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Abstract

The discovery of logical paradoxes in set theory during the last part of the nineteenth century called into question the claim to certainty that mathematical objects and relations had traditionally enjoyed, presenting mathematicians and philosophers with the difficult challenge of reevaluating the epistemological status of mathematics. A foundationalist movement in the recently formed field of the philosophy of mathematics attempted to establish the certainty of mathematical theories by showing that they could be constructed from epistemologically justifiable principles. This paper provides a critical overview and evaluation of the methods and philosophies used by the three main schools of foundationalism—logicism, intuitionism, and formalism—followed by a critique of the attempt to establish maximal epistemological justification of mathematics in general.
What is the epistemological status of mathematical objects and relations? This question seems at first so abstract that it is hard to imagine how it would have significant practical consequences. However, the seriousness with which the question of certainty in mathematics has been met by many philosophers and mathematicians cannot be underestimated. A well-known story from antiquity relates how an individual’s challenge to the notion of mathematical certainty resulted in his death. The Pythagoreans believed that mathematics were so certain that they described the structure of reality. After showing that numerical ratios in the musical scale bring the chaos of sound into order, the Pythagoreans argued that all indefinite and chaotic aspects of the universe could be ordered and rendered intelligible through the concept of number (Curd 17). The *apeiron* could be ordered and understood in terms of finite positive integers. However, this metaphysical view was called into question by the discovery of numbers that are not finite. Story has it that one day, while Pythagoras and his followers were sailing on the coast of Southern Italy, Hippasus showed Pythagoras that the ratio of the hypotenuse to the sides of a right triangle is the number that, when squared, equals two (Rucker 57). This number cannot be expressed as a ratio of positive integers. In fact, the number is not describable in terms of any finite entities, since it is infinite in its decimal expansion. Through their discovery of an irrational number, the Pythagoreans were forced to acknowledge the existence of numbers that are essentially infinite and therefore fail to organize the cosmos in finite categories (Moore 2). This discovery was so troublesome that, it is said, Pythagoras threw Hippasus overboard to his death and swore all of his students to secrecy.
Controversy over the certainty of mathematical objects has not waned since the days of Pythagoras. The discovery of a number of paradoxes in set theory at the end of the nineteenth century brought the problem of mathematical certainty to the center stage of mathematical and philosophical investigation. The foundationalist movement was thus born. Comprised by three schools of thought—logicism, intuitionism, and formalism—this movement sought to rebuild mathematics on a rigorous and certain theoretical foundation, thus establishing once and for all that mathematics is an edifice of indubitable truths. In the spirit of Poincaré’s assertion that “[t]o foresee the future of mathematics, the true method is to study its history and its present state” (Kline 3), this paper will explore and critique the philosophies and methods used by the three foundationalist schools with the aim of contributing to an understanding of the current state of debates over mathematical certainty and to the development of an informed position concerning the nature, validity, and future of mathematical activity.

Infinity and the Paradoxes: from the Greeks to Cantor

Numerous difficulties associated with the concept of infinity have been known to mathematicians and philosophers for millennia. The oldest recorded paradoxes concerning the infinite were discovered by the Greeks. Parmenides’ argument against the possibility of change and Zeno’s paradoxes are logical difficulties that result from pondering the infinite. Parmenides argued that change is a contradictory notion because it entails that things that exist come in and out of being. What has being at one point in time may not have being before or after that point. However, it is contradictory to say that something does and does not have being. Therefore, change is unintelligible and
ultimately false. "How could what is be in the future? How could it come to be? For if
it came into being, it is not, nor if it is ever going to be. In this way, coming to be has
been extinguished and destruction is unheard of" (Curd 47). An unstated assumption of
Parmenides’ argument against change is that time can be conceptualized as a finite whole
composed of infinite parts. Only within this notion of time does it become possible to
say that change entails that a thing at once does and does not have being, and is therefore
contradictory.

Zeno also reached a paradoxical result by considering the concept of the infinite.
Assuming that space is an infinitely divisible whole, Zeno showed that it would be
impossible for the swift demi-god Achilles to win a race against a tortoise if the latter has
a head start. Before Achilles can catch up with the tortoise, he must reach the point from
which she started. But by the time he reaches this point, she will have advanced. When
he tries to overtake her again, the same will happen. Therefore, Achilles can never pass
the tortoise. This argument can be extended to refute the possibility of motion. Suppose,
for example, that an object can move from point A to point B, where these points
determine a line segment. The object must first cover half the distance between A and B.
Then it must cover half the distance left, and so on ad infinitum. Since there are infinite
halves to be covered before the object can move from A to B, the whole trajectory can
never be completed. The object gets closer and closer to its destination but can never
reach it.

Aristotle, like much of the Greek world, was troubled by the paradoxes that
resulted from thinking about the infinite. One way of solving the contradictions could be
to simply reject the notion of the infinite. However, it is impossible to ignore the
existence of at least a kind of infinity. Time and space seem to be infinitely divisible. Numbers increase without limit. How, then, can the concept of the infinite be acknowledged and yet prevented from leading to paradoxical conclusions? Aristotle answered this question by distinguishing between two kinds of infinity: potential and actual. Potential infinity is the characteristic of a process that has no end. Counting and the division of space and time into smaller and smaller parts are examples of potential infinity. This kind of infinity, Aristotle claimed, is a fundamental aspect of reality and cannot be denied. Actual infinity, on the other hand, is the conceptualization of the infinite as a completed whole. Parmenides' treatment of time and Zeno's understanding of space are both examples of actual infinity. Aristotle claimed that this concept is illegitimate and that its rejection abolishes the paradoxes.

For much of the Middle Ages, and even through the modern period, Aristotle's distinction between potential and actual infinity, as well as his rejection of the latter, prevailed. However, discussion of the contradictions caused by the notion of actual infinity continued. A famous new paradox was articulated by Galileo in the early sixteen hundreds. He showed that the counting numbers can be paired in one-to-one correspondence with their squares. If equinumerosity between two collections implies that the collections have an equal number of members, then the fact that the counting numbers can be paired off with their squares entails that the set of counting numbers has the same number of elements as that of their squares. However, the set of squared counting numbers is a proper subset of the set of counting numbers. The notion that the proper subset of a larger set has the same number of elements as the larger collection
violates Euclid's principle that the whole must always be greater than its parts: thus the paradox.

The concept of actual infinity was not dead in the field of speculative philosophy either. In fact, it was central to a number of metaphysical theories of the modern period. Descartes' notion of God was that of a being containing infinite attributes — in other words, a personification of actual infinity. Spinoza, an important rationalist philosopher of the seventeenth century, went so far as to reject potential infinity as a false infinity, while embracing the actually infinite as that which ultimately describes reality. "For Spinoza the infinite is not the fixing of a limit and then passing beyond the limit fixed [potential infinity]...but absolute infinity, the positive, which has complete and present in itself an absolute multiplicity which has no Beyond" (Hegel 262).

