6-1-1966

Unsteady Heat Transfer from a Rotating Disk Due to a Step Change in Local Wall Heat Flux with Large Suction UARI Research Report No. 27

D. R. Jeng

Y. V. Subba Rao

Masters

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UNSTEADY HEAT TRANSFER FROM A ROTATING DISK DUE TO A STEP CHANGE IN LOCAL WALL HEAT FLUX WITH LARGE SUCTION

by

D. R. Jeng and Y. V. Subba Rao

The research for this report was supported by the
National Aeronautics and Space Administration
under research grant NsG-381

UNIVERSITY OF ALABAMA RESEARCH INSTITUTE
Huntsville, Alabama
June 1966

Masters

October 14, 1966

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Attention: Dr. T. L. K. Smull

Dear Dr. Smull:

Enclosed are five (5) copies of the University of Alabama Research Institute Research Report No. 27 entitled, "Unsteady Heat Transfer From a Rotating Disk Due to a Step Change in Local Wall Heat Flux with Large Suction", by D. R. Jeng and Y. V. Subba Rao. The research for this report was supported by the National Aeronautics and Space Administration Grant NsG-381.

Sincerely yours,

W. P. W.

W.P. Watts
Administrative Manager

Enclosures (5)

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October 12, 1966

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Attention: AMSMI-G/Mr. R. R. Finney

References: (a) Final Technical Report on Army Missile Command Contract DA-01-021-AMC-14042(Z) entitled "Multi-Rail Launcher with Six Degrees of Freedom"

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Dear Sir:

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Pursuant to the requirements of reference (b), we are forwarding two copies of the report.

Sincerely,

W. P. Watts
Administrative Manager

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cc: Chief of Information, Department of the Army
Attn: Civil Liaison Division

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Abstract

An analysis is made for the unsteady heat-transfer, due to a time dependent wall heat flux, from an infinite disk, rotating in still fluid with large suction. The procedure begins with a consideration of the thermal response caused by a step change in the local wall heat flux. It is found that the response time varies inversely with the Prandtl number, the angular velocity and the second power of the suction parameter.

Incompressible flow with dissipation is considered first. For an ideal gas with constant Prandtl number whose viscosity varies linearly with temperature, the incompressible results may be, if dissipative effect can be ignored, directly applied provided that all properties in the solution are replaced by their wall values.

Nomenclature

a_o	Mass transfer parameter, $-\frac{w_w}{(\nu\,\omega)^{1/2}}$, dimensionless
c_p	Specific heat at constant pressure, J/kg ^oK
erf x	error function, $\frac{2}{\sqrt{\pi}}\int_o^{\infty} e^{-\xi^2} d\xi$
F, G, H	function defined by (6)
erfc x	1 - erf x
k	thermal conductivity, J/m sec ^oK
p	parameter in Laplace transform
p	static pressure, n/m^2
P_r	Prandtl number, dimensionless
q_w	Wall surface heat flux, J/sec m^2
$Q(\eta)$	function defined in (11) or (15)
r	radial coordinate for rotating disk, m
R	dimensionless radial coordinate, $r(\frac{\omega}{\nu})^{1/2}$
$S(\eta)$	function defined in (11) or (15)
t	time, sec.
T	temperature, ^oK
u	velocity parallel to the surface of a disk, m/sec
v	velocity in φ direction, m/sec
w	velocity normal to the surface of a disk, m/sec

z	coordinate measuring distance normal to a disk, m	
η	dimensionless coordinate	$\eta = Z(\frac{\omega}{\nu})^{1/2}$
θ	dimensionless temperature defined by (11) or (15)	
μ	viscosity, kg/m sec	
ν	kinematic viscosity, m^2/sec	
ρ	density, kg/m^3	
τ	dimensionless time, $\frac{\omega t}{P_r}$	
ω	angular velocity, rad/sec	
φ	the angular position	
$1(t)$	Heaviside unit operator; $= 0$ for $t < 0$, and $= 1$ for $t \geq 0$	

Subscripts

i	initial conditions
ss	steady state
w	condition at wall surface
∞	free stream condition

1.0 Introduction

Unsteady heat transfer due to a time prescribed wall temperature or heat flux has been a subject of interest for many years. Sparrow and Gregg [1,2] investigated the laminar forced convection heat transfer from a compressible fluid to a flat plate with uniform, but time dependent, surface temperature. Cess [3] and Riley [4] examined the same problem for incompressible flow, Goodman [5] and Adams and Gebhart [6] have employed the heat balance integral technique to obtain approximate solutions for the flat plate problem. Sparrow [7] reported an approximate analysis for the unsteady, two dimensional stagnation point heat transfer. He employed the integral technique for which a third degree polynomial was chosen for the unsteady temperature profile. Subsequently Chao and Jeng [8] published an analysis for the unsteady heat transfer at a two-dimensional and axisymmetrical front stagnation point due to an arbitrarily prescribed wall temperature or heat flux. The analysis was extended by Jeng [9] to a three-dimensional magnetohydrodynamic stagnation point flow with simultaneous suction or blowing. The heat transfer from rotating bodies is of technological interest and has attracted the attention of several researchers. The steady heat transfer from a rotating disk was first studied by Millsaps and Pohlhausen [10] and later by Sparrow and Gregg [11] who also investigated the effect of blowing and suction [12]. Cess and Sparrow [13] were probably the first to analyze the unsteady heat transfer from a rotating disk. The problem was later re-examined by Jeng [9] who also included the effects of mass transfer. Two asymptotic solutions, respectively valid for small and large times, are found and satisfactorily joined to cover a wide range of Prandtl numbers. When the suction velocity becomes sufficiently large, a closed-form solution can be obtained. The analysis for this case was also made in [9] but for a step change in wall temperature. In the present report, we investigate the same unsteady heat transfer problem but with the step change in wall heat flux. The method of solution adopted in the present analysis closely parallels to that used in [9] but the analysis becomes more complicated. We first consider a case of incompressible flow. A solution describing the entire time history of the non-steady temperature field has been obtained. For an ideal gas with constant Prandtl number whose viscosity varies linearly with temperature, it is shown that the incompressible result may be directly applicable to compressible case.

2.0 Governing Equations for Incompressible Flow

Consider an infinite disk rotating in an infinite mass of fluid about an axis normal to its own plane and at a constant angular velocity ω. Fig. 1 illustrates the cylindrical coordinate (r, φ, z) and the corresponding velocity components (u, v, w) appropriate for the problem. From physical considerations, one sees that the velocity and temperature field would be independent of φ, if the thermal condition at the disk surface is also independent of φ. Under the assumption of steady, incompressible flow with constant properties, the governing equations are (with dissipation included):

Continuity:

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0 \tag{1}$$

Momentum:

$$u \frac{\partial u}{\partial r} - \frac{v^2}{r} + w \frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu \left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{\partial r}\left(\frac{u}{r}\right) + \frac{\partial^2 u}{\partial z^2} \right] \tag{2.a}$$

$$u \frac{\partial v}{\partial r} + \frac{uv}{r} + w \frac{\partial v}{\partial z} = \nu \left[\frac{\partial^2 v}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{v}{r}\right) + \frac{\partial^2 v}{\partial z^2} \right] \tag{2.b}$$

$$u \frac{\partial w}{\partial r} + w \frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu \left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2} \right] \tag{2.c}$$

Energy:

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial r} + w \frac{\partial T}{\partial z} = \alpha \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial z^2} \right] + \frac{2\nu}{c_p}\left[\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{u}{r}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 \right]$$

$$+ \frac{\nu}{c_p}\left[\left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial r} + \frac{\partial u}{\partial z}\right)^2 + \left\{ r \frac{\partial}{\partial r}\left(\frac{v}{r}\right)\right\}^2 \right] \tag{3}$$

The velocity boundary conditions are:

$$\begin{aligned} &u(r,0) = 0 && u(r,\infty) = 0 \\ &v(r,0) = r\omega && v(r,\infty) = 0 \\ &w(r,0) = w_w && \end{aligned} \tag{4}$$

In the above, w_w is the fluid velocity at the wall. Positive values of w_w correspond to blowing, and negative values correspond to suction.

For time $t < 0$, there will be a steady temperature field in the fluid due to frictional dissipation. Its precise nature would depend on conditions prevailing in the disk. As an example, one may consider a disk of infinite thermal conductivity with the consequence that its temperature rise due to the frictional dissipation is uniform. On the other hand, the disk may have large resistance to heat flow in the radial direction. This situation contrasts with that for flow over a flat plate. With these observations, we consider two cases of simple initial and boundary conditions for the temperature field as follows:

Case (i) For disks initially at adiabatic wall temperature.

$$\begin{aligned} &T(r,z,0) = T_{i,1}(r,z) \\ &-\frac{\partial T(r,0,t)}{\partial z} = \frac{q_w}{k}\, 1(t) \\ &T(r,\infty,0) = T_\infty \\ &\frac{\partial T(r,0,0)}{\partial z} = 0 \end{aligned} \tag{5.a}$$

In the above $T_{i,1}(r,z)$ represents the initial temperature distribution in the fluid consistent with an adiabatic wall temperature T_{aw}. Here the problem is to examine the transient response of the wall temperature due to a step change in wall heat flux.

Case (ii) For disks initially of uniform wall temperature

$$T(r,z,0) = T_{i,2}(r,z)$$

$$-\frac{\partial T(r,0,t)}{\partial z} = \frac{q_{iw}(r)}{k} + \frac{1}{k}\left[q_w(r) - q_{iw}(r)\right] 1(t) \qquad (5.b)$$

$$T(r,\infty,t) = T_\infty$$

$$T(r,0,0) = T_{iw} = \text{constant}$$

Where $q_{iw}(r)$ is the wall flux due to frictional dissipation.

The initial temperature field $T_{i,2}$ is clearly different from that in Case (i). It is to be noted that for a disk of uniform temperature, the wall flux q_{iw} is not uniform. Here we are examining the transient response of the wall temperature due to a uniform, step change of the local wall flux. While $q_w(r)$ and $q_{iw}(r)$ both vary with r, their difference is a constant being independent of the radius.

3.0 Method of Solution

The solution of (1), (2), and (3) with the initial and the boundary conditions (4) and (5) for moderate suction is given in [9]. In this report we consider large values of suction. Physical considerations suggest expressing the velocity components u and v as:

$$u = r\,\omega\,F(\eta) \qquad , \qquad v = r\,\omega\,G(\eta) \qquad (6.a)$$

wherein $\eta = z\left(\frac{\omega}{\nu}\right)^{1/2}$.

It follows, then, from the continuity requirement that

$$w = (\omega\,\nu)^{1/2}\,H(\eta) \qquad (6.b)$$

Upon introducing the foregoing expressions for the velocity components into the momentum equations and observing that the pressure must necessarily approach a constant value at infinity, one is led to the conclusion that P is independent of r and may be expressed as

$$p = \rho\,\omega\,\nu\,P(\eta) \qquad (6.c)$$

Substituting (6.a, b, and c) into (1) and (2.a, b, and c) yields, after some rearrangement,

$$G'' = HG' - H'G \tag{7.a}$$ *

$$H''' = HH'' - \frac{1}{2}(H')^2 + 2G^2 \tag{7.b}$$ *

and $$F = -\frac{H'}{2} \tag{7.c}$$ *

with

$$G(0) = 1, \qquad H(0) = -a_0 = \frac{w_w}{(\omega\,\nu)^{1/2}}, \qquad H'(0) = 0$$
$$G(\infty) = 0, \qquad H'(\infty) = 0 \tag{8}$$

In the foregoing, the prime denotes differentiation with respect to η ; a_o is the mass transfer parameter. Numerical solutions of the above set of equations for an impervious wall, i.e. $a_o = 0$, were given by Cochran [14]. Stuart presented an analytical solution for large suction. His result will be used later for the integration of the unsteady energy equation in the present analysis.

