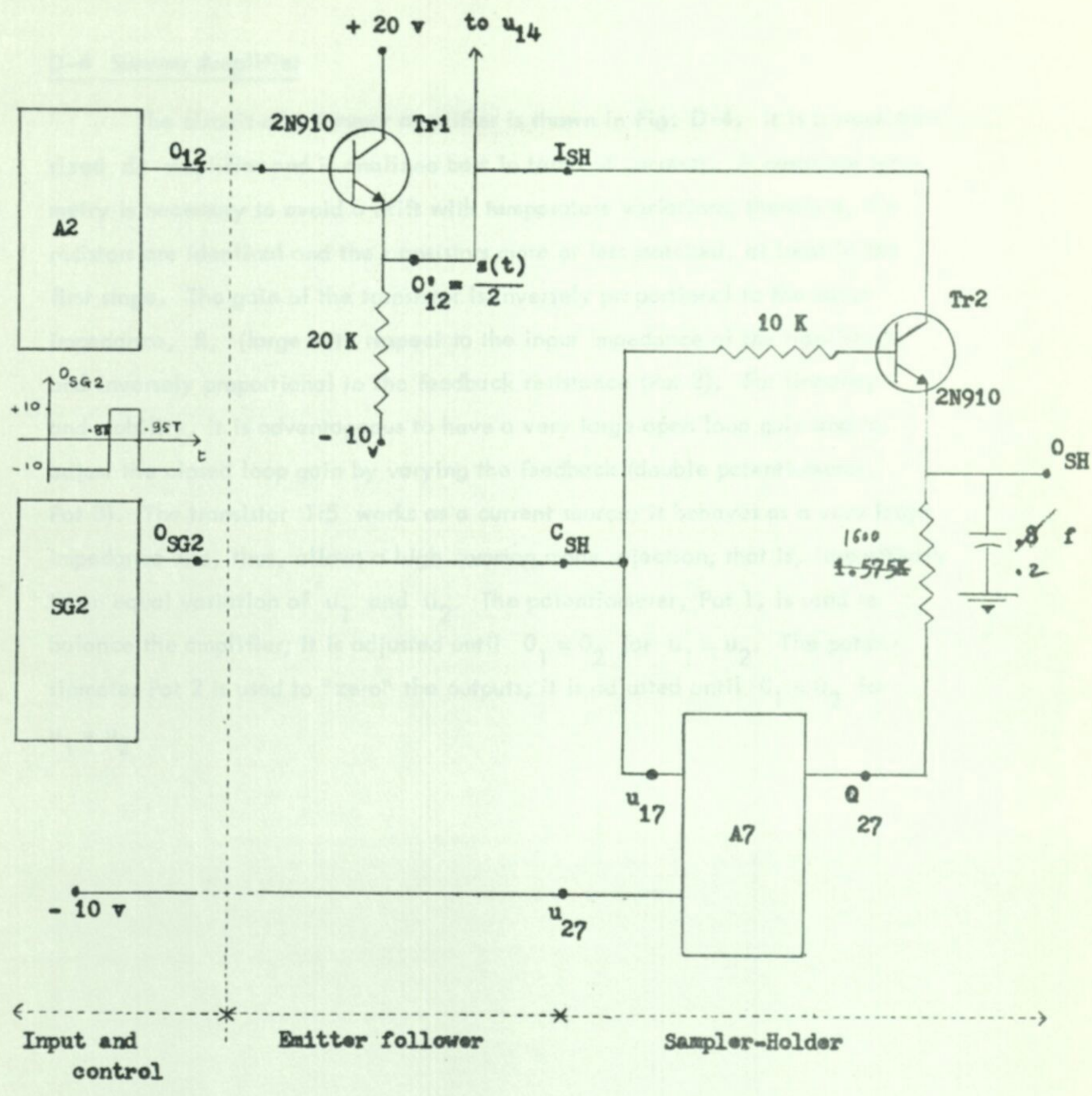


Fig. D-3 : Sampler-Holder

D-4 Summer Amplifier

The circuit of a summer amplifier is shown in Fig. D-4. It is a transistorized dc amplifier and is analized best in terms of currents. A complete symmetry is necessary to avoid a drift with temperature variations; therefore, the resistors are identical and the transistors more or less matched, at least in the first stage. The gain of the transistor is inversely proportional to the input impedance, R, (large with respect to the input impedance of the transistor), and inversely proportional to the feedback resistance (Pot 3). For linearity and stability, it is advantageous to have a very large open loop gain and to adjust the closed loop gain by varying the feedback (double potentiometer, Pot 3). The transistor Tr5 works as a current source; it behaves as a very large impedance and, thus, allows a high common mode rejection; that is, insensitivity to an equal variation of u_1 and u_2. The potentiometer, Pot 1, is used to balance the amplifier; it is adjusted until $0_1 = 0_2$ for $u_1 = u_2$. The potentiometer Pot 2 is used to "zero" the outputs; it is adjusted until $0_1 = 0_2$ for $u_1 = u_2$.

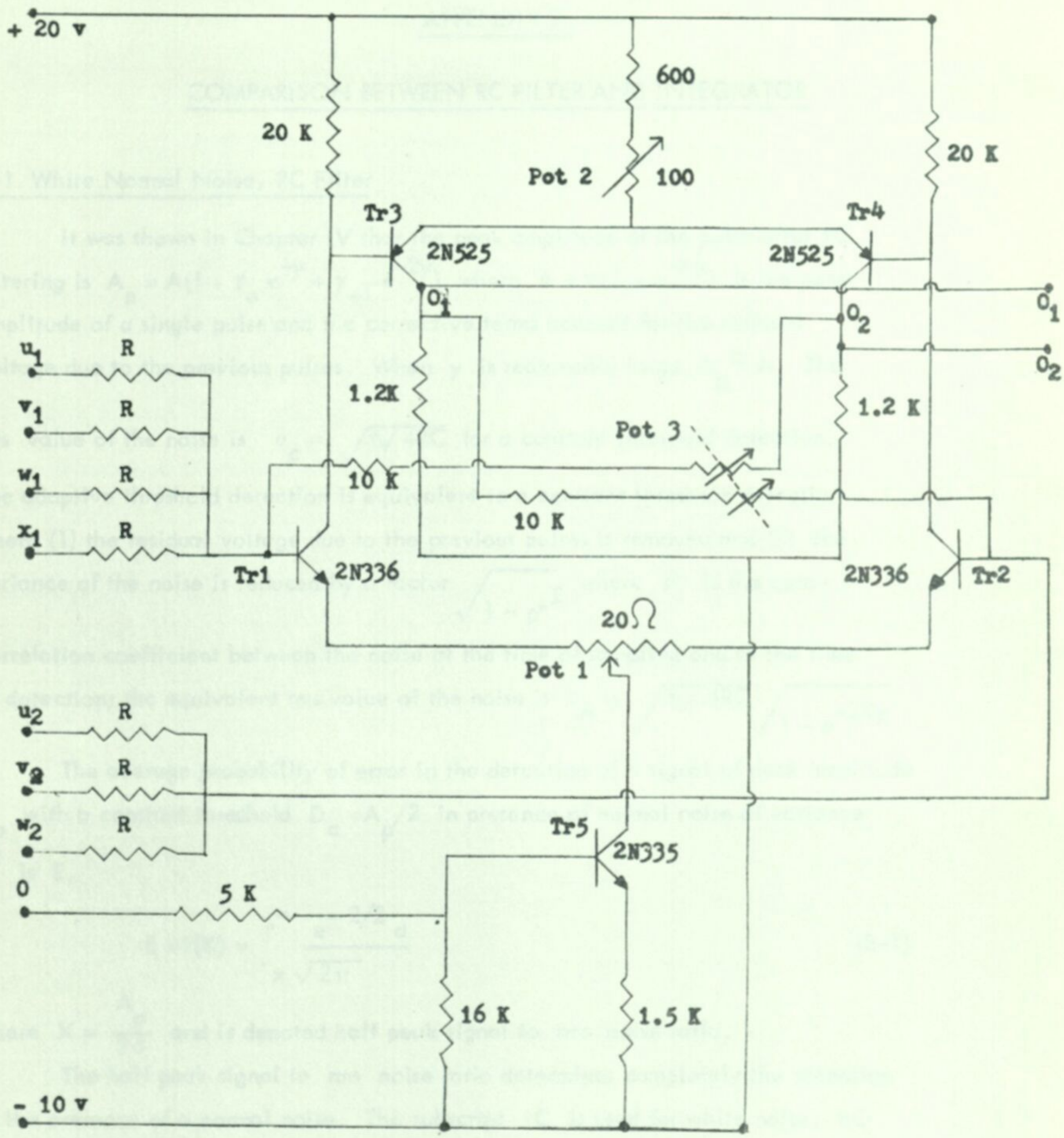


Fig. D-4 : Summer Amplifier Circuit

APPENDIX E

COMPARISON BETWEEN RC FILTER AND INTEGRATOR

E-1 White Normal Noise, RC Filter

It was shown in Chapter IV that the peak amplitude of the pulse after RC filtering is $A_p = A(1 + \gamma_o e^{-y} + \gamma_{-1} e^{-2y})$ where $A = V(1 - e^{-Ky})$ is the peak amplitude of a single pulse and the corrective terms account for the residual voltage due to the previous pulses. When y is reasonably large $A_p \approx A$. The rms value of the noise is $\sigma_c = \sqrt{\eta / 4RC}$ for a constant threshold detection. The adaptive threshold detection is equivalent to a constant threshold detection where (1) the residual voltage due to the previous pulses is removed and (2) the variance of the noise is reduced by a factor $\sqrt{1 - \rho^{*2}}$ where ρ^* is the auto-correlation coefficient between the noise at the time of sampling and at the time of detection; the equivalent rms value of the noise is $\sigma_A = \sqrt{\eta / 4RC} \sqrt{1 - e^{-2Ky}}$

The average probability of error in the detection of a signal of peak amplitude A_p with a constant threshold $D_c = A_p/2$ in presence of normal noise of variance σ^2 is E,

$$E = I(X) = \int_x^{\infty} \frac{e^{-2/2}}{\sqrt{2\pi}} d \qquad \text{(E-1)}$$

where $X = \frac{A_p}{2\sigma}$ and is denoted half peak signal to rms noise ratio.

The half peak signal to rms noise ratio determines completely the detection in the presence of a normal noise. The subscript 1C is used for white noise, RC filter and constant threshold; the subscript 1A is used for white noise, RC filter and adaptive threshold:

$$X_{1C} = \frac{A}{2\sigma_c} = \frac{V(1 - e^{-Ky})}{2\sqrt{\eta/4RC}} = \frac{\sqrt{Q}\,\sqrt{2}\,(1 - e^{-Ky})}{\sqrt{Ky}} \tag{E-2}$$

$$X_{1A} = \frac{A}{2\sigma_A} = \frac{V(1 - e^{-Ky})}{2\sqrt{\eta/4RC}\,\sqrt{1 - e^{-2Ky}}}$$

$$= \frac{\sqrt{Q}\,\sqrt{2}\,(1 - e^{-Ky})}{\sqrt{Ky}\,\sqrt{1 - e^{-2Ky}}} = \sqrt{Q}\,\sqrt{\frac{2(1 - e^{Ky})}{Ky\,(1 + e^{-Ky})}} \tag{E-3}$$

E-2 White Normal Noise, Integrator

It was shown in Chapter V that the amplitude of the pulse after integration is V at time $T + KT$ and that the mean square noise is given by formula (V-15). In the case of white noise of power density $\eta/2$, the autocorrelation function is a δ-function (Fig. II-3 c), $R_i(\tau) = \eta/2\ \delta(\tau)$. Substituting $R_i(\tau) = \eta/2\ \delta(\tau)$ in formula (V-15) yields the mean square noise at the output of the integrator,

$$\sigma_2^{\,2} = \frac{2}{K^2T^2} \int_0^{KT} (KT - \tau)\,\eta/2\ \delta(\tau)\,d\tau \tag{E-4}$$

The integral of $\delta(x)$ from $-Kt$ to $+KT$ is one by definition, but the integral form 0 to KT is only one half. Therefore,

$$\sigma_2^{\,2} = \frac{\eta}{2KT} \tag{E-5}$$

The half amplitude to noise ratio for white noise and integrator is denoted by X_2,

$$X_2 = \frac{V}{2\sigma_2} = \frac{V}{2\sqrt{\frac{\eta}{2KT}}} = \sqrt{Q} \tag{E-6}$$

E-3 Comparison Between RC Filter and Integrator for White Noise

The average probability of error for white noise and constant threshold, E_{1C}, is approximately

$$E_{1C} = I(X_{1C}) \tag{E-7}$$

where X_{1C} is given by formula (E-1). The exact value of E_{1C} is given by formula (IV-58). The average probability of error E_{1C} is minimum and denoted by $E_{Cmin}(Q)$ when $K = .1$ and $y = 12.5$. The plot of $-\log E_{Cmin}(Q)$ is shown in Fig. IV-1.

