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CERTIFIABLE HIGHER ORDER SLIDING MODE
CONTROL: PRACTICAL STABILITY MARGINS
APPROACH

by

CHANDRASEKHARA BHARATH PANATHULA

A DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy
in
The Department of Electrical and Computer Engineering
to
The School of Graduate Studies
of
The University of Alabama in Huntsville

HUNTSVILLE, ALABAMA

2016
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\textit{Chandrasekhar Bharath Panathula} \\
\textit{08/15/2016 (date)}
DISSEETATION APPROVAL FORM

Submitted by Chandrasekhara Bharath Panathula in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Electrical Engineering and accepted on behalf of the Faculty of the School of Graduate Studies by the dissertation committee.

We, the undersigned members of the Graduate Faculty of The University of Alabama in Huntsville, certify that we have advised and/or supervised the candidate of the work described in this dissertation. We further certify that we have reviewed the dissertation manuscript and approve it in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Electrical Engineering.

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The Higher Order Sliding Mode (HOSM) controllers are well known for their robustness/insensitivity to bounded perturbations and for handling any given arbitrary relative degree system. The HOSM controller is to be certified for robustness to unmodeled dynamics, before deploying the controller for practical applications. Phase Margin ($PM$) and Gain Margin ($GM$) are the classical characteristics used in linear systems to quantify the linear controller robustness to unmodeled dynamics, and certain values of these margins are required to certify the controller. These conventional margins ($PM$ and $GM$) are extended to Practical Stability Phase Margin ($PSPM$) and Practical Stability Gain Margin ($PSGM$) in this dissertation, and are used to quantify the HOSM control robustness to unmodeled dynamics, presiding the tool to close the gap for HOSM control certification. The proposed robustness metrics ($PSPM$ and $PSGM$) are identified by developing tools/algorithms based on Describing Function-Harmonic Balance method. In order for the HOSM controller to achieve the prescribed values on robustness metrics ($PSPM$ and $PSGM$), the HOSM controller is cascaded with a linear compensator. A case study of the application of the
proposed metrics (PSPM and PSGM) for the certification of F-16 aircraft HOSM attitude control robustness to cascade unmodeled dynamics is presented. In addition, several simulation examples are presented to verify and to validate the proposed methodology.

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ACKNOWLEDGMENTS

I would like to express my sincere gratitude to my advisor Dr. Yuri B. Shtessel for his endless support and motivation during my Ph.D. pursuit. It has been a great honor to work with him, and I could not imagine in accomplishing this dissertation without his guidance. An unwavering enthusiasm he has for his research was contagious and motivational for me and kept me engaged with my research even during tough times. More than a technical mentor, Dr. Shtessel is one of my role models for scientific research and development. I always feel determined every time I step out of his office door, and I sincerely thank him for having insightful discussions with me.

Many thanks to Dr. Michael V. Basin, Professor at Autonomous University of Nuevo Leon, Mexico, for providing me an opportunity to involve in his research, when he was on sabbatical leave to the University of Alabama in Huntsville. I would like to thank my other advisory committee members Dr. Farbod Fahimi, Dr. Wenzhang Huang, Dr. Laurie L. Joiner, and Dr. W. David Pan for their overwhelming support, encouragement, and for hard questions that facilitated me to widen my research from various perspectives.

I appreciate my colleagues Dr. Antonio Rosales and Dr. Leonid Fridman for their generous contributions to this dissertation and for their support.

I would like to thank my friends, my beloved parents Sudhakar Panathula and Amruthavalli Panathula, my brother Venkata Sarath Panathula, and my fiance Rekha Rasa Manjari Paleti for supporting me throughout my course of Ph.D. study.
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designed HOSM controllers
In loving memory of my grandparents, Venkata Krishnaiah Panathula (1925 – 2000)
and Nagaratnamma Panathula (1931 – 2008)
CHAPTER 1

INTRODUCTION

It is well known that Sliding Mode/Higher Order Sliding Mode (SM/HOSM) control [1–3] technique is robust/insensitive to matched bounded perturbations. The HOSM controllers provide finite-time convergence for the sliding set (sliding variable and consecutive sliding variable derivative(s)) of any given arbitrary relative degree system to the origin in the presence of bounded perturbations. Specifically, these controllers include, but are not limited to, Nested (N-HOSM) [4], Quasi-continuous (Q-HOSM) [5], and Continuous (C-HOSM) [6] controllers. A fixed-time convergent second order SM (2-SM) controller [7] that is robust/insensitive to unbounded Lipschitzian disturbances is proposed in this dissertation. The insensitivity of the SM/HOSM control technique to matched bounded perturbations are taken advantage for various applications, such as [8–16].

It is worth noting that the SM/HOSM controller is to be certified for robustness to unmodeled dynamics, prior to be implemented in practical application. Therefore, a major challenge persists in defining robustness metrics for systems controlled by SM/HOSM controllers to unmodeled dynamics, presiding the tool for SM/HOSM control certification. Phase Margin ($PM$) and Gain Margin ($GM$) are the classi-
cal characteristics used in linear systems to certify linear controller robustness to unmodeled dynamics. It is understood that certain values of these robustness metrics are prescribed for most of the practical applications (for instance, refer page 33 of [17]: \( PM = 30^\circ \) and \( GM = 12\text{dB} \)). \( PM \) and \( GM \) are identified using frequency-domain techniques such as Bode plots, Nyquist plots, etc. Such kind of state-of-the-art methodology is hard to define for certifying nonlinear HOSM controllers.

Another challenge of SM/HOSM control certification is in addressing the side effect of the finite-/fixed-time convergence provided by the SM/HOSM control, i.e., chattering (self-sustained oscillations) that may exhibit due to unmodeled dynamics. Unmodeled dynamics are inescapable in any mathematical model that is used to design a controller and are fractal in nature [18]. Most lately, many researchers are intrigued with chattering due to cascade unmodeled dynamics of physical actuators in systems controlled by SM/HOSM controllers [19–29].

The certification of HOSM controllers for robustness to cascade unmodeled dynamics is the subject of this dissertation. The robustness metrics for SM/second order SM (2-SM) controller to unmodeled dynamics in terms of Practical Stability Phase Margin (\( PSPM \)) and Practical Stability Gain Margin (\( PSGM \)) were newly defined and proposed in [30] by characterizing the chattering parameters (frequency and amplitude of the fundamental harmonic). These practical stability margins are extended to HOSM controllers in this dissertation, and are used to certify the HOSM controllers. The parameters of the chattering can be predicted using Poincare Maps [31, 32], Tsypkin Locus [33], and LPRS method [26]. Note that these methods are confined mostly for the analysis of second order systems and are hard to implement.
Describing Function - Harmonic Balance (DF-HB) method is the only engineering method that is heavily used for estimating the parameters of predicted limit cycles in nonlinear systems [34,35]. Note that the DF-HB method is an approximation method, nevertheless, the definiteness of method in predicting the chattering parameters in dynamically perturbed systems controlled by SM/HOSM control has been verified by several works [20,26,36]. The DFs for 2-SM control algorithms, in particular, twisting [37], super-twisting [38], and sub-optimal [39] control algorithms were obtained analytically [1,20,26,36] by transforming the algorithms into an equivalent nonlinearity or combination of nonlinearities, whose DF(s) is/are readily available in DF tables [34,35]. Note that the DFs for HOSM controllers are very difficult to obtain analytically, and thus, there is a requirement for a numerical algorithm that generates DFs for HOSM controllers. Moreover, a numerical algorithm is essential to compute the predicted chattering parameters by solving HB equation.

The ultimate goal of this dissertation is to close the gap of HOSM control certification for robustness to cascade unmodeled dynamics. To achieve this goal the following tasks and sub-tasks have been formulated.

- To introduce robustness metrics for HOSM controllers: $PSPM$ and $PSGM$.
- To develop tools/algorithms to identify the proposed robustness metrics.
  - To develop a numerical algorithm to identify DFs for HOSM controllers.
  - To develop a numerical algorithm to estimate the predicted chattering parameters by solving HB eq. using Newton Raphson (NR) method.
  - To develop algorithms for the identification of $PSPM$ and $PSGM$. 

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• To develop an algorithm for designing a cascade linear compensator to HOSM control for achieving prescribed values of practical stability margins.

• To apply the proposed technique to certify HOSM attitude controller of F-16 aircraft for robustness to cascade unmodeled dynamics as a case study.

The main contributions of this dissertation are as follows:

► The metrics of robustness for N-HOSM, Q-HOSM, C-HOSM, and fixed-time convergent 2-SM controllers to cascade unmodeled dynamics are proposed.

► A fixed-time convergent 2-SM controller, whose upper bound for convergence time is independent of initial conditions, is proposed.

► A case study of the application of the proposed metrics (PSPM and PSGM) for the certification of F-16 aircraft HOSM attitude control robustness to cascade unmodeled dynamics is presented.

The structure of this dissertation is as follows. The fundamentals of HOSM controllers are discussed in Chapter 2. The fundamentals of DF-HB method are presented in Chapter 3. The problem is formulated in Chapter 4. In Chapter 5, the notion of PSPM and PSGM for SM/HOSM control robustness to cascade unmodeled dynamics are introduced, and the tools to identify the metrics are developed and proposed. Several simulation examples are shown in Chapter 6 validating the proposed method. In Chapter 7, the proposed robustness metrics (PSPM and PSGM) are applied for the certification of HOSM attitude controller of an F-16 aircraft to cascade unmodeled dynamics as a case study. Chapter 8 concludes this dissertation.
CHAPTER 2

FUNDAMENTALS OF HIGHER ORDER SLIDING MODE CONTROLLERS AND DIFFERENTIATORS

Consider a Single-Input-Single-Output (SISO) dynamic system

\[ \dot{x}(t) = a(x, t) + b(x, t)u_a, \quad (2.1) \]
\[ \sigma(t) = \sigma(x, t), \]

where \( x(t) \in \mathbb{R}^n \) represents a vector of system states, \( a(x, t) \) and \( b(x, t) \in \mathbb{R}^n \) are partially known smooth enough and known Lipschitz vector fields, respectively, \( u_a \in \mathbb{R}^1 \) represents an actuator control input, and \( \sigma(t) \in \mathbb{R}^1 \) represents a system output or a sliding variable. Let \( r_d \) be relative degree of system (2.1).

The input-output dynamics of system (2.1) are derived as

\[ \sigma^{(r_d)} = g(x, t) + h(x, t)u_a, \quad r_d \leq n, \quad (2.2) \]

where \( g(x, t) = \sigma^{(r_d)}|_{u_a=0} \) is an unknown smooth function and \( h(x, t) = \frac{\partial}{\partial u_a} (\sigma^{(r_d)}) \neq 0 \) is a known smooth function. While assuming \( g(x, t) \) is independent of \( x \), \( g(x, t) = \zeta(t) \) and \( \zeta(t) \) is bounded, i.e., \( |\zeta(t)| \leq L, L > 0, h(x, t)u_a = u(t), \) and internal/zero
dynamics of system (2.1) are stable, the input-output dynamics (2.2) are rewritten as

$$\sigma^{(r_d)} = u(t) + \zeta(t).$$

(2.3)

Let \( r \) be the order of Higher Order Sliding Mode (HOSM) controller. Note that for a properly designed HOSM controller, \( r \) is considered to be equal to relative degree \( r_d \) of system (2.1). The HOSM control algorithms that can drive the sliding set \((\sigma, \dot{\sigma}, \ddot{\sigma}, ..., \sigma^{(r-1)}) \rightarrow 0\) in finite-/fixed-time in the presence of bounded disturbance \( \zeta(t) \) (or sometimes the disturbance \( \zeta(t) \) with bounded derivative \(|\dot{\zeta}(t)| \leq L_1\)) are given in the following subsections.