Sparrow and Gregg [11] reported computer results for a_o ranging from -5 to 4.

To transform the energy equation, we further let

$$R = r\left(\frac{\omega}{\nu}\right)^{1/2}, \qquad \tau = \frac{\omega\, t}{P_r} \tag{9}$$

and obtain

$$\frac{\partial T}{\partial \tau} + P_r \cdot R \cdot F\frac{\partial T}{\partial R} + P_r \cdot H \cdot \frac{\partial T}{\partial \eta} = \frac{1}{R}\frac{\partial}{\partial R}\left(R \cdot \frac{\partial T}{\partial R}\right) + \frac{\partial^2 T}{\partial \eta^2}$$

$$+ \frac{P_r\,\omega\,\nu}{C_p} R^2\left[(G')^2 + (F')^2\right] + \frac{2\,P_r\,\omega\,\nu}{C_p}\left[2F^2 + (H')^2\right] \tag{10}$$

---
* A fourth equation which governs the pressure distribution in the fluid is not of our concern and is thus omitted.

We shall first consider the case for the adiabatic wall temperature with initial and boundary conditions prescribed by (5.a). For convenience, we introduce a dimensionless temperature defined by

$$\frac{T - T_\infty}{\frac{qw}{k}\left(\frac{\nu}{\omega}\right)^{1/2}} = \frac{\omega\ \nu}{C_p \frac{qw}{k}\left(\frac{\nu}{\omega}\right)^{1/2}} \ (R^2 S + Q) + \theta(\eta,\tau) \tag{11}$$

The functions $S(\eta)$, $Q(\eta)$, and $\theta(\eta,\tau)$ respectively, satisfy

$$S'' - P_r \cdot H \cdot S' + P_r \cdot H' \cdot S = - P_r \left[(G')^2 + (F')^2 \right] \tag{12.a}$$

$$\text{with} \quad S'(0) = 0 \quad , \quad S(\infty) = 0 \tag{12.b}$$

$$Q'' - P_r \cdot H \cdot Q' = - 4\ (S + 3\ P_r \cdot F^2) \tag{13.a}$$

$$\text{with} \quad Q'(0) = 0 \quad , \quad Q(\infty) = 0 \tag{13.b}$$

and

$$\frac{\partial\theta}{\partial\tau} + P_r \cdot H \frac{\partial\theta}{\partial\eta} = \frac{\partial^2\theta}{\partial\eta^2} \tag{14.a}$$

$$\text{with} \quad \theta(\eta,0) = 0 \qquad \frac{\partial\theta(0,\tau)}{\partial\eta} = -1(\tau), \qquad \theta(\infty,\tau) = 0 \tag{14.b}$$

For disks initially at the uniform wall temperature, T_{iw} subjected to a uniform, step change in wall flux, we let

$$\frac{T - T_\infty}{\left(\frac{\nu}{\omega}\right)^{1/2} \frac{\Delta q_w}{k}} = \frac{\omega\ \nu}{C_p \frac{\Delta q_w}{k}\left(\frac{\nu}{\omega}\right)^{1/2}} \left[R^2 S(\eta) + Q(\eta) \right] + \theta(\eta,\tau) \tag{15}$$

Here $\Delta q_w = q_w(R) - q_{iw}(R)$ which is a constant. The functions S and Q, respectively, satisfy (12.a) and (13.a) but with boundary conditions altered to

$$S(0) = 0 \quad , \quad S(\infty) = 0 \tag{16.a}$$

and

$$Q(0) = \frac{cp}{\omega\nu}\ (T_{iw} - T_\infty) \quad , \quad Q(\infty) = 0 \tag{16.b}$$

It is easy to show that θ of Eqn. 15 satisfies Eqn. 14.a with initial and boundary conditions of Eqn. 14.b.

Finally, we note that for negligible dissipation the fluid will initially have a uniform temperature T_∞, which is also the wall temperature prior to any thermal disturbance. If we now re-define $\theta(\eta,\tau)$ according to

$$\theta = \frac{T - T_\infty}{\frac{q_w}{k}\left(\frac{\nu}{\omega}\right)^{1/2}}$$

it is easy to demonstrate that θ again satisfies (14.a) with initial and boundary conditions (14.b).

To integrate (14.a) with (14.b) for large values of the suction parameter (say a_o = 1.5 or greater), we employ the Stuart's solution [15] for the velocity field in inverse powers of a_o. In particular, the H function is given as,

$$H(\xi) = -a_o + \frac{1}{a_o^3}\left(-\frac{1}{2} + e^{-\xi} - \frac{1}{2}e^{-2\xi}\right)$$
$$+ \frac{1}{a_o^7}\left[\frac{201}{288} + \left(\frac{-1}{2}\xi - \frac{95}{72}\right)e^{-\xi} + \left(\frac{1}{2}\xi + \frac{13}{24}\right)e^{-2\xi}\right.$$
$$\left. + \frac{1}{12}e^{-3\xi} - \frac{1}{288}e^{-4\xi}\right] + O\left(a_o^{-11}\right) \qquad (17)$$

wherein $\xi = a_o\eta$

Application of Laplace transform to (14.a) with initial and boundary conditions (14.b) yields

$$\bar{\theta}'' - P_r \,.\, H \,.\, \bar{\theta}' = p\bar{\theta} \qquad (18)$$

with $\bar{\theta}'(0) = -\frac{1}{p}$ and $\bar{\theta}(\infty) = 0$. We now introduce a new variable $\bar{Y}(\eta,p)$ defined by

$$\bar{Y} = p\bar{\theta}\,\exp\left(-\frac{P_r}{2}\int_0^\eta H(\eta)d\eta\right) \qquad (19)$$

and obtain from (18)

$$\bar{Y}'' + \left(-p + \frac{P_r}{2} H' - \frac{P_r^{\,2}}{4} H^2\right) \bar{Y} = 0 \tag{20}$$

with

$$-\frac{a_o P_r}{2} \bar{Y}(0) + a_o \bar{Y}'(0) = -1, \quad \bar{Y}(\infty) = 0$$

In (18) and (20) the prime denotes differentiation with respect η. Transforming the independent variable from η to ξ and making use of (17), for large a_o, we arrive at,

$$\frac{d^2\bar{Y}(\xi)}{d\xi^2} - \left(C_0^{\,2} + C_1 e^{-\xi} + C_2 e^{-2\xi} + C_3 e^{-3\xi} + \ldots\right) \bar{Y}(\xi) = 0 \tag{21}$$

wherein

$$C_0^{\,2} = \frac{p}{a_o^{\,2}} + \frac{P_r^{\,2}}{4}\left[1 + \frac{1}{a_o^{\,4}} + 0\left(\frac{1}{a_o^{\,8}}\right)\right], \quad C_1 = \frac{P_r(1 - P_r)}{2\, a_o^{\,4}}\left[1 + 0\left(\frac{1}{a_o^{\,4}}\right)\right]$$

$$C_2 = \frac{P_r\left(P_r - 2\right)}{4a_o^{\,4}}\left[1 + 0\left(\frac{1}{a_o^{\,4}}\right)\right], \quad C_3 = \frac{P_r\left(1 - \frac{7}{3} P_r\right)}{8a_o^{\,8}}\left[1 + 0\left(\frac{1}{a_o^{\,4}}\right)\right] \text{ etc.}$$

The solution of (21) satisfying the boundary condition

$-\frac{a_o P_r}{2} \bar{Y}(0) + a_o \bar{Y}'(0) = -1$ and $\bar{Y}(\infty) = 0$ may be appropriately represented by a series of the form

$$\bar{Y} = K e^{-C_0\xi}\left(1 + A_1 e^{-\xi} + A_2 e^{-2\xi} + A_3 e^{-3\xi} + \ldots\right) \tag{22}$$

where

$$K = \frac{1}{a_o\left[\left(\frac{P_r}{2} + C_0\right)\left(1 + A_1 + A_2 + A_3 + \ldots\right) + \left(A_1 + 2A_2 + 3A_3 + \ldots\right)\right]}$$

$$A_1 = \frac{C_1}{1 + 2C_0}, \qquad A_2 = \frac{C_1^{\,2} + C_2(1 + 2\,C_0)}{4\,(1 + 2\,C_0)\,(1 + C_0)}$$

$$A_3 = \frac{C_1^{\,3} + C_1C_2(5 + 6\,C_0) + 4C_3(1 + 2\,C_0)\,(1 + C_0)}{12\,(1 + 2C_0)\,(1 + C_0)\,(3 + 3\,C_0)}, \text{ etc.}$$

The latter are obtained by substituting (22) into (21) and equating coefficients of terms like $e^{-(C_0 + 1)\xi}$, $e^{-(C_0 + 2)\xi}$ . . . . , to zero. Since $C_1 = 0\,(\frac{1}{a_0^{\,4}})$, $C_2 = 0\,(\frac{1}{a_0^{\,4}})$, but $C_3 = 0\,(\frac{1}{a_0^{\,8}})$, it is a valid approximation to ignore terms involving $C_1^{\,2}$, C_1C_2, C_3, and $C_1^{\,3}$. Accordingly, after substituting the values of the A_n, K may also be written as

$$K = \frac{\frac{1}{2a_0}\left(1 + 2\,C_0\right)\left(1 + C_0\right)}{C_0^{\,3} + \left(\frac{3}{2} + \frac{C_1}{2} + \frac{C_2}{4} + \frac{P_r}{2}\right)C_0^{\,2} + \left(\frac{1}{2} + C_1 + \frac{5}{8}C_2 + \frac{3P_r}{4} + \frac{C_1P_r}{4} + \frac{C_2P_r}{8}\right)C_0}$$

$$\overline{\qquad + \left(\frac{C_1}{2} + \frac{C_2}{4}\right) + \frac{P_r}{4}\left(1 + C_1 + \frac{C_2}{4}\right)} \quad \text{(Note denominator only continued to this line)}$$

(23)

and thus (22) may be approximated by

$$\bar{Y} = e^{-C_0\xi}\left[\frac{C_0^{\,2} + C_0\left(\frac{3}{2} + \frac{C_1}{2}e^{-\xi} + \frac{C_2}{4}e^{-2\xi}\right) + \frac{1}{2} + \frac{C_1}{2}e^{-\xi} + \frac{C_2}{8}e^{-2\xi}}{a_0\,(C_0 - \Lambda_1)\,(C_0 - \Lambda_2)\,(C_0 - \Lambda_3)}\right] \tag{24}$$

where Λ_1; Λ_2 and Λ_3 are the three roots of the cubic equation in the denominator of (23). These roots are characterized by the determinant of this cubic equation. For convenience, let

$$p_1 = \frac{3}{2} + \frac{C_1}{2} + \frac{C_2}{4} + \frac{P_r}{2} \qquad q = \frac{1}{2} + C_1 + \frac{5C_2}{8} + \frac{3P_r}{4} + \frac{C_1P_r}{4} + \frac{C_2P_r}{8}$$

$$r = \frac{C_1}{2} + \frac{C_2}{4} + \frac{P_r}{4}\left(1 + C_1 + \frac{C_2}{4}\right)$$

$$a = \frac{1}{3}\left(3q - p_1^{\,2}\right) \qquad b = \frac{1}{27}\left(2\,p_1^{\,3} - 9qp_1 + 27\,r\right)$$

Then if $\frac{b^2}{4} + \frac{a^3}{27} > 0$, one real and two complex conjugate roots will be obtained as follows;

let $A = \left[-\frac{b}{2} + \left(\frac{b^2}{4} + \frac{a^3}{27}\right)^{1/2}\right]^{1/3}$ $\qquad B = \left[-\frac{b}{2} - \left(\frac{b^2}{4} + \frac{a^3}{27}\right)^{1/2}\right]^{1/3}$

we may write,

$$\Lambda_1 = \beta_1 = A + B - \frac{p_1}{3}$$

$$\Lambda_2 = \beta_2 - i\beta_3 = -\frac{A+B}{2} + \frac{A-B}{2}\sqrt{-3} - \frac{p_1}{3}$$

$$\Lambda_3 = \beta_2 + i\beta_3 = -\frac{A+B}{2} - \frac{A-B}{2}\sqrt{-3} - \frac{p_1}{3}$$

In the above, β_1 is the real root and β_2 is the real part and the β_3 is the imaginary part of the two conjugate roots.