The controversy over the legitimacy of actual infinity saw its most drastic development since Aristotle in the work of the German mathematician Georg Cantor, who not only accepted the actually infinite as a valid notion but also created a theory that aimed to rigorously define this slippery concept. Cantor was the first person to introduce the concept of a set, that is, the primitive notion of a collection of elements. However, a collection need not be finite. In fact, Cantor asserted the existence of infinite sets and defined them as those that can be put in one-to-one correspondence with a proper subset of themselves (Kline 201). According to Cantor, one-to-one correspondence between two sets entails that the two collections have the same number of elements (i.e., they have the same cardinality). Consequently, an infinite set is one that has the same number of elements as a proper subset of itself. Cantor's conception of infinite sets was
revolutionary since it not only used the controversial notion of actual infinity but also explicitly denied Euclid's maxim that a whole must be greater than its parts.

However, making use of actual infinity is not the boldest move that Cantor made. After asserting that infinite sets could be considered completed wholes and that their sizes could be compared through the notion of one-to-one correspondence, he still needed to determine whether all infinite sets could be put into that relation to one another—in other words, whether all infinite sets contain the same number of elements. Cantor's most startling discovery was that they do not. The set of real numbers cannot be put into one-to-one correspondence with the natural numbers. This result can be proved by a *reductio ad absurdum*. Consider the open interval of real numbers from zero to one. Suppose that all the real numbers in this interval can be put into one-to-one correspondence with the natural numbers in the following way:

\[
1 \leftrightarrow 0 . \ a_{11} \ a_{12} \ a_{13} \ a_{14} \ldots \\
2 \leftrightarrow 0 . \ a_{21} \ a_{22} \ a_{23} \ a_{24} \ldots \\
3 \leftrightarrow 0 . \ a_{31} \ a_{32} \ a_{33} \ a_{34} \ldots \\
\vdots
\]

\(a_{nn}\) stands for any of the ten digits from zero to nine, so that this model for the pairing up of natural numbers with the reals from zero to one includes any possible way of arranging the decimals in order for the relation of one-to-one correspondence to hold. If the correspondence is possible, then all the real numbers between zero and one must be included among the numbers in the list. However, it is possible to construct a new number that is not in the list by choosing for its first decimal place one that is different from \(a_{11}\), for the second place a digit different from \(a_{22}\), and so on, so that the \(n^{th}\) decimal place of the new number is different from \(a_{nn}\). The number thus constructed is a real
number between zero and one that could not have possibly been included in the numbers that were put into one-to-one correspondence with the naturals since it differs from the first entry in its first decimal place, from the second in its second decimal place, and more generally from the $n^{th}$ entry in its $n^{th}$ decimal place. Therefore, the model above does not put all real numbers between zero and one in a relation of one-to-one correspondence with the natural numbers. Moreover, since the model exemplifies all possible arrangements that could lead to the pairing off of reals and naturals, it follows that no such relation can be established.

Through this simple but consequential proof, Cantor showed that there are more elements in the set of real numbers between zero and one than there are natural numbers. It follows that there are infinities of different sizes: the infinite number of elements contained in the set of real numbers is larger than the infinite number of elements contained in the set of natural numbers. This conclusion baffles the most imaginative of minds. The concept of the infinite, often intuitively understood as applicable only to things that exceed all limits, is shown to apply to collections that have specific "sizes". As if this result were not astounding enough, Cantor also shows that infinite collections can have comparatively larger or smaller "sizes".

Cantor’s work paved the way for a great deal of development in set theory, which fostered an atmosphere of hope for the attainment of unprecedented rigor and accuracy in mathematics. However, the discovery of antinomies in set theory soon led to a crisis. As Frege later pointed out, "just as the building was completed, the foundations collapsed" (Kline 197). In 1899, Cantor discovered a paradox that resulted from his treatment of the notion of set. Consider the set of all sets; call it $S$. Now think about the set of all
subsets of the set of all sets (in other words, the power set of S); call it P. P must be larger than S since a power set is always larger than the original set. Therefore, the cardinality of P must be larger than the cardinality of S. However, P is a set. Since S is “the set of all sets,” and P is a set, P is a proper subset of S (by definition). But the cardinality of the proper subset of a set must be less than or equal to the cardinality of the latter (in other words, the size of the subset must be smaller or equal to the size of the collection of which it is a subset). Therefore, the cardinality of P must be less than or equal to the cardinality of S. We have established both that $\text{card}(P) > \text{card}(S)$ and $\text{card}(P) \leq \text{card}(S)$, which is a contradiction.

Cantor’s solution to this paradox (known as Cantor’s Paradox) was to deny the existence of the set of all sets. Though for any given set there always exists a still larger set (namely the power set of the original set), it is impossible to consider the ultimate set to which all others belong. The number of sets that exist increases indefinitely, but the totality is never completed. In other words, this number is potentially rather than actually infinite. Though necessary to escape contradiction, this ad hoc solution to the paradox is hardly satisfactory.

Another troublesome paradox that originated in Cantor’s theory was discovered by Burali-Forti. This antinomy is associated with the concept of ordinal numbers, which was first introduced by Cantor. Cardinal numbers represent the size of a set independently of the ordering of the set. Ordinal numbers, on the other hand, denote the size of well-ordered sets only. The ordinal number associated with the set \{0\} is 1, the one associated with \{0,1\} is 2, the one associated with \{0,1,2\} is 3, and so on. The transfinite ordinal associated with the ordered set of all natural numbers was denoted $\omega$.
by Cantor. In the theory of transfinite ordinal numbers, the following three points hold true:

1. Every well-ordered set is associated with a unique ordinal number.

2. Every set of well-ordered ordinals is associated with an ordinal number that is greater than any ordinal number in the set.

3. The set of all ordinals is well ordered.

From statements 1 and 3 it follows that the set of all ordinals is associated with a unique ordinal number. Suppose that the set of all ordinals is associated with the ordinal number α. But α is a member of the set. As a member of the set, α must be smaller than the ordinal number associated with the set (according to statement 2). Therefore α < α, which is a contradiction.

The easiest way to dispense with Cantor's antinomy and Burali-Forti's paradox would be to find a fallacy in the way in which they are derived. Russell and Poincaré proposed that the fallacy was the use of impredicative definitions, which are those such that an object is defined in terms of a set of objects that contains it. Cantor's paradox contains an impredicative definition because it refers to the "set of all sets": a totality that is itself one of the objects contained in it. Burali-Forti's paradox also makes use of an impredicative definition since it requires the notion of the transfinite ordinal associated with the set of all ordinals, to which it itself must belong. Since the paradoxes depend on the use of impredicative definitions, they could be solved by making the latter illegitimate. However, doing so would not only do away with the antinomies, but also with large parts of classical mathematics, particularly analysis. For example, the notion of a least upper bound can only be defined in terms of a set of upper bounds that contains the upper
bound being defined. Herman Weyl tried to redefine the least upper bound in order to
avoid impredicative definitions, but his efforts were fruitless, and he concluded that
analysis is not well-founded and parts of it must be rejected. Like the definition of a least
upper bound, that of the maximum value of a function relies on all the values that the
function takes, of which the maximum value is one (Kline 208). Since eliminating the
use of impredicative definitions would entail rejecting parts of classical mathematics that
most mathematicians want to retain, this method of dealing with the paradoxes has not
been widely supported.