The average probability of error for white noise and adaptive threshold, E_{1A}, is

$$E_{1A} = I(X_{1A}) \tag{E-8}$$

where X_{1A} is given by formula (E-3). E_{1A} is minimum for Ky minimum and denoted by $E_{1Amin}(Q)$ when $Ky = .1$. The plot of $-\log E_{Amin}(Q)$ is shown in Fig. IV-1.

The average probability of error for white noise and integrator is

$$E_2 = I(X_2) \tag{E-9}$$

where $X_2 = \sqrt{Q}$ as given by formula (E-6).

The optimum value for X_{1C} is obtained by substituting $Ky = 1.25$ in formula (E-2) and is denoted $X_{1C\ opt}$,

$$X_{1C\ opt} = \frac{\sqrt{Q}\ \sqrt{2}\ (1 - .287)}{\sqrt{1.25}} = .902\sqrt{Q} \tag{E-10}$$

and

$$E_{Cmin}(Q) \approx I(.902\ \sqrt{Q}\ ) \tag{E-11}$$

The realizable optimum value for X_{1A} is obtained by substituting $Ky = .3$ in formula (E-2) and is denoted $X_{1A\ opt}$.

$$X_{1A\ opt} = \sqrt{Q}\ \sqrt{\frac{2(1 - .741)}{.3(1 + .741)}} = .996\sqrt{Q} \qquad \text{(E-12)}$$

and

$$E_{1A\ min} = \Gamma(.996\sqrt{Q}\ ) \qquad \text{(E-13)}$$

The average probabilities of error using an integrator or an adaptive threshold are theoretically equal. Indeed, it was shown in paragraph (IV-9) that the theoretical optimum for X_{1A} is

$$\sqrt{Q} \text{ since } \lim_{Ky \to 0} \sqrt{\frac{2(1 - e^{-Ky})}{Ky(1 + e^{-Ky})}} = 1.$$

Practically, any error in the switching on and off of the integrator will decrease X_2. Finally, $X_2 \approx X_{1A} = .996\sqrt{Q}$ (E-14)

The optimum average probabilities of error for white noise as a function of the fictitious signal-to-noise ratio Q are compared in Fig. E-1. Curve (1) is for constant threshold and curve (2) is either for adaptive threshold or integrator.

E-4 RC Noise, RC Filter, Constant Threshold

As in Chapter III the RC noise can be assumed to be white noise filtered by a fictitious RC network, $R_f C_f$. The normal RC noise is defined by its auto-correlation function $R_i(\tau) = \sigma_i^2 e^{\frac{-\beta|\tau|}{T}}$ where σ_i^2 is the variance of the noise. Let $\eta/2$ be the power density of the fictitious white noise. Using formulas IV-7 and IV-12 it follows: $R_f C_f = \frac{T}{\beta}$ and $\sigma_i^2 = \frac{\eta}{4 R_f C_f}$. The block

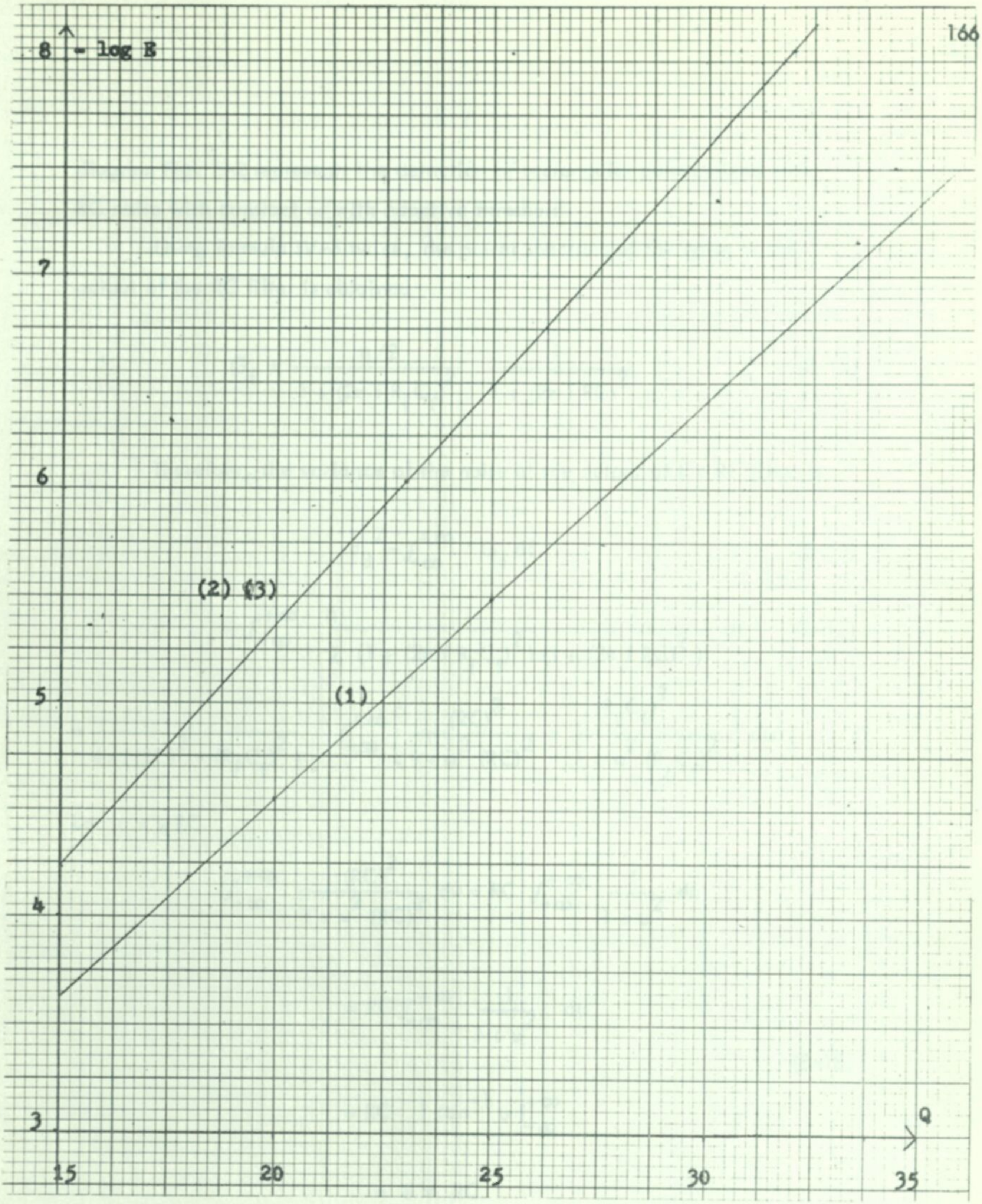


Fig. E-1 : Probability of Error versus Q, for Pulses mixed with White Noise
(1) RC Filter, Constant Threshold, (2) RC Filter, Adaptive Threshold,
(3) Integrator, Constant Threshold

diagram Fig. E-2 represents the detection of rectangular pulses mixed with RC noise with constant or with adaptive threshold.

Using formula IV-3 twice, the power spectrum of the noise at the output of the RC filter is obtained

$$G_f = \frac{\eta/2}{1 + (2\pi\ f\, R_f C_f)^2} \quad \frac{1}{1 + (2\pi\ f\, RC)^2} \tag{E-15}$$

Therefore, the variance of the noise at the output of the RC filter is

$$\sigma_3{}^2 = \int_{-\infty}^{+\infty} G_f\, df \tag{E-16}$$

$$= \int_{-\infty}^{+\infty} \frac{\eta}{2} \frac{1}{(1 + (2\pi\ f\, R_f C_f)^2)\,(1 + (2\pi\ f\, RC)^2\,)}\, df \tag{E-17}$$

$$= \frac{\eta}{4\pi\,((RC)^2 - (R_f C_f)^2\,)} \int_{-\infty}^{+\infty} \left[\frac{(RC)^2}{1 + \omega^2\ (RC)^2} - \frac{(R_f C_f)^2}{1 + \omega^2\ (R_f C_f)^2} \right] d\omega$$

Let $u = \omega\, RC$

$$\int_{-\infty}^{+\infty} \frac{(RC)^2}{1 + \omega^2\ (RC)^2}\, d\omega = RC \int_{-\infty}^{+\infty} \frac{1}{1 + u^2}\, du$$

$$= RC \int_{-\infty}^{+\infty} \frac{1}{1 + u^2}\, du$$

$$= RC\ \left[\tan^{-1} u\right]_{-\infty}^{+\infty} \tag{E-18}$$

$$= +\pi\ RC$$

Rc noise

$$R_i(\tau) = \sigma_i^2 \; e^{-\beta|\tau|/T}$$

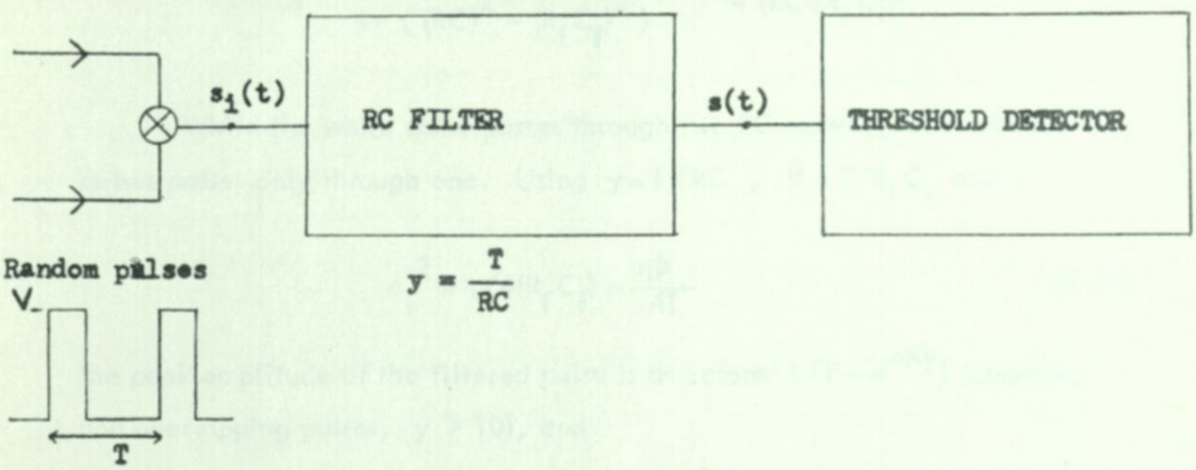


Fig. E-2 a

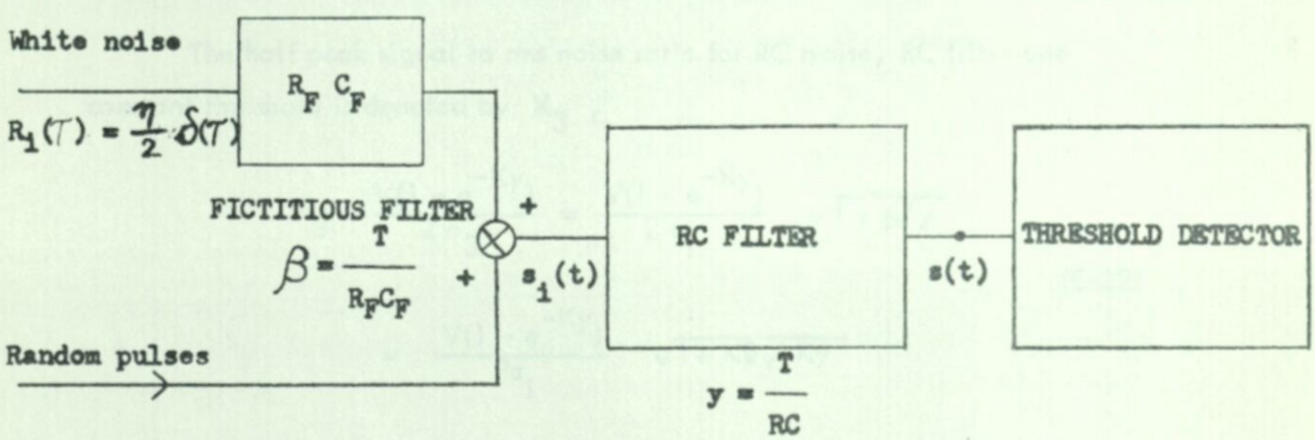


Fig. E-2 b

Fig. E-2 : Threshold detection of RC noise using an RC filter. (a) actual block diagram ; (b) equivalent block diagram