If $\frac{b^2}{4} + \frac{a^3}{27} < 0$, three unequal real roots will be obtained as follows:

let $$\cos \varphi = - \frac{b}{2} \Big/ \sqrt{- \frac{a^3}{27}}$$

$$\Lambda_1 = 2\sqrt{\frac{-a}{3}} \cos \frac{\varphi}{3} - \frac{p_1}{3}$$

$$\Lambda_2 = 2\sqrt{\frac{-a}{3}} \cos \left(\frac{\varphi}{3} + \frac{2\pi}{3}\right) - \frac{p_1}{3}$$

$$\Lambda_3 = 2\sqrt{\frac{-a}{3}} \cos \left(\frac{\varphi}{3} + \frac{4\pi}{3}\right) - \frac{p_1}{3}$$

If $\frac{b^2}{4} + \frac{a^3}{27} = 0$, three real roots will be obtained but at least two are equal.

In view of the foregoing analysis, we substitute (24) into (19); revert back to the η variable, consistently ignore terms of small order of magnitude, take the inverse transform and obtain for

$$\frac{b^2}{4} + \frac{a^3}{27} > 0 \quad \text{and} \quad \alpha \neq \alpha_0{}^2\beta_1{}^2$$

the dimensionless temperature function as:

$$\theta(\eta,\tau) = \exp\left[- \frac{P_r}{2}\left(a_0 + \frac{P_r}{2a_0{}^3}\right)\eta - \frac{P_r}{2a_0{}^4}\left(e^{-a_0\eta} - \frac{1}{4}e^{-2a_0\eta} - \frac{3}{4}\right)\right]$$

$$\times \left\{\frac{\psi_1}{2}\left[\frac{1}{-a_0\beta_1 + \alpha^{1/2}} e^{-\alpha^{1/2}\eta} \operatorname{erfc}\left[\frac{\eta}{2\alpha^{1/2}} - (\alpha\,\tau)^{1/2}\right]\right.\right.$$

$$+ \frac{e^{\alpha^{1/2}\eta}}{-a_0\beta_1 - \alpha^{1/2}} \operatorname{erfc}\left[\frac{\eta}{2\tau^{1/2}} + (\alpha\,\tau)^{1/2}\right]$$

(continued next page)

$$+ \frac{\Psi_1 a_0 \beta_1}{a_0^2 \beta_1^2 - \alpha} \exp\left[-a_0\beta_1\eta + (a_0^2\beta_1^2 - \alpha)\tau\right] \operatorname{erfc}\left(\frac{\eta}{2\tau^{1/2}} - a_0\beta_1\tau^{1/2}\right)$$

$$+ 2\,\mathrm{Re}\left(\frac{\Psi_2}{2}\left[\frac{e^{-\alpha^{1/2}\eta}}{-a_0\left(+\beta_2 - i\beta_3\right) + \alpha^{1/2}} \quad \operatorname{erfc}\left[\frac{\eta}{2\tau^{1/2}} - (\alpha\,\tau)^{1/2}\right]\right.\right.$$

$$+ \frac{e^{\alpha^{1/2}\eta}}{a_0\left(-\beta_2 + i\beta_3\right) - \alpha^{1/2}} \quad \operatorname{erfc}\left[\frac{\eta}{2\tau^{1/2}} + (\alpha\,\tau)^{1/2}\right]$$

$$+ e^{-\alpha\tau} \cdot \frac{\Psi_2 a_0 (\beta_2 - i\beta_3)}{a_0^2(\beta_2 - i\beta_3)^2 - \alpha} \exp\left[-a_0(\beta_2 - i\beta_3)\eta + a_0^2(\beta_2 - i\beta_3)^2\tau\right]$$

$$\left.\left.\operatorname{erfc}\left[\frac{\eta}{2\tau^{1/2}} - a_0(\beta_2 - i\beta_3)\tau^{1/2}\right]\right)\right\} . \qquad (25.a)$$

For $\frac{b^2}{4} + \frac{a^3}{27} > 0$, but $\alpha = a_0^2\beta_1^2$ or $\alpha^{1/2} = -a_0\beta_1$ we obtain

$$\theta(\eta,\tau) = \exp\left[-\frac{P_r}{2}\left(a_0 + \frac{P_r}{2a_0^3}\right)\eta - \frac{P_r}{2a_0^4}\left(e^{-a_0\eta} - \frac{1}{4}e^{-2a_0\eta} - \frac{3}{4}\right)\right]$$

$$\left\{ \frac{\Psi_1}{2\alpha^{1/2}} \left\{2\alpha^{1/2}\left(\frac{\tau}{\pi}\right)^{1/2} e^{-\frac{\eta^2}{4\tau} - \alpha\tau} - \left(1 + \alpha^{1/2}\eta + 2\alpha\tau\right) e^{+\alpha^{1/2}\eta} \quad \operatorname{erfc}\left[\frac{\eta}{2\tau^{1/2}} + (\alpha\tau)^{1/2}\right]\right.\right.$$

$$\left. + \frac{1}{2} e^{+\alpha^{1/2}\eta} \operatorname{erfc}\left[\frac{\eta}{2\tau^{1/2}} + (\alpha\tau)^{1/2}\right] + \frac{1}{2} e^{-\alpha^{1/2}\eta} \quad \operatorname{erfc}\left[\frac{\eta}{2\tau^{1/2}} - (\alpha\tau)^{1/2}\right]\right\}$$

$$\left. + 2\,\mathrm{Re}\left(\text{----- Repeat 25.a}\right)\right\} \qquad (25.b)$$

The quantity in the bracket after 2Re in the equation (25.b) is same as that in equation (25.a). In the above the symbols α, Ψ_1 and Ψ_2 are defined as:

$$\alpha = \frac{P_r^2 a_0^2}{4}\left(1 + \frac{1}{a_0^4}\right)$$

$$\Psi_1(\eta) = \frac{\beta_1^2 + \frac{1}{2}(3\beta_1 + 1) + \frac{P_r(1 - P_r)}{4a_0^4}(\beta_1 + 1)e^{-a_0\eta} + \frac{P_r(P_r - 2)}{16a_0^4}(\beta_1 + \frac{1}{2})e^{-2a_0\eta}}{(\beta_2 - \beta_1)^2 + \beta_3^2}$$

$$\Psi_2(\eta) = \frac{\left[(\beta_2-\beta_1)i-\beta_3\right]\left[(\beta_2-i\beta_3)^2 + \frac{3}{2}(\beta_2-i\beta_3) + \frac{1}{2} + \frac{P_r(1-P_r)}{4a_0^4}(\beta_2-i\beta_3+1)e^{-a_0\eta}\right.}{2\beta_3\left[(\beta_1 - \beta_2)^2 + \beta_3^2\right]}$$

$$\left. + \frac{P_r(P_r-2)}{16a_0^4}(\beta_2-i\beta_3+\frac{1}{2})e^{-2a_0\eta}\right]$$

(Numerator only continued to this line)

for $\frac{b^2}{4} + \frac{a^3}{27} < 0$, and for $\alpha \neq a_0^2 \Lambda_1^2$; $\alpha \neq a_0^2 \Lambda_2^2$; $\alpha \neq a_0^2 \Lambda_3^2$ the dimensionless temperature θ takes the following form:

$$\theta(\eta,\tau) = \exp\left[-\frac{P_r}{2}\left(a_0 + \frac{P_r}{2a_0^3}\right)\eta - \frac{P_r}{2a_0^4}\left(e^{-a_0\eta} - \frac{1}{4}e^{-2a_0\eta} - \frac{3}{4}\right)\right]$$

$$\left\{\frac{1}{2}\left[\frac{\Phi_1(\eta)}{\alpha^{1/2} - a_0\Lambda_1} + \frac{\Phi_2(\eta)}{\alpha^{1/2} - a_0\Lambda_2} + \frac{\Phi_3(\eta)}{\alpha^{1/2} - a_0\Lambda_3}\right] e^{-\alpha^{1/2}\eta} \quad \operatorname{erfc}\left[\frac{\eta}{2\tau^{1/2}} - (\alpha\tau)^{1/2}\right]\right.$$

$$- \frac{1}{2}\left[\frac{\Phi_1(\eta)}{\alpha^{1/2} + a_0\Lambda_1} + \frac{\Phi_2(\eta)}{\alpha^{1/2} + a_0\Lambda_2} + \frac{\Phi_3(\eta)}{\alpha^{1/2} + a_0\Lambda_3}\right] e^{\alpha^{1/2}\eta} \quad \operatorname{erfc}\left[\frac{\eta}{2\tau^{1/2}} + (\alpha\tau)^{1/2}\right]$$

(continued next page)

$$+ a_0 e^{-\alpha\tau}\left[\frac{\Lambda_1\Phi_1(\eta)}{a_0^2\Lambda_1^2 - \alpha} \exp\left(-a_0\Lambda_1\eta + a_0^2\Lambda_1^2\tau\right) \operatorname{erfc}\left(\frac{\eta}{2\tau^{1/2}} - a_0\Lambda_1\tau^{1/2}\right)\right.$$

$$+ \frac{\Lambda_2\Phi_2(\eta)}{a_0^2\Lambda_2^2 - \alpha} \exp\left(-a_0\Lambda_2\eta + a_0^2\Lambda_2^2\tau\right) \operatorname{erfc}\left(\frac{\eta}{2\tau^{1/2}} - a_0\Lambda_2\tau^{1/2}\right)$$

$$\left.+ \frac{\Lambda_3\Phi_3(\eta)}{a_0^2\Lambda_3^2 - \alpha} \exp\left(-a_0\Lambda_3\eta + a_0^2\Lambda_3^2\tau\right) \operatorname{erfc}\left(\frac{\eta}{2\tau^{1/2}} - a_0\Lambda_3\tau^{1/2}\right)\right]\Bigg\} \qquad (25.c)$$

also for $\frac{b^2}{4} + \frac{a^3}{27} < 0$, and $\alpha = a_0^2\Lambda_1^2$ which automatically implies $\alpha \neq a_0^2\Lambda_2^2$ and $\alpha \neq a_0^2\Lambda_3^2$ ** the temperature function is expressed as:

$$\theta(\eta,\tau) = \exp\left[-\frac{P_r}{2}\left(a_0 + \frac{P_r}{2a_0^3}\right)\eta - \frac{P_r}{2a_0^4}\left(e^{-a_0\eta} - \frac{1}{4}e^{-2a_0\eta} - \frac{3}{4}\right)\right]$$

$$\left\{\left[\frac{\Phi_1}{4\alpha^{1/2}} + \frac{\Phi_2}{2\left(\alpha^{1/2} - a_0\Lambda_2\right)} + \frac{\Phi_3}{2\left(\alpha^{1/2} - a_0\Lambda_3\right)}\right] e^{-\alpha^{1/2}\eta} \operatorname{erfc}\left[\frac{\eta}{2\tau^{1/2}} - (\alpha\tau)^{1/2}\right]\right.$$

$$- \left[\frac{\Phi_1}{2\alpha^{1/2}}\left(\frac{1}{2} + \alpha^{1/2}\eta + 2\alpha\tau\right) + \frac{\Phi_2}{2\left(\alpha^{1/2} + a_0\Lambda_2\right)} + \frac{\Phi_3}{2\left(\alpha^{1/2} + a_0\Lambda_3\right)}\right] e^{\alpha^{1/2}\eta}$$

$$\operatorname{erfc}\left[\frac{\eta}{2\tau^{1/2}} + (\alpha\tau)^{1/2}\right] + \Phi_1\left(\frac{\tau}{\pi}\right)^{1/2} \exp\left(-\frac{\eta^2}{4\tau} - \alpha\tau\right)$$