The discovery of antinomies in the foundations of mathematics and the difficulty
found in attempting to eliminate them caused an intellectual crisis for mathematics and
philosophy. Three main schools of thought attempted to deal with these problems by
reconstructing mathematics in accordance with principles that, within the framework of
their particular philosophical positions, would provide it with a secure and indubitable
foundation. The next section of this paper is an overview and brief evaluation of the
philosophies and methods used by the logicist, intuitionist, and formalist movements.

**Logicism**

The late nineteenth-century and early twentieth-century school of logicism is
often described as a movement that adheres to mathematical Platonism. Plato believed
that mathematical objects were eternal, objective entities that were more real and the
properties of which could be known with more certainty than the entities and properties
of the physical world. Plato's theory states that the world of images, constituted by
shadows and reflections of physical objects and reached through the faculty of
imagination, is the realm of least reality. The physical world, which encompasses the originals of images grasped by the imagination, is more real. However, neither images nor physical objects are as real as entities of the intelligible realm. Of the latter there are two kinds: those grasped through the faculty of thought, and those apprehended through noēsis or unmediated understanding. The former are mathematical objects and relations. They are the original entities of which the physical world is an imperfect copy. Finally, the objects grasped through pure understanding are the forms, which are the epistemological foundations of all knowledge.

<table>
<thead>
<tr>
<th>Metaphysical Realm</th>
<th>Faculty for Grasping Objects</th>
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<tbody>
<tr>
<td>The Forms</td>
<td>Understanding (noēsis)</td>
</tr>
<tr>
<td>Mathematical Objects</td>
<td>Thought (dianoia)</td>
</tr>
<tr>
<td>The Physical World</td>
<td>Belief (pistis)</td>
</tr>
<tr>
<td>Images, Shadows, Reflections</td>
<td>Imagination (eikasia)</td>
</tr>
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Mathematical objects, for Plato, are more clear and real than the objects of the physical world. In fact, the latter are no more than inadequate copies of things in the intelligible realm in which mathematical objects exist. Furthermore, in the opinion of the Greek philosopher, mathematical objects are objective, eternal, and mind-independent. They are discovered—not created—through the correct exercise of thought.

Logicism adhered to mathematical Platonism insofar as it advocated that mathematical objects and relations are objective and that their existence and properties can be grasped through reason. Gottlob Frege, the first important figure of the
movement, claims in his *Foundations of Arithmetic* that the "incorruptible truth of number is common to me and to any reasoning person whatsoever" (Hersh 145). Frege, like Plato, believed that mathematical objects, such as numbers, are objective and discoverable through the exercise of reason. In order to prove the objective status of mathematical entities, members of the logicist school appealed to the notion of analytcity. The main tenet of the logicist school was that all mathematical truths are derivable from the axioms and rules of inference of formal logic. If this could be demonstrated, then the metaphysical and epistemological status of mathematical truths would be equivalent to that of logical truths. Since the latter were held to be analytic (and therefore necessarily true), the reduction of mathematics to logic would put the edifice of mathematical theorems on a firm ground of absolute certainty.

In his *Foundations of Arithmetic*, Frege says that his goal in constructing gapless proofs for arithmetical laws from primitive logical principles is "to have made it probable that the laws of arithmetic are analytic judgments and consequently a priori" (Weiner 119). Though this evaluation of Frege's philosophical program has been challenged, I agree with Weiner that his project should be understood as a metaphysical and epistemological inquiry into the nature of mathematical truth and knowledge. By constructing a list of axioms from which all mathematical theorems could be derived, Frege hoped to isolate the principles upon which the epistemological justification of these theorems depended. In his *Basic Laws*, Frege says, "Because there are no gaps in the chains of inference, every 'axiom', every 'assumption', 'hypothesis', or whatever you wish to call it, upon which a proof is based is brought to light; and in this way we gain a basis upon which to judge the epistemological nature of the law that is proved" (Weiner
Since the members of the logicist school believed that logical principles were analytic and thus a priori, showing that mathematical truths could be derived strictly from logical principles would demonstrate that their epistemological status is also analytic and a priori.

Frege was the first proponent of logicism who attempted to systematically show how mathematical truths could be derived from formal logic. However, something in the system that he developed for this derivation was quickly shown to be problematic. Shortly before the publication of the second volume of his *Fundamental Laws of Mathematics*, in which he tried to demonstrate that arithmetic could be constructed from set theory (which, he believed, was a branch of formal logic), Russel wrote Frege to inform him of a paradox originating in Frege’s definition of set. Frege’s axiom of set abstraction asserts that “given any property there exists a set whose members are just those entities having that property” (Suppes 5). If no object satisfies the property, then the set is the empty set. However, a contradiction arises if one considers the set of all things sharing the property of not being members of themselves. Does this set belong to itself? If it does, then it must have the property of not being a member of itself. Therefore, it does not. However, if it does not belong to itself, then it cannot have the property that characterizes its elements—that is, it cannot have the property of not belonging to itself—which means that it must belong to itself.

Russell’s paradox becomes extremely clear when it is expressed in symbolic language. The unrestricted axiom of set abstraction is expressed in logical notation in the following way:

\[ \exists y \forall x (x \in y \leftrightarrow \varphi(x)) \]
where $\varphi(x)$ is a formula containing the variable $x$. Russell considered the case in which this formula is satisfied only by sets that are not members of themselves, that is, the case in which $\varphi(x)$ is the formula $\neg(x \in x)$. Substituting $\neg(x \in x)$ for $\varphi(x)$ in the axiom of abstraction results in the following expression:

$$\exists y \forall x \ (x \in y \leftrightarrow \neg(x \in x))$$

Letting $a$ denote the object referred to by this existential quantifier, we get:

$$\forall x \ (x \in a \leftrightarrow \neg(x \in a))$$

Applying universal instantiation, it follows that

$$a \in a \leftrightarrow \neg(a \in a)$$

Whence

$$\exists x \ (x \in a \leftrightarrow \neg(x \in a))$$

This statement expresses Russell’s paradox. It asserts that there exists an object such that it is a member of itself if, and only if, it is not. The discovery of this paradox was so devastating for Frege’s project that the German philosopher added a postscript to his book remarking, “Hardly anything more unfortunate can befall a scientific writer than to have one of the foundations of his edifice shaken after the work is finished. This was the position I was placed in by a letter of Mr. Bertrand Russell” (Aczel 182).

However, logicism was not dead. Russell inherited Frege’s project, and brought it to fruition in a work that he co-authored with Alfred North Whitehead and published in three volumes in 1910, 1912, and 1913: The *Principia Mathematica*. Since this work is the landmark of logicism, it is worthwhile to outline the procedure whereby it attempts to establish that mathematical truths are reducible to formal logic. The book begins with the development of propositional logic. Undefined notions and axioms are presented. The
authors then proceed to construct functional logic. In order to avoid the paradoxes (in particular, Russell's paradox), Russell and Whitehead introduce the *ramified theory of types*. Simply put, the theory of types requires that "whatever involves all members of a collection must not itself be a member of the collection" (Kline 221). According to the theory, individual elements are of type 0; sets of elements are of type 1; sets of sets of elements (classes) are of type 2, and so on. Under this classification, it is permissible to say that an object $a$ belongs to an object $b$ only if $b$ is of a higher type than $a$. Consequently, the theory of types makes it illegitimate to state a formula of the type $(\exists x)\in x$, and thus avoids Russell's paradox. Basically, what the theory of types does is to exclude the use of impredicative definitions. Consequently, it is also successful in avoiding Cantor's and Burali-Forti's paradoxes.