Hence,

$$\sigma_3^2 = \frac{\eta\pi\,(RC - R_f C_f)}{4\pi\,(\,(RC)^2 - (R_f C_f)^2\,)} = \frac{\eta}{4\,(RC + R_f C_f)} \tag{E-19}$$

While the white noise passes through two RC network, the rectangular pulses passes only through one. Using $y = T/RC$, $\beta = T/R_f C_f$ and

$$\sigma_i^2 = \eta/4(R_f C_f) = \frac{\eta\beta}{4T} \tag{E-20}$$

the peak amplitude of the filtered pulse is as before $V(1 - e^{-Ky})$ (assuming non overlapping pulses, $y > 10$), and

$$\sigma_3^2 = \frac{\eta}{4T\,(\frac{1}{y} + \frac{1}{\beta})} = \frac{\sigma_i^2}{(1 + \frac{\beta}{y})} \tag{E-21}$$

The half peak signal to rms noise ratio for RC noise, RC filter and constant threshold is denoted by X_3 ,

$$X_3 = \frac{V(1 - e^{-Ky})}{2\,\sigma_3} = \frac{V(1 - e^{-Ky})}{2\,\sigma_i}\,\sqrt{1 + \beta/y}$$

$$= \frac{V(1 - e^{-Ky})}{2\sigma_i}\,\sqrt{1 + K\beta\,/\,Ky} \tag{E-22}$$

The average probability of error for RC noise, RC filter and constant threshold is E_3. For non-overlapping pulses (i.e. $y > 10$),

$$E_3 = I(X_3) \tag{E-23}$$

A more general formula which takes into account the residual values of the two previous pulses can be obtained as in Chapter IV.

$$E_3 = \sum_{\gamma_{-1} = 0,1}\ \sum_{\gamma_o = 0,1}\ \sum_{\ell = -1,1} \frac{1}{8}\, I\left[X_3(1 + 2\ell(\gamma_o e^{-y} + \gamma_{-1} e^{-2y})\right] \tag{E-24}$$

E_3 is a function of (1) the half peak signal to rms noise ratio at the input of the filter $(V/2\sigma_i)$, (2) the product $Ky = \frac{KT}{RC}$, i.e. the choice of the pulse width and the RC filter, (3) the product $K\beta$, i.e. the auto-correlation function of the RC type noise, and (4) of y if $\frac{RC}{T}$ is large enough to produce overlap. Fig. E-3 shows $-\log E_3$ versus $V/2\sigma_i$ for $K\beta = 2$ and $Ky = 1.4$.

E-5 RC Noise, RC Filter, Adaptive Threshold

The autocorrelation function is the Fourier cosine transform of the power density spectrum (Wiener's theorem). Using formula (E-15), the auto-correlation function at the output of the filter is

$$R(\tau) = \int_{-\infty}^{+\infty} \frac{\eta/2}{(1 + (2\pi f R_f C_f)^2)(1 + 2\pi f RC)^2} \cos \omega\tau \, df \qquad \text{(E-25)}$$

$$= \frac{\eta}{4\pi\,((RC)^2 - (RC)^2)} \int_{-\infty}^{+\infty} \left[\frac{(RC)^2}{1 + \omega^2 (RC)^2} - \frac{(R_f C_f)^2}{1 + (R_f C_f)^2} \right] \cos \omega\tau \, df$$

$$= \frac{\eta}{4\,((RC)^2 - (R_f C_f)^2)} \left[\frac{(RC)^2 e^{-|\tau|/RC}}{2(RC)} - \frac{(R_f C_f)^2 e^{-|\tau|/R_f C_f}}{2(R_f C_f)} \right] \qquad \text{(E-26)}$$

The normalized autocorrelation function is $\rho(\tau)$,

$$\rho(\tau) = \frac{R(\tau)}{R(o)}$$

$$= \frac{RC\, e^{-|\tau|/RC} - R_f C_f\, e^{-|\tau|/R_f C_f}}{RC - R_f C_f} \qquad \text{(E-27)}$$

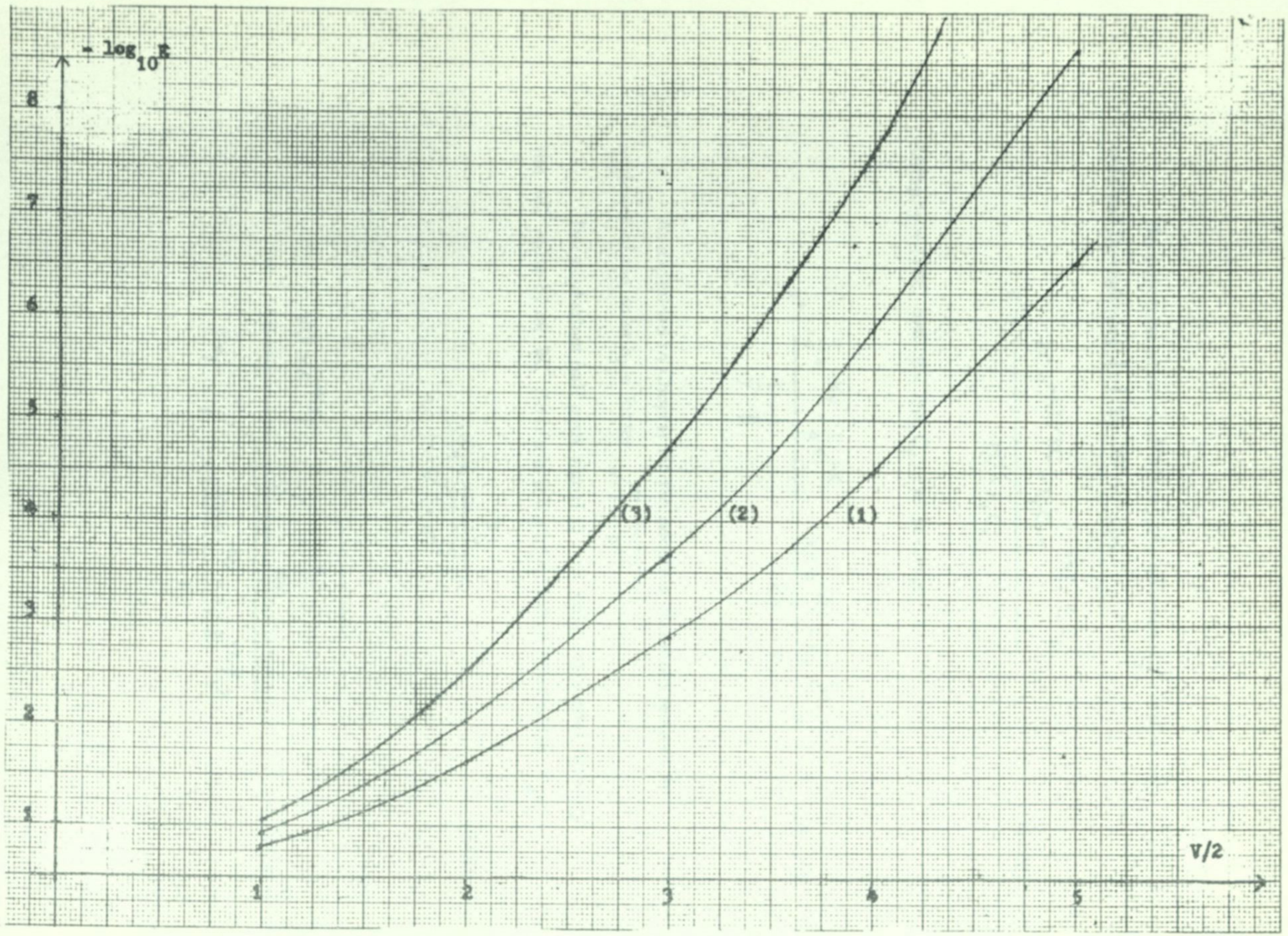


Fig. E-3 : Probability of Error versus $V/2\sigma_i$, RC Noise ($K\beta = 2$) : (1) No Filter and Constant Threshold, (2) Optimum RC Filter and Constant Threshold, (3) Optimum RC Filter and Adaptive Threshold.

The autocorrelation coefficient, ρ^*, between $s_1{}^*$ and s_1 is obtained by replacing $|\tau|$ by KT in $\rho(\tau)$

$$\rho^* = \frac{RC\, e^{-KT/RC} - R_f C_f\, e^{-KT/R_f C_f}}{RC - R_f C_f} \qquad \text{(E-28)}$$

Using $y = T/RC$ and $\beta = T/R_f C_f{}'$, ρ^* becomes

$$\rho^* = \frac{\frac{1}{y} e^{-Ky} - \frac{1}{\beta} e^{-K}}{\frac{1}{y} - \frac{1}{\beta}} = \frac{K\beta\, e^{-Ky} - Ky\, e^{-K}}{K\beta - Ky} \qquad \text{(E-29)}$$

As explained in Chapter IV the adaptive threshold reduces the variance of the noise by a factor $\sqrt{1 - \rho^{*2}}$.

The equivalent half peak signal to rms noise ratio after filtering the RC noise by an RC filter and using an adaptive threshold is X_4,

$$X_4 = \frac{X_3}{\sqrt{1 - \rho^{*2}}} \qquad \text{(E-30)}$$

$$= \frac{V(1 - e^{-Ky})\sqrt{1 + \frac{K\beta}{Ky}}}{2\,\sigma_i \sqrt{1 - \left(\frac{K\beta\, e^{-Ky} - Ky\, e^{-K\beta}}{K\beta - Ky}\right)^2}} \qquad \text{(E-31)}$$

The average probability of error for RC noise, RC filter and adaptive threshold is E_4 and

$$E_4 = I(X_4) \tag{E-32}$$

E_4 is a function of the same variables as E_3. $E_4 < E_3$ because $\left(X_4 = \frac{X_3}{\sqrt{1 - \rho^{*2}}}\right) > X_3$. Therefore, the average probability of error is reduced when an adaptive threshold is used as seen in Fig. E-3, which shows $-\log E_4$ and $-\log E_3$ versus $V/2\sigma_i$.

The increase of signal-to-noise ratio in decibels is equal to twice the increase in half peak signal-to-rms noise ratio. In the case of RC noise and RC filter the increase of signal-to-noise ratio is a function of Ky, Kβ and the type of threshold; let G_3 and G_4 be the increase in decibels for constant or adaptive threshold, respectively; then,

$$G_3 = 40 \log_{10} \frac{X_3}{(V/2\sigma_i)}$$

$$G_4 = 40 \log_{10} \frac{X_4}{(V/2\sigma_i)}$$

For example, if $K\beta = 2$, G_3 is maximum for $Ky \approx 1.4$ and equal to 2.88 db; G_4 is maximum for $Ky \approx .8$ and equal to 5.6 db. Those results can be compared to the increase in signal-to-noise ratio using an integrator which is shown in Fig. V-2. The conclusions of the comparison between RC filter and integrator are part of Chapter IX, (Paragraph IX-2).

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PART II

Chapter I

DETECTION AND INFORMATION PROCESSING WITH ADAPTIVE DECISION CIRCUITS

1. Introduction

A substantial amount of the work performed in this report depends upon the ideas and theorem of decision theory. Decision theory treats in a general way the problem relating to the detection of signals in noise and the estimation of the structures of signal and noise. The approach is based upon the fact that in testing hypotheses of all decisions involving doubt and uncertainty, various costs and risks are associated with them. These may be measured in any way appropriate to the problem at hand. The cost could be a function both of the true hypothesis and the hypothesis as the observer decides it to be. The conditional risk is the average value of the cost over all possible decisions the observer can make given a particular hypothesis. The average risk is the average of the conditional risk over all possible hypothesis. Decision theory assumes that the observer wishes to decide in the way which will minimize this conditional or average risk. On this assumption, the decision theory shows us how to choose a decision rule for processing the received data. This decision rule will yield decisions which minimize risk for the particular physical situation and the cost involved. We have described the essence of the decision theory. Now we can proceed with a more detailed study of this theory,in particular, the application of this theory to the binary detection problem.