(equation continued on next page)

** When $\alpha = a_0^2\Lambda_2^2$ (But $\alpha \neq a_0^2\Lambda_1^2$), interchange Φ_1 and Φ_2 and replace Λ_2 by Λ_1. Similary rule may be applied when $\alpha = a_0^2\Lambda_3^2$.

$$+\frac{a_0\Lambda_2\Phi_2}{a_0^2\Lambda_2^2-\alpha}\exp\left[\left(a_0^2\Lambda_2^2-\alpha\right)\tau-a_0\Lambda_2\eta\right]\operatorname{erfc}\left(\frac{\eta}{2\tau^{1/2}}-a_0\Lambda_2\tau^{1/2}\right)$$

$$+\frac{a_0\Lambda_3\Phi_3}{a_0^2\Lambda_3-\alpha}\exp\left[\left(a_0^2\Lambda_3^2-\alpha\right)\tau-a_0\Lambda_3\eta\right]\operatorname{erfc}\left(\frac{\eta}{2\tau^{1/2}}-a_0\Lambda_3\tau^{1/2}\right)\Bigg\}$$

where

$$\Phi_1(\eta)=\frac{\Lambda_1^2+\frac{3}{2}\Lambda_1+\frac{1}{2}+\frac{P_r(1-P_r)}{4a_0^4}(1+\Lambda_1)e^{-a_0\eta}+\frac{P_r(P_r-2)}{16a_0^4}\left(\Lambda_1+\frac{1}{2}\right)e^{-2a_0\eta}}{(\Lambda_1-\Lambda_2)(\Lambda_1-\Lambda_3)} \qquad (25.d)$$

$$\Phi_2=\frac{\Lambda_2^2+\frac{3}{2}\Lambda_2+\frac{1}{2}+\frac{P_r(1-P_r)}{4a_0^4}(1+\Lambda_2)e^{-a_0\eta}+\frac{P_r(P_r-2)}{16a_0^4}\left(\Lambda_2+\frac{1}{2}\right)e^{-2a_0\eta}}{(\Lambda_2-\Lambda_1)(\Lambda_2-\Lambda_3)}$$

$$\Phi_3=\frac{\Lambda_3^2+\frac{3}{2}\Lambda_3+\frac{1}{2}+\frac{P_r(1-P_r)}{4a_0^4}(1+\Lambda_3)e^{-a_0\eta}+\frac{P_r(P_r-2)}{16a_0^4}\left(\Lambda_3+\frac{1}{2}\right)e^{-2a_0\eta}}{(\Lambda_3-\Lambda_1)(\Lambda_3-\Lambda_2)}$$

for $\frac{b^2}{4}+\frac{a^3}{27}=0$, and for $\alpha \neq a_0^2\ \Lambda_1^2$, and $\alpha \neq a_0^2\ \Lambda_2^2$

the function θ is given as:

$$\theta(\eta,\tau)=\exp\left[-\frac{P_r}{2}\left(a_0+\frac{P_r}{2a_0^3}\right)\eta-\frac{P_r}{2a_0^4}\left(e^{-a_0\eta}-\frac{1}{4}e^{-a_0\eta}-\frac{3}{4}\right)\right]$$

$$\left\{\left[\frac{\chi_1 a_0}{2\left(\alpha^{1/2}-a_0\Lambda_2\right)^2}+\frac{\chi_2}{2\left(\alpha^{1/2}-a_0\Lambda_2\right)}+\frac{\chi_3}{2\left(\alpha^{1/2}-a_0\Lambda_1\right)}\right]e^{-\alpha^{1/2}\eta}\operatorname{erfc}\left(\frac{\eta}{2\tau^{1/2}}-(\alpha\tau)^{1/2}\right)\right.$$

$$+\left[\frac{\chi_1 a_0}{2\left(\alpha^{1/2} - a_0\Lambda_2\right)^2} - \frac{\chi_2}{2\left(\alpha^{1/2} + a_0\Lambda_2\right)} - \frac{\chi_3}{2\left(\alpha^{1/2} + a_0\Lambda_1\right)}\right] e^{\alpha^{1/2}\eta} \operatorname{erfc}\left[\frac{\eta}{2\tau^{1/2}} + (\alpha\tau)^{1/2}\right]$$

$$+\left[\chi_1 a_0 \left(\frac{1 - a_0\Lambda_2\eta + 2a_0^2\Lambda_2^2\tau}{a_0^2\Lambda_2^2 - \alpha} - \frac{a_0\Lambda_2}{(a_0^2\Lambda_2^2 - \alpha)^2}\right) + \frac{\chi_2 a_0\Lambda_2}{(a_0^2\Lambda_2^2 - \alpha)}\right]\cdot$$

$$\exp\left(-a_0\Lambda_2\eta + (a_0^2\Lambda_2^2 - \alpha)\,\tau\right) \operatorname{erfc}\left(\frac{\eta}{2\tau^{1/2}} - a_0\Lambda_2\tau^{1/2}\right) +$$

$$+\frac{\chi_1 a_0}{\pi^{1/2}} \exp\left(-\frac{\eta^2}{4\tau} - \alpha\tau\right)\left[\frac{2a_0\Lambda_2\tau^{1/2}}{a_0^2\Lambda_2^2 - \alpha} - \frac{1}{(a_0^2\Lambda_2^2 - \alpha)^2\,\tau^{1/2}}\right]$$

$$+\frac{\chi_3 a_0\Lambda_1}{(a_0^2\Lambda_1^2 - \alpha)} \exp\left(-a_0\Lambda_1\eta + (a_0^2\Lambda_1^2 - \alpha)\,\tau\right) \operatorname{erfc}\left(\frac{\eta}{2\tau^{1/2}} - a_0\Lambda_1\tau^{1/2}\right)\Bigg\} \qquad (25.e)$$

for $\frac{b^2}{4} + \frac{a^3}{27} = 0$ and $\alpha = a_0^2\,\Lambda_1^2$ the temperature function is:

$$\theta(\eta,\tau) = \exp\left[-\frac{P_r}{2}\left(a_0 + \frac{P_r}{2a_0^3}\right)\eta - \frac{P_r}{2a_0^4}\left(e^{-a_0\eta} - \frac{1}{4}e^{-2a_0\eta} - \frac{3}{4}\right)\right]\cdot$$

$$\left\{\left[\frac{\chi_1 a_0}{2\left(\alpha^{1/2} - a_0\Lambda_2\right)^2} + \frac{\chi_2}{2\left(\alpha^{1/2} - a_0\Lambda_2\right)} + \frac{\chi_3}{4\alpha^{1/2}}\right] e^{-\alpha^{1/2}\eta} \operatorname{erfc}\left[\frac{\eta}{2\tau^{1/2}} - (\alpha\tau)^{1/2}\right]\right.$$

$$+\left[\frac{\chi_1 a_0}{2(\alpha^{1/2} - a_0\Lambda_2)^2} - \frac{\chi_2}{2(\alpha^{1/2} + a_0\Lambda_2)} - \frac{\chi_3}{2\alpha^{1/2}}\left(\frac{1}{2} + \alpha^{1/2}\eta + 2\,\alpha\,\tau\right)\right] e^{\alpha^{1/2}\eta}$$

$$\operatorname{erfc}\left[\frac{\eta}{2\tau^{1/2}} + (\alpha\tau)^{1/2}\right] + \left[\left(\frac{1 - a_0\Lambda_2\eta + 2a_0^{\,2}\Lambda_2^{\,2}\tau}{a_0^{\,2}\Lambda_2^{\,2} - \alpha} - \frac{a_0\Lambda_2}{(a_0^{\,2}\Lambda_2^{\,2} - \alpha)^2}\right)\chi_1 a_0\right.$$

$$\left. + \frac{\chi_2 a_0\Lambda_2}{(a_0^{\,2}\Lambda_2^{\,2} - \alpha)}\right] \exp\left(-a_0\Lambda_2\eta + (a_0^{\,2}\Lambda_2^{\,2} - \alpha)\,\tau\right) \operatorname{erfc}\left(\frac{\eta}{2\tau^{1/2}} - a_0\Lambda_2\tau^{1/2}\right)$$

$$+ \frac{\chi_1 a_0}{\pi^{1/2}}\left[\frac{2a_0\Lambda_2\tau^{1/2}}{a_0^{\,2}\Lambda_2^{\,2} - \alpha} - \frac{1}{(a_0^{\,2}\Lambda_2^{\,2} - \alpha)^2\tau^{1/2}} + \frac{\chi_3\tau^{1/2}}{\chi_1 a_0}\right] \exp\left(-\frac{\eta^2}{4\tau} - \alpha\tau\right)\Bigg\} \tag{25.f}$$

For $\frac{b^2}{4} + \frac{a^3}{27} = 0$, and $\alpha = a_0^{\,2}\Lambda_2^{\,2}$, $\alpha \neq a_0^{\,2}\Lambda_1^{\,2}$

$$\theta(\eta,\tau) = \exp\left[-\frac{P_r}{2}\left(a_0 + \frac{P_r}{2a_0 3}\right)\eta - \frac{P_r}{2a_0 4}\left(e^{-a_0\eta} - \frac{1}{4}e^{-2a_0\eta} - \frac{3}{4}\right)\right].$$

$$\Bigg\{\chi_1 a_0 \int_0^{\tau}\left[-2\left(\frac{\alpha}{\pi}\right)^{1/2}(S)^{1/2} \exp\left(-\frac{\eta^2}{4S} - \alpha S\right) + \left(1 + \alpha^{1/2}\eta + 2\alpha S\right)e^{\alpha^{1/2}\eta}\,.\right.$$

$$\left. .\ \operatorname{erfc}\left(\frac{\eta}{2S^{1/2}} + (aS)^{1/2}\right)\right] dS$$

$$+\left[\frac{\chi_2}{4(\alpha)^{1/2}} + \frac{\chi_3}{2\left((\alpha)^{1/2} - a_0\Lambda_1\right)}\right] \exp\left(-\alpha^{1/2}\eta\right) \operatorname{erfc}\left(\frac{\eta}{2(\tau)^{1/2}} - (\alpha\tau)^{1/2}\right)$$

$$+ \chi_2 \left(\frac{\tau}{\pi}\right)^{1/2} \exp\left(- \frac{\eta^2}{4\tau} - \alpha\tau\right)$$

$$-\left[\frac{\chi_2}{2(\alpha)^{1/2}}\left(\frac{1}{2} + \alpha^{1/2}\eta + 2\,\alpha\tau\right) + \frac{\chi_3}{2(\alpha^{1/2} + a_0\Lambda_1)}\right] e^{\alpha^{1/2}\eta} \operatorname{erfc}\left(\frac{\eta}{2\,\tau^{1/2}} + (\alpha\tau)^{1/2}\right)$$

$$+ \frac{\chi_3 a_0 \Lambda_1}{a_0{}^2 \Lambda_1 - \alpha} \exp\left(-a_0\Lambda_1\eta + a_0{}^2\Lambda_1{}^2\tau - \alpha\tau\right) \operatorname{erfc}\left(\frac{\eta}{2\tau^{1/2}} - a_0\Lambda_1\,\tau^{1/2}\right)\Bigg\} \qquad (25.g)$$