Though successful in avoiding the antinomies, the theory of types poses new problems. The development of mathematical concepts within the framework of the theory is extremely convoluted. For example, the fact that two objects satisfy all the same predicates and predicate functions becomes insufficient for establishing identity if the objects are not of the same type. Additionally, proving theorems for the real numbers becomes a very complicated process. Within the real number system, irrational numbers are defined in terms of the rational numbers, and the latter are in turn defined in terms of the integers. Therefore, the real number system consists of sub-systems of different types. Irrational numbers are of a higher type than the rational numbers, and rational numbers of a higher type than the integers. Without the theory of types, showing that a particular theorem holds true for the real numbers is sufficient to establish that it also holds true for rational numbers, irrational numbers, and integers. However, with the
theory of types, any theorem that is claimed to hold for the real numbers must be proved to hold for irrational numbers, rational numbers, and integers separately (Kline 222).

In order to avoid the complexities that result from the theory of types, Russell and Whitehead introduce the axiom of reducibility, which states that any proposition of higher type is equivalent to one of the first type. This axiom is not only instrumental in avoiding the difficulties of the theory of types, but it is also necessary for introducing the concept of mathematical induction. The remaining steps in the *Principia Mathematica* are relatively straightforward. The theory of relations is introduced. Using this theory and the language of propositional functions, set theory is introduced. From the latter, the authors define the natural numbers, which in turn serve as a basis for the construction of the real number system, the complex number system, and all of analysis. Finally, geometry is introduced through the use of coordinates and equations of curves.

The project of the *Principia Mathematica* is now widely considered a failure partly because of its inclusion of the axiom of reducibility. Use of the axiom of reducibility is problematic because it is not an axiom of formal logic. Even though there is no proof of its falsity, there is no logical evidence for its validity. Since Russell and Whitehead use this non-logical “axiom” in the derivation of mathematics from logic, this derivation fails to establish that logic alone is sufficient for the derivation of mathematical theorems. Russell and Whitehead recognized this problem, and in the second edition of the *Principia Mathematica* stated, “One point in regard to which improvement is obviously desirable is the axiom of reducibility. This axiom has a purely pragmatic justification: it leads to the desired results, and to no others. But clearly it is not the sort of axiom with which we can rest content” (xiv).
Other important problems with the *Principia Mathematica* are the authors' use of the axiom of infinity and the axiom of choice. The axiom of infinity asserts the existence of the natural number series. In effect, the axiom of infinity posits the existence of an infinity contained in a completed whole—that is, of what Aristotle called an actual infinity. The introduction of this axiom therefore carries with it the notable controversy that surrounds acceptance of the actually infinite. Moreover, this axiom is not a proposition of formal logic.

The other problematic axiom used by Russell and Whitehead in the *Principia* is the axiom of choice. This seemingly self-evident axiom states that given a finite or infinite collection of sets, it is possible to create a new set by picking one element from each set of the collection. In symbolic notation, the axiom states that for any set $A$ there is a function $f$ such that for any non-empty subset $B$ of $A$, $f(B) \in B$ (Suppes 239). Most criticisms of the axiom of choice maintain that, unless an effective procedure specifies which element must be chosen from each set—that is, unless a mechanism for choosing is provided—no real choice can be made. Without such a rule, no element is picked, and therefore no new set can be formed. Borel voiced this concern when he said that a choice which is not guided by a law is an act of faith and as such lies outside the scope of mathematics (Kline 210). Brouwer objected to the axiom of choice in its application to infinite sets because he rejected the existence of actual infinity. Russell also expressed concern about the legitimacy of the axiom of choice. He stated that a set has to be defined by a property that all and only the members of the set possess. However, the axiom of choice allows for the creation of a set whose elements do not necessarily share a common property. The only argument that can be given in favor of accepting the axiom
of choice is pragmatic: it is necessary for the construction of the real numbers from Peano's axioms and for the proof of important mathematical theorems, such as the one stating that in a bounded infinite set it is always possible to select a sequence of numbers that converges to a limit point of the set. However, this argument in favor of the axiom of choice does not show that it is true, much less certain or necessary.

The logicist school was unable to show that mathematical truths were just a subset of logical truths because it had to rely on non-logical principles (the theory of types and the axioms of reducibility, infinity, and choice) in order to build mathematics from formal logic. However, logicism ultimately failed not only as a result of its inability to show how its objective could be carried out, but also because it failed to address deep philosophical questions crucial to its project. Suppose that logicists had been able to derive mathematics from formal logic: the metaphysical and epistemological status of mathematical theorems would then be equivalent to that of logical truths. But what is the status of logical truths? Frege, Russell, and Whitehead, like the logical positivists, would argue that logical truths are analytic statements, and as such are necessarily true. However, as Quine showed in his "Two Dogmas of Empiricism," the concept of analycity is far from clear. In order for reductionism to formal logic to lend absolute certainty to mathematical truths it would be necessary to provide an account of the meaning of analycity and to demonstrate that analytic statements are necessarily true. In My Philosophical Development (1959), a disenchanted Russell acknowledges the obstacles that logicism met and says, "The splendid certainty which I had always hoped to find in mathematics was lost in a bewildering maze. . . . It is truly a complicated conceptual labyrinth" (Kline 230).
Intuitionism

The main tenet of the intuitionist school is that mathematical truths obtain epistemological justification from intuition rather than deduction from formal logic. In order to introduce the notion that intuition must be the ultimate source of epistemological validation, I will discuss Aristotle's epistemic regress problem as it is described in his *Posterior Analytics* and his solution to the quandary.

The epistemic regress problem is a paradox that arises through the following *reductio ad absurdum* argument: Suppose that it is possible to have knowledge of a proposition—that is, that there exists a proposition that can be known. This assumption entails that there exists a proposition that is supported by other known propositions (since the definition of knowledge states that a proposition is known only if it is supported by other known propositions). Two possibilities arise. Either there is a finite sequence of propositions supporting each other, where a single proposition appears twice in the sequence, or there is an infinite sequence of distinct propositions such that every proposition is supported by its predecessor. However, the first possibility describes a circular regress and the second an infinite regress. Neither a circular nor an infinite regress can lend support to a proposition—the former because circular reasoning results in a question-begging argument, and the latter because, since the regress is never completed, it is ultimately ungrounded. Therefore, no proposition can be supported. This conclusion is a direct contradiction of the second premise, which states that "there exists a proposition that is supported by other known propositions." By *reductio ad absurdum*, we conclude the negation of the initial assumption, which asserted that it is
possible to have knowledge of a proposition. The inevitable paradoxical conclusion of this argument is that knowledge in general is impossible.

In order to avoid the paradox, one of the premises upon which it is built must be rejected. Aristotle rejects the premise that a proposition is known only if it is supported by other known propositions. He says, “We on the other hand maintain that (1) not all knowledge is demonstrable but that (2) knowledge of immediate premises is indemonstrable. And it is evident that this [i.e., (2)] is necessary; for if it is necessary to know the prior [premises] from which a demonstration proceeds, and if these [premises] eventually stop when they are immediate, they must be indemonstrable” (101). The only way to make knowledge through demonstration possible is by basing it on what Aristotle calls “first principles”—propositions that are known to be true without need of further proof.