2.1 Nested Higher Order Sliding Mode (N-HOSM) controllers

The family of N-HOSM control algorithms for any arbitrary value of \( r \) is given by [4]

$$u = -\lambda \Psi_{r-1,r} (\sigma, \dot{\sigma}, ..., \sigma^{(r-1)}),$$

(2.4)

where

$$E_{i,r} = (|\sigma|^{\kappa/r} + |\dot{\sigma}|^{\kappa/(r-1)} + ... + |\sigma^{(i-1)}|^{\kappa/(r-i+1)}),$$

$$\Psi_0,r = \text{sign}(\sigma), \quad \Psi_{i,r} = \text{sign}(\sigma^{(i)} + \chi_i E_{i,r} \Psi_{i-1,r}),$$

\( \lambda > 0 \) is a controller gain, \( \kappa > 1 \) and \( \chi_i > 0 \) (\( i = 1, 2, ..., r - 1 \)) represent the control parameters.
For instance, the N-HOSM control algorithms for $r \leq 4$ can be written as [4]

\begin{align*}
& r = 1; \quad u = -\lambda \text{sign } (\sigma), \\
& r = 2; \quad u = -\lambda \text{sign } (\sigma + |\sigma|^{1/2} \text{sign } (\sigma)), \quad (2.5) \\
& r = 3; \quad u = -\lambda \text{sign } (\dot{\sigma} + 2 (|\sigma|^3 + |\sigma|^2)^{1/6} \text{sign } (\dot{\sigma} + |\sigma|^{2/3} \text{sign } (\sigma))), \\
& r = 4; \quad u = -\lambda \text{sign } (\ddot{\sigma} + 3 (\dot{\sigma}^6 + \dot{\sigma}^4 + |\sigma|^3)^{1/12} \text{sign } (\ddot{\sigma} + (\dot{\sigma}^4 + |\sigma|^3)^{1/6} \times \\
& \quad \text{sign } (\dot{\sigma} + 0.5|\sigma|^{3/4} \text{sign } (\sigma))))),
\end{align*}

where $\lambda > 0$ is a controller gain.

### 2.2 Quasi-continuous HOSM (Q-HOSM) controllers

The family of Q-HOSM control algorithms for any arbitrary value of $r$ is given by [5]

\begin{equation}
\hat{u} = -\lambda \Psi_{r-1,r}(\sigma, \dot{\sigma}, ..., \sigma^{(r-1)}), \quad (2.6)
\end{equation}

where

\begin{align*}
E_{0,r} &= \sigma, \quad D_{0,r} = |\sigma|, \quad \Psi_{0,r} = \frac{E_{0,r}}{D_{0,r}} = \text{sign } (\sigma), \quad \Psi_{i,r} = \frac{E_{i,r}}{D_{i,r}}, \\
E_{i,r} &= \sigma^{(i)} + \chi_i D_{i-1,r}^{(r-i)/(r-i+1)} \Psi_{i-1,r}, \quad D_{i,r} = |\sigma^{(i)}| + \chi_i D_{i-1,r}^{(r-i)/(r-i+1)},
\end{align*}

$\lambda > 0$ is the controller gain, and $\chi_i > 0$ ($i = 1, 2, ..., r - 1$) represents the control parameters.
For instance, the Q-HOSM control algorithms for \( r \leq 4 \) can be written as [5]

\[
\begin{align*}
    r &= 1; \quad u = -\lambda \text{sign} (\sigma), \\
    r &= 2; \quad u = -\lambda \left( \dot{\sigma} + |\sigma|^{1/2} \text{sign} (\sigma) \right) / (|\dot{\sigma}| + |\sigma|^{1/2}), \\
    r &= 3; \quad u = -\lambda \left( \ddot{\sigma} + 2 \left( |\dot{\sigma}| + |\sigma|^{2/3} \right)^{-1/2} \left( \dot{\sigma} + |\sigma|^{2/3} \text{sign} (\sigma) \right) \right) / \\
    &\quad \left( |\dot{\sigma}| + 2 \left( |\dot{\sigma}| + |\sigma|^{2/3} \right)^{1/2} \right), \\
    r &= 4; \quad u = -\lambda E_{3,4}/D_{3,4},
\end{align*}
\]

\[
\begin{align*}
    E_{3,4} &= \dddot{\sigma} + 3 \left( \ddot{\sigma} + (|\dot{\sigma}| + 0.5|\sigma|^{3/4})^{-1/3} \left( \dot{\sigma} + 0.5|\sigma|^{3/4} \text{sign} (\sigma) \right) \right) \times \\
    &\quad \left( |\dot{\sigma}| + (|\dot{\sigma}| + 0.5|\sigma|^{3/4})^{2/3} \right)^{1/2}, \\
    D_{3,4} &= |\dddot{\sigma}| + 3 \left( \ddot{\sigma} + (|\dot{\sigma}| + 0.5|\sigma|^{3/4})^{2/3} \right)^{1/2},
\end{align*}
\]

where \( \lambda > 0 \) is a controller gain.

**Remark 1.** Note that the N-HOSM (2.4) and Q-HOSM (2.6) controllers are applicable to system dynamics (2.2) with bounded \( g(x, t) \) and \( h(x, t) \), in addition to system dynamics (2.3) [1].

**Remark 2.** It is to be worth noting that the N-HOSM (2.4) and Q-HOSM (2.6) controllers generate continuous functions of control for any given arbitrary relative degree system (2.1) by artificially increasing the order of control by one.
2.3 Continuous HOSM (C-HOSM) controllers

The family of C-HOSM control algorithms for any arbitrary value of \( r \) is given by [6]

\[
\begin{align*}
  u &= -\gamma_1|\sigma|^{\kappa_1} \text{sign}(\sigma) - \ldots - \gamma_r|\sigma^{(r-1)}|^{\kappa_r} \text{sign}(\sigma^{(r-1)}) - v, \\
  v &= \lambda|s|^{1/2} \text{sign}(s) + v_1, \quad \dot{v}_1 = \chi \text{sign}(s),
\end{align*}
\]

where \( \gamma_1, \gamma_2, \ldots, \gamma_r \) are the scalar coefficients of Hurwitz polynomial \( p^r + \gamma_r p^{r-1} + \ldots + \gamma_2 p + \gamma_1; s = \sigma^{(r-1)} - y, \dot{y} = u + v; \lambda = 1.5L^{1/2}, \chi = 1.1L \). There exists \( \epsilon \in (0, 1) \) such that for every \( \kappa \in (1 - \epsilon, 1) \) the sliding set driven to zero in finite-time and \( \kappa_1, \kappa_2, \ldots, \kappa_r \) satisfy \( \kappa_{i-1} = \frac{\kappa_i \kappa_{i+1}}{2\kappa_{i+1} - \kappa_i} \), \( i = 2, 3, \ldots, r \) with \( \kappa_{r+1} = 1 \) and \( \kappa_r = \kappa \) [6, 40].

For instance, considering the root(s) of Hurwitz polynomial all equal to -2, the C-HOSM control algorithms for \( r \leq 4 \) can be written as

\[
\begin{align*}
  r &= 1, \quad \kappa = 0.5; \quad u = -2|\sigma|^{5/10} \text{sign}(\sigma) - v, \\
  r &= 2, \quad \kappa = 0.6; \quad u = -4|\sigma|^{6/14} \text{sign}(\sigma) - 4|\dot{\sigma}|^{6/10} \text{sign}(\dot{\sigma}) - v, \\
  r &= 3, \quad \kappa = 0.7; \quad u = -8|\sigma|^{7/16} \text{sign}(\sigma) - 12|\dot{\sigma}|^{7/13} \text{sign}(\dot{\sigma}) - 6|\ddot{\sigma}|^{7/10} \text{sign}(\ddot{\sigma}) - v, \\
  r &= 4, \quad \kappa = 0.8; \quad u = -16|\sigma|^{8/16} \text{sign}(\sigma) - 32|\dot{\sigma}|^{8/14} \text{sign}(\dot{\sigma}) - 24|\dddot{\sigma}|^{8/12} \text{sign}(\dddot{\sigma}) \\
  &\quad\quad - 8|\dddot{\sigma}|^{8/10} \text{sign}(\dddot{\sigma}) - v,
\end{align*}
\]

where \( v \) is given in eq. (2.8).
Remark 3. Unlike the N-HOSM (2.4) and Q-HOSM (2.6) controllers, the C-HOSM (2.8) controller generates a continuous control function for any given arbitrary relative degree system (2.1) without artificially increasing the order \( r \) of control.

Remark 4. Note that the N-HOSM (2.4), Q-HOSM (2.6), and C-HOSM (2.8) controllers drive the sliding set \((\sigma, \dot{\sigma}, \ddot{\sigma}, .., \sigma^{(r-1)})\) \(\to 0\) for any given arbitrary relative degree system (2.1) in the presence of bounded perturbations \(\zeta\) in finite-time, and the convergence time depends on initial conditions.

2.4 Fixed-time convergent second order SM (2-SM) controller

The input-output dynamics (2.3) of system (2.1) for relative degree one are written as

\[
\dot{\sigma}(t) = u(t) + \zeta(t),
\]

where \(\zeta\) is bounded smooth perturbation, i.e., \(|\zeta| \leq L\) and \(|\dot{\zeta}| \leq L_1\).

Definition 1. [41] The control system (2.10) is said to be a fixed-time convergent to the origin, if there exists a time moment \(T\) such that the system state \(\sigma(t)\) is equal to zero, \(\sigma(t) = 0\), for all \(t \geq T\), starting from any initial condition \(\sigma_0\).

A fixed-time convergent 2-SM controller that is robust to bounded perturbations (and bounded derivative of perturbations) is proposed in [42] as

\[
u(t) = -\lambda_1 \left( |\sigma(t)|^{1/2} \text{sign}(\sigma(t)) + \chi |\sigma(t)|^{3/2} \text{sign}(\sigma(t)) \right)
- \lambda_2 \int_{t_0}^{t} \left( (1/2) \text{sign}(\sigma(\tau)) + 2\chi \sigma(\tau) + (3/2)\chi^2 |\sigma(\tau)|^2 \text{sign}(\sigma(\tau)) \right) d\tau,
\]
where \( \lambda_1, \lambda_2, \chi > 0 \).

Note that the term \(-\lambda_2(2\chi\sigma + (3/2)\chi^2\sigma^2)\) in eq. (2.11) is necessary for proving the fixed-time convergence, and this term increases the complexity of the controller. Moreover, the upper bound estimate for the convergence time obtained is too conservative in the sense that the upper bound estimate exceeds the real convergence time by 100 times [7]. A fixed-time convergent 2-SM control law, which is simple and less conservative in upper bound estimate of convergence time, is proposed in this dissertation.

2.5 HOSM differentiators

The sliding variable derivatives required for the HOSM controllers (2.4)-(2.9) can be estimated in real-time using the following robust HOSM differentiator [4,38]:

\[
\dot{z}_0 = v_0, \quad v_0 = -\lambda_k K^{1 \over k+1} |z_0 - \sigma(t)|^{2 \over k+1} \text{sign} (z_0 - \sigma(t)) + z_1,
\]

\[
\dot{z}_1 = v_1, \quad v_1 = -\lambda_{k-1} K^{1 \over k} |z_1 - v_0|^{2 \over k+1} \text{sign} (z_1 - v_0) + z_2,
\]

\[
\vdots
\]

\[
\dot{z}_{k-1} = v_{k-1}, \quad v_{k-1} = -\lambda_1 K^{1 \over 2} |z_{k-1} - v_{k-2}|^{1 \over 2} \text{sign} (z_{k-1} - v_{k-2}) + z_k,
\]

\[
\dot{z}_k = -\lambda_0 K \text{sign} (z_k - v_{k-1}),
\]

where \( \lambda_0, \lambda_1, ..., \lambda_k > 0 \) and \(|\sigma^{k+1}(t)| \leq K\).
2.6 Summary

The overall summary of this chapter is as follows:

- The fundamentals of finite-time (the time that is dependent on initial conditions) convergent HOSM controllers, specifically, N-HOSM (2.4), Q-HOSM (2.6), and C-HOSM (2.8) controllers that are robust/insensitive to matched bounded perturbations were discussed.

- A fixed-time (the time that is independent of initial conditions) convergent 2-SM controller (2.11) that is robust/insensitive to matched bounded perturbations with bounded derivative was discussed. It was mentioned that this controller is complex, and is too conservative in estimating the upper bound for convergence time.