Where
$$\chi_1 = \frac{\Lambda_2{}^2 + \frac{3}{2}\Lambda_2 + \frac{1}{2} + \frac{P_r(1 - P_r)}{4a_0{}^4}(1 + \Lambda_2)e^{-a_0\eta} + \frac{P_r(P_r - 2)}{16a_0{}^4}(\Lambda_2 + \frac{1}{2})e^{-2a_0\eta}}{\Lambda_2 - \Lambda_1}$$

$$\chi_2 = \frac{\Lambda_2^2 - 2\Lambda_1\Lambda_2 + \frac{3}{2}(\Lambda_2 - \Lambda_1) - \frac{1}{2} + \frac{P_r(1 - P_r)}{4a_0{}^4}(\Lambda_2 - \Lambda_1 - 1)e^{-a_0\eta}}{(\Lambda_2 - \Lambda_1)^2}$$

$$+ \frac{P_r(P_r - 2)}{16a_0{}^4}(\Lambda_2 - \Lambda_1)e^{-2a_0\eta}$$ (Numerator only continued to this line)

$$\chi_3 = \frac{\Lambda_1{}^2 + \frac{3}{2}\Lambda_1 + \frac{1}{2} + \frac{P_r(1 - P_r)}{4a_0{}^4}(1 + \Lambda_1)e^{-a_0\eta} + \frac{P_r(P_r - 2)}{16a_0{}^4}(\Lambda_1 + \frac{1}{2})e^{-2a_0\eta}}{(\Lambda_1 - \Lambda_2)^2}$$

Eq. (25) represents the dimensionless transient temperature profile due to a step change in local heat flux. Since we are primarily interested in evaluating the wall temperature variation, the quantity desired is the dimensionless temperature at the wall $\theta(0,\tau)$ which can be readily formed from (25) by letting $\eta = 0$. Thus, for disks initially at adiabatic wall temperature, the wall temperature as a function dimensionless time is:

$$T_w(\tau) = T_\infty + \frac{\omega\nu}{c_p}\left[R^2 S(0) + Q(0)\right] + \theta(0,\tau) \tag{26.a}$$

and for disks initially at uniform temperature, the same can be written:

$$T_w(\tau) = T_{iw} + \left(\frac{\nu}{\omega}\right)^{1/2} \cdot \frac{\Delta q_w}{k}\,\theta(0,\tau) \tag{26.b}$$

The values of S(0) and Q(0) were obtained from the numerical integration of (12) and (13) by electronic computer, the Univac 1107. Their value for Prandtl numbers ranging from 0.001 to 100 and suction parameter ranging from -1.2 to -4.0 were obtained.

Inasmuch (25) is valid for the entire time domain, the corresponding steady state solution at the wall can be obtained by simply letting $\tau \to \infty$, and $\eta \to 0$. The result is

$\frac{b^2}{4} + \frac{a^3}{27} > 0$ and $\alpha \neq a_0^{\,2}\beta_1^{\,2}$

$$\theta_{ss}(0) = \frac{1}{a_0}\left[\frac{\Psi_1(0)}{\frac{\alpha^{1/2}}{a_0} - \beta_1} + 2\,\mathrm{Re}\,\frac{\Psi_2(0)\left(\frac{\alpha^{1/2}}{a_0} - \beta_2 - i\beta_3\right)}{\left(\frac{\alpha^{1/2}}{a_0} - \beta_2\right)^2 + \beta_3^{\,2}}\right] \tag{27.a}$$

for $\frac{b^2}{4} + \frac{a^3}{27} > 0$ and $\alpha = a_0^{\,2}\beta_1^{\,2}$ or

$\alpha^{1/2} = -\,a_0\beta_1$

$$\theta_{ss}(0) = \frac{1}{a_0}\left[\frac{\Psi_1(0)}{2\alpha^{1/2}} - 2\,\mathrm{Re}\,\frac{\Psi_2(0)\left(\frac{-\alpha^{1/2}}{a_0} + \beta_2 + i\beta_3\right)}{\left(\frac{\alpha^{1/2}}{a_0} - \beta_2\right)^2 + \beta_3^{\,2}}\right] \tag{27.b}$$

for $\frac{b^2}{4} + \frac{a^3}{27} < 0$ and $\alpha \neq a_0{}^2\Lambda_1{}^2$

$\alpha \neq a_0{}^2\Lambda_2{}^2$

$\alpha \neq a_0\,\Lambda_3{}^2$

$$\theta_{ss}(0) = \frac{\Phi_1(0)}{\alpha^{1/2} - a_0\Lambda_1} + \frac{\Phi_2(0)}{\alpha^{1/2} - a_0\Lambda_2} + \frac{\Phi_3(0)}{\alpha^{1/2} - a_0\Lambda_3} \qquad (27.c)$$

for $\frac{b^2}{4} + \frac{a^3}{27} < 0$ and $\alpha = a_0{}^2\Lambda_1{}^2$ (which automatically implies $\alpha \neq a_0{}^2\Lambda_2{}^2$, $\alpha \neq a_0{}^2\Lambda_3{}^2$)

$$\theta_{ss}(0) = \frac{\Phi_1(0)}{2\alpha^{1/2}} + \frac{\Phi_2(0)}{\alpha^{1/2} - a_0\Lambda_2} + \frac{\Phi_3(0)}{\alpha^{1/2} - a_0\Lambda_3} \qquad (27.d)$$

For the cases $\alpha = a_0{}^2\Lambda_2{}^2$ or $\alpha = a_0{}^2\Lambda_3{}^2$ see footnote on page 15

for $\frac{b^2}{4} + \frac{a^3}{27} = 0$ and $\alpha \neq a_0{}^2\Lambda_1{}^2$, $\alpha \neq a_0{}^2\Lambda_2{}^2$

$$\theta_{ss}(0) = \frac{a_0\chi_1(0)}{\left(\alpha^{1/2} - a_0\Lambda_2\right)^2} + \frac{\chi_2(0)}{\alpha^{1/2} - a_0\Lambda_2} + \frac{\chi_3(0)}{\alpha^{1/2} - a_0\Lambda_1} \qquad (27.e)$$

For $\frac{b^2}{4} + \frac{a^3}{27} = 0$ and $\alpha = a_0{}^2 \Lambda_1{}^2$

$$\theta_{ss}(0) = \frac{a_0 \chi_1(0)}{\left(\alpha^{1/2} - a_0\Lambda_2\right)^2} + \frac{\chi_2(0)}{\alpha^{1/2} - a_0\Lambda_2} + \frac{\chi_3(0)}{2\,\alpha^{1/2}} \tag{27.f}$$

We have not worked the steady state case corresponding to Equation 25.g

In the above equations, the expressions $\Psi_n(0)$, Φ_n and $\chi_n(0)$ are obtained by letting $\eta = 0$ in the equations defining $\Psi\,(\eta)$ $\Phi_n(\eta)$ and $\chi_n(\eta)$.

For very large suction, i.e., $a_0 \to \infty$, the foregoing results simplify to:

$$\lim_{a_0 \to \infty} \theta_{ss}(0) = \frac{1}{a_0 P_r} \tag{28}$$

The numerical values of $\theta_{ss}(0)$ as evaluated from (27) are listed in Table 1 for the several Prandtl numbers and suction parameter shown. For comparison purposes, we have included the results for $P_r = 0.7$ obtained from the relation

$$\theta_{ss}(0) = -\frac{1}{\theta_{ss}'(0)}$$ (Prime indicates first derivative with respect to η)

in which the values of $\theta_{ss}'(0)$ are taken from the results reported by Sparrow and Gregg [11]. The maximum discrepancy is about 2% for $a_0 = 1.2$ and the agreement is progressively better for large values of a_0. It may also be shown that if one uses the limiting expression (28), the error remains within a few per cent when $a_0 > 2$.

Finally, we note that when dissipative effects are negligible, the steady wall temperature under the limiting condition of $a_0 \to \infty$ becomes

$$\lim_{a_0 \to \infty} T_{w,ss} = T_\infty + \frac{q_w}{k}\left(\frac{\nu}{\omega}\right)^{1/2} \frac{1}{P_r a_0} = T_\infty + \frac{q_w}{C_p \rho \, |W_w|} \tag{29}$$

The above results are to be expected on physical grounds because, when the suction is very large, the heat transfer at the wall would be completely dominated by the convective process. The effect of fluid injection $(a_0 < 0)$ is to decrease the heat transfer by blanketing the surface with the injecting fluid of the same temperature as T_w. On the other hand, suction increases heat transfer, because fluid at free stream temperature is effectively brought to the disk surface.

4.0 Numerical Results and Discussion

The values of the roots of the cubic equation in the denominator of K are tabulated in Table III for various values of the Prandtl number and the blowing parameter.

The values of $S(0)$ and $Q(0)$ obtained from (12) and (13) are tabulated in Table II for the range of the Prandtl number and the blowing parameter as shown. It is seen from Table II that the Prandtl number and the blowing parameter has little effect on the values of $S(0)$, which has almost a constant value of 0.5 in the range of P_r and positive a_0 under consideration. By contrast, $Q(0)$ is a rapidly varying function of P_r and positive a_0. It decreases with increasing P_r as well as a_0.

The ratio of the dimensionless temperature at the wall, $\theta(0,\tau) / \theta_{ss}(0)$, is shown plotted against $\tau = \omega t / P_r$ in Fig. 2 to 7 for the several Prandtl numbers indicated. If one replots the $\theta(0,\tau)/\theta_{ss}(0)$ against $a_0^2 P_r^2 \tau (= a_0^2 P_r \omega t)$ the data can, for all practical purposes, be brought to lie on a single curve for Prandtl number ranging from 0.1 to 100 and a_0 from 1.5 to 4.0. This is illustrated in Fig. 8. This result indicates that for sufficiently large suction the response time varies inversely with the Prandtl number, the angular velocity and the second power of the suction parameter. The same conclusion has also been drawn for the case of a step change in wall temperature [9]. Due to the linearity of the energy equation, the previous results can be readily generalized for any arbitrary wall heat flux using the Duhamel's theorem.

5.0 A Special Case of Compressible Flow

For an ideal gas of constant Prandtl number, exhibiting a linear variation of viscosity with temperature, Ostrach and Thornton [16] has shown that, if dissipative effects can be ignored, the steady heat transfer solution for the compressible flow over a rotating disk can be obtained from the corresponding incompressible solutions. It is natural to examine if the same could be stated for the unsteady heat transfer processes. Using the same transformations for both the dependent and independent variables in the governing conservation equations as those given in [16], except for difference noted below, one may show that the answer to the equation posed above is affirmative. Now the velocity component normal to the disk is given by

$$w = \frac{\rho_\infty}{\rho}\left[(A\,\omega\,\nu_\infty)^{1/2}\,H(\eta) - \frac{\partial}{\partial t}\int_0^z \frac{\rho}{\rho_\infty}\,dz\right]$$

Where the subscript ∞ refers to the free stream condition and A is the proportionality constant in the linear viscosity - temperature relation, i.e.

$$\frac{\mu}{\mu_\infty} = A\frac{T}{T_\infty}$$

With the transformation stated above, the solution obtained for the incompressible case, may be directly applied for the compressible case. Expressions given for the transient wall temperature for the incompressible case remain valid provided that all properties are replaced by wall values.