The next question that Aristotle must answer is how knowledge of the first principles is justified. In section 19 of the Posterior Analytics, the Greek philosopher asserts, “Clearly, then, we must come to know the primary [universals] by induction; for it is in this way that [the power of] sensation, too, produces in us the universal” (139). For Aristotle, knowledge of the first principles is arrived at through induction, where the latter is understood as the discovery of universals through the analysis of particulars. However, induction requires a separation of the essential characteristics of particulars from their contingent characteristics. The cluster of essential characteristics constitutes the universal. Therefore, knowledge of first principles depends on one’s ability to distinguish essential characteristics from contingent ones. In order for this distinction to not be arbitrary, there must be a criterion that directs it. Aristotle claims that this
criterion is *intuition*. "[S]ince scientific knowledge and intuition are always true and no genus [of knowledge] exists which is more accurate than scientific knowledge except intuition; since the principles of demonstration are [by nature] more known than [what is demonstrated], and all scientific knowledge is knowledge by means of reasoning whereas there could be no scientific knowledge of the principles; and since nothing can be more true than scientific knowledge except intuition; it follows from a consideration of these facts that intuition would be [the habit or faculty] of principles, and that a principle of a demonstration could not be a demonstration and so [the principles] of scientific knowledge could not be scientific knowledge. Accordingly, if we have no genus of a true [habit] other than scientific knowledge, intuition would be the principle [or starting point] of scientific knowledge" (139, my emphasis).

The intuitionist school of thought believed that certain mathematical truths had the epistemological status of Aristotle’s first principles. These statements are known immediately through intuition, and can be used to infer further statements. They constitute the rightful basis of scientific knowledge. Mathematical principles are intuitively justified and can lend justification to deductive arguments built upon them but do not need themselves to be justified by logic or any other presupposition. Kleene quotes from Heyting, “According to Brouwer [the leading figure of intuitionism], mathematics is identical with the exact part of our thinking. ... no science, in particular not philosophy or logic, can be a presupposition for mathematics. It would be circular to apply any philosophical or logical principles as means of proof, since mathematical conceptions are already presupposed in the formulation of such principles” (51). For the intuitionists, mathematical principles are justified in the same fashion as Aristotle’s first
principles: they are known immediately and constitute a starting point for knowledge rather than an end that must be reached through deductive reasoning.

However, Brouwer and other intuitionists claim that not all the results of classical and modern mathematics are justified. In accordance once again with Aristotle, the intuitionists reject the conceptualization of infinity as a completed whole (actual infinity). For the members of the intuitionist school, the only kind of infinity that exists is the potential infinity of the progression of natural numbers. Given any number, it is always possible to find a higher one. However, the progression is never completed, and it is therefore illegitimate to talk about an infinite set whose members can all stand simultaneously in front of the intellect's gaze. “According to Weyl 1946, 'Brouwer made it clear...that there is no evidence supporting the belief in the existential character of the totality of all natural numbers. ... The sequence of numbers which grows beyond any stage already reached by passing to the next number, is a manifold of possibilities open towards infinity; it remains forever in the status of creation, but is not a closed realm of things existing in themselves. That we blindly converted one into the other is the true source of our difficulties, including the antinomies...’”(Kleene 49). The rejection of actual infinity forces intuitionists to also dismiss important parts of mathematics, in particular analysis, since the real numbers are infinite decimal expansions. Additionally, they also reject Cantor’s theory of transfinite numbers and all theorems that rely on the application of the axiom of choice to infinite sets.

Though intuitionist beliefs are similar in many ways to Aristotle’s philosophy, as has been shown, they differ from the latter in a fundamental way, namely in the evaluation of the field of logic. Aristotle is generally acknowledged as the father of the
discipline of formal logic. He was the first philosopher to explicitly articulate the principles of excluded middle and non-contradiction. The Greek philosopher believed that logical principles are first principles and therefore always necessarily true. The intuitionists, on the other hand, are suspicious of logic. They do not consider the principles of logic to be first principles. According to Brouwer, the accordance of universal validity to logical principles was a historical error. The axioms of logic were abstracted from experience and then unjustifiably ascribed a priori validity. However, according to Brouwer, this validity was assigned arbitrarily. It can only be justified when it coincides with intuitive validity. The difference between Aristotle and Brouwer is that, while Aristotle believed that logical principles always met with intuitive validity and could therefore be abstracted from experience and universally applied, Brouwer argued that the principles are only warranted by intuition in certain cases; therefore, their validity must be intuitively justified in each application. They cannot be abstracted from specific applications in experience. An example of a principle about whose application Aristotle and Brouwer disagree is excluded middle. Whereas Aristotle (sometimes)¹ argues that the principle of excluded middle is always intuitively true, Brouwer distinguishes between cases in which it can be legitimately applied and cases in which it cannot.

Brouwer argues that excluded middle is only intuitively applicable to finite sets. Based on this intuitive applicability to finite collections, an alleged universal applicability is often unjustifiably inferred. This unwarranted inference is responsible for the invalid application of excluded middle to infinite sets. In Weyl’s words, “classical logic was abstracted from the mathematics of finite sets and their subsets. … Forgetful of this

¹ Aristotle usually claims says that excluded middle holds universally, but he also says (in different writings) that it does not apply to future contingent propositions.
limited origin, one afterwards mistook that logic for something above and prior to all mathematics, and finally applied it, without justification, to the mathematics of infinite sets" (Kleene 46). The intuitionist position is that many principles that are legitimate when applied to finite sets cannot be validly applied to infinite sets, and that one such principle is excluded middle. The first premise is obvious. For example, Euclid’s maxim that a whole must always be greater than any of its parts is always true of finite sets but not of infinite collections (Kleene 46). The second premise (namely that the principle of excluded middle is one of those principles whose application holds only for finite sets) has proved extremely controversial. The acceptance of this statement hinges on the metaphysical and epistemological status of logical principles. If these maxims are, as Brouwer argues, justified solely on their derivation from experience, then their application to infinite collections (which are not found in experience) is unwarranted. On the other hand, if logical principles are analytic and a priori, as logicism maintained, then their application is always justified.

The intuitionists’ rejection of actual infinity and of the application of excluded middle to infinite sets underpins their rejection of proofs for the existence of mathematical entities through the method of *reductio ad absurdum*. In both classical and modern mathematics, numerous existence proofs follow this method by assuming, for example, that there does not exist an entity with a particular property, showing that this assumption leads to a logical contradiction, and then concluding the negation of the assumption—namely that an entity with the property in question must exist. Members of the intuitionist movement argue that such a proof is insufficient for establishing existence unless it provides an example of a specific entity that satisfies the property. This view on
existence explains the intuitionist approach to sets. Since the full characterization of a mathematical entity is insufficient to determine existence, and an example of the entity must be exhibited, a set of entities must be defined in constructive terms. The definition of a characteristic shared by all the members of a collection is not enough to establish the existence of that collection. Instead, a method must be given through which it is possible to find the elements of the set in a finite number of steps.