The robustness study of the HOSM controllers including fixed-time convergent 2-SM controller to cascade unmodeled dynamics is the subject of this dissertation. This robustness study is accomplished using Describing Function-Harmonic Balance method, whose fundamentals are discussed in the next Chapter 3.
CHAPTER 3

FUNDAMENTALS OF DESCRIBING FUNCTION-HARMONIC BALANCE METHOD

Describing Function-Harmonic Balance (DF-HB) method is a popular engineering frequency-domain technique and an approximation tool to predict limit cycle parameters, amplitude and frequency of main harmonic, in nonlinear system [34,35].

![Block diagram of a system with DF of nonlinearity](image)

**Figure 3.1**: Block diagram of a system with DF of nonlinearity

3.1 Describing Function (DF)

Figure 3.1 represents a zero input negative feedback closed-loop system with the combined loop nonlinearity and the combined loop linearity are separated apart. Assume that a periodic motion of the form \( \sigma(t) = A \sin(\omega t) \) exists in the loop. Then,
DF for nonlinearity is defined as [34,35]:

\[
N(A, \omega) = \frac{1}{\pi A} \int_0^{2\pi} u(A \sin \psi, A\omega \cos \psi) [\sin \psi + j \cos \psi] \, d\psi, \quad \psi = \omega t. \quad (3.1)
\]

In Fig. 3.1, \(N(A, \omega)\) represents DF of nonlinearity, and \(W(j\omega)\) represents frequency characteristics of linearity.

The DF method is based on the hypothesis that the linear element has a low-pass filter properties. In other words, except the fundamental harmonic of \(u\), all other sub-harmonics are assumed to be filtered out by the linear element \(W(j\omega)\).

### 3.2 Existence of limit cycles

The HB equation for the system represented in Fig. 3.1 is written as [34, 35]

\[
1 + N(A, \omega)W(j\omega) = 0. \quad (3.2)
\]

The eq. (3.2) is rewritten as

\[
W(j\omega) = -N^{-1}(A, \omega). \quad (3.3)
\]

Therefore, the parameters, amplitude \(A\) and frequency \(\omega\), of limit cycle in the system shown in Fig. 3.1 must satisfy the eq. (3.3) [34,35]. In other words, the solution of HB eq. (3.3) gives the parameters, amplitude \(A\) and frequency \(\omega\), of predicted limit cycle for system represented in Fig. 3.1.
3.3 Stability of limit cycles

The stability of predicted limit cycles can be studied either graphically using Nyquist criterion of stability extension [43] or analytically using Loeb’s criterion [34].

3.3.1 Graphical analysis

The characteristic equation of linear system represented in Fig. 3.2 is

\[ 1 + NG(s)H(s) = 0, \]  

(3.4)

where \( N \) is a constant gain (could be a complex number) that has been added in the forward path.

**Proposition 1.** [43] Assuming that there are no poles or zeros on the \( j\omega \)-axis of complex plane, the stability of closed-loop system in Fig. 3.2 can be identified using Nyquist criterion extension

\[ Z = O + P, \]  

(3.5)
where \( Z \) is the number of unstable poles of closed-loop system, \( P \) is the number of unstable poles of \( G(s)H(s) \), and \( O \) is the number of clockwise encirclements of the \( G(s)H(s) \) plot around \((-1/N, 0)\). The closed-loop system is stable, if \( Z = 0 \) or \( O = -P \).

This Proposition 1 can be used to graphically interpret/analyze the stability of predicted limit cycle in system represented in Fig. 3.1.

**Example 1.** The stability of limit cycle at \( C_0 \), as shown in Fig. 3.3, for a system is graphically analyzed. If the input amplitude of nonlinearity is perturbed such that the operating point moves to \( C_1 \), then the system becomes unstable and the amplitude of system signals start growing along \(-N^{-1}(A, \omega)\) curve until the point reaches \( C_0 \). On contrary, if the operating point moves to \( C_2 \), then the system becomes stable and the amplitude starts decreasing along \(-N^{-1}(A, \omega)\) curve until the point reaches \( C_0 \). This shows that the limit cycle at \( C_0 \) is stable.
3.3.2 Analytical analysis

Consider $A_0$ and $\omega_0$ are the parameters of predicted limit cycle. Then, the eq. (3.2) can be rewritten as

$$1 + N(A_0, \omega_0)W(j\omega_0) = 0. \quad (3.6)$$

The above eq. (3.6) can be rewritten as

$$U(A_0, \omega_0) + jV(A_0, \omega_0) = 0, \quad (3.7)$$

where $U$ and $V$ are the real and imaginary parts of the left hand side of eq. (3.6), respectively.

**Proposition 2.** [34] For the predicted limit cycle to be stable, it is necessary that:

$$\frac{\partial U}{\partial A} \frac{\partial V}{\partial \omega} - \frac{\partial U}{\partial \omega} \frac{\partial V}{\partial A} > 0, \quad (3.8)$$

according to Loeb’s criterion [34].

3.4 Summary

In this chapter, the fundamentals of DF-HB method that is used to estimate the predicted limit cycle parameters (amplitude and frequency of main harmonic) in nonlinear system were discussed. In the next Chapter 4, the motivation behind the robustness study of HOSM controllers to cascade unmodeled dynamics is discussed.
The following example motivates the necessity of the robustness study of HOSM controlled system to cascade unmodeled dynamics.

\[ \sigma \]

\[ u_1 \]

**Figure 4.1:** a) $\sigma$ and b) $u_1$ in the HOSM controlled dynamically unperturbed system
Figure 4.2: a) $\sigma$ and b) $u_1$ in the HOSM controlled dynamically perturbed system

Example 2. A nonlinear SISO system is considered as

\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= x_3, \\
\dot{x}_3 &= -50x_1 - 180x_2 - 20x_3 + 35 \sin(x_1) + 50u_1, \\
\sigma &= x_1,
\end{align*}
\]

(4.1)

where $u_1$ is an actuator control input, and $\sigma$ is a system output.
Then, input-output dynamics for system (4.1) are obtained as

\[
\sigma^{(3)} = -50x_1 - 180x_2 - 20x_3 + 35\sin(x_1) + 50u_1 \\
= -50\sigma - 180\dot{\sigma} - 20\ddot{\sigma} + 35\sin(\sigma) + 50u_1. \tag{4.2}
\]

The dynamics in eq. (4.2) are rewritten as

\[
\frac{1}{50} \times (\sigma^{(3)} + 20\dot{\sigma} + 180\dot{\sigma} + 50\sigma) = u_1(t) + \zeta, \tag{4.3}
\]

where \( \zeta = 0.7\sin(\sigma) \), and \(|\zeta| \leq 0.7\).

Note that the system (4.1) has relative degree \( r_d = 3 \). By taking \( u_1 \) from Q-HOSM control in eq. (2.7) with \( r = 3 \) and \( \lambda = 1 \), the system (4.1) is simulated for \( \sigma(0) = 0.5 \). The HOSM controller has efficiently driven \( \sigma \) to zero in finite-time in the presence of bounded disturbance \( \zeta \) as shown in Fig. 4.1.

Then, cascade unmodeled actuator dynamics are introduced as

\[
\begin{align*}
\dot{u}_1 &= u_2, \\
\dot{u}_2 &= -\frac{1}{\eta^2}u_1 - \frac{1}{\eta}u_2 + \frac{1}{\eta^2}u,
\end{align*}
\tag{4.4}
\]

where \( \eta \) is a time constant.

By defining the same Q-HOSM control to \( u \), the sliding variable \( \sigma \) in the dynamically perturbed system (4.1), (4.4) with \( \eta = 0.2 \) exhibits chattering as shown in Fig. 4.2a. The corresponding control input \( u_1 \) is shown in Fig. 4.2b.
Remark 5. Note that the order $r$ of Q-HOSM controller is taken equal to the relative degree $r_d$ of the system (4.1), i.e., $r = r_d = 3$. The increase in the relative degree of the system (4.1) due to unmodeled actuator dynamics (4.4) has yielded chattering. In other words, when $r < r_d$, the HOSM control may not drive the sliding set $(\sigma, \dot{\sigma}, \ddot{\sigma}, \ldots, \sigma^{(r-1)})$ to zero in finite-/fixed-time and may cause chattering.

The main intent of this dissertation is to close the gap of HOSM control certification for robustness to cascade unmodeled dynamics. To achieve this goal the following tasks and sub-tasks have been formulated.

- To introduce robustness metrics for HOSM controllers: $PSPM$ and $PSGM$.
- To develop tools/algorithms to identify the proposed robustness metrics.
  - To develop a numerical algorithm for the identification of DFs for the N-HOSM (2.5), Q-HOSM (2.7), C-HOSM (2.9), and fixed-time convergent 2-SM controllers.
  - To develop a numerical algorithm to estimate the predicted limit cycle parameters by solving HB eq. using Newton Raphson (NR) method.
  - To develop algorithms for the identification of $PSPM$ and $PSGM$.
- To develop an algorithm for designing a cascade linear compensator to HOSM control for achieving prescribed values of practical stability margins.
- To apply the proposed technique to certify HOSM attitude controller of F-16 aircraft for robustness to cascade unmodeled dynamics as a case study.
CHAPTER 5

MAIN RESULTS

The ultimate goal of this dissertation is to close the gap of certification for HOSM controllers (2.4)-(2.9) including fixed-time convergent 2-SM controller, which is proposed in this chapter, robustness to cascade unmodeled dynamics.

5.1 Fixed-time convergent 2-SM controller

As seen in Chapter 3, the fixed-time convergent 2-SM controller (2.11) is complex, and is too conservative in estimating the upper bound for convergence time.

A fixed-time convergent 2-SM control, which is simple and less conservative in estimating upper bound of convergence time, is proposed in this dissertation [7] as

\begin{align*}
\begin{aligned}
 r = 2; \quad & u = u_1 + u_2 + u_3, \\
 u_1 &= -\lambda_1|\sigma(t)|^{1/2}\text{sign}(\sigma(t)), \quad u_2 = -\lambda_2|\sigma(t)|^{\kappa}\text{sign}(\sigma(t)), \\
 u_3 &= -\chi \int_{t_0}^{t} \text{sign}(\sigma(\tau)) \, d\tau, \quad u_3(0) = 0,
\end{aligned}
\end{align*}

(5.1)

where \(\lambda_1, \lambda_2, \chi > 0\) are the controller gains, and \(\kappa > 1\).
Remark 6. Note that the fixed-time convergent 2-SM control (5.1) drives the sliding set \((\sigma, \dot{\sigma}) \to 0\) in fixed-time for a relative degree one system (2.10) to the origin in the presence of bounded derivative of perturbations \(\zeta\), and the convergence time does not depend on initial conditions.

Then, the system’s input-output dynamics in eqs. (2.3), (5.1) are reduced to

\[
\begin{align*}
\dot{\sigma}(t) &= -\lambda_1 |\sigma(t)|^{1/2} \operatorname{sign}(\sigma(t)) - \lambda_2 |\sigma(t)|^\kappa \operatorname{sign}(\sigma(t)) + y(t), \quad \sigma(t_0) = \sigma_0, \\
\dot{y}(t) &= -\chi \operatorname{sign}(\sigma(t)) + \zeta_1(t), \quad y(t_0) = 0,
\end{align*}
\] (5.2)

where \(\zeta_1(t) = \dot{\zeta}(t)\) is bounded by the Lipschitz constant \(L_1\).

The following result is obtained:

**Theorem 1.** Consider a dynamic system (5.2) in the presence of a disturbance \(\zeta_1(t)\) bounded by a constant \(L_1\). Then, both states \(\sigma(t)\) and \(y(t)\) converge to the origin uniformly in fixed time

\[
T_f \leq \left( \frac{1}{\lambda_2(\kappa - 1)\epsilon^{p-1}} + \frac{2\epsilon^{1/2}}{\lambda_1} \right) \left( 1 + \frac{1}{m\left( \frac{1}{M} - \frac{k(\lambda_1)}{\lambda_1} \right)} \right),
\] (5.3)

where \(\epsilon > 0\), \(M = \chi + L_1\), \(m = \chi - L_1\), \(k(\lambda_1) = 1/\lambda_1 + (2\epsilon/m\lambda_1)^{1/3}\), and \(e\) is the base of natural logarithms, provided that the following conditions hold for control gains:

\(\chi > L_1, \lambda_1 k^{-1}(\lambda_1) > M\). The minimum value of \(T_f(\epsilon)\) is reached for \(\epsilon = (\lambda_1/\lambda_2)^{1/1/2}\).