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Table 1

Values of the Dimensionless Temperature, $\theta_{ss}(0)$, for Large Suction

a_0 \ P_r	0.01	0.1	0.7		1.0	10.0	100
			From Analysis	From Ref. [11]			
1.2	68.6931	7.0703	1.0892	1.1114	0.7784	0.07745	0.00747
1.5	61.0106	6.1733	0.9151	0.9200	0.6466	0.07095	0.00632
2.0	48.5301	4.8711	0.7068	0.7052	0.4949	0.05015	0.00487
2.5	39.5055	3.9566	0.5685	0.5685	0.3983	0.03999	0.00399
3.0	33.1328	3.3157	0.4750	0.4748	0.3327	0.03333	0.00334
3.5	28.4784	2.8490	0.4076		0.2854	0.02857	0.00286
4.0	24.9523	2.4958	0.3568	0.3569	0.2498	0.02500	0.00250

Table 2

Values of S(0)

P_r	a_0					
	1.2	1.5	2.0	2.5	3.0	4.0
0.01	0.4466	0.4657	0.4851	0.4939	*	*
0.1	0.4509	0.4720	0.4890	0.4952	0.4975	0.4992
0.7	0.4894	0.4942	0.4980	0.4992	0.4995	0.4998
1.00	0.4991	0.4998	0.5002	0.5001	0.5000	0.5000
10.0	0.5441	0.5228	0.5087	0.5038	0.5017	0.5005
100.0	0.5503	0.5258	0.5097	0.5042	*	*

*s failed to converge

Values of Q(0)

	a_0					
	1.2	1.5	2.0	2.5	3.0	4.0
0.01	9633	7106	4560	3093	*	*
0.1	96.9393	75.1505	48.3896	32.4440	22.9324	13.0515
0.7	2.9213	2.1302	1.3088	0.8624	0.6055	0.3431
1.0	1.6548	1.1874	0.7199	0.4718	0.3307	0.1872
10.0	0.1159	0.0660	0.0329	0.0200	0.1362×10^{-1}	0.756×10^{-2}
100.0	0.4598×10^{-1}	0.1924×10^{-1}	0.5911×10^{-2}	0.2594×10^{-2}	*	*

*s failed to converge

Table 3

Roots of the Cubic Equation:

		a_o (Blowing Parameter)						
P_R	Root	1.2*	1.5*	2.0	2.5	3.0	3.5	4.0
0.01	Λ_1	- .0062	- .0055	- .0052	- .0051	- .0050	- .0050	- .0050
	Λ_2	- 1.0006	- 1.0002	- 1.0001	- 1.0000	- 1.0000	- 1.0000	- 1.0000
	Λ_3	- .4988	- .4995	- .4998	- .4999	- .5000	- .5000	- .5000
0.10	Λ_1	- .0623	- .0550	- .0516	- .0506	- .0503	- .0502	- .0501
	Λ_2	- 1.0062	- 1.0026	- 1.0008	- 1.0003	- 1.0002	- 1.0001	- 1.0001
	Λ_3	- .4865	- .4945	- .4983	- .4993	- .4997	- .4998	- .4999
0.70	Λ_1	- 1.0500	- 1.0220	- .3625	- .3547	- .3522	- .3512	- .3507
	Λ_2	- .3990	- .4136	- 1.0072	- 1.0030	- 1.0015	- 1.0008	- 1.0005
	Λ_3	- .0967	- .0312	- .4800	- .4922	- .4963	- .4980	- .4988
1.00	Λ_1	- .3951	- .4538	- .4847	- .4937	- .4969	- .4983	- .4990
	Λ_2	- 1.0747	- 1.0339	- 1.0114	- 1.0047	- 1.0023	- 1.0012	- 1.0007
	Λ_3	- .5000	- .5000	- .5000	- .5000	- .5000	- .5000	- .5000
10.00	Λ_1	8.5302	3.3904	.8414	.0924	- .1974	- .3300	- .3978
	Λ_2	- 5.4190	- 5.2757	- 5.1242	- 5.0578	- 5.0293	- 5.0162	- 5.0096
	Λ_3	- 1.1717	- 1.1578	- 1.1234	- 1.0866	- 1.0572	- 1.0371	- 1.0242
100.00	Λ_1	898.4965	368.1564	116.5402	47.6720	22.8684	12.2049	7.0155
	Λ_2	- 50.6339	- 50.5876	- 50.4663	-50.3256	-50.2104	-50.1330	-50.0851
	Λ_3	- 1.1634	- 1.1662	- 1.1670	- 1.1660	- 1.1638	- 1.1601	- 1.1547

*For Prandtl number 0.7 and blowing parameters 1.2 and 1.5, the names Λ_1, Λ_2, and Λ_3 are to be replaced by β_1, β_2, and β_3 respectively.