A complete evaluation of the success or failure of the intuitionist school must be twofold. First, with respect to their practical success in reconstructing mathematics, the final assessment is at best mixed. Though the intuitionists were able to re-create a large part of classical mathematics using only those principles and methods that they found acceptable, there are important mathematical theorems that they had to discard, especially in analysis and other transcendental branches. Additionally, there has been significant disagreement within the school with respect to the distinction between acceptable and unacceptable methods. Some intuitionists have rejected all set theoretical notions, while others have accepted a primitive notion of set and, from that basis, accepted only constructionist procedures. Some intuitionists have gone so far as to admit the existence of a class of real numbers—one that does not extend over the entire continuum—while others have only accepted the existence of the integers.

The second pertinent dimension of evaluation for the intuitionist position concerns its defense of certain mathematical concepts as first principles. The assertion that such principles exist and that they make themselves evident to the intellect in an unmediated manner demands a criterion for the distinction of these principles and an explication of how they are immediately known and what metaphysical theory warrants
their indubitability. Brouwer comes close to addressing these important questions in his *Cambridge Lectures on Intuitionism*, where he claims that “[M]athematics is an essentially languageless activity of the mind having its origin in the perception of a move of time. This perception of a move of time may be described as the falling apart of a life moment into two distinct things, one of which gives way to the other, but is retained by memory. If the twoity thus born is divested of all quality, it passes into the empty form of the common substratum of all twoities. And it is this common substratum, this empty form, which is the basic intuition of mathematics” (Hersh 154). The claim that the progression of the natural numbers is given to the intellect through a sort of pure intuition of the passing of time is reminiscent of Kant’s position, which maintains that mathematical truths derive their a priori validity through their accordance with the form of the pure intuitions of space and time. Kant’s claim that Euclidean geometry is validated a priori by its derivation from the pure intuition of space has been discredited by the development of non-Euclidean geometry and its use in the special and general theories of relativity. However, his claim that the notion of number proceeds from the pure intuition of time remains an interesting philosophical position, similar to the one advocated by Brouwer. The framework of transcendental idealism may thus be able to provide a metaphysical foundation for Brouwer’s philosophy of mathematics, in which the concept of the progression of natural numbers is known a priori through its derivation from the pure intuition of time, and further mathematical results are reached through “mental processes that can be built up by an unlimited sequence of steps repeating primitive mathematical acts indefinitely” (Bar-Hillel 221).
Formalism

The formalist school of thought, championed by the German mathematician David Hilbert, agreed with the intuitionist criticism that the concept of actual infinity and the application of certain methods of inference to infinite collections are unwarranted by intuitive evidence, but argued that this criticism does not entail that the non-intuitive elements of mathematical thinking need to be relinquished. Hilbert claims that the statements of classical mathematics can be divided into two kinds: real and ideal (Kleene 55). The real statements have an intuitive meaning and coincide with statements accepted by the intuitionists. Ideal statements, on the other hand, do not have an intuitive meaning. Nothing in experience provides evidence for their truth or falsity. However, ideal statements have significant pragmatic import because, in conjunction with real statements, they result in the formation of theoretical constructs that prove helpful in the understanding and development of physical theories (both applied and theoretical).

Hilbert's view emphasizes the pragmatic consequences rather than the truth of the parts of mathematics that are not justified by intuitive evidence. The criterion for accepting these elements of mathematical reasoning is twofold: first, they must combine with the intuitive parts in a way that is useful for the development of new theories. Second, they must be consistent with the "real" mathematical statements (those that are intuitively true). This second requirement brought the issue of the consistency to the forefront of the theoretical investigations of the formalist school. According to Hilbert, proving the consistency of mathematical statements would be sufficient to justify their acceptance.
The philosophical underpinnings of the formalist position are similar to the views advocated by the American philosopher W. V. Quine. In his "Two Dogmas of Empiricism," Quine argues that the propositions that are (and should be) incorporated into a generally accepted conceptual scheme are those that organize experience in a coherent and efficient way. "Epistemologically, these [the abstract entities which are the substance of mathematics] are myths on the same footing with physical objects and gods, neither better nor worse except for differences in the degree to which they expedite our dealings with sense experiences. ... Total science, mathematical and natural and human, is similarly but more extremely underdetermined by experience. The edge of the system must be kept squared with experience; the rest, with all its elaborate myths or fictions, has its objective the simplicity of laws" (Quine 45). Hilbert, like Quine, argues that many parts of mathematics are not warranted by experience or intuition, but these elements should not be rejected if they are useful and do not conflict with the parts that are empirically or intuitively justified.

Since numerous parts of mathematics that make use of actual infinity and other ideal concepts have proved to be useful in theoretical physics, they are pragmatically justified. It only remained to show that they were consistent with the real statements of mathematics. Devising proofs for consistency thus became Hilbert's main goal. In order to carry out this project, the German mathematician used both indirect and direct proofs. An indirect proof of consistency gives a model for an axiomatic mathematical theory by assigning to each axiom and object of the theory a corresponding element in another theory. The consistency of the original theory thus becomes dependent upon the consistency of the theory with which its objects and axioms are correlated.
A direct proof of consistency, on the other hand, is one that takes the entire mathematical theory whose consistency is in question and makes it an object of mathematical study. Such a study (called meta-mathematical) must be able to prove a proposition about all the possible proofs of theorems of the theory in order to establish the latter's consistency (Kleene 55). The meta-mathematical investigation of a theory begins with its formalization. The system of propositions that constitutes the theory must be made explicit and arranged deductively, so that it becomes possible to isolate a finite number of axioms from which all other statements of the theory can be derived. Additionally, the rules of inference that allow for the deduction of all theorems from the axioms must be stated explicitly. When the theory has been reduced to a system of axioms and rules of inference that can be stated symbolically, its meaningful content has been stripped out and only its form remains: the theory has been formalized. It becomes possible to characterize all valid statements of the theory in terms of symbols and rules that make no reference to meaning. Meta-mathematics is a mathematical study of the formalized theory (i.e., of the symbols and rules devoid of meaning) through the application of intuitive and informal methods. The new field of meta-mathematics thus allows mathematics to turn its inquisitive gaze on itself.

In order to establish the consistency of different branches of mathematics, Hilbert first used indirect proofs. He demonstrated that arithmetic can be used as a model for numerous mathematical theories, and thereby reduced the question of the consistency of these theories to the question of the consistency of arithmetic. It only remained to prove the latter through a direct proof of consistency—that is, through a meta-mathematical study of formalized arithmetic. In 1931, a mathematician and logician named Kurt Gödel
used the meta-mathematical method to reach an important result concerning the consistency of arithmetic. Alas, this result could not have been further from what Hilbert expected.

Kurt Gödel's incompleteness theorems proved that Hilbert's goal of establishing the consistency of arithmetic with the direct method was doomed to failure. In order to demonstrate this startling result, Gödel showed that it was possible, using the formalized language of arithmetic, to construct a sentence that denies its own provability. The sentence is of a logical form equivalent to that of the liar's paradox, which is the antinomy entailed by the sentence "This sentence is false;" call it $S$. If $S$ is false, then what it says must not be the case; in other words, $S$ must not be false. However, if $S$ is not false, then it must be true (by the law of excluded middle), which means that what it says must be the case. Therefore, $S$ must be false. In other words, if $S$ is false, then it must be true, and if it is true, then it must be false. The following paragraphs will provide an outline of Nagel and Newman's version of Gödel's proof in order to explain how a statement similar to the liar's sentence can be constructed in the language of arithmetic and how the consequences of constructing such a statement were fatal to the formalist project.