2. The Decision Problem

In order to determine the decision problem, a loss function $\mathcal{F}(S, \gamma)$ is assigned to each combination of decision γ and signal s in accordance with some prior judgment of the relative importance of the various correct and incorrect decisions. Each decision rule may then be rated by adopting an evaluating function which takes into consideration both the probabilities of correct and incorrect decisions and the losses associated with them. We may now state the reception problem in the following general terms:

Given the family of distribution functions $F_n(v/S)$, the a priori signal probability distribution $\sigma(S)$, the class of possible decisions γ, and the loss function $\mathcal{F}$, the problem is to determine the best rule $\delta(\gamma/v)$ for using the data to make decisions. The decision γ is not restricted to a finite number m of alternatives

$$\gamma = (\gamma_1, \gamma_2, \dots \gamma_m) .$$

An infinite number may be used equally well. In fact, the extension to a continuum of possible alternatives is a matter of reinterpretation. The decision rule $\delta(\gamma/v)$ which used to be a discrete probability distribution must in this case be interpreted as a probability density function; i.e., $\delta(\gamma/v)\, d\gamma$ is the probability that γ lies between γ and $\gamma + d\gamma$, given v. To represent a nonrandomized decision rule, we interpret $\delta(\gamma/v)$ as a Dirac δ function. Usually, the family of distribution functions is not given directly and must be found from a given noise distribution W(N) and the mode of combining signal and noise.

Now let us explain a little about the notation of the reception situation. As shown in Fig. 1, a decision γ is to be made about a signal S, based on data v, in accordance with a decision rule $\delta(\gamma/v)$. Here, $\gamma = (\gamma_1, \gamma_2 \dots \gamma_m)$, $S = (S_1, S_2 \dots S_m)$ $v = (v_1, v_2, \dots v_m)$ are vectors. The subscripts on the components of S and v are ordered in time so that $S_k = S(t_k)$, $v_k = v(t_k)$ etc., with $0 \le t_1 \le t_2 \le \dots \le t_k \le \dots \le t_m \le T$.

In Fig. 1 and Fig. 2, each of the quantities S, N, V, γ can be represented by a point in an abstract space of appropriate dimensionality, and the occurrence of particular values is governed in each instance by an appropriate probability density function. Here $\sigma(S)$, W(N), and $F_n(v/S)$ are the probability density functions for signal, noise, and the data v respectively when S is given. These are multidimensional density functions which are discrete or continuous depending on the discrete or continuous nature of the spaces and of corresponding dimensionality.

3. Functions of Evaluation

$\mathcal{F}(S, \gamma)$ is a generalized loss function, adopted in advance of any optimization procedure which assigns a loss, or cost to every combination of system input and decision (system output) in a way that may or may not depend on the system's operation. Actual evaluation of system performance is made by adopting an evaluation function $\mathcal{E}(\mathcal{F})$ which takes into account all possible modes of system behavior and their relative frequencies of occurrence and assigns an overall loss rating to each system or decision rule. One obvious choice of $\mathcal{E}$ is the mathematical expectation or average value of $\mathcal{F}$ and it is on this reasonable choice that the present theory is based.

The conditional loss rating $\mathcal{L}(S, \sigma)$ of σ is defined as the conditional expectation of loss. That is, for given S,

$$\mathcal{L}(S,\delta) = E_{v/S} \{ \mathcal{F}[S, \gamma(v)] \}$$

$$= \int_{\Gamma} dv \int_{\Delta} d\gamma \, \mathcal{F}(S,\gamma) \, F_n(v/S) \, \delta(\gamma/v) \tag{1}$$

where

$\mathcal{F}(S,\gamma)$ = generalized loss function

$F_n(v/S)$ = conditional probability density function of the observed quantity v given signal S

$\delta(\gamma/v)$ = conditional probability density function of the decision γ given v.

Equation (1) can be applied to discrete as well as continuous spaces Δ. For the discrete space, the integral over Δ is to be interpreted as a sum and $\sigma(\gamma/v)$ as a probability, rather than as a probability density.

However, when the signal distribution $\sigma(S)$ is known, we use the above information to rate the system by averaging the loss over both the sample and the signal distributions. The average loss rating $\mathcal{L}(\sigma, \delta)$ of δ is defined as the (unconditional) expectation of loss when the signal distribution is $\sigma(S)$.

$$\mathcal{L}(\sigma, \delta) = E_{v,s} \{ \mathcal{F}(S, \gamma) \}$$

$$= \int_{\Omega} ds \int_{\Gamma} dv \int_{\Delta} d\gamma \ \mathcal{F}(S,\gamma) \ \sigma(S) \ F_n(v/S) \ \delta(\gamma/v) \qquad (2)$$

$\mathcal{F}$ is usually a function which assigns to each combination of signal and decision a certain loss, or cost, which is independent of δ:

$$\mathcal{F} = C \ (S,\gamma) \qquad (3)$$

However, a more general type of loss function is suggested by information theory.

$$\mathcal{F} = - \log \ P(S/\gamma) \qquad (4)$$

By substituting (3) into (1) and (2), we have

Conditional risk $\gamma(S,\delta)$

$$r \ (S,\delta) = \int_{\Gamma} dv \ F_n(v/S) \int_{\Delta} d\gamma \ C(S,\gamma) \ \delta(\gamma/v) \qquad (5)$$

Average risk $R(\sigma, \delta)$

$$R(\sigma,\delta) = E \ \{\gamma(S,\delta)\} = \int_{\Omega} \gamma(S, \delta) \ \sigma(S) \ dS \qquad (6)$$

or

$$R(\sigma, \delta) = \int_{\Omega} \sigma(S) \ dS \int_{\Gamma} dv \ F_n(v/S) \int_{\Delta} d\gamma \ C(S,\gamma) \ \delta(\gamma/v) \qquad (7)$$

By substituting (4) into (1) and (2), we get,

Conditional information loss h (S, δ)

$$h\,(S,\delta) \;=\; -\int_{\Gamma} dv\, F_n(v/S) \int_{\Delta} d\gamma\, [\log P\,(S/\gamma)]\,\delta(\gamma/v) \qquad (8)$$

Average infomation loss $H(\sigma, \delta)$

$$H(\sigma,\delta) \;=\; E\,\{h\,(S,\delta)\}$$

$$= \int_{\Omega} h(S,\delta)\,\sigma(S)\,dS$$

$$= -\int_{\Omega} \sigma(S)\,dS \int_{\Gamma} dv\, F_n\,(v/S) \int_{\Delta} d\gamma\; [\log P\,(S/\gamma)]\,\delta(\gamma/v) \qquad (9)$$

When S is a function of a set of random parameters θ, frequently it is the parameters θ about which decisions are to be made rather than about S itself. Similar to Eqs. (5) and (6), the conditional and average risks for this situation may be expressed as:

$$r\,(\theta, S) \;=\; \int_{\Gamma} dv\, F_n\,[v/S\,(\theta)] \int_{\Delta} d\gamma\, C\,(\theta,\gamma)\,\delta\,(\gamma/v) \qquad (10)$$

and

$$R\,(\sigma,\delta)_\theta \;=\; \int_{\Omega\theta} \gamma(\theta,\delta)\,\sigma\,(\theta)\,d\theta$$

$$= \int_{\Omega\theta} \sigma(\theta)\,d\theta \int_{\Gamma} dv\, F_n\,[v/S\,(\theta)] \int_{\Delta} d\gamma\, C\,(\theta,\gamma)\,\delta(\gamma/v) \qquad (11)$$

Here $r(\theta, \delta)$ and $R\,(\sigma,\delta)_\theta$ are not necessarily the same as $r(S, \delta)$, $R(\sigma,\delta)$ above, nor is the form of γ either. Notice that the cost function $C(\theta, \gamma)$ is usually a different function of θ from $C\,[\,S(\theta),\ \gamma\,]$ also.

4. Bayes Systems and Minimax Systems

Average and conditional loss ratings may be assigned to any system once the evaluation and cost functions have been selected. We now describe two kinds of optimum decision systems. We consider one system is better than another if its average loss rating is smaller for the same application (and criterion) and that the best or optimum system is the one with the smallest average loss rating. We call this optimum system a Bayes system. A Bayes system obeys a Bayes decision rule δ^*, where δ^* is a decision rule whose average loss rating $\mathcal{L}$ is smallest for a given a priori distribution σ. For the risk and information criteria of Eqs. (7) and (9), we have

$$R^* = \min_{\delta} R(\sigma, \delta) = R(\sigma, \delta^*) \text{ Bayes Risk} \tag{12}$$

and

$$H^* = \min_{\delta} H(\sigma, \delta) = H(\sigma, \delta^*) \text{ Bayes equivocation} \tag{13}$$

R^* minimizes the average risk while H^* minimizes the equivocation. For a given $\mathcal{F}$, Bayes decision rules form a Bayes class, each member of which corresponds to a different a priori distribution $\sigma(S)$.

When the a priori signal probabilities are now known or are only incompletely given, a possible criterion for optimization in such cases is provided by the Minimax decision rule δ^*_M which is a Bayes rule associated with conditional risk $r(S,\delta)$. Roughly speaking, the Minimax rule is the decision rule which reduces the maximum risk as far as possible. The Minimax decision rule δ^*_M is the rule for which the maximum conditional loss rating $\mathcal{L}(S, \delta)_{max}$, as the signal S ranges over all possible values, is not greater than the maximum conditional loss rating of any other decision rule δ.

In terms of conditional risk r, we have

$$\max_{S} r(S, \delta^*_M) = \max_{S} \min_{\delta} r(S, \delta) \leq \max_{S} r(S, \delta) \tag{14}$$

In terms of conditional information loss h, we may write

$$\max_{S} h(S, \delta^*_M) = \max_{S} \min_{\delta} h(S, \delta) \leq \max_{S} h(S, \delta) \tag{15}$$

We may also express the Minimax decision process in terms of the resulting average risk.

$$R^*_M(\sigma_o, \delta^*_M) = \underset{\sigma}{\text{Max}}\ R^*(\sigma, \delta^*)$$
$$= \underset{\sigma}{\text{Max}}\ \underset{\delta}{\text{Min}}\ R(\sigma, \delta) \qquad (16)$$

Thus the Minimax average risk is the largest of all the Bayes risks, considered over the class of a priori signal distribution $\{\sigma(S)\}$. Geometrically, the Minimax situation of $\sigma \rightarrow \sigma_o$, $\delta \rightarrow \delta^*_M$ $R(\sigma, \delta) \rightarrow R^*_M(\delta_o, \delta^*_M)$ is represented by a saddle point of the average-risk surface over the (σ, δ) plane as shown in Fig. 3.

5. The Average Risk for Binary Detection System

In binary detection, we test the hypothesis H_o (noise alone) against the alternative H_1 (signal and noise). Therefore, there are only two points $\gamma = (\gamma_o, \gamma_1)$ in decision space Δ. Let $\delta(\gamma_o/v)$ and $\delta(\gamma_1/v)$ be the probabilities that γ_1 and γ_o are decided given v. Since definite terminal decisions are postulated, some decision is always made and therefore

$$\delta(\gamma_o/v) + \delta(\gamma_1/v) = 1 \qquad (17)$$

Denoting by S the input signal that may occur during the observation interval, we may express the two hypotheses concisely as $H_o : S \notin \Omega_o$ and $H_1 : S \in \Omega_1$, where Ω_o and Ω_1 are the appropriate non-overlapping hypothesis classes. It is now convenient to describe the occurrence of signals within the non-overlapping classes Ω_o, Ω_1 by density functions $\omega_o(S)$, $\omega_1(S)$, normalized over the corresponding spaces, e.g.

$$\int_{\Omega_o} \omega_o(S)\, dS = 1 \qquad (18)$$

$$\int_{\Omega_1} \omega_1 (S)\, dS = 1 \tag{19}$$

Let q and P (= 1 − q) are the a priori probabilities that signal from Ω_o and Ω_1 respectively will occur. The a priori probability distribution $\sigma(S)$ over the total signal space $\Omega = \Omega_o + \Omega_1$ becomes

$$\sigma(S) = q\, \omega_o(S) + p\, \omega_1(S) \tag{20}$$

If there is only a single signal in class Ω_1 , then Eq. (20) becomes

$$\sigma(S) = q\, \delta(S - 0) + p\, \delta(S - S_1) \tag{21}$$

In this simple alternative situation and the more general case, we have

$$\int \sigma(S)\, dS = 1 \tag{22}$$

For the one sided alternative case, we can assign a set of costs $C(S, \gamma) = \mathcal{F}$ to each possible combination of signal input and decision.