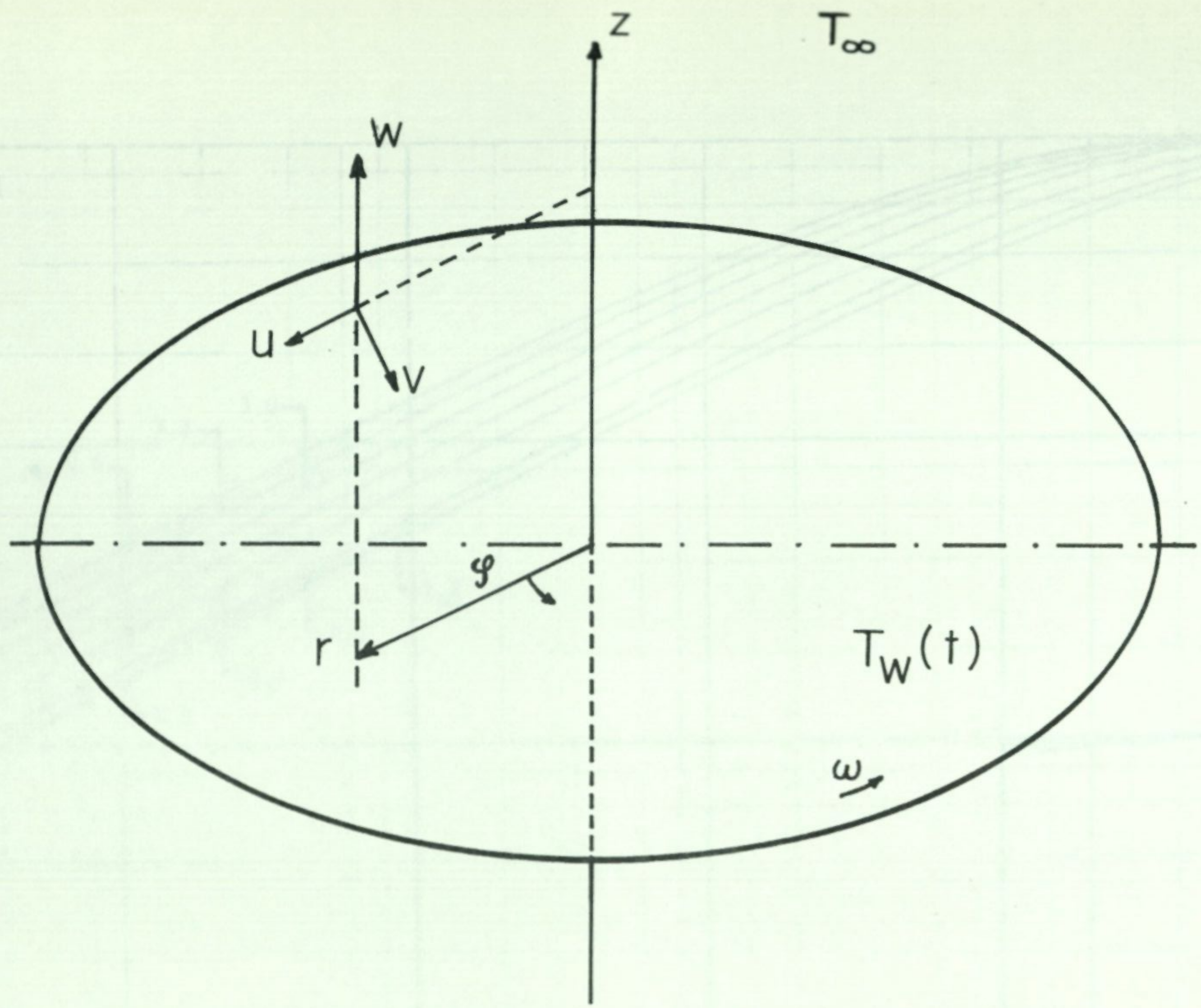

FIG. 1

COORDINATE AND VELOCITY NOMENCLATURE

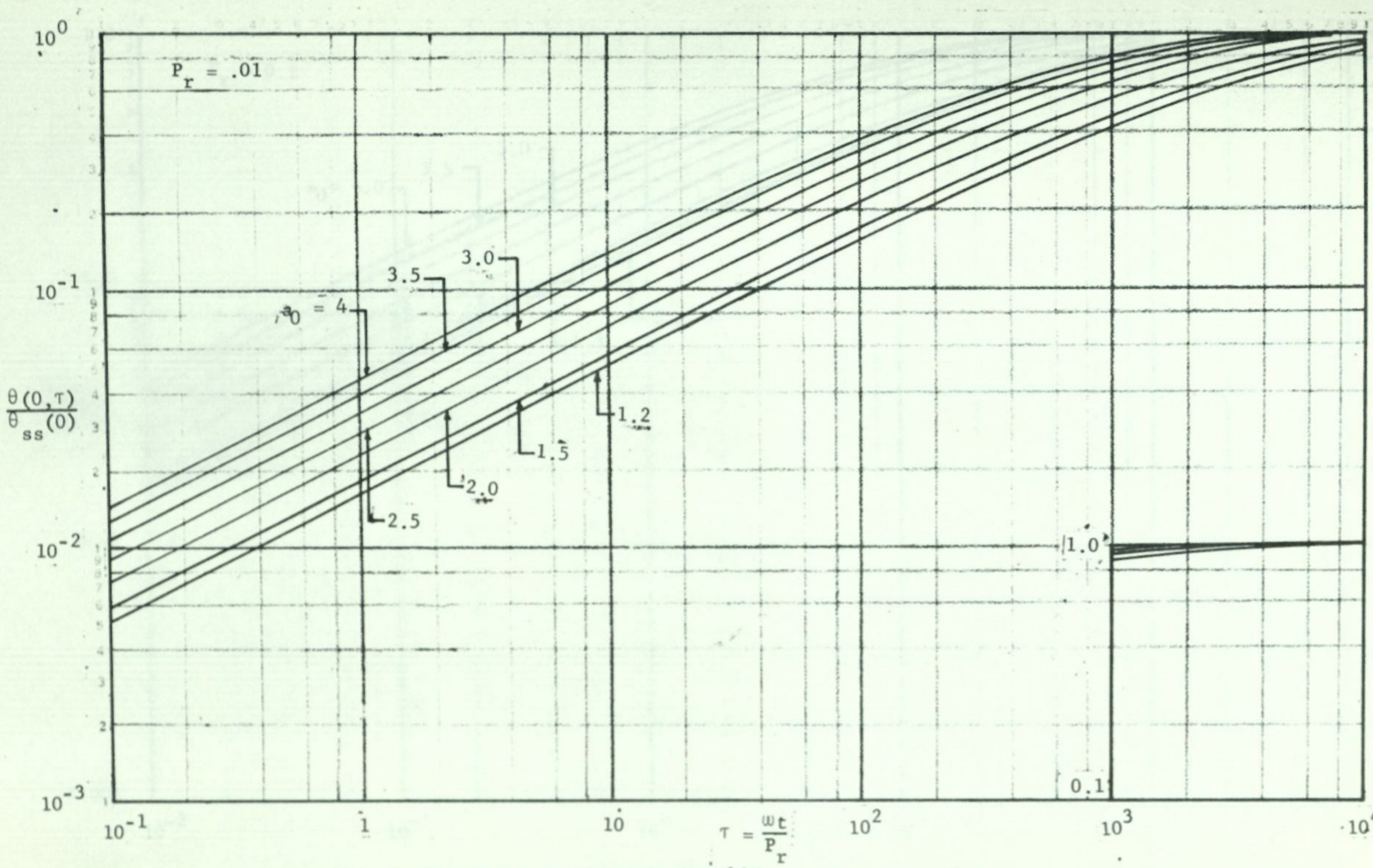

FIG. 2 DIMENSIONLESS TEMPERATURE RESPONSE AT THE SURFACE OF A ROTATING DISK DUE TO A STEP CHANGE IN LOCAL HEAT FLUX

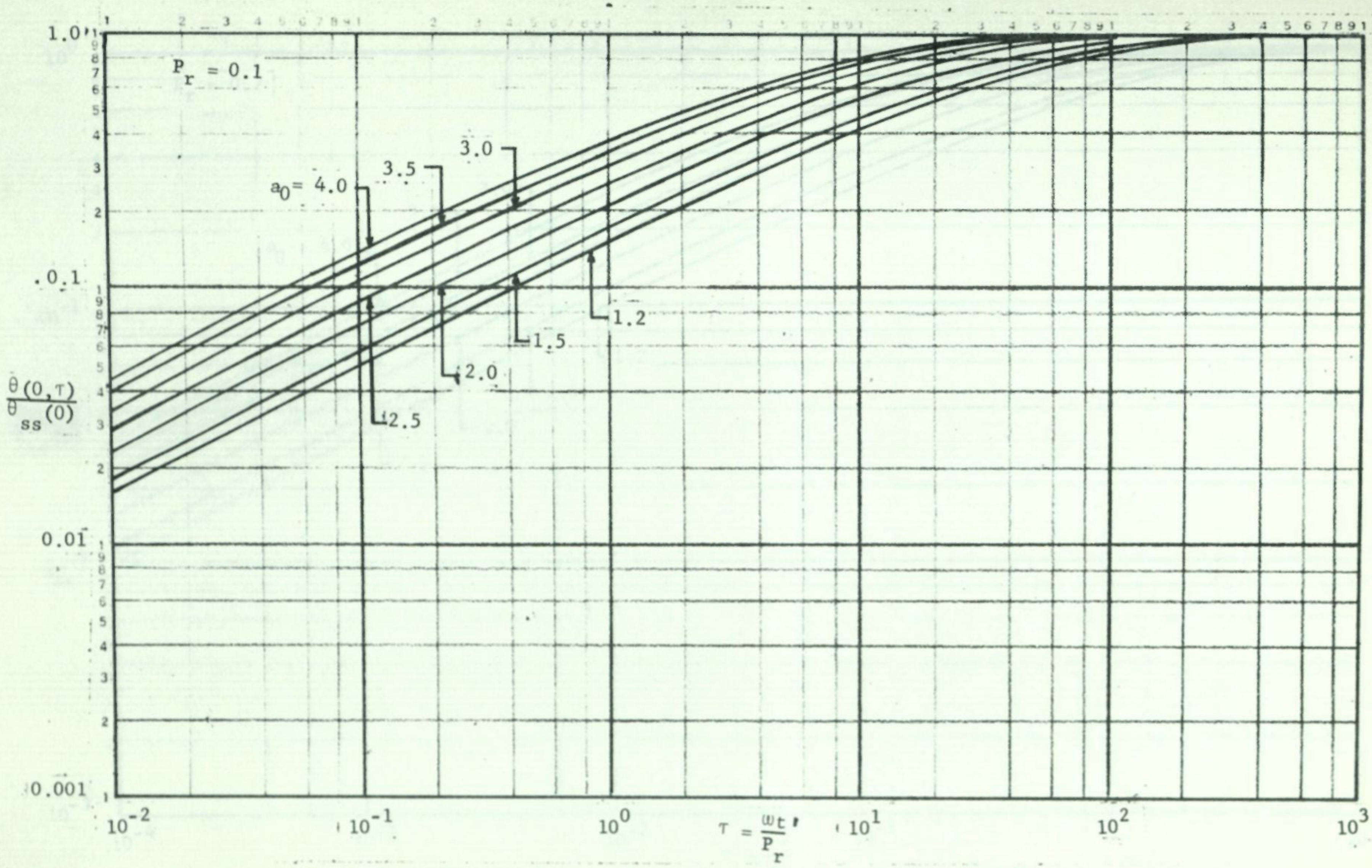

FIG. 3 DIMENSIONLESS TEMPERATURE RESPONSE AT THE SURFACE OF A ROTATING DISK DUE TO A STEP CHANGE IN LOCAL HEAT FLUX

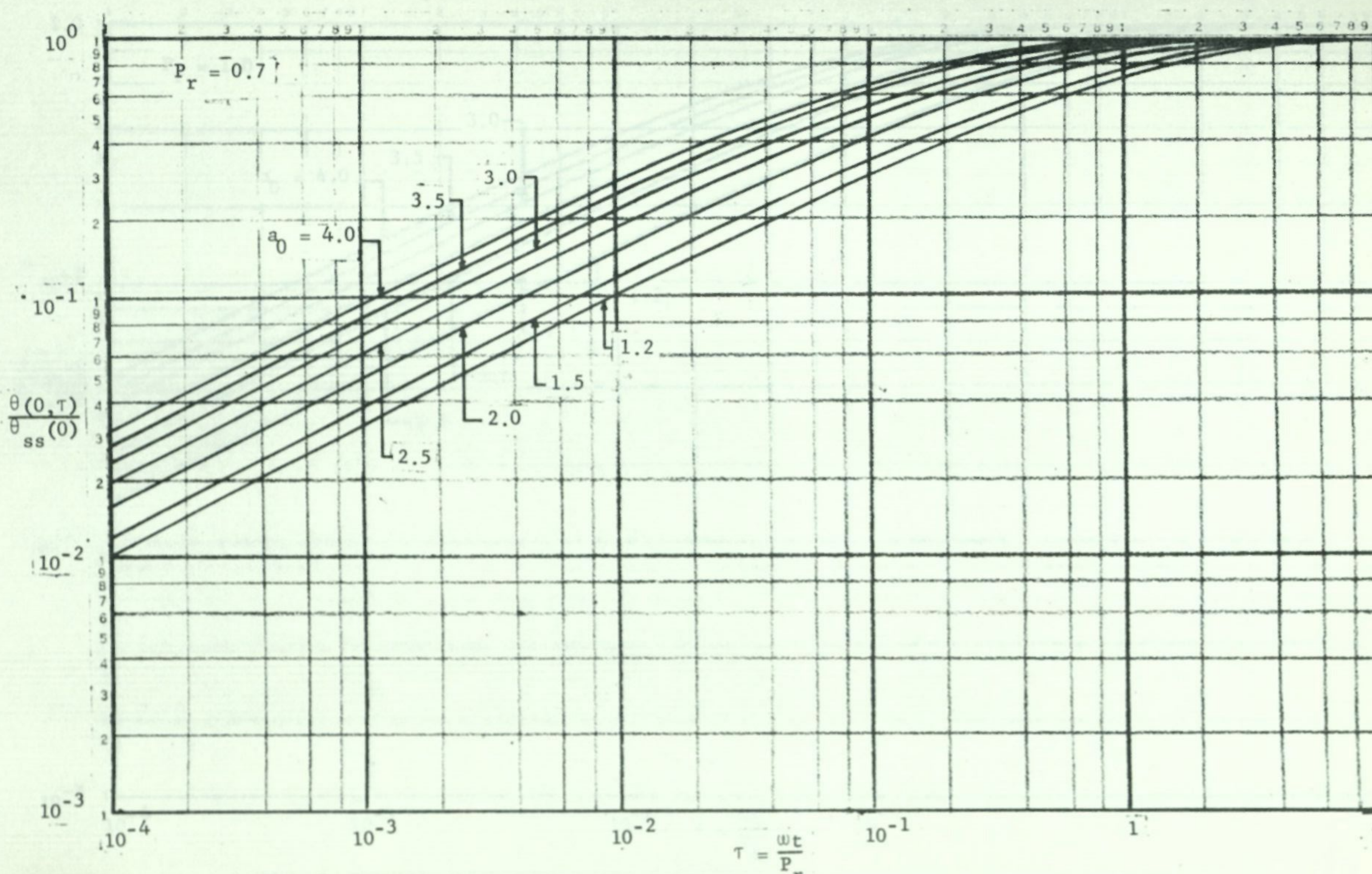

FIG. 4 DIMENSIONLESS TEMPERATURE RESPONSE AT THE SURFACE OF A ROTATING DISK DUE TO A STEP CHANGE IN LOCAL HEAT FLUX

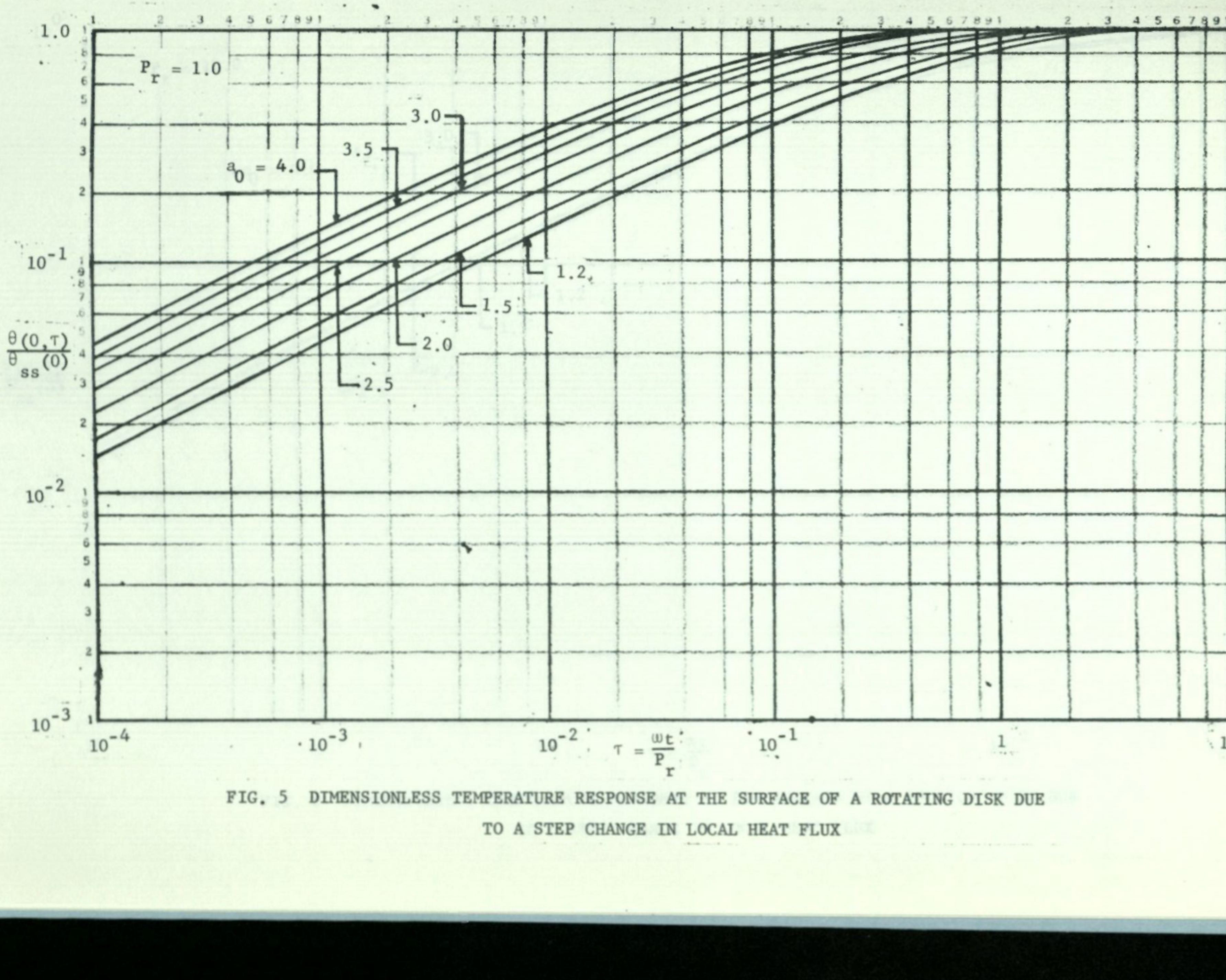