The first step in understanding Gödel's proof consists in understanding the method of "Gödel numbering" that he used in order to map the meta-mathematics of arithmetic back into the arithmetical calculus. First, Gödel assigned a unique number to each elementary arithmetical sign in the following way:
Not only individual signs but also arithmetical formulas are associated with a specific number. The Gödel number of a formula is found from the Gödel numbers of the elementary signs that constitute it by multiplying prime numbers, in order of magnitude, each raised to a power equal to the Gödel number of the corresponding elementary sign.

The same procedure can be followed in order to assign a specific number to a sequence of formulas. The Gödel number of a formula or sequence of formulas is always unique (given the number, it is possible to determine exactly what expression it is associated with) since, according to the fundamental theorem of arithmetic, every integer greater than one can be decomposed as the unique product of prime numbers.

In order to proceed with Gödel's proof, it only remains to define two notations.

Let Sub(m,13,m) denote the Gödel number of the formula obtained from the formula with Gödel number m, by substituting for the variable with Gödel number 13, the numeral for m (definition 1). Also, let Dem(x,y) be the assertion that the sequence of formulas with Gödel number x is a proof for the formula with Gödel number y (definition 2). With

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<th>Sentential Variable</th>
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<td>q</td>
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<td>17^2</td>
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these definitions in mind, follow steps (1) through (4) in order to construct a particular
arithmetical formula.

(1) Consider the formula \((\forall x)\neg \text{Dem}(x, \text{sub}(13, y))\).

Let \(n\) be the G"odel number associated with this formula.

(2) Meta-mathematically, this formula states that the formula obtained from the
formula with G"odel number \(y\), by substituting for the variable with G"odel
number 13 (which is \(y\)) the numeral for \(y\) (which is 13), is not demonstrable.
In other words, the result of taking the formula with G"odel number \(y\) and
substituting all instances of the variable with G"odel number \(y\) with the
number 13 is a formula that cannot be proved to be true. Keep in mind that
whether the formula in (1) is true or not is irrelevant, since it is presently
being used merely as a basis for the construction of a new formula whose
truth or falsity will be analyzed separately.

(3) In the formula in (1), substitute \(n\) for \(y\), to obtain a new formula \(G\).

(4) \(G\) is the following formula: \((\forall x)\neg \text{Dem}(x, \text{sub}(n, 13, n))\).

Now consider the following two questions: First, what is the meta-mathematical meaning
of \(G\)? According to definition 2, the meaning of \(G\) is that the formula with G"odel number
\(\text{sub}(n, 13, n)\) is not demonstrable. Second, what is the G"odel number associated with \(G\)?
This number is \(\text{sub}(n, 13, n)\) by definition 1, since \(G\) was constructed from the formula
with G"odel number \(n\) by substituting the variable with G"odel number 13 with \(n\).
Therefore, in the meta-language, \(G\) says of itself that it is not demonstrable.

Is \(G\) a true formula of the arithmetical calculus, or is it false? Assume that \(G\) can
be proved in the formalized language of arithmetic. Let \(x\) be the G"odel number
associated with the sequence of formulas of the proof of $G$. The assumption can then be
written in symbolic language as follows: $(\exists x)\text{Dem}(x,\text{sub}(n,13,n))$. This formula is
equivalent to the formula $(\forall x)\neg\text{Dem}(x,\text{sub}(n,13,n))$. However, this formula is the
negation of $G$. Therefore, if $G$ is demonstrable in the language of arithmetic, then $G$ is
false. This state of affairs would entail that the arithmetical calculus is inconsistent (since
it could be used to prove the validity of a formula that is false). In order to maintain the
consistency of arithmetic, it must not be the case that $G$ is demonstrable. However, to
say that $G$ is unprovable is to restate, in the meta-language, what $G$ says of itself.
Therefore, $G$ is an unprovable true statement.

The possibility of constructing a statement in the language of arithmetic that is
true but cannot be derived from the formalized arithmetical calculus shows that
arithmetic is fundamentally incomplete (if consistent). Within a formalized system that
attempts to characterize all arithmetical theorems with a finite set of consistent axioms
and rules of inference it is always possible to find a statement that is true but cannot be
derived from the formal language. In other words, an axiomatized theory of arithmetic is
incapable of characterizing all arithmetical truths. This result is known as Gödel’s first
incompleteness theorem. It is responsible for showing the futility of Hilbert’s call for the
formalization of all mathematical theories.

However, it was Gödel’s second incompleteness theorem that dealt a final blow to
the formalist project. The first theorem shows that if arithmetic is consistent, then it is
incomplete. Let $p$ be the statement ‘Arithmetic is consistent’ and $q$ be ‘Arithmetic is
incomplete.’ Thus $p \rightarrow q$ is true. By definition, arithmetic is incomplete if, and only if,
there exists a true arithmetical statement that is not formally demonstrable in the
formalized calculus. The formula $G$ asserts the existence of precisely such a statement. Therefore, $q \iff G$. Since $p \rightarrow q$ and $q \iff G$, $p \rightarrow G$. Now, suppose that the consistency of arithmetic were provable. Then $p$ would be provable, and, by *modus ponens*, so would $G$.

However, according to the first incompleteness theorem, if $G$ were provable, then arithmetic could be shown to be inconsistent. In other words, the assumption that arithmetic can be proved to be consistent results in a logical contradiction. Therefore, it is impossible to establish the consistency of arithmetic through the meta-mathematical method.

Gödel's first incompleteness theorem marked the end of the effort to formalize all mathematical truths since it showed that it is impossible for an axiomatic system to characterize all the theorems of a mathematical theory similar to the arithmetical calculus—that is, any theory that is finitely describable, consistent, and strong enough to prove the theorems of natural numbers (Rucker 178). However, the consequences of Gödel's result for the formalist movement were more devastating than the destruction of the goal of absolute formalization. Gödel's second incompleteness theorem demonstrated that it is impossible to prove the consistency of arithmetic through the direct methods of meta-mathematics. Hilbert had used the indirect method to reduce the question of consistency for most mathematical theories to the question of the consistency of arithmetic. Establishing the consistency of every one of the theories that had been mapped into arithmetic depended on demonstrating the consistency of the latter through a direct proof. However, Gödel showed that such a proof is impossible if arithmetic is consistent, thus extinguishing any hope for the completion of Hilbert's goal of establishing the consistency of all mathematical truths.
A Final Appraisal: The Three Schools of Foundationalism

A philosophical evaluation of logicism, intuitionism, and foundationalism must look at the method and purpose of their programs. In this section, I discuss the notions of truth and certainty that the three schools advocated and the goals that they wanted to accomplish through the use of these notions in the analysis of the foundations of mathematics.