Let

$$C(S \in \Omega_o\, ;\ \gamma_o) = C_{1-\alpha}$$
$$C(S \in \Omega_o\, ;\ \gamma_1) = C_\alpha$$
$$C(S \in \Omega_1\, ;\ \gamma_o) = C_\beta$$
$$C(S \in \Omega_1\, ;\ \gamma_1) = C_{1-\beta} \tag{23}$$

$C_{1-\alpha}$ and $C_{1-\beta}$ are the costs associated with correct decision (success) while C_α and C_β are associated with the possible incorrect decisions (failure). Physically, we require

$$C_\alpha > C_{1-\alpha}$$

$$C_\beta > C_{1-\beta} \tag{24}$$

We can also postulate that

$$C_{1-\alpha} \geq 0$$

$$C_{1-\beta} \geq 0 \tag{25}$$

The best we can expect here is that success may cost us nothing ($C_{1-\alpha} = C_{1-\beta} = 0$). Substituting Eq. (23) into Eq. (11), we see that the integration over decision space $\triangle$ is replaced by an appropriate summation. Therefore, the average risk of Eq. (11) becomes

$$R(\sigma, \delta) = \int_{\Gamma} dv \; \{ \delta(\gamma_o/v) \, [q \, C_{1-\alpha} \, F_n(v/0) + p \, C_\beta \langle F_n(v/S) \rangle_s \,]$$

$$+ \delta(\gamma_1/v) \, [p \, C_{1-\beta} \langle F_n(v/S) \rangle_s + q \, C_\alpha \, F_n(v/0) \,] \, \} \tag{26}$$

where

$$P \langle F_n(v/S) \rangle_s = \int_{\Omega_1} \sigma(S) \, F_n(v/S) \, dS$$

$$= p \int_S \omega_1(S) \, F_n(v/S) \, dS \tag{27}$$

$$q \, F_n(v/0) = \int_{\Omega_o} \sigma(S) \, F_n(v/S) \, dS$$

$$= q \int_S \omega_o(S) \, F_n(v/S) \, ds \tag{28}$$

When $S = S(\theta)$ these become

$$P \langle F_n(v/S) \rangle_S = P \int_{\theta} \omega_1(\theta) \, F_n[v/S(\theta)] \, d\theta \tag{29}$$

$$q\,F_n\,(v/0) \;=\; q \int_\theta \omega_o(\theta)\,F_n\,[v/S\,(\theta)]\;d\theta \tag{30}$$

There are two possible classes of error:

Type I: Noise ⟶ signal + noise

Type II: signal + noise ⟶ noise

The class conditional probabilities of these two types of error are:

$$\alpha = \int_\Gamma F_n\,(v/0)\;\delta\,(\gamma_1/v)\,dv$$
$$\beta = \int_\Gamma \left\langle F_n\,(v/S)\right\rangle_S\;\delta(\gamma_o/v)\,dv \tag{31}$$

Substituting (31) into (26) and using relation (17), we get

$$R = q\,C_{1-\alpha} + p\,C_{1-\beta} + q\,\alpha\,(C_\alpha - C_{1-\alpha}) + p\,\beta\,(C_\beta - C_{1-\beta})$$
$$= R_o + q\,\alpha\,(C_\alpha - C_{1-\alpha}) + p\beta\,(C_\beta - C_{1-\beta}) \tag{32}$$

where

$$R_o = q\,C_{1-\alpha} + p\,C_{1-\beta} \tag{33}$$

Similarly, the conditional risk corresponds to Eq. (5) and becomes

$$r\,(S) = \begin{cases} (1-\alpha')\;C_{1-\alpha} + \alpha'\,C_\alpha & (S = 0) \\ [1-\beta'\,(S)]\;C_{1-\beta} + \beta'(S)\,C_\beta & (S \neq 0) \end{cases} \tag{34}$$

where

$$\alpha' = \int_\Gamma F_n\,(v/0)\;\delta\,(\gamma_1/v)\,dv = \alpha \tag{35}$$

$$\beta' = \int_\Gamma F_n\,(v/S)\;\delta\,(\gamma_o/v)\,dv \;\neq\; \beta \tag{36}$$

6. Optimum Detection

The optimum decision rule δ^* (Bayes decision rule) can now be found. From Eqs. (26) and (17) we get

$$R(\sigma, \delta) = q\,C_{1-\alpha} + P\,C_{1-\beta} + \int_{\Gamma} S\ (\gamma_o/v)\ [\,p\,(C_\beta - C_{1-\beta}) \left\langle F_n(v/S) \right\rangle_S - q\,(C_\alpha - C_{1-\alpha})\,F_n(v/0)\ ]\ dv \qquad (37)$$

The problem is to choose $\delta(\gamma_o/v)$ and $\delta(\gamma_1/v)$ in such a way as to minimize the average risk. For each $v \in \Gamma$, both δ's must be positive and (equal to or) less than 1. The optimum choice of δ's is as follows:

when $p\,(C_\beta - C_{1-\beta}) \left\langle F_n(v/S) \right\rangle_S \geq q\,(C_\alpha - C_{1-\alpha})\,F_n(v/0)$

$$\text{choose} \quad \left.\begin{aligned} \delta^*(\gamma_1/v) &= 1 \\ \delta^*(\gamma_o/v) &= 0 \end{aligned}\right\} \qquad (38)$$

when

$$p\,(C_\beta - C_{1-\beta}) \left\langle F_n(v/S) \right\rangle_S < q\,(C_\alpha - C_{1-\alpha})\,F_n(v/0)$$

$$\text{choose} \quad \left.\begin{aligned} \delta^*(\gamma_1/v) &= 0 \\ \delta^*(\gamma_o/v) &= 1 \end{aligned}\right\} \qquad (39)$$

It is now convenient to introduce the generalized likelihood ratio.

$$\Lambda = \frac{p \left\langle F_n(v/S) \right\rangle_S}{q\,F_n(v/0)} \quad (\geq 0) \qquad (40)$$

In the binary case, Eqs. (38) and (39) may be stated more compactly as:

$$\text{Decide } \gamma_1 \text{ when } \Lambda \geq K$$
$$\text{Decide } \gamma_o \text{ when } \Lambda < K$$

where

$$K = \frac{C_\alpha - C_{1-\alpha}}{C_\beta - C_{1-\beta}} > 0 \tag{41}$$

K is called the threshold and depends only on the preassigned costs. Thus, the Bayes rule essentially amounts to a division of observation space Γ into two regions separated by the v satisfying the equation.

$$\Lambda(v) = K \tag{42}$$

In general, the optimum detector is a computer which processes the received data v in a nonlinear fashion. Its precise form depends on the statistics of the background noise and signal structure, as well as on the a priori probabilities.

7. Detection for a priori distributed signals

From Eqs. (40), (41), and (42), we have the decision equation.

$$\Lambda = \frac{P \left\langle F_n (v/S) \right\rangle_S}{q\ F_n (v/0)} = \frac{C_\alpha - C_{1-\alpha}}{C_\beta - C_{1-\beta}} \tag{43}$$

P is the a priori distribution of received signal S in a signal vector space Ω. v is the received data vector in vector space Γ, and F_n is the conditional probability distribution of v given S. v is taken to be the sum of signal S and noise N. A signal is said to be present whenever Λ exceeds some preset threshold level K. The value of K is dependent upon the various costs involved. One reasonable assumption for the costs are:

$$C_\alpha = C_\beta$$

$$C_{1-\alpha} = C_{1-\beta}$$

With the above condition, K equals unity. An observer who makes a decision in this way is called an ideal observal.

It is convenient to represent the received signal S(t) by an expansion of the form

$$S(t) = \sum_{k=1}^{\infty} S_k \, \phi_k (t) \tag{44}$$

The ϕ_k are orthonormal on an interval which completely spans the interval (0,T) in which all the S(t) of interest are assumed to exist. Here we are interested in detecting single pulses received from a source which emits pulses of a constant shape but not necessarily with constant repetition rate. The received pulses need not have the same shape as the transmitted pulses. The set of ϕ_k (t) is a solution of the integral equation.

$$\int_0^T R_N (t, u) \, \phi_k (u) \, du = \sigma_k^2 \, \phi_k(t) \tag{45}$$

where

$$0 \le (t, u) \le T$$

$$k = 1, 2, 3, \ldots\ldots$$

The Kernel R_N (t,u) is the autocorrelation function of the noise N(t). For example, for band limited white noise, with frequency interval $(-\frac{1}{2\tau_o}, \frac{1}{2\tau_o})$ the autocorrelation function of the noise is

$$R_N (t, u) = N \frac{\sin 2\pi f_o (t - u)}{2\pi f_o (t - u)} \tag{46}$$

where

$$f_o = \frac{1}{2\tau_o}$$

τ_o is the separation of the equally spaced pulses.

Thus the integral equation (45) becomes

$$\int_{-\infty}^{+\infty} R_N(t,u)\, \phi_k(u)\, du = \sigma_k^2\, \phi_k(t) \tag{47}$$

Equation (47) is satisfied by the set of functions defined by

$$\phi_k(t) = \frac{\sin 2\pi f_o (t - k\tau_o)}{2\pi f_o (t - k\tau_o)} \tag{48}$$

where $k = 0, \pm 1, \pm 2, \pm 3 \ldots\ldots$

The $\phi_k(t)$ are orthogonal over the interval $(-\infty, \infty)$. The characteristic values of the integral equation are equal and given by

$$\sigma_k^2 = N_o = \frac{N}{2 f_o} .$$

The noise N(t) can then be written as

$$N(t) = \sum_{k=-\infty}^{\infty} n_k \frac{\sin 2\pi f_o (t - k\tau_o)}{2\pi f_o (t - k\tau_o)} \tag{49}$$

and

$$N(k\tau_o) = n_k \tag{50}$$

The values of N(t) at the sampling instants $k\tau_o$ are the coefficients of the cardinal functions in the orthogonal expansion of N(t), $(-\infty < t < \infty)$.

Now back to the discussion of the likelihood ratio Λ. Assume $\langle F_n(v/S)\rangle_S$ and $F_n(v/0)$ are two N-variate normal distributions with the same covariance matrices but different means. The space Γ of observation v is N-dimensional vector space. Thus we put

$$\langle F_n(v/S)\rangle_S = \frac{1}{(2\pi)^{\frac{N}{2}} \sigma_1 \sigma_2 \ldots \sigma_N}\, e^{-\frac{1}{2} \sum_{k=1}^{N} \frac{(v_k - S_k)^2}{\sigma_k^2}} \tag{51}$$

$$F_n(v/0) = \frac{1}{(2\pi)^{\frac{N}{2}}\,\sigma_1\sigma_2\ldots\sigma_N}\; e^{-\frac{1}{2}\sum_{k=1}^{N}\frac{(v_k-0)^2}{\sigma_k^2}} \tag{52}$$

Substitute Eqs. (51) and (52) into Equation (40), and we get

$$\Lambda = \frac{P\left\langle F_n(v/S)\right\rangle_S}{q\,F_n(v/0)}$$

$$= \frac{P}{q}\ \mathrm{Exp} - \left[\frac{1}{2}\sum_{k=1}^{N}\frac{(v_k-S_k)^2}{\sigma_k^2} - \frac{1}{2}\sum_{k=1}^{N}\frac{(v_k-0)^2}{\sigma_k^2}\right]$$

$$= \frac{P}{q}\ \mathrm{Exp.}\left[-\frac{1}{2}\sum_{k=1}^{N}\frac{S_k^2}{\sigma_k^2} + \sum_{k=1}^{N} v_k\frac{S_k}{\sigma_k^2}\right] \tag{53}$$

It is more convenient mathematically and physically to express the likelihood ratio in terms of $\log\Lambda$. The detection threshold for $\log\Lambda$ is a monotonically increasing function of Λ.