FIG. 5 DIMENSIONLESS TEMPERATURE RESPONSE AT THE SURFACE OF A ROTATING DISK DUE TO A STEP CHANGE IN LOCAL HEAT FLUX

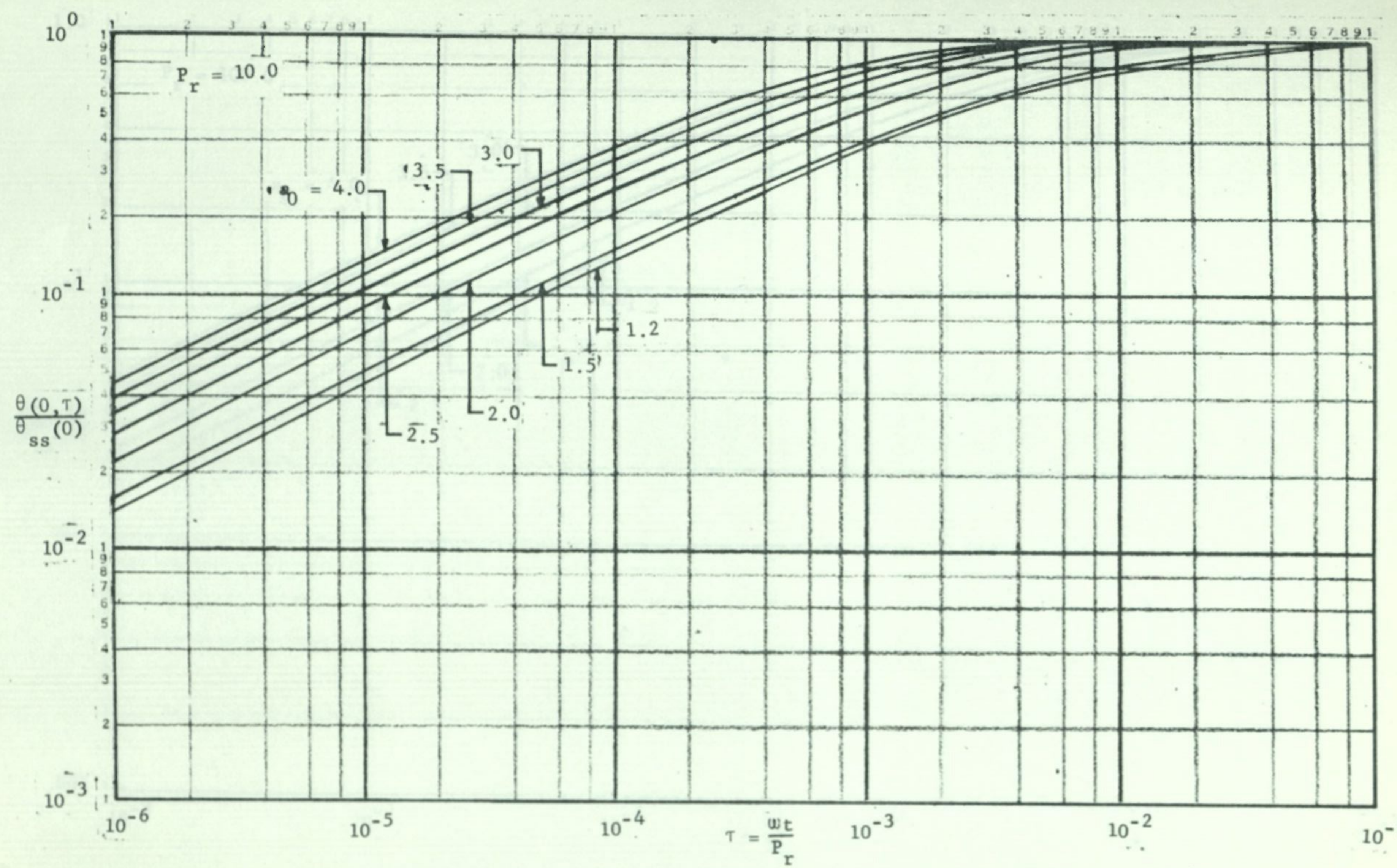

FIG. 6 DIMENSIONLESS TEMPERATURE RESPONSE AT THE SURFACE OF A ROTATING DISK DUE TO A STEP CHANGE IN LOCAL HEAT FLUX

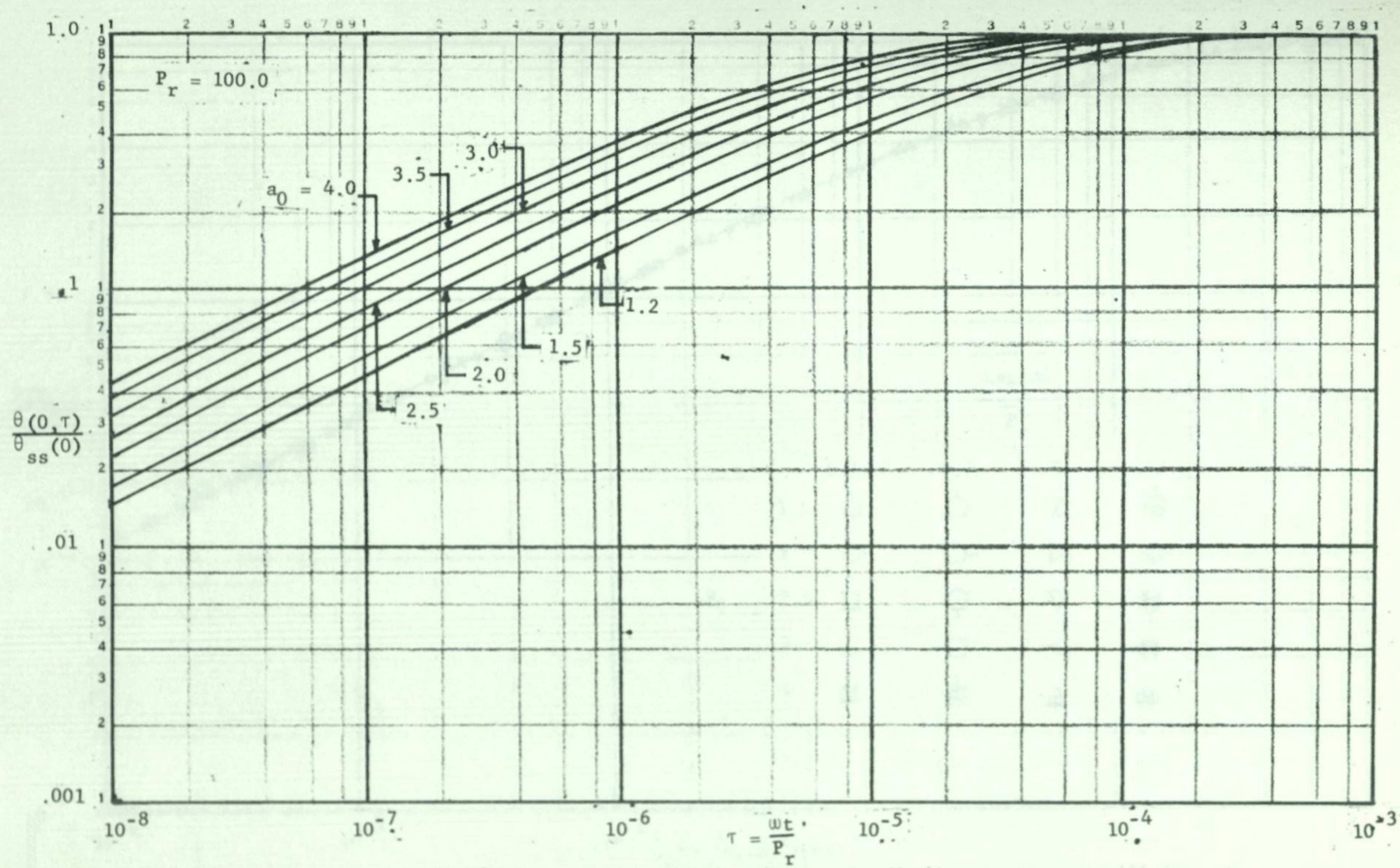

FIG. 7 DIMENSIONLESS TEMPERATURE RESPONSE AT THE SURFACE OF A ROTATING DISK DUE TO A STEP CHANGE IN LOCAL HEAT FLUX

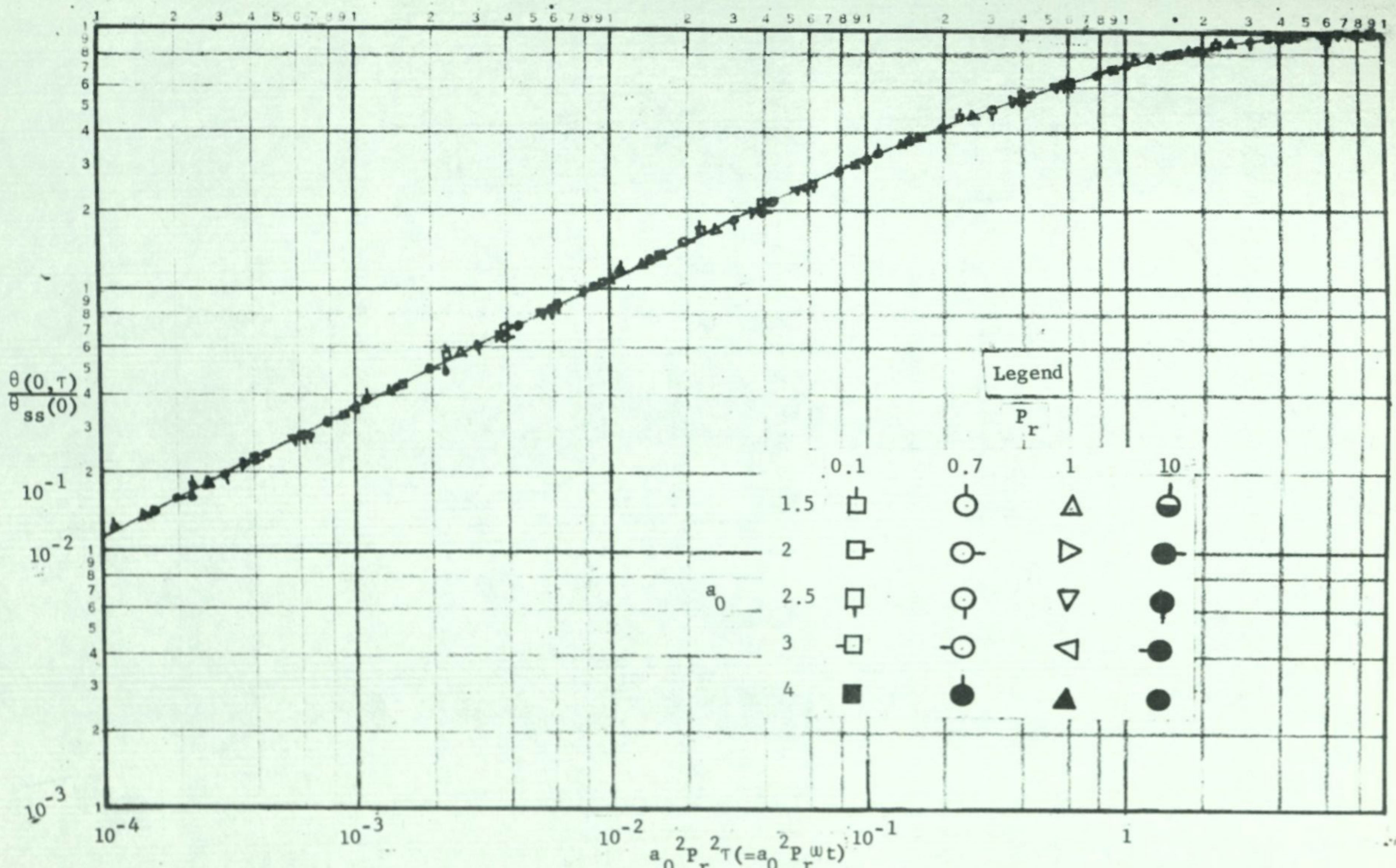

FIG. 8 SIMILARITY PRESENTATION OF THE RESULTS SHOWN IN FIGURES 2 to 7 FOR $P_r = 0.1$ and $a_0 = 1.5$ to 4.0

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