**Proof.** is given in Appendix. \(\square\)
5.2 Robustness study of HOSM control to unmodeled dynamics

As discussed earlier, the unmodeled dynamics are inevitable in any mathematical model that is used for designing a controller and are fractal in nature [18]. The presence of unmodeled dynamics may cause chattering in SM/HOSM controlled systems. The chattering in 2-SM controlled systems due to unmodeled physical actuator dynamics has been studied by many authors [19–29]. In this dissertation, the robustness of HOSM control (2.4)-(2.9) including fixed-time convergent 2-SM control (5.1) to unmodeled actuator dynamics is studied, while chattering is analyzed.

Consider cascade unmodeled actuator dynamics for system (2.1)

\[
\dot{z} = f(z, u), \quad u_a = u_a(z),
\]

where \( z \in \mathbb{R}^m \), \( u \in \mathbb{R}^1 \) is a HOSM control input, \( \eta > 0 \) is a small parameter, \( u_a(z) \) is a continuous function of actuator output, and \( f(z, u) \) is a Lipschitz function.

The dynamically perturbed system (2.1), (5.4) is linearized in the vicinity of the origin as

\[
\begin{align*}
\dot{x}(t) & = Px(t) + Qu(t) + \xi, \\
\sigma(t) & = Bx(t),
\end{align*}
\]

where \( x(t) \in \mathbb{R}^{n+m} \), \( u(t) \in \mathbb{R}^1 \), and \( \sigma(t) \in \mathbb{R}^1 \); the matrix \( P \) and vectors \( Q \) and \( B \) have appropriate dimensions; \( \xi \in \mathbb{R}^{n+m} \) represents a vector of perturbations and uncertainties including the differences between original and linearized systems.
It is worth noting that a HOSM controller in eqs. (2.4)-(2.9), (5.1) can completely compensate the perturbations/uncertainties in sliding mode that is achieved in finite-/fixed-time, when the order \( r \) of HOSM controller is taken equal to the relative degree \( r_d \) of system (2.1). This compensation is guaranteed with a price of chattering in a dynamically perturbed system controlled by HOSM control. In this dissertation, the term \( \xi \) that represents perturbations/uncertainties is assumed to be exactly compensated by HOSM controller, while chattering (limit cycle) is analyzed. Then, the frequency characteristics of linearized system (5.5) are written as

\[
W(j\omega) = \frac{\sigma(s)}{u(s)} \bigg|_{s=j\omega} = B(I\omega - P)^{-1}Q \bigg|_{s=j\omega}.
\] (5.6)

DF-HB method is the only engineering method that is heavily used to estimate the predicted limit cycle parameters, amplitude and frequency of main harmonic, in a nonlinear system [34, 35]. Although DF-HB method is an approximation tool, the accuracy of this method in estimating the parameters of chattering in dynamically perturbed systems controlled by SM/2-SM has been confirmed by many works including [20, 26, 36]. In this dissertation, the robustness of HOSM control to cascade unmodeled dynamics is studied using DF-HB method.

Note that the DF-HB method is applied, while assuming the following:

\textbf{A1.} \( W(j\omega) \) is available and has the low-pass filter properties i.e., for \( d = 2, 3, \ldots \),

\(|W(j\omega)| >> |W(jd\omega)|\).
A2. The magnitude and the phase frequency characteristics of $W(j\omega)$ are monotonously decreasing functions, i.e., for $\omega_1 < \omega_2$, $|W(j\omega_1)| > |W(j\omega_2)|$ and $\text{arg} \ W(j\omega_1) > \text{arg} \ W(j\omega_2)$.

5.2.1 Robustness metrics for HOSM control to unmodeled dynamics: practical stability margins

$PM$ and $GM$ are the classical characteristics used in linear control systems to quantify/certify the linear controller robustness to unmodeled dynamics. For the first time, these classical stability margins ($PM$ and $GM$) are extended to practical stability margins ($PSPM$ and $PSGM$) to quantify the robustness of SM/2-SM controlled systems to unmodeled dynamics in [30] by characterizing the parameters of chattering/limit-cycle, i.e., amplitude $A$ and frequency $\omega$ of fundamental harmonic.

Definition 2. [30, 44] The frequency $0 < \omega_c < \infty$ and the amplitude $A_c > 0$ are said to be the Tolerance Limits ($TL$) for the limit cycle of sliding variable, if the amplitude $A \leq A_c$ and the frequency $\omega \geq \omega_c$ of the limit cycle correspond to the acceptable performance of the SM/HOSM controlled dynamically perturbed closed-loop system.

Definition 3. [30, 44] The maximum additional phase lag that can be added to the frequency characteristics of the linear (linearized) plant $W(j\omega)$, while the sliding variable exhibits a limit cycle with an amplitude and a frequency satisfying the $TL$, is said to be the Practical Stability Phase Margin ($PSPM$) in the SM/HOSM controlled closed-loop system.
**Definition 4.** [30, 44] The maximum additional gain that can be added to the frequency characteristics of the linear (linearized) plant $W(j\omega)$, while the sliding variable exhibits a limit cycle with an amplitude and a frequency satisfying the $TL$, is said to be the Practical Stability Gain Margin ($PSGM$) in the SM/HOSM controlled closed-loop system.

**Figure 5.1:** a) Stability margins ($PM$ and $GM$) b) practical stability margins ($PSPM$ and $PSGM$)

Fig. 5.1a depicts the classical stability margins for a closed-loop linear control system. On the other hand, Fig. 5.1b represents practical stability margins for a nonlinear system controlled by classical SM control [19]. The robustness of SM/2-SM control to unmodeled dynamics is quantified in [30]. In this dissertation, the HOSM control is certified/quantified for robustness to cascade unmodeled dynamics in terms of $PSPM$ and $PSGM$. The algorithms to identify the proposed robustness metrics ($PSPM$ and $PSGM$) for HOSM control to unmodeled dynamics are developed later in this chapter.
5.2.2 Development of tools/algorithms for the identification of the practical stability margins

Consider that in Fig. 3.1, $N(A, \omega)$ represents DF of HOSM controller in eqs. (2.5), (2.7), (2.9), (5.1), and $W(j\omega)$ represents the frequency characteristics of linearized system (5.5) at the vicinity of the origin. Note that the DF-HB method has been applied with the assumptions A1 and A2 hold.

5.2.2.1 Identification of DFs for HOSM controllers

In this section, the DFs for HOSM controllers (2.5), (2.7), (2.9), (5.1) are computed, and a database of corresponding DFs is developed.

A. Numerical identification of DFs for HOSM controllers

For the reason that the HOSM control algorithms in eqs. (2.4), (2.6), (2.8) for $r \geq 2$ are complex, obtaining DFs in an analytic way are very difficult. Therefore, corresponding DFs are computed numerically using the following procedure [44, 45]:

1. Isolate the HOSM control algorithm in eqs. (2.5), (2.7), (2.9) from the frequency characteristics of the linearized dynamically perturbed system (5.5), i.e., $W(j\omega)$.

2. Excite the HOSM controller in eqs. (2.5), (2.7), (2.9) by a set of sinusoidal inputs $E_{k,i} = A_k \sin(\omega_i t)$ of the known amplitude $A_k$ and the known frequency $\omega_i$, where $k, i = 1, 2, 3, \ldots$

3. Compute the output $u_{k,i}$ of the HOSM controller.
Figure 5.2: $-N^{-1}(A_k, \omega_i)$ for N-HOSM control with a) $r = 2$, b) $r = 3$, and c) $r = 4$
Figure 5.3: $-N^{-1}(A_k, \omega_i)$ for Q-HOSM control with a) $r = 2$, b) $r = 3$, and c) $r = 4$
Figure 5.4: $-N^{-1}(A_k, \omega_i)$ for C-HOSM control with a) $r = 2$, b) $r = 3$, and c) $r = 4$
Using the Fourier series technique, compute the DF of the HOSM controller in eqs. (2.5), (2.7), (2.9) as follows [34, 35],

\[
N_{k,i}(A_k, \omega_i) = \frac{2}{TA_k} \int_{0}^{T} f_{k,i}(A_k \sin(\omega_i t), t) \left( \sin(\omega_i t) + j \cos(\omega_i t) \right) dt, \tag{5.7}
\]

where \(T = \frac{2\pi}{\omega_i}\) and \(f_{k,i}(A_k \sin(\omega_i t), t) = u_{k,i}\).

**Table 5.1:** \(-N^{-1}(A_k, \omega_i)\) look-up table representation

<table>
<thead>
<tr>
<th>(\omega)</th>
<th>(A)</th>
<th>(A_1)</th>
<th>(\ldots)</th>
<th>(A_k)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\omega_1)</td>
<td>Re((-N^{-1}(A_1, \omega_1))) + j Im((-N^{-1}(A_1, \omega_1)))</td>
<td>(\ldots)</td>
<td>Re((-N^{-1}(A_k, \omega_1))) + j Im((-N^{-1}(A_k, \omega_1)))</td>
<td></td>
</tr>
<tr>
<td>(\omega_2)</td>
<td>Re((-N^{-1}(A_1, \omega_2))) + j Im((-N^{-1}(A_1, \omega_2)))</td>
<td>(\ldots)</td>
<td>Re((-N^{-1}(A_k, \omega_2))) + j Im((-N^{-1}(A_k, \omega_2)))</td>
<td></td>
</tr>
<tr>
<td>(\vdots)</td>
<td>(\vdots)</td>
<td>(\vdots)</td>
<td>(\vdots)</td>
<td></td>
</tr>
<tr>
<td>(\omega_i)</td>
<td>Re((-N^{-1}(A_1, \omega_i))) + j Im((-N^{-1}(A_1, \omega_i)))</td>
<td>(\ldots)</td>
<td>Re((-N^{-1}(A_k, \omega_i))) + j Im((-N^{-1}(A_k, \omega_i)))</td>
<td></td>
</tr>
</tbody>
</table>

Negative reciprocals of DFs, \(-N^{-1}_{i,k}(A_k, \omega_i)\), for N-HOSM, Q-HOSM, and C-HOSM are represented in the form of a look-up Table 5.1, where each element \(\text{Re}(\!-\!N^{-1}(A_k, \omega_i)) + j \text{Im}(\!-\!N^{-1}(A_k, \omega_i))\) corresponds to the sinusoidal input \(E_{k,i} = A_k \sin(\omega_i t)\).