From Eq. (53), when the noise is Graussian, the log of the likelihood ratio is

$$\log\Lambda = \log\frac{P}{q} - \left[\frac{1}{2}\sum_{k=1}^{N}\frac{S_k^2}{\sigma_k^2} - \sum_{k=1}^{N} v_k\frac{S_k}{\sigma_k^2}\right]$$

$$= \log\Lambda_o + \sum_{k=1}^{N} v_k\frac{S_k}{\sigma_k^2} \tag{54}$$

where

$$\log\Lambda_o = \log u - \frac{1}{2}\sum_{k=1}^{N}\frac{S_k^2}{\sigma_k^2} \tag{55}$$

and

$$u = \frac{P}{q}$$

8. The adoptive circuit arrangement

A proposed adoptive circuit arrangement is shown in Fig. 4. We know that the received signal can be expanded as:

$$S(t) = \sum_{k=1}^{\infty} S_k \, \phi_k(t) \tag{56}$$

Therefore, the radative energy ratio of signal to noise can be optimized by passing the observed quantity v(t) through a time invariant linear filter systems. The linear filters have the impulses responses $\phi_k(-t)$. In any real system there are only finite number of K of these orthogonal filters. The output of each filter passes through an operational amplifier with a fixed gain of $\frac{S_k}{\sigma_k}$. Then a summing circuit is employed to add all the components at the output of all the filter-amplifier units. Another term $\log \Lambda_o$ should be added into the sum of this point. From equation (55) we know that

$$\log \Lambda_o = \log u - \frac{1}{2} \sum_{k=1}^{\infty} \frac{S_k^2}{\sigma_k^2} \tag{57}$$

where $u = \frac{P}{q}$

since P, q, σ_k, and S_k^2 are known quantities. Therefore, $\log \Lambda_o$ is just a constant term. Thus the output of the summing circuit represents the log likelihood ratio. This then goes to an amplitude comparator which is controlled by the threshold bias K. The bias K is determined by the costs C_α, C_β, $C_{1-\alpha}$, and $C_{1-\beta}$. As mentioned before, one reasonable decision rule is let $K = 1$.

Therefore, the comparator produces a detected pulse when the output of the summing circuit exceeds that of the threshold bias K. Likewise, there will be no detected pulse when the output of the summing circuit is below the level of the threshold bias.

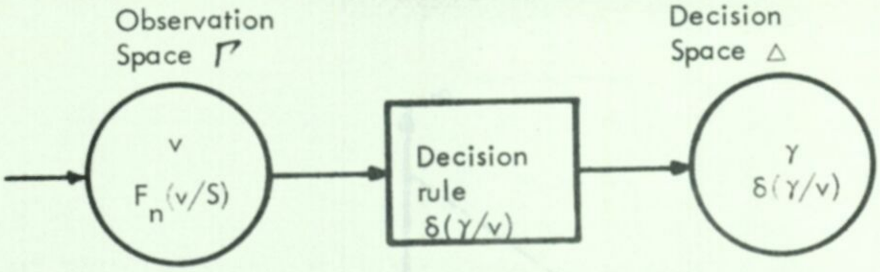


Fig. 1 - Observation and decision space

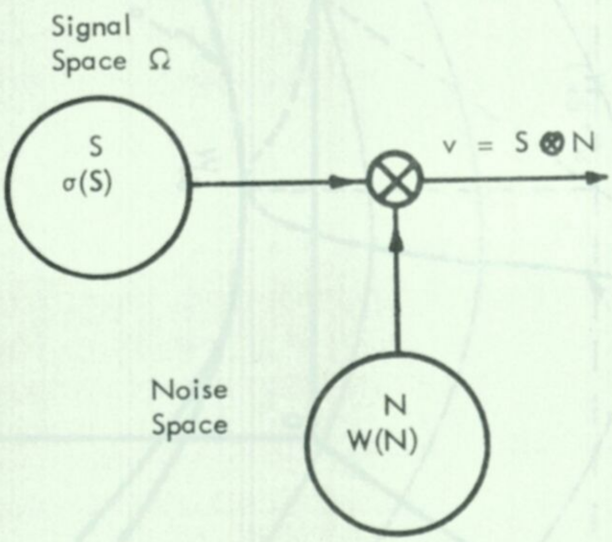


Fig. 2 - Signal Space and Noise Space

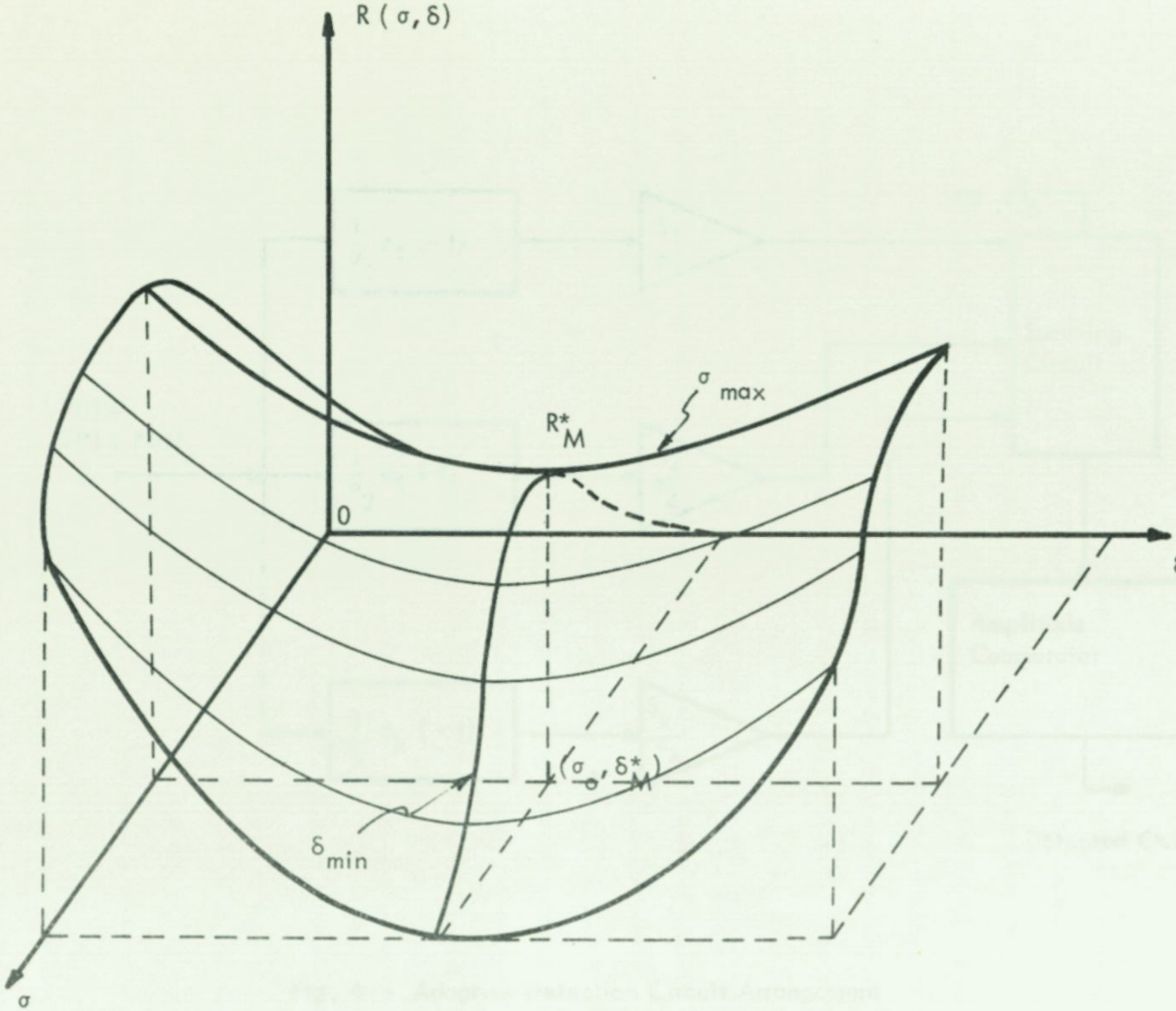


Fig. 3 - Average risk as a function of decision rule and a priori distribution.

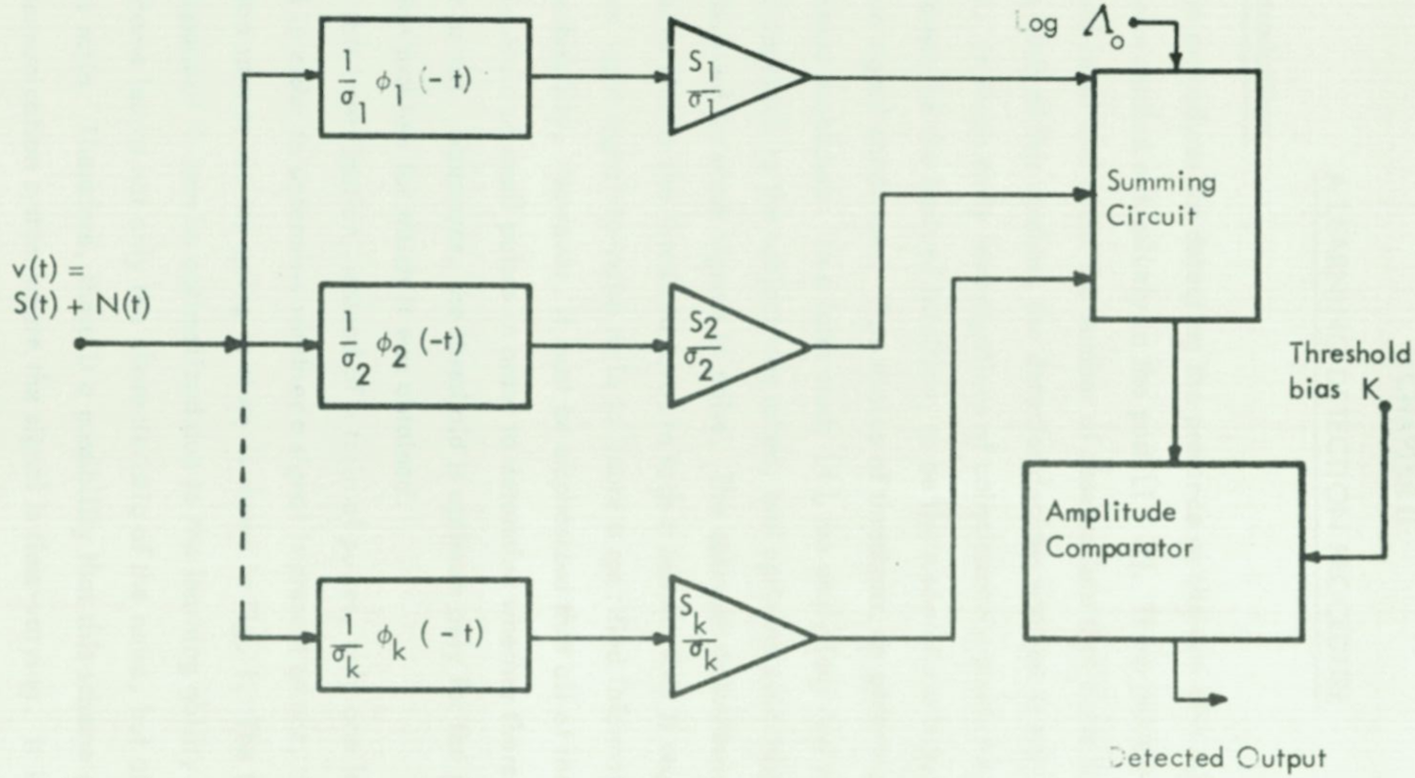


Fig. 4 - Adoptive Detection Circuit Arrangement

PART II

CHAPTER II

A LEARNING DETECTION PROCEDURE

I. Introduction

The procedures for detecting the presence or absence of a signal in noise have been studied extensively in the past [1 - 3]. These dectors choose a threshold level and count the number of observations that exceed this level. On the basis of this number, the detector decides whether or not there is a signal present. In these early investigations of coincidence procedures, the threshold was chosen, on the basis of intuition, to be the mean of the input waveform under no signal conditions. This choice of threshold, in general, leads to a suboptimum procedure. In a later study [4], an analytical and more sophisticated approach to the subject was taken, and optimum coincidence procedures were obtained for weak signals in noise. The optimum coincidence detection procedures choose the threshold level in such a manner that it requires the minimum input signal-to-noise ratio to insure a specified information rate and error probability. However, it must be emphasized that all of these papers studies a train of small pulses in order to determine whether there is a signal present or not. Moreover, the threshold is optimum only for the particular detection problem for which it was obtained.