Logicism, which was the first movement to advocate a reconstruction of the foundational principles of mathematics, emerged as a direct response to the discovery of logical antinomies in set theory. The existence of paradoxes was extremely troublesome because it challenged the previously (mostly) uncontested claim to truth of mathematical principles. The approach that logicism took to restoring certainty to mathematical theorems was to try to show their reducibility to purely logical principles. The most sophisticated attempt to carry out this reduction, namely the one developed in the Principia Mathematica, failed because it ultimately had to rely on certain ad hoc non-logical principles. However, had it succeeded, it would not have automatically established an indubitable foundation for mathematical truths. It would have only transformed the issue of certainty in mathematics into that of certainty in formal logic.

A crucial question for the logicist project, then, concerns the epistemological justification of logical truths. Where do the principles of formal logic acquire their alleged certainty? There are two possibilities. Logical principles are either known a priori or a posteriori. In both cases, their epistemological justification is difficult to establish. If logical truths are valid a priori (that is, supported independently of
experience), then it becomes difficult to explain exactly what is meant by saying that they are legitimate. The claim that they are necessarily true entails that they accurately describe reality. But if they are independent of experience and yet still valid tools for the description of reality, then at least their alleged structural isomorphism to that reality must be justified. The question of epistemological justification is thus not abolished but rather moved a step back.

The other possibility is that logical principles are justified a posteriori—that is, that they derive their legitimacy from experience (an empirical approach to logical truth) or from the efficiency with which they organize experience (a pragmatic approach). However, neither of these approaches provides a ground of certainty for mathematical principles. Consider the empirical approach, which states that logical principles are learned from sensory data. The problem with this account is that it is vulnerable to the problem of induction. Even if logical principles accurately describe past and present experiences, there is no way to establish their validity for the description of future experiences. Therefore, the empirical approach cannot infuse logical principles with certainty. Now consider the pragmatic position. Such an approach to the justification of logic is eloquently articulated by Nelson Goodman in "The New Riddle of Induction." He argues that, while the deductive validity of logical arguments is ensured by their conforming to the rules of logic, the validity of the rules is in turn grounded on their agreement with accepted deductive arguments. This system of circular justification between valid arguments and logical rules is what Goodman calls "a virtuous circle." He further claims that the existence of this circle is evidenced in the dual adjustment that occurs between rules of logical inference and arguments generally accepted to be
deductively valid. When manipulation of accepted rules yields an argument that we are unwilling to accept, we conclude that the rules need to be adjusted. On the other hand, when an argument violates a rule that we are unwilling to give up, we reject the argument, however plausible it may appear. Logical principles are therefore dictated by accepted logical practice; in other words, they are justified by convention. However, Goodman's view of the epistemological status of logical truths clearly does not procure them the analytic necessity in which logicists believed.

The intuitionist school, which was largely a response to the shortcomings of the logicist school, criticized the latter's acceptance of logical principles as necessarily true on bases similar to Goodman's. Brouwer argued that logical principles were justified by experience and therefore could be applied only to experienceable objects. Intuitionist thinkers strongly criticized the abstraction of logical truths from experience to a realm of independent self-justified objectivity. However, the intuitionist account of how mathematical truths derive epistemological justification is not without problems of its own. Brouwer maintained that mathematical principles make themselves evident to the intellect in an intuitive, unmediated manner. However, Brouwer's criterion for which principles are mathematical (and therefore self-evident) and which are not is not one that most mathematicians would readily accept. According to Brouwer, large parts of classical and modern mathematics are not legitimate because they cannot be derived from the only legitimate mathematical principles—those that are given to the pure mathematical intuition of the mind. This problem is not only pragmatically taxing (since it leads to the rejection of important parts of theories that are widely accepted in mathematical practice), but also theoretically unclear. How does one differentiate
between true mathematical (intuitive) principles and illegitimate ones? The fact that the answer to this question has met widespread disagreement even among members of the intuitionist movement reveals that a criterion for distinguishing valid from invalid statements of mathematics is not itself intuitively evident. If intuition cannot lead to a clear differentiation between legitimate and illegitimate principles, then it cannot be the ultimate ground of epistemological justification.

The third and last foundationalist school of thought in the philosophy of mathematics tried to address the intuitionist criticism of logic by defending the use of mathematical principles that are not warranted by intuition in a novel manner. The formalists argued that, though the meaning of these principles is not intuitively justified, their use in mathematical reasoning can be supported if it is defined without reference to meaning. In order to define the use of unintuitive mathematical principles without reference to meaning it was only necessary to describe their formal structure and to prove that they were consistent with intuitive principles. However, Gödel's first incompleteness theorem shattered all hope for the first requirement—that of absolute formalization—and his second theorem proved the impossibility of establishing consistency (at least with the resources of first-order logic). The work of the Austrian logician Kurt Gödel thus ended the foundationalist agenda of the formalist school.

Even if completing Hilbert's project had not proved impossible, it remains questionable what its consequences would have been with regard to the question of certainty in mathematics. Hilbert seems to have believed that proving the consistency of mathematical principles would have been enough to establish their validity. He wrote in a letter to Frege, "If the arbitrarily given axioms do not contradict each other with all
their consequences, then they are true and the things defined by the axioms exist. For me this is the criterion of truth and existence” (Kaplan 51). Hilbert thus appears to have adhered to a weak version of the coherence theory of truth. According to such a theory, a statement is true if, and only if, it is part of a system of statements that is coherent—that is, consistent and containing an additional explanatory unity. Though most philosophers would probably agree that coherence, hence consistency, is a necessary condition for truth, the claim that it is sufficient is dubitable. “On the face of it, there could well exist several conflicting, yet internally coherent, systems of belief—suggesting that although coherence may confer plausibility, it is no guarantee of truth” (Horwich 64).

Closing Comments on Foundationalism

The movements of logicism, intuitionism, and formalism were unable to reconstruct mathematical principles from a rigorous foundation that would ensure their epistemological status as unshakable, indubitable, objective truths. Is mathematics today in need of such a foundational reconstruction? This question, concerning the need for epistemological justification of mathematical concepts, leads back to a more fundamental question: What is mathematics? If one considers mathematical objects and relations to be objective entities that exist independently of human activity, then there must be a way to substantiate their claim to truth. Only under this assumption, which entails that mathematical objects are discovered and not created, does it become necessary to seek an epistemic foundation for mathematical certainty.

If, on the other hand, one judges mathematical objects and reasoning to be products of human activity, the need for a foundational epistemological justification
vanishes. However, it is still necessary to have an account of whether or not one can justifiably believe mathematical results in a way that does not require maximal, truth-entailing justification. A consideration of the utility of mathematics provides such an account. There is no doubt that mathematical methods have provided science with powerful tools for the investigation and conceptual organization of experience. There are also branches of mathematics which, though not clearly associated with experience of the physical world, have proved useful in the development of theoretical constructs, for example in the field of abstract physics. Additionally, mathematics constitutes an infinite source of intellectual stimulation and enjoyment. All of these pragmatic considerations are sufficient for a validation of mathematical activity. Relinquishing a foundationalist approach to mathematics entails leaving behind a belief that mathematical objects and relations are independent of human activity and constitute an avenue to objective truth; however, the abandonment of this belief does not diminish the pragmatic and intellectual elegance and power of mathematical reasoning.
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