**Example 3.** DFs for N-HOSM (2.5), Q-HOSM (2.7), and C-HOSM (2.9) controllers with \(r = 2, 3, 4\) and \(\lambda = \chi = 1\) are computed and tabulated using the proposed procedure in Section 5.2.2.1. The controllers are excited by a set of sinusoidal inputs \(E_{k,i} = A_k \sin(\omega_i t)\) with \(A_k \in [0.0001, 0.01]\) and \(\omega_i \in [30, 300]\). The corresponding negative inverse of DFs are shown in Figs. 5.2 – 5.4.
### Table 5.2: $-N^{-1}(A_k, \omega_i)$ look-up table for N-HOSM ($r = 2$)

<table>
<thead>
<tr>
<th>$\omega$</th>
<th>$A$</th>
<th>0.0001</th>
<th>$\cdots$</th>
<th>0.00998</th>
<th>0.01</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>$-7.821 \times 10^{-3} + j 0.046 \times 10^{-3}$</td>
<td>$-2.555 \times 10^{-3} + j 7.406 \times 10^{-4}$</td>
<td>$-0.002 560 \times 10^{-3} + j 7.421 \times 10^{-4}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>30.541</td>
<td>$-7.819 \times 10^{-3} + j 7.339 \times 10^{-3}$</td>
<td>$-2.510 \times 10^{-3} + j 7.421 \times 10^{-4}$</td>
<td>$-0.002 515 \times 10^{-3} + j 7.436 \times 10^{-4}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>298.917</td>
<td>$-2.667 \times 10^{-3} + j 7.333 \times 10^{-3}$</td>
<td>$-0.445 \times 10^{-3} + j 7.772 \times 10^{-4}$</td>
<td>$-0.000 446 \times 10^{-3} + j 7.787 \times 10^{-4}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>299.458</td>
<td>$-2.736 \times 10^{-3} + j 7.343 \times 10^{-3}$</td>
<td>$-0.380 \times 10^{-3} + j 7.811 \times 10^{-4}$</td>
<td>$-0.000 381 \times 10^{-3} + j 7.826 \times 10^{-4}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>300</td>
<td>$-2.586 \times 10^{-3} + j 7.382 \times 10^{-3}$</td>
<td>$-0.298 \times 10^{-3} + j 7.799 \times 10^{-4}$</td>
<td>$-0.000 298 \times 10^{-3} + j 7.815 \times 10^{-4}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Table 5.3: $-N^{-1}(A_k, \omega_i)$ look-up table for N-HOSM ($r = 3$)

<table>
<thead>
<tr>
<th>$\omega$</th>
<th>$A$</th>
<th>0.0001</th>
<th>$\cdots$</th>
<th>0.00998</th>
<th>0.01</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>$-4.826 \times 10^{-3} + j 0.192 \times 10^{-3}$</td>
<td>$7.777 \times 10^{-4} + j 0.974 \times 10^{-3}$</td>
<td>$7.792 \times 10^{-4} + j 0.976 \times 10^{-3}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>30.541</td>
<td>$-3.867 \times 10^{-3} + j 7.989 \times 10^{-3}$</td>
<td>$7.779 \times 10^{-4} + j 0.951 \times 10^{-3}$</td>
<td>$7.795 \times 10^{-4} + j 0.953 \times 10^{-3}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>298.917</td>
<td>$7.8247 \times 10^{-3} + j 6.243 \times 10^{-3}$</td>
<td>$7.825 \times 10^{-3} + j 0.393 \times 10^{-3}$</td>
<td>$7.841 \times 10^{-3} + j 0.394 \times 10^{-3}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>299.458</td>
<td>$7.8280 \times 10^{-3} + j 6.155 \times 10^{-3}$</td>
<td>$7.831 \times 10^{-3} + j 0.390 \times 10^{-3}$</td>
<td>$7.846 \times 10^{-3} + j 0.390 \times 10^{-3}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>300</td>
<td>$7.8233 \times 10^{-3} + j 6.539 \times 10^{-3}$</td>
<td>$7.831 \times 10^{-3} + j 0.390 \times 10^{-3}$</td>
<td>$7.846 \times 10^{-3} + j 0.390 \times 10^{-3}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Table 5.4: $-N^{-1}(A_k, \omega_i)$ look-up table for N-HOSM ($r = 4$)

<table>
<thead>
<tr>
<th>$\omega$</th>
<th>$A$</th>
<th>0.0001</th>
<th>$\cdots$</th>
<th>0.00998</th>
<th>0.01</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>$2.597 \times 10^{-3} - j 7.432 \times 10^{-3}$</td>
<td>$0.302 \times 10^{-3} - j 7.852 \times 10^{-3}$</td>
<td>$0.302 \times 10^{-3} - j 7.852 \times 10^{-3}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>30.541</td>
<td>$2.507 \times 10^{-3} - j 7.462 \times 10^{-3}$</td>
<td>$0.297 \times 10^{-3} - j 7.852 \times 10^{-3}$</td>
<td>$0.297 \times 10^{-3} - j 7.852 \times 10^{-3}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>298.917</td>
<td>$4.698 \times 10^{-3} - j 8.020 \times 10^{-3}$</td>
<td>$0.468 \times 10^{-3} - j 8.005 \times 10^{-3}$</td>
<td>$0.469 \times 10^{-3} - j 8.020 \times 10^{-3}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>299.458</td>
<td>$4.011 \times 10^{-3} - j 8.062 \times 10^{-3}$</td>
<td>$0.278 \times 10^{-3} - j 8.051 \times 10^{-3}$</td>
<td>$0.278 \times 10^{-3} - j 8.067 \times 10^{-3}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>300</td>
<td>$3.137 \times 10^{-3} - j 8.055 \times 10^{-3}$</td>
<td>$0.313 \times 10^{-3} - j 8.059 \times 10^{-3}$</td>
<td>$0.278 \times 10^{-3} - j 8.067 \times 10^{-3}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Table 5.5: $-N^{-1}(A_k, \omega_i)$ look-up table for Q-HOSM ($r = 2$)

<table>
<thead>
<tr>
<th>$\omega$</th>
<th>$A$</th>
<th>0.0001</th>
<th>$\cdots$</th>
<th>0.00998</th>
<th>0.01</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>$-8.289 \times 10^{-3} + j 2.837 \times 10^{-3}$</td>
<td>$-4.433 \times 10^{-3} + j 7.774 \times 10^{-3}$</td>
<td>$-4.460 \times 10^{-3} + j 7.794 \times 10^{-3}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>30.541</td>
<td>$-8.285 \times 10^{-3} + j 2.876 \times 10^{-3}$</td>
<td>$-4.408 \times 10^{-3} + j 7.790 \times 10^{-3}$</td>
<td>$-4.415 \times 10^{-3} + j 7.806 \times 10^{-3}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>298.917</td>
<td>$-4.527 \times 10^{-3} + j 7.672 \times 10^{-3}$</td>
<td>$-1.038 \times 10^{-3} + j 7.956 \times 10^{-3}$</td>
<td>$-1.040 \times 10^{-3} + j 7.972 \times 10^{-3}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>299.458</td>
<td>$-4.560 \times 10^{-3} + j 7.696 \times 10^{-3}$</td>
<td>$-1.035 \times 10^{-3} + j 7.994 \times 10^{-3}$</td>
<td>$-1.036 \times 10^{-3} + j 8.010 \times 10^{-3}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>300</td>
<td>$-4.498 \times 10^{-3} + j 7.706 \times 10^{-3}$</td>
<td>$-1.015 \times 10^{-3} + j 7.978 \times 10^{-3}$</td>
<td>$-1.016 \times 10^{-3} + j 7.994 \times 10^{-3}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 5.6: $-N^{-1}(A_k, \omega_i)$ look-up table for Q-HOSM ($r = 3$)

<table>
<thead>
<tr>
<th>$\omega$</th>
<th>$A$</th>
<th>0.0001</th>
<th>...</th>
<th>0.00998</th>
<th>0.01</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>$9.441 \times 10^{-8} + j0.155 \times 10^{-3}$</td>
<td>...</td>
<td>$8.515 \times 10^{-3} + j2.468 \times 10^{-3}$</td>
<td>$8.332 \times 10^{-3} + j2.471 \times 10^{-3}$</td>
<td></td>
</tr>
<tr>
<td>30.541</td>
<td>$1.375 \times 10^{-8} + j0.155 \times 10^{-3}$</td>
<td>...</td>
<td>$8.504 \times 10^{-3} + j2.419 \times 10^{-3}$</td>
<td>$8.517 \times 10^{-3} + j2.422 \times 10^{-3}$</td>
<td></td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>298.917</td>
<td>$8.069 \times 10^{-8} + j1.345 \times 10^{-3}$</td>
<td>...</td>
<td>$7.852 \times 10^{-3} + j0.513 \times 10^{-3}$</td>
<td>$7.868 \times 10^{-3} + j0.514 \times 10^{-3}$</td>
<td></td>
</tr>
<tr>
<td>299.458</td>
<td>$8.082 \times 10^{-8} + j1.302 \times 10^{-3}$</td>
<td>...</td>
<td>$7.860 \times 10^{-3} + j0.435 \times 10^{-3}$</td>
<td>$7.876 \times 10^{-3} + j0.435 \times 10^{-3}$</td>
<td></td>
</tr>
<tr>
<td>300</td>
<td>$8.071 \times 10^{-8} + j1.295 \times 10^{-3}$</td>
<td>...</td>
<td>$7.858 \times 10^{-3} + j0.447 \times 10^{-3}$</td>
<td>$7.873 \times 10^{-3} + j0.448 \times 10^{-3}$</td>
<td></td>
</tr>
</tbody>
</table>

Table 5.7: $-N^{-1}(A_k, \omega_i)$ look-up table for Q-HOSM ($r = 4$)

<table>
<thead>
<tr>
<th>$\omega$</th>
<th>$A$</th>
<th>0.0001</th>
<th>...</th>
<th>0.00998</th>
<th>0.01</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>$5.268 \times 10^{-8} - j9.213 \times 10^{-3}$</td>
<td>...</td>
<td>$0.942 \times 10^{-3} - j8.083 \times 10^{-3}$</td>
<td>$0.943 \times 10^{-3} - j8.099 \times 10^{-3}$</td>
<td></td>
</tr>
<tr>
<td>30.541</td>
<td>$5.112 \times 10^{-8} - j9.181 \times 10^{-3}$</td>
<td>...</td>
<td>$0.918 \times 10^{-3} - j8.076 \times 10^{-3}$</td>
<td>$0.920 \times 10^{-3} - j8.092 \times 10^{-3}$</td>
<td></td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>298.917</td>
<td>$5.813 \times 10^{-8} - j8.045 \times 10^{-3}$</td>
<td>...</td>
<td>$0.480 \times 10^{-3} - j8.007 \times 10^{-3}$</td>
<td>$0.481 \times 10^{-3} - j8.023 \times 10^{-3}$</td>
<td></td>
</tr>
<tr>
<td>299.458</td>
<td>$4.961 \times 10^{-8} - j8.090 \times 10^{-3}$</td>
<td>...</td>
<td>$0.310 \times 10^{-3} - j8.053 \times 10^{-3}$</td>
<td>$0.311 \times 10^{-3} - j8.069 \times 10^{-3}$</td>
<td></td>
</tr>
<tr>
<td>300</td>
<td>$4.906 \times 10^{-8} - j8.079 \times 10^{-3}$</td>
<td>...</td>
<td>$0.335 \times 10^{-3} - j8.042 \times 10^{-3}$</td>
<td>$0.335 \times 10^{-3} - j8.058 \times 10^{-3}$</td>
<td></td>
</tr>
</tbody>
</table>

Table 5.8: $-N^{-1}(A_k, \omega_i)$ look-up table for C-HOSM ($r = 2$)

<table>
<thead>
<tr>
<th>$\omega$</th>
<th>$A$</th>
<th>0.0001</th>
<th>...</th>
<th>0.00998</th>
<th>0.01</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>$-0.468 \times 10^{-4} + j0.446 \times 10^{-3}$</td>
<td>...</td>
<td>$-1.043 \times 10^{-3} + j3.412 \times 10^{-3}$</td>
<td>$-1.043 \times 10^{-3} + j3.415 \times 10^{-3}$</td>
<td></td>
</tr>
<tr>
<td>30.541</td>
<td>$-0.456 \times 10^{-4} + j0.439 \times 10^{-3}$</td>
<td>...</td>
<td>$-1.019 \times 10^{-3} + j3.383 \times 10^{-3}$</td>
<td>$-1.019 \times 10^{-3} + j3.386 \times 10^{-3}$</td>
<td></td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>298.917</td>
<td>$-2.194 \times 10^{-8} + j0.136 \times 10^{-3}$</td>
<td>...</td>
<td>$-7.681 \times 10^{-3} + j0.973 \times 10^{-3}$</td>
<td>$-7.685 \times 10^{-3} + j0.974 \times 10^{-3}$</td>
<td></td>
</tr>
<tr>
<td>299.458</td>
<td>$-2.289 \times 10^{-8} + j0.137 \times 10^{-3}$</td>
<td>...</td>
<td>$-7.719 \times 10^{-3} + j0.977 \times 10^{-3}$</td>
<td>$-7.723 \times 10^{-3} + j0.978 \times 10^{-3}$</td>
<td></td>
</tr>
<tr>
<td>300</td>
<td>$-2.154 \times 10^{-8} + j0.135 \times 10^{-3}$</td>
<td>...</td>
<td>$-7.549 \times 10^{-3} + j0.964 \times 10^{-3}$</td>
<td>$-7.553 \times 10^{-3} + j0.965 \times 10^{-3}$</td>
<td></td>
</tr>
</tbody>
</table>

Table 5.9: $-N^{-1}(A_k, \omega_i)$ look-up table for C-HOSM ($r = 3$)