In this investigation, instead of a train of pulses, only one large pulse is studied in order to determine whether a signal is present or not. The particular detection scheme we are going to study is shown in Fig. 1. The threshold of the comparator I can be optimalized due to the learning ability of the scheme. This scheme learns not only the characteristic of the noise, but also the signal to noise ratio. Therefore, there is a possibility that this scheme can be applied to a communication system where the signal is time-varying. It is for sure that this scheme can be applied to a quasi-stationary communication system.

II. The Detection Scheme

As shown in Fig. 1, the input of the detector consists of signal plus random noise. The gate opens only at the interval where the signal pulse would appear. Therefore, the function of the gate is to eliminate part of the noise which is not contaminated in the signal. Now, we only consider the signal plus noise combination as input to the threshold comparator I during the interval where the signal pulse would appear. We have,

$$i(t) = s(t) + n(t) \tag{1}$$

where
$i(t)$ = input time function
$n(t)$ = unknown random noise
$s(t)$ = unknown signal (may be time varying)

The output of the threshold comparator I is associated with the input $i(t)$ by the following relations:

$$O_1(t) = \begin{cases} A & \text{for } i(t) \geq \frac{S}{2} \\ o & \text{for } i(t) < \frac{S}{2} \end{cases} \tag{2}$$

$O_1(t)$ is then sent into the ideal integrating circuits. The output of the ideal integrator is:

$$O_2(t) = \int_o^t O_1(t)\ dt \tag{3}$$

Since the duration of the pulse, which is being investigated is T. Therefore, the integrator output at the end of the pulse interval is:

$$O_2(T) = \int_o^T O_1(t)\ dt \tag{4}$$

$O_2(T)$ is then sent into the comparator II. The output $O(t)$ of the threshold comparator II obeys the following relation:

$$O(t) = \begin{cases} 1 & \text{when } O_2(T) \geq K \\ O & \text{when } O_2(T) < K \end{cases} \qquad (5)$$

The value of K will be determined later.

III. Derivations and Calculations

Let us assume that the signal is quasi-stationary. That is, the signal amplitude does not change in a reasonable length of time. But, within a long period of time, the constant signal amplitude may change from time to time, due to the variation of transmission distance or numerous other reasons. Due to the same reasons, the random noise characteristics may change from time to time, too. Let us investigate and see what happens when a series of pulses or non-pulses is sent at the transmitting end. We want to find the variance of the output $O_2(T)$ of the integrator.

The variance of $O_2(T)$ is defined by the following equation:

$$\sigma^2_{O_2(T)} = \left\langle O_2^{\,2}(T) \right\rangle - \left\langle O_2(T) \right\rangle^2 \qquad (6)$$

From Eq. 4, we know that $O_2(T)$ is determined by:

$$O_2(T) = \int_o^T O_1(S, t)\, dt \qquad (7)$$

The probability of error of $O_1(t)$ at the output of the threshold comparator I is:

$$P\left(\frac{S}{2}\right) = P\left\{ n(t) \geq \frac{S}{2} \right\} = P\left\{ n(t) \leq -\frac{S}{2} \right\}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{\frac{S}{2}}^{\infty} e^{-\frac{n^2}{2\sigma^2}}\, dn \qquad (8)$$

where σ = variance of the noise

Therefore, the mean value of $O_2(T)$ is,

$$\left\langle O_2(T) \right\rangle = A\,T\,\frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\frac{S}{2}} e^{-\frac{n^2}{2\sigma^2}}\, dn$$

$$= A\,T\,P\left(\frac{S}{2}\right) \tag{9}$$

Hence, the square of the mean value is:

$$\left\langle O_2(T) \right\rangle^2 = A^2\,T^2\,P^2\left(\frac{S}{2}\right) \tag{10}$$

The mean square value of $O(T)$ is :

$$\left\langle O^2(T) \right\rangle = \left\langle \int_0^T O_1(t_1)\, dt_1 \int_0^T O_1(t_2)\, dt_2 \right\rangle$$

$$= \int_0^T \int_0^T \left\langle O_1(t_1)\, O_1(t_2) \right\rangle dt_1\, dt_2$$

$$= \int_0^T \int_0^T \phi_{O_1}(t_2 - t_1)\, dt_1\, dt_2 \tag{11}$$

$$= 2 \int_0^T (T - \tau)\, \phi_{O_1}(\tau)\, d\tau$$

Substituting Eqs. 11 and 10 into Eq. 6, we get:

$$\sigma^2_{O_2(T)} = \left\langle O_2^{\,2}(T) \right\rangle - \left\langle O_2(T) \right\rangle^2$$

$$= 2 \int_0^T (T - \tau)\, \phi_{O_1}(\tau)\, d\tau - A^2\,T^2\,P^2\left(\frac{S}{2}\right) \tag{12}$$

$\phi_{O_1}(\tau)$ can be determined from the bivariate normal density distribution function $W(X, X_\tau)$. We have,

$$\phi_{O_1}(\tau) = \int_{-\infty}^{\frac{S}{2}} \int_{-\infty}^{\frac{S}{2}} A^2 \, W(X , X_\tau) \, dx \, dx_\tau \qquad (13)$$

where

$$W(X , X_\tau) = \frac{1}{2\pi\sigma^2\sqrt{1 - C(\tau)}} \; e^{-\left\{ \frac{X^2 + X_\tau{}^2 - 2C(\tau)\,X\,X_\tau}{2\sigma^2\,[1 - C(\tau)]} \right\}} \qquad (14)$$

Where $C(\tau)$ is the correlation coefficient. Eq. 13 can be expanded into a series in powers of $C(\tau)$, [5].

$$\phi_{O_1}(\tau) = A^2 \sum_{0}^{\infty} \frac{\left[\Phi^{(\nu - 1)}\left(\frac{S}{2\sigma}\right)\right]^2}{\nu!} \left[C(\tau)\right]^{\nu} \qquad (15)$$

where

$$\Phi^{\nu}\left(\frac{S}{2\sigma}\right) = \left[\frac{1}{\sqrt{2\pi}} \; e^{-\frac{1}{2}\left(\frac{S}{2\sigma}\right)^2}\right] H_\nu\left(\frac{S}{2\sigma}\right)(-1)^{\nu} \qquad (16)$$

$H_\nu(X)$ are Hermite polynomials. We have

$$H_o(X) = 1$$

$$H_1(X) = (X)$$

$$H_2(X) = (X)^2 - 1$$

$$H_3(X) = (X)^3 - 3(X)$$

$$H_4(X) = (X)^4 - 6(X)^2 + 3$$

$$H_5(X) = (X)^5 - 10(X)^3 + 15(X)$$

$$H_6(X) = (X)^6 - 15(X)^4 + 45(X)^2 - 15$$

$$H_7(X) = (X)^7 - 21(X)^5 + 105(X)^3 \;\; 105(X)$$

$$H_8(X) = (X)^8 - 28(X)^6 + 210(X)^4 - 420(X)^2 + 105$$

$$H_9(X) = (X)^9 - 36(X)^7 + 378(X)^5 - 1260(X)^3 + 945(X)$$

Now, Eq. 12 can be written as:

$$\sigma^2_{O_2(T)} = 2\int_o^T (T-\tau)\ \phi_{01}(\tau)\,dt - A^2 T^2 P^2\left(\frac{S}{2}\right)$$

$$= A^2\left[2\int_o^T (T-\tau)\sum_o^\infty \frac{\left[\Phi(\nu-1)\left(\frac{S}{2\sigma}\right)\right]^2}{\nu!}\left[C(\tau)\right]^\nu d\tau - T^2 P^2\left(\frac{S}{2}\right)\right. \tag{17}$$

for $\nu = 0$

$$2\int_o^T (T-\tau)\ \frac{\Phi(\nu-1)\left(\frac{S}{2\sigma}\right)}{\nu!}\ \left[C(\tau)\right]^\nu d\tau$$

$$= 2\int_o^T (T-\tau)\left[\frac{1}{\sqrt{2\pi}\,\sigma}\ e^{-\frac{1}{2}\left(\frac{S}{2\sigma}\right)^2}\right]^2 d\tau \tag{18}$$

$$= 2\int_o^T (T-\tau)\,P^2\left(\frac{S}{2}\right) d\tau$$

$$= T^2 P^2\left(\frac{S}{2}\right)$$

Substituting Eq. 18 into Eq. 17, we get:

$$\sigma^2_{O_2(T)} = 2A^2\int_o^T (T-\tau)\left\{\sum_{\nu=1}^{\infty}\frac{\left[\Phi(\nu-1)\left(\frac{S}{2\sigma}\right)\right]^2}{\nu!}\left[C(\tau)\right]^\nu\right\} d\tau \tag{19}$$

The correlation coefficient $C(\tau)$ of the random noise is usually in the following form:

$$C(\tau) = e^{-\beta|\tau|} \tag{20}$$

where β can be determined from the received data of the random noise.

Substituting Eq. 20 into Eq. 19, we get,

$$\sigma^2_{O_2(T)} = 2A^2 \sum_{\nu=1}^{\infty} \left\{ \frac{\left[\Phi(\nu-1)\left(\frac{S}{2\sigma}\right)\right]^2}{\nu!} \int_o^T (T-\tau)\, e^{-\nu\beta|\tau|}\, dt \right\} \tag{21}$$

Since,

$$\int_o^T (T-\tau)\, e^{-\nu\beta|\tau|}\, d\tau$$

$$= \frac{1}{\beta\nu} \left[(T-\tau)\, e^{-\beta\nu|\tau|} \Big/_o^T + \int_o^T e^{-\beta\nu|\tau|}\, d\tau \right]$$

$$= \frac{1}{-\beta\nu} \left[(T-\tau)\, e^{-\beta\nu|\tau|} - \frac{1}{\beta\nu}\, e^{-\beta\nu|\tau|} \right]_o^T \tag{22}$$

$$= \frac{1}{\beta^2\nu^2} \left[\beta\nu T + e^{-\beta\nu T} - 1 \right]$$

Substituting Eq. 22 into Eq. 21, we get,

$$\sigma^2_{O_2(T)} = \frac{2A^2}{\beta^2} \sum_{\nu=1}^{\infty} \left\{ \frac{\left[\Phi(\nu-1)\left(\frac{S}{2\sigma}\right)\right]^2}{\nu!} \; \frac{\left[\beta\nu T + {}^{-\beta\nu T} - 1\right]}{\nu!} \right\} \tag{23}$$

A reasonable threshold level for the comparable II is $\frac{AT}{2}$. This is also the appropriate value of K in Eq. 5. Therefore, the equivalent output signal to noise ratio is:

$$\left(\frac{S}{N}\right)_o = \frac{\frac{AT}{2}}{\sigma_{O_2(T)}}$$

$$= \frac{\beta T}{2\sqrt{2} \sum_{V=1}^{\infty} \left\{ \frac{\left[{}_{\Phi}(\nu-1)\left(\frac{S}{2\sigma}\right)\right]^2}{\nu\,!} \frac{\left[\beta\nu T + e^{-\beta\nu T} - 1\right]}{\nu^2} \right\}^{1/2}} \tag{24}$$

A computer program has been compiled to compute Eq. 24 using the first ten terms (i.e. from $\nu = 1$ to $\nu = 10$) as an approximation of $\left(\frac{S}{N}\right)_o$. The accuracy of the approximating calculation is within 1%. Fig. 2 is a plot of a family of curves of signal to noise ratio of output $\left(\frac{S}{N}\right)_o$ versus signal to noise ratio of input $\left(\frac{S}{2\sigma}\right)$ using βT as a parameter. Figure 3 is a family of curves which plots $\left(\frac{S}{N}\right)_o$ versus βT using $\left(\frac{S}{2\sigma}\right)$ as parameter.

IV. Conclusion

For a given communication system, the pulse interval T is already a fixed value designed to meet the requirement of the system. β is a quantity which we can readily measure at the receiving end. Therefore, the only unknown variable of Eq. 24 is the input signal to noise ratio. The equivalent output signal to noise ratio can be expressed as:

$$\left(\frac{S}{N}\right)_o = \frac{\frac{AT}{2}}{\sigma_{O_2(T)}} \tag{25}$$

The exact value of $\sigma_{O_2(T)}$ can be measured at the integrator output $O_2(T)$. Therefore, $\left(\frac{S}{N}\right)_o$ can be easily calculated. Thus, the input signal to noise can be determined. This means that the detection scheme is learning all the time the input signal to noise ratio. In other words, there must be an optimum setting of the threshold level of comparator I. The optimum setting is determined by the learning scheme such that it gives a minimum variance $\sigma_{O_2(T)}$ at the output of the integrator. If the learning process is very fast, then this scheme can be applied to detect the time-varying signal. Conservatively speaking, the scheme can certainly apply to quasi-stationary signal detection.

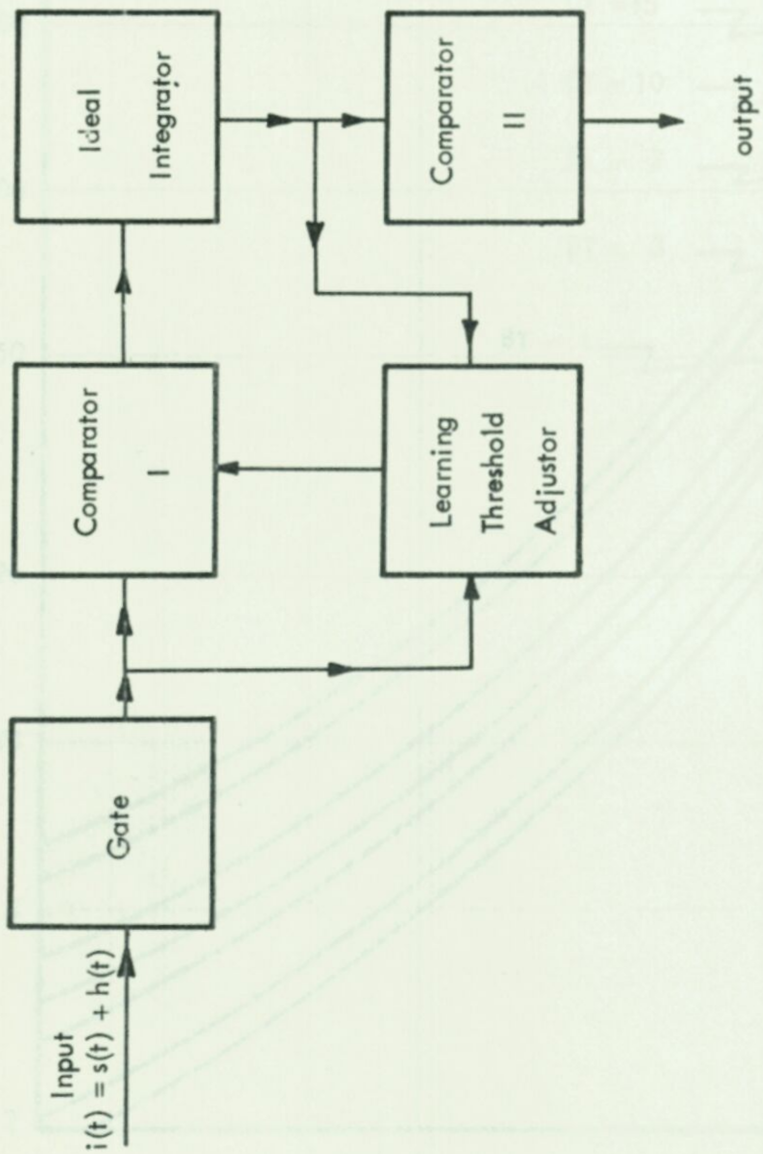
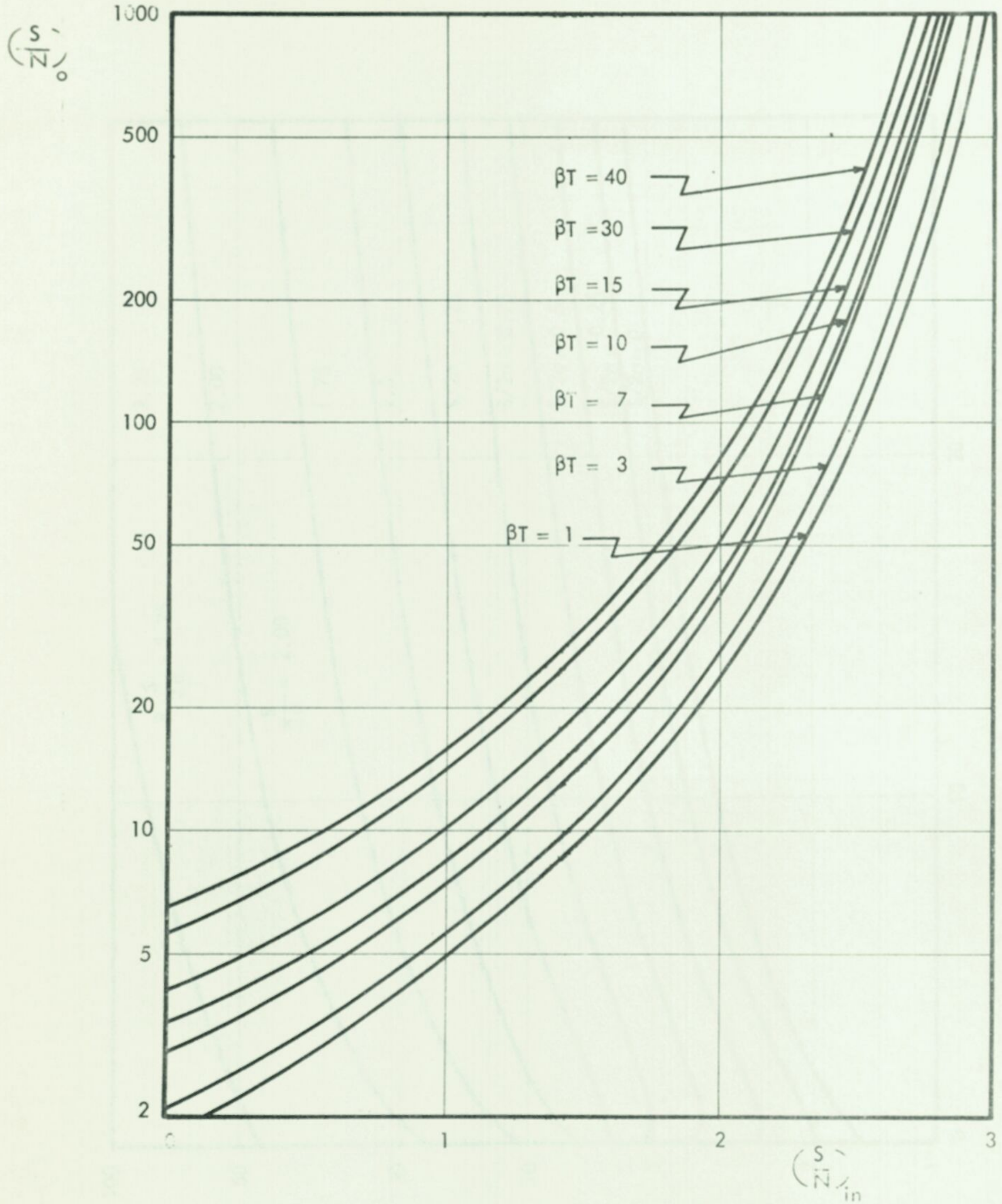
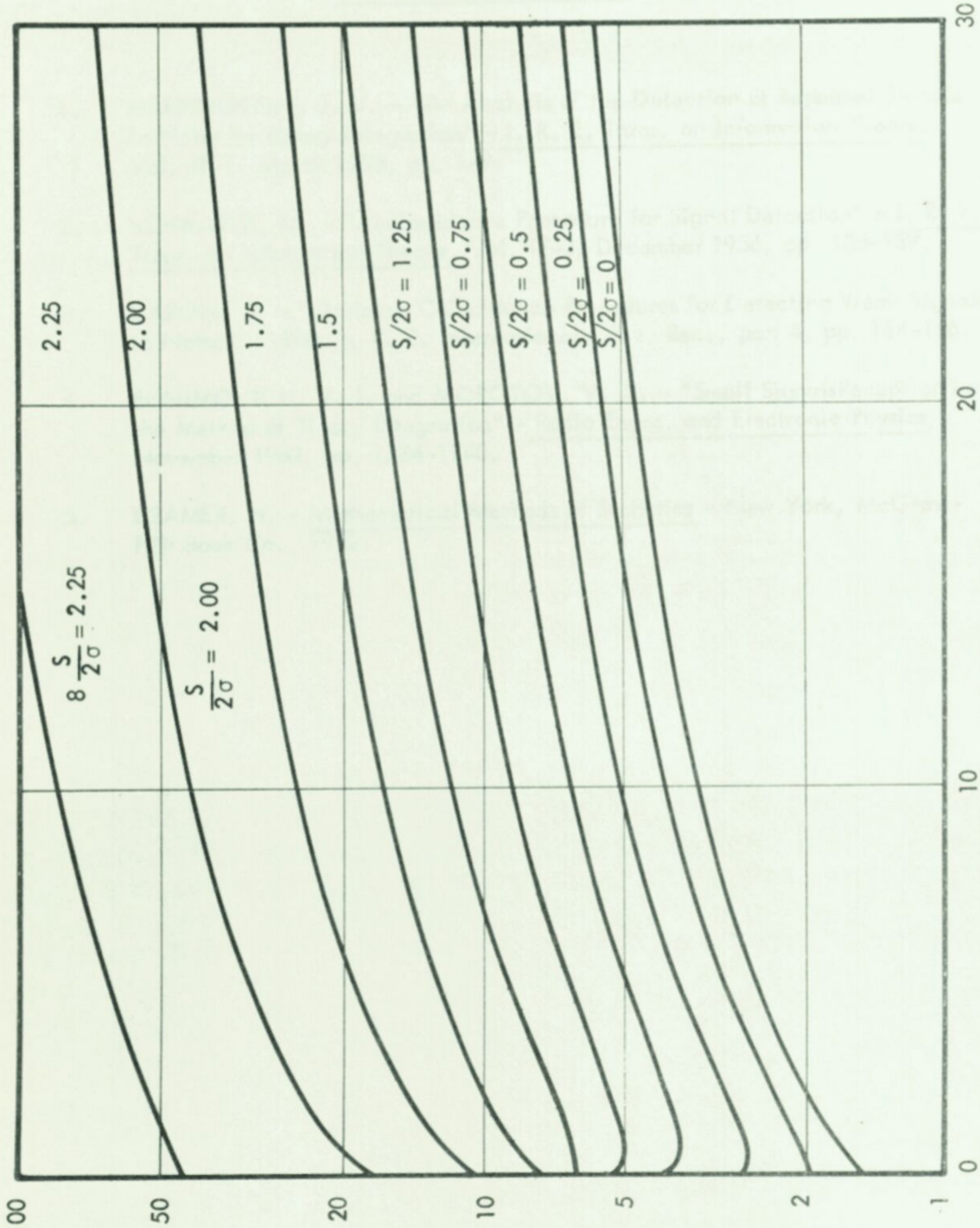

Input
i(t) = s(t) + h(t)
Gate
Comparator
I
Learning
Threshold
Adjustor
Ideal
Integrator
Comparator
II
output

Figure 1



$\left(\frac{S}{N}\right)_o$
1000
500
200
100
50
20
10
5
2
βT = 40
βT = 30
βT = 15
βT = 10
βT = 7
βT = 3
βT = 1
0
1
2
3
$\left(\frac{S}{N}\right)_{in}$

Figure 2


2.25
2.00
1.75
1.5
S/2σ = 1.25
S/2σ = 0.75
S/2σ = 0.5
S/2σ = 0.25
S/2σ = 0
8 S/2σ = 2.25
S/2σ = 2.00
100
50
20
10
5
2
1
0
10
20
30

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