<table>
<thead>
<tr>
<th>$\omega$</th>
<th>$A$</th>
<th>0.0001</th>
<th>...</th>
<th>0.00998</th>
<th>0.01</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>$5.604 \times 10^{-8} + j3.640 \times 10^{-3}$</td>
<td>...</td>
<td>$0.291 \times 10^{-3} + j8.281 \times 10^{-3}$</td>
<td>$0.291 \times 10^{-3} + j8.284 \times 10^{-3}$</td>
<td></td>
</tr>
<tr>
<td>30.541</td>
<td>$5.505 \times 10^{-8} + j3.495 \times 10^{-3}$</td>
<td>...</td>
<td>$0.284 \times 10^{-3} + j7.989 \times 10^{-3}$</td>
<td>$0.284 \times 10^{-3} + j7.992 \times 10^{-3}$</td>
<td></td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>298.917</td>
<td>$3.035 \times 10^{-8} + j5.836 \times 10^{-3}$</td>
<td>...</td>
<td>$1.294 \times 10^{-3} + j2.110 \times 10^{-3}$</td>
<td>$1.295 \times 10^{-3} + j2.111 \times 10^{-3}$</td>
<td></td>
</tr>
<tr>
<td>299.458</td>
<td>$3.028 \times 10^{-8} + j5.797 \times 10^{-3}$</td>
<td>...</td>
<td>$1.291 \times 10^{-3} + j2.097 \times 10^{-3}$</td>
<td>$1.292 \times 10^{-3} + j2.098 \times 10^{-3}$</td>
<td></td>
</tr>
<tr>
<td>300</td>
<td>$2.946 \times 10^{-8} + j5.656 \times 10^{-3}$</td>
<td>...</td>
<td>$1.255 \times 10^{-3} + j2.051 \times 10^{-3}$</td>
<td>$1.256 \times 10^{-3} + j2.052 \times 10^{-3}$</td>
<td></td>
</tr>
</tbody>
</table>
in the Section 5.2.2.1 is also applicable for the identification of DF for fixed-time
since the controller (5.1) is less complex. Note that the proposed numerical algorithm
B. Analytic DF for fixed-time convergent 2-SM controller

The DF for fixed-time convergent 2-SM controller (5.1) is obtained analytically,
since the controller (5.1) is less complex. Note that the proposed numerical algorithm
in the Section 5.2.2.1 is also applicable for the identification of DF for fixed-time
convergent 2-SM controller (5.1).

Table 5.10: $-N^{-1}(A_k, \omega_i)$ look-up table for C-HOSM ($r = 4$)

<table>
<thead>
<tr>
<th>(\omega)</th>
<th>(A)</th>
<th>0.0001</th>
<th>...</th>
<th>0.00998</th>
<th>0.01</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>$1.580 \times 10^{-6} - j\cdot 4.855 \times 10^{-6}$</td>
<td>...</td>
<td>$2.489 \times 10^{-6} - j\cdot 1.327 \times 10^{-5}$</td>
<td>$2.490 \times 10^{-6} - j\cdot 1.327 \times 10^{-5}$</td>
<td></td>
</tr>
<tr>
<td>30.541</td>
<td>$1.490 \times 10^{-6} - j\cdot 4.662 \times 10^{-6}$</td>
<td>...</td>
<td>$2.351 \times 10^{-6} - j\cdot 1.271 \times 10^{-5}$</td>
<td>$2.352 \times 10^{-6} - j\cdot 1.272 \times 10^{-5}$</td>
<td></td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>298.917</td>
<td>$3.270 \times 10^{-6} - j\cdot 2.258 \times 10^{-6}$</td>
<td>...</td>
<td>$7.671 \times 10^{-6} - j\cdot 5.726 \times 10^{-6}$</td>
<td>$7.674 \times 10^{-6} - j\cdot 5.731 \times 10^{-6}$</td>
<td></td>
</tr>
<tr>
<td>299.458</td>
<td>$3.292 \times 10^{-6} - j\cdot 2.660 \times 10^{-6}$</td>
<td>...</td>
<td>$7.723 \times 10^{-6} - j\cdot 5.734 \times 10^{-6}$</td>
<td>$7.726 \times 10^{-6} - j\cdot 5.736 \times 10^{-6}$</td>
<td></td>
</tr>
<tr>
<td>300</td>
<td>$3.148 \times 10^{-6} - j\cdot 2.150 \times 10^{-6}$</td>
<td>...</td>
<td>$7.401 \times 10^{-6} - j\cdot 5.454 \times 10^{-6}$</td>
<td>$7.404 \times 10^{-6} - j\cdot 5.457 \times 10^{-6}$</td>
<td></td>
</tr>
</tbody>
</table>

Table 5.11: $-N^{-1}(A_k, \omega_i)$ look-up table for fixed-time convergent 2-SM

<table>
<thead>
<tr>
<th>(\omega)</th>
<th>(A)</th>
<th>0.0001</th>
<th>...</th>
<th>0.00998</th>
<th>0.01</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>$-0.573 \times 10^{-3} - j\cdot 2.207 \times 10^{-4}$</td>
<td>...</td>
<td>$-7.792 \times 10^{-4} - j\cdot 2.953 \times 10^{-4}$</td>
<td>$-7.802 \times 10^{-4} - j\cdot 2.954 \times 10^{-4}$</td>
<td></td>
</tr>
<tr>
<td>30.541</td>
<td>$-0.595 \times 10^{-3} - j\cdot 2.240 \times 10^{-4}$</td>
<td>...</td>
<td>$-7.824 \times 10^{-4} - j\cdot 2.911 \times 10^{-4}$</td>
<td>$-7.833 \times 10^{-4} - j\cdot 2.912 \times 10^{-4}$</td>
<td></td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>298.917</td>
<td>$-7.861 \times 10^{-4} - j\cdot 3.101 \times 10^{-4}$</td>
<td>...</td>
<td>$-8.900 \times 10^{-4} - j\cdot 0.349 \times 10^{-2}$</td>
<td>$-8.908 \times 10^{-4} - j\cdot 0.349 \times 10^{-2}$</td>
<td></td>
</tr>
<tr>
<td>299.458</td>
<td>$-7.814 \times 10^{-4} - j\cdot 3.466 \times 10^{-4}$</td>
<td>...</td>
<td>$-8.892 \times 10^{-4} - j\cdot 0.343 \times 10^{-2}$</td>
<td>$-8.901 \times 10^{-4} - j\cdot 0.343 \times 10^{-2}$</td>
<td></td>
</tr>
<tr>
<td>300</td>
<td>$-7.862 \times 10^{-4} - j\cdot 3.009 \times 10^{-4}$</td>
<td>...</td>
<td>$-8.893 \times 10^{-4} - j\cdot 0.335 \times 10^{-2}$</td>
<td>$-8.902 \times 10^{-4} - j\cdot 0.335 \times 10^{-2}$</td>
<td></td>
</tr>
</tbody>
</table>

**Figure 5.5:** Block diagram for the fixed-time convergent 2-SM controlled system

B. Analytic DF for fixed-time convergent 2-SM controller
Considering $N_1$, $N_2$, and $N_3$ are the respective DFs for $u_1$, $u_2$, and $u_3$ nonlinearities in eq. (5.1), the block diagram for closed-loop control system is represented in Fig. 5.5.

Since $u_2$ in eq. (5.1) is odd and static (memoryless), the DF in eq. (3.1) is rewritten as [29]

$$
N_2(A) = \frac{2}{\pi A} \int_0^\pi \lambda_2(A \sin \psi)^\kappa \sin \psi \, d\psi = \frac{2\lambda_2 A^{\kappa-1} \Gamma(\frac{\kappa}{2} + 1)}{\sqrt{\pi} \Gamma(\frac{\kappa}{2} + 1.5)}, \quad \kappa > 1,
$$

where $\Gamma$ is a gamma-function [46]. From eq. (5.8), DFs for $u_1$ (with $\kappa = 1/2$) [29] and $u_2$ (with $\kappa = 3/2$) in eq. (5.1) are obtained as

$$
N_1(A) = \frac{2\lambda_1}{\sqrt{A} \sqrt{\pi} \Gamma(\frac{1}{4} + 1)} \approx \frac{1.1128 \lambda_1}{\sqrt{A}},
$$

$$
N_2(A) = \frac{2\lambda_2 \sqrt{A}}{\sqrt{\pi} \Gamma(\frac{3}{4} + 1)} \approx 0.9153 \lambda_2 \sqrt{A},
$$

respectively. DF for $u_3$ in eq. (5.1) is written as [29]

$$
N_3(A, \omega) = \frac{4\chi}{\pi A j\omega} \approx -j \frac{1.2732 \chi}{A \omega}.
$$

Therefore, the DF for fixed-time convergent 2-SM controller in eq. (5.1) is written as

$$
N(A, \omega) = N_1(A) + N_2(A) + N_3(A, \omega)
= \frac{1.1128 \lambda_1}{\sqrt{A}} + 0.9153 \lambda_2 \sqrt{A} - j \frac{1.2732 \chi}{A \omega}.
$$
Remark 7. Note that the minus signs for $u_1$, $u_2$, and $u_3$ in eq. (5.1) are not considered in the computation of DF for the fixed-time convergent controller (5.1) due to the negative feedback.

The negative reciprocal of DF in the eq. (5.12) is obtained as

$$-N^{-1}(A, \omega) = \frac{1}{0.9153^2 \Lambda^2 A} \left( \frac{1.1128 \Lambda}{\sqrt{A}} + 0.9153 \lambda \sqrt{A} + j \frac{1.2732}{A \omega} \right).$$

(5.13)

As per the eq. (5.13), the negative reciprocal of DF for fixed-time convergent controller (5.1) converges to the origin as $A \to \infty$ for a constant value of $\omega$. Moreover,

$$\lim_{A \to 0} \arg \{-N^{-1}(A)\} = -\pi/2.$$

Figure 5.6 depicts the negative inverse of DF for the fixed-time convergent 2-SM controller (5.1) with $\omega_1 < \omega_2 < \omega_3$. 

Figure 5.6: $-N^{-1}(A, \omega)$ for fixed-time convergent 2-SM control
Remark 8. Note that the negative reciprocals of DFs in Figs. 5.2–5.6 are obtained for a specific value(s) of control gain(s), i.e., $\lambda = \lambda_1 = \lambda_2 = \chi = 1$. For different values of control gains, corresponding DFs are to be identified. Moreover, smaller stepsizes for amplitudes $A_k$ and frequencies $\omega_i$ in $E_{k,i} = A_k \sin(\omega_i t)$ yield better results in identifying the parameters of predicted chattering.

A database of DFs for the HOSM controllers, specifically for N-HOSM (2.5), Q-HOSM (2.7), C-HOSM (2.9), and fixed-time convergent 2-SM (5.1) controllers is developed in the form of Tables 5.2–5.11, respectively.

5.2.2.2 Identification of parameters for predicted limit cycles

In this section, the parameters of the predicted chattering/limit-cycle in the dynamically perturbed system (2.1), (5.4) controlled by HOSM controller in eqs. (2.4)–(2.9), (5.1) are computed by solving HB eq. (3.3).

A. Numerical identification of parameters for predicted limit cycles in HOSM controlled system

The parameters of the predicted chattering in the dynamically perturbed system (2.1), (5.4) controlled by HOSM controller in eqs. (2.4)–(2.9) are computed by solving HB eq. (3.3) numerically as follow:

From the eq. (3.2),

\begin{align*}
F_1(\omega_i, A_k) &= \text{Re} \left\{ W(j\omega_i) \right\} + \text{Re} \left\{ N^{-1}(A_k, \omega_i) \right\} = 0, \quad (5.14) \\
F_2(\omega_i, A_k) &= \text{Im} \left\{ W(j\omega_i) \right\} + \text{Im} \left\{ N^{-1}(A_k, \omega_i) \right\} = 0. \quad (5.15)
\end{align*}
The unknown limit cycle parameters, amplitude $A_k$ and frequency $\omega_i$, are predicted by solving eqs. (5.14) and (5.15) using the Newton-Raphson (NR) method [47] as follows:

$$
\omega_{i+1} = \omega_i - \frac{1}{J_{i,k}} \left[ F_1 \frac{\partial F_2}{\partial A_k} - F_2 \frac{\partial F_1}{\partial A_k} \right], \quad (5.16)
$$

$$
A_{k+1} = A_k + \frac{1}{J_{i,k}} \left[ F_1 \frac{\partial F_2}{\partial \omega_i} - F_2 \frac{\partial F_1}{\partial \omega_i} \right], \quad (5.17)
$$

where $J_{i,k} = \frac{\partial F_1}{\partial \omega_i} \frac{\partial F_2}{\partial A_k} - \frac{\partial F_1}{\partial A_k} \frac{\partial F_2}{\partial \omega_i} \neq 0$. Due to the unavailability of analytic expressions for $F_1$ and $F_2$, the partial derivatives are computed numerically as follows:

$$
\frac{\partial F_{1,2}}{\partial \omega} = \frac{F_{1,2}(\omega_{i-1}, A_k) - F_{1,2}(\omega_i, A_k)}{\omega_{i-1} - \omega_i}, \quad (5.18)
$$

$$
\frac{\partial F_{1,2}}{\partial A} = \frac{F_{1,2}(\omega_i, A_{k-1}) - F_{1,2}(\omega_i, A_k)}{A_{i-1} - A_i}.
$$

Given an initial solution guess, $A_k = A_0$ and $\omega_i = \omega_0$, the eqs. (5.16) and (5.17) are solved iteratively. The iteration stops when the conditions,

$$
\left| \frac{\omega_{i+1} - \omega_i}{\omega_{i+1}} \right| < \epsilon_\omega \quad \text{and} \quad \left| \frac{A_{k+1} - A_k}{A_{k+1}} \right| < \epsilon_A, \quad (5.19)
$$

are satisfied, where $\epsilon_\omega > 0$ and $\epsilon_A > 0$ are the tolerance errors for frequency and amplitude, respectively, and the solution of eq. (3.3) is approximately identified. If the error conditions in eq. (5.19) are not satisfied in a reasonable amount of iterations, then the iteration is to start with a new initial solution guess, $A_0$ and $\omega_0$. 

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B. Analytic identification of parameters for predicted limit cycle in fixed-time convergent 2-SM controlled system

In this section, the parameters of chattering in dynamically perturbed system (2.1), (5.4) controlled by fixed-time convergent 2-SM control (5.1) are predicted analytically. Note that the proposed numerical algorithm in the Section 5.2.2.2 is also applicable for estimating the parameters of chattering in dynamically perturbed system (2.1), (5.4) controlled by the fixed-time convergent 2-SM controller (5.1).

The following is obtained by substituting eq. (5.12) in HB eq. (3.3):

\[
\frac{1.1128\lambda_1}{\sqrt{A}} + 0.9153\lambda_2\sqrt{A} - j\frac{1.2732\chi}{A\omega} = -\text{Re } W^{-1}(j\omega) - \text{Im } W^{-1}(j\omega). \quad (5.20)
\]

From the above eq. (5.20),

\[
\frac{1.1128\lambda_1}{\sqrt{A}} + 0.9153\lambda_2\sqrt{A} = -\text{Re } W^{-1}(j\omega), \quad (5.21)
\]

\[
\frac{1.2732\chi}{A\omega} = \text{Im } W^{-1}(j\omega). \quad (5.22)
\]

Equation (5.22) is rewritten as

\[
A = \frac{1.2732\chi}{\omega \text{ Im } W^{-1}(j\omega)}. \quad (5.23)
\]

Substituting eq. (5.23) in eq. (5.21) produces

\[
0.9862\lambda_1\sqrt{\frac{\omega \text{ Im } W^{-1}(j\omega)}{\alpha}} + 1.0328\lambda_2\sqrt{\frac{\alpha}{\omega \text{ Im } W^{-1}(j\omega)}} = -\text{Re } W^{-1}(j\omega). \quad (5.24)
\]
Then, the frequency $\omega$ of limit cycle is obtained by solving eq. (5.24). Next, the amplitude $A$ of limit cycle is obtained by substituting $\omega$ in eq. (5.23).

### 5.2.2.3 Stability of predicted limit cycles

The Loeb’s criterion in Proposition 2 (see Chapter 3) is applied to analyze the stability of predicted limit cycle in dynamically perturbed system (2.1), (5.4) controlled by HOSM control in eqs. (2.4)–(2.9), (5.1).

**Proposition 3.** [44] If the predicted limit cycle in dynamically perturbed linearized system (5.5), whose frequency characteristics satisfy the assumptions A1 and A2, controlled by HOSM controller (2.4)–(2.9), (5.1) is locally stable, then the following inequality holds:

$$
(\rho_1 \rho_4 + \rho_2 \rho_3)\big|_{(A_0, \omega_0)} > |W(j\omega_0)|^2 \Xi\big|_{(A_0, \omega_0)} ,
$$

where $A_0$ and $\omega_0$ are the parameters of limit cycle,

$$
\rho_1 = \text{Re}\{W(j\omega)\} \text{Re}\{N(A, \omega)\} + \text{Im}\{W(j\omega)\} \text{Im}\{N(A, \omega)\},
$$

$$
\rho_2 = \text{Re}\{W(j\omega)\} \text{Im}\{N(A, \omega)\} - \text{Im}\{W(j\omega)\} \text{Re}\{N(A, \omega)\},
$$

$$
\rho_3 = \frac{\partial(\text{Re}\{N(A, \omega)\})}{\partial A} \frac{\partial(\text{Re}\{W(j\omega)\})}{\partial \omega} + \frac{\partial(\text{Im}\{N(A, \omega)\})}{\partial A} \frac{\partial(\text{Im}\{W(j\omega)\})}{\partial \omega},
$$

$$
\rho_4 = \frac{\partial(\text{Re}\{N(A, \omega)\})}{\partial A} \frac{\partial(\text{Im}\{W(j\omega)\})}{\partial \omega} - \frac{\partial(\text{Im}\{N(A, \omega)\})}{\partial A} \frac{\partial(\text{Re}\{W(j\omega)\})}{\partial \omega},
$$

$$
\Xi = \frac{\partial(\text{Im}\{N(A, \omega)\})}{\partial A} \frac{\partial(\text{Re}\{N(A, \omega)\})}{\partial \omega} - \frac{\partial(\text{Re}\{N(A, \omega)\})}{\partial A} \frac{\partial(\text{Im}\{N(A, \omega)\})}{\partial \omega}.
$$

$N(A, \omega)$ is the DF of HOSM controller (2.4)–(2.9), (5.1), and $W(j\omega)$ represents the frequency characteristics of linearized system (5.5).
5.2.2.4 Identification of practical stability margins

In this section, a step-by-step procedure for the identification of the introduced robustness metrics (PSPM and PSGM) for the system (2.1), (5.4) controlled by HOSM controller in eqs. (2.4)-(2.9), (5.1) is presented.

A. PSPM computation method

The phase shift $\theta \geq 0^\circ$ is introduced into the linearized system (5.5) due to unmodeled dynamics. Then, the HB eq. (3.3) becomes

$$e^{-j\theta}W(j\omega_i) = -N^{-1}(A_k, \omega_i).$$ (5.26)

From the eq. (5.26),

$$|W(j\omega_i)| = |N^{-1}(A_k, \omega_i)|,$$ (5.27)

$$\arg\{W(j\omega_i)\} - \theta = \arg\{-N^{-1}(A_k, \omega_i)\},$$ (5.28)

where $\theta$ is identified based on the given system TL, amplitude $A_c$ and frequency $\omega_c$.

The PSPM computational algorithm consists of the following steps:

Step 1. Assume that $A_k = A_c$, then a solution $\omega_i = \omega_{A_c}$ of eq. (5.27) is to be obtained. If $\omega_{A_c} \geq \omega_c$, then the PSPM is to be identified from eq. (5.28) as

$$PSPM = \arg\{W(j\omega_{A_c})\} - \arg\{-N^{-1}(A_c, \omega_{A_c})\}.$$ (5.29)
If $\omega_{A_c} < \omega_c$, then proceed to Step 2.

Step 2. Assume that $\omega_i = \omega_c$, then a solution $A_k = A_{\omega_c}$ of eq. (5.27) is to be obtained. Here, $A_{\omega_c} \leq A_c$ and the $PSPM$ is to be identified from eq. (5.28) as

$$PSPM = \arg\{W(j\omega_c)\} - \arg\{-N^{-1}(A_{\omega_c}, \omega_c)\}. \quad (5.30)$$

**B. PSGM computation method**

The gain $K \neq 1$ is introduced into the linearized system (5.5) due to unmodeled dynamics. Then, the HB eq. (3.3) becomes

$$KW(j\omega_i) = -N^{-1}(A_k, \omega_i). \quad (5.31)$$

From the eq. (5.31),

$$K|W(j\omega_i)| = |N^{-1}(A_k, \omega_i)|, \quad (5.32)$$

$$\arg\{W(j\omega_i)\} = \arg\{-N^{-1}(A_k, \omega_i)\}, \quad (5.33)$$

where $K$ is identified based on the given system $TL$, amplitude $A_c$ and frequency $\omega_c$. The $PSPM$ computational algorithm consists of the following steps:

Step 1. Assume that $A_k = A_c$, then a solution $\omega_i = \omega_{A_c}$ of eq. (5.27) is to be obtained. If $\omega_{A_c} \geq \omega_c$, then the $PSPM$ is to be identified from eq. (5.32) as

$$PSPM = \frac{|N^{-1}(A_c, \omega_{A_c})|}{|W(j\omega_{A_c})|}. \quad (5.34)$$
If $\omega_{Ak} < \omega_c$, then proceed to Step 2.

Step 2. Assume that $\omega_i = \omega_c$, then a solution $A_k = A_{\omega_c}$ of eq. (5.27) is to be obtained. Here, $A_{\omega_c} \leq A_c$ and the $PSGM$ is to be identified from eq. (5.32) as

$$PSGM = \frac{|N^{-1}(A_{\omega_c}, \omega_c)|}{|W(j\omega_c)|}.\quad (5.35)$$

### 5.2.3 Design of cascade linear compensator to HOSM control for achieving required practical stability margins

The practical stability margins ($PSPM$ and $PSGM$) for system (2.1) controlled by HOSM controller (2.4)–(2.9), (5.1) may not satisfy the prescribed values due to the presence of unmodeled dynamics (5.4). For instance, the prescribed values can be (see page 33 of [17]): $PSPM = 30^\circ$ and $PSGM = 12\,\text{dB}$. Then, the prescribed values on practical stability margins can be achieved in two ways.

1. By cascading HOSM control (2.4)–(2.9), (5.1) with linear compensator, whose design procedure is discussed in this Chapter, when the transfer function of linearized system (5.5) is available.

2. By artificially increasing the order of HOSM controller (2.4)–(2.9).

Following the option 1, the cascade linear compensator to HOSM controller takes the form

$$W_c(s) = \frac{s + \frac{1}{\mu}}{s + \frac{1}{\mu}},\quad (5.36)$$

where $\mu$ is the attenuation parameter. If $0 < \mu < 1$, then the eq. (5.36) is a phase-lag compensator. If $\mu > 1$, then the eq. (5.36) is a phase-lead compensator.
The design procedure for a phase-lead dynamic compensator [48], which is based on the classical Bode plots methodology [49], is as follows:

Step 1. Obtain the \( PSPM \) of dynamically perturbed system (2.1), (5.4) controlled by HOSM controller in eqs. (2.4)–(2.9), (5.1): \( PSPM_{un} \).

Step 2. Determine the maximum phase-lead angle of the compensator as

\[
\phi_m = PSPM_c - PSPM_{un} + \langle 5, 12 \rangle^\circ,
\]

where \( PSPM_c \) is the required \( PSPM \) and \( \langle 5, 12 \rangle \) is an interval.

Step 3. Obtain the parameter \( \mu \), which satisfies

\[
\sin \phi_m = \frac{\mu - 1}{\mu + 1}.
\]

Step 4. From the Bode magnitude plot of the dynamically perturbed system (2.1), (5.4), identify the frequency \( \omega_m \) corresponding to the magnitude

\[
- [10 \log \mu + 20 \log |N(A_c, \omega_c)|].
\]

Step 5. Find the pole and zero of the compensator \( W_c(s) \) using

\[
\text{Pole: } \frac{1}{\mu} = \omega_m \sqrt{\mu}; \text{ Zero: } \frac{1}{\mu \mu} = \frac{\text{Pole}}{\mu}.
\]

Step 6. Compute the \( PSPM_{new} \), after cascading the dynamic compensator with the HOSM controller in eqs. (2.4)–(2.9), (5.1) for verification.
5.3 Summary

In this Chapter, a fixed-time convergent 2-SM controller (5.1) that is robust to bounded derivative of perturbations was proposed. The tools for the certification of HOSM control robustness to cascade unmodeled dynamics were developed as follows:

- The HOSM control robustness metrics to cascade unmodeled dynamics were introduced: \( PSPM \) and \( PSGM \).

- The tools/algorithms for the identification of introduced robustness metrics based on DF-HB method were developed.
  
  - A numerical algorithm to compute DFs for HOSM controllers (specifically, N-HOSM, Q-HOSM, and C-HOSM controllers) was proposed, and the analytic DF for fixed-time convergent 2-SM controller was obtained. A database of DFs for these controllers was developed.
  
  - A numerical algorithm that solves HB equation using NR method to obtain predicted chattering parameters was proposed.
  
  - The computational algorithms to identify the HOSM control robustness metrics (\( PSPM \) and \( PSGM \)) were proposed.

- When the obtained values of \( PSPM \) and \( PSGM \) do not satisfy the prescribed values, a cascade linear compensator to HOSM controller was suggested to achieve the prescribed values.

In the next Chapter, several simulation examples are presented validating the proposed tools for the certification of HOSM robustness to unmodeled dynamics.
CHAPTER 6

SIMULATION EXAMPLES

In this chapter, the robustness study of N-HOSM (2.4), Q-HOSM (2.6), C-HOSM (2.8), and fixed-time convergent 2-SM (5.1) controllers to cascade unmodeled dynamics is accomplished by validating the proposed tools/algorithms presented in Chapter 5 on a series of examples. The robustness of HOSM control to cascade unmodeled dynamics is quantified/certified using the robustness metrics $PSPM$ and $PSGM$. Dynamic compensators that cascade with the HOSM controllers are designed to achieve prescribed values (as in page 33 of [17]) on $PSPM$ and $PSGM$.

The simulation scenario is that the HOSM control is designed for a system, which does not include physical actuator dynamics. Then, the robustness of HOSM control is studied by cascading unmodeled actuator dynamics and is quantified.

6.1 Robustness study of N-HOSM control to unmodeled dynamics

In this section, the robustness of system (4.1) controlled by N-HOSM control (2.4) to cascade unmodeled dynamics (4.4) is studied. Similar to Example 2, the sliding variable $\sigma$ in dynamically unperturbed system (4.1) is driven to zero from $\sigma(0) = 0.5$ by applying N-HOSM control ($r = 3$) in eq. (2.5) with $\lambda = 1$ as $u_1$. By
defining the same N-HOSM control to $u$, the sliding variable $\sigma$ in the dynamically perturbed system (4.1), (4.4) with $\eta = 0.2$ exhibits chattering/limit-cycle.

![Graph](image)

**Figure 6.1**: a) $\sigma$ and b) $u_1$ in the N-HOSM controlled dynamically unperturbed system
The evolution of $\sigma$ and $u_1$ over time for dynamically unperturbed system (4.1) and dynamically perturbed system (4.1), (4.4) are shown in Figs. 6.1 and 6.2, respectively. The robustness metrics for N-HOSM control to cascade unmodeled dynamics are obtained, while analyzing the chattering, as follows.
6.1.1 Computing parameters of limit cycle

DF is numerically obtained for N-HOSM (r=3) in eq. (2.5) with $\lambda = 1$ based on the proposed procedure in the Section 5.2.2.1. Note that the DF method is applied by assuming $\zeta(t)$ in eq. (4.3) has exactly been compensated by the designed N-HOSM control and the assumptions A1 and A2 are hold.

Then, the transfer function for the system dynamics (4.3) are written as

$$W(s) = \frac{\sigma(s)}{u(s)} = \frac{50}{s^3 + 20s^2 + 180s + 50}.$$  \hfill (6.1)

Next, the transfer function for dynamically perturbed system (4.1), (4.4) is

$$W(s) = \left(\frac{1}{\eta^2s^2 + \eta s + 1}\right) \cdot \left(\frac{50}{s^3 + 20s^2 + 180s + 50}\right),$$  \hfill (6.2)

where $\eta = 0.2$.

**Table 6.1**: Parameters of $\sigma$-limit cycle in N-HOSM controlled system

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Simulations</th>
<th>Numerical solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>$5.81 \times 10^{-3}$</td>
<td>$4.98 \times 10^{-3}$</td>
</tr>
<tr>
<td>$\omega$ (rad/sec)</td>
<td>11.220</td>
<td>11.473</td>
</tr>
</tbody>
</table>

The chattering parameters, amplitude $A$ and frequency $\omega$ of main harmonic, are predicted by solving HB eq. (3.3) via NR method, as shown in the Section 5.2.2.2, for the dynamically perturbed system (4.1), (4.4) controlled by N-HOSM control. Table 6.1 shows the comparison of $\sigma$-limit cycle parameters that are obtained numerically using NR method with the simulations and the accuracy of the
proposed method. Note that the predicted limit cycle using DF-HB method is stable in accordance with the Loeb’s criterion in Proposition 3.

6.1.2 Practical stability margins

The Tolerance Limits (TL), reasonable amplitude and frequency of chattering, for the system (4.1) controlled by N-HOSM control are defined as $A_c = 0.01$ and $\omega_c = 8$ rad/sec. Then, the proposed robustness metrics ($PSPM$ and $PSGM$) are identified for the N-HOSM controlled dynamically perturbed system (4.1), (4.4) by following the steps presented in the Section 5.2.2.4.

To compute $PSPM$: Step 1. The solution of eq. (5.27) for $A_c = 0.01$ is obtained as: $\omega_{A_c} = 9.537$ rad/sec $> \omega_c$. Therefore,

$$PSPM = 1.078 - 0.6607 = 0.4168 \text{rad} = 23.88^\circ.$$

To compute $PSGM$: Step 1. The solution of eq. (5.33) for $A_c = 0.01$ is obtained as: $\omega_{A_c} = 13.476 > \omega_c$. Therefore,

$$PSGM = \frac{7.852 \times 10^{-3}}{2.050 \times 10^{-3}} = 11.67 \text{dB}.$$ 

The prescribed values of practical stability margins for the system (4.1) are defined as: $PSPM \geq 35^\circ$ and $PSGM \geq 6 \text{dB}$. Note that the computed $PSPM$ has not satisfied the required value. Then, the prescribed value on $PSPM$ is achieved by cascading a phase-lead compensator with the designed N-HOSM controller.
6.1.3 Cascade compensator for N-HOSM controlled system

The cascaded phase-lead compensator is designed by following the steps presented in the Section 5.2.3 as follows.

Step 1. $PSPM_{un} = 23.88^\circ$; Step 2. $\phi_m = 35^\circ - 23.88^\circ + 5^\circ = 16.12^\circ$; Step 3. $\mu = 1.77$; Step 4. $|W(j\omega_m)| = -(2.476 + 42.10) = -44.576$dB $\Rightarrow \omega_m = 10.3$rad/sec; Step 5. Pole = 13.70 and Zero = 7.745. Therefore, the phase-lead compensator to N-HOSM control is obtained as

$$W_c(s) = \frac{13.70}{7.745} \times \frac{s + 7.745}{s + 13.70};$$

Step 6. Then, $PSPM_{new} = 35.01^\circ$ is obtained. Note that the $PSPM_{new}$ satisfies the required specification on $PSPM$, i.e., $PSPM \geq 35^\circ$.

6.2 Robustness study of Q-HOSM control to unmodeled dynamics

In this section, the robustness of system (4.1) controlled by Q-HOSM control (2.6) to cascade unmodeled dynamics (4.4) is studied. As seen in Example 2, the sliding variable $\sigma$ in the dynamically perturbed system (4.1), (4.4) with $\eta = 0.2$ controlled by Q-HOSM control exhibits chattering/limit-cycle. The evolution of $\sigma$ and $u_1$ over time for dynamically unperturbed system (4.1) and dynamically perturbed system (4.1), (4.4) are shown in Figs. 4.1 and 4.2, respectively. The robustness metrics for Q-HOSM control to cascade unmodeled dynamics are obtained, while analyzing the chattering, as follows.
6.2.1 Computing parameters of limit cycle

DF is numerically obtained for Q-HOSM (r=3) in eq. (2.7) with \( \lambda = 1 \) based on the proposed procedure in the Section 5.2.2.1. Note that the DF method is applied by assuming \( \zeta(t) \) in eq. (4.3) has exactly been compensated by the designed Q-HOSM control and the assumptions A1 and A2 are hold.

Then, the transfer function of dynamically perturbed system (4.1), (4.4) is obtained as eq. (6.2), where \( \eta = 0.2 \).

Table 6.2: Parameters of \( \sigma \)-limit cycle in Q-HOSM controlled system

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Simulations</th>
<th>Numerical solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A )</td>
<td>6.81 \times 10^{-3}</td>
<td>6.78 \times 10^{-3}</td>
</tr>
<tr>
<td>( \omega ) (rad/sec)</td>
<td>8.825</td>
<td>8.859</td>
</tr>
</tbody>
</table>

The chattering parameters, amplitude \( A \) and frequency \( \omega \) of main harmonic, are predicted by solving HB eq. (3.3) via NR method, as shown in the Section 5.2.2.2, for the dynamically perturbed system (4.1), (4.4) controlled by Q-HOSM control. Table 6.2 shows the comparison of \( \sigma \)-limit cycle parameters that are obtained numerically using NR method with the simulations and the accuracy of the proposed method. Note that the predicted limit cycle using DF-HB method is stable in accordance with the Loeb’s criterion in Proposition 3.

Table 6.3: Parameters of \( \sigma \)-limit cycle in C-HOSM controlled system

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Simulations</th>
<th>Numerical solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A )</td>
<td>8.91 \times 10^{-3}</td>
<td>9.76 \times 10^{-3}</td>
</tr>
<tr>
<td>( \omega ) (rad/sec)</td>
<td>8.726</td>
<td>8.546</td>
</tr>
</tbody>
</table>
6.2.2 Practical stability margins

The Tolerance Limits $TL$, reasonable amplitude and frequency of chattering, for the system (4.1) controlled by Q-HOSM control are defined as $A_c = 0.01$ and $\omega_c = 5\text{rad/sec}$. Then, the proposed robustness metrics ($PSPM$ and $PSGM$) are identified for the Q-HOSM controlled dynamically perturbed system (4.1), (4.4) by following the steps presented in the Section 5.2.2.4.

To compute $PSPM$: Step 1. The solution of eq. (5.27) for $A_c = 0.01$ is obtained as: $\omega_{A_c} = 7.94\text{rad/sec} > \omega_c$. Therefore,

$$PSPM = 1.464 \times 1.199 = 0.2645 \text{rad} = 15.16^\circ.$$  

To compute $PSGM$: Step 1. The solution of eq. (5.33) for $A_c = 0.01$ is obtained as: $\omega_{A_c} = 10.739 > \omega_c$. Therefore,

$$PSGM = \frac{1.285 \times 10^{-2}}{5.039 \times 10^{-3}} = 8.134\text{dB}.$$  

The prescribed values of practical stability margins for the system (4.1) are defined as: $PSPM \geq 35^\circ$ and $PSGM \geq 6\text{dB}$. Note that the computed $PSPM$ has not satisfied the required value. The prescribed value on $PSPM$ is achieved by cascading a phase-lead compensator with the designed Q-HOSM controller.
6.2.3 Cascade compensator for Q-HOSM controlled system

The cascaded phase-lead compensator is designed by following the steps presented in the Section 5.2.3 as follows.

Step 1. \( P_{SPM_{wn}} = 15.16^\circ \); Step 2. \( \phi_m = 35^\circ - 15.16^\circ + 8^\circ = 27.84^\circ \); Step 3. \( \mu = 2.75 \); Step 4. \(|W(j\omega_m)| = -(4.397 + 36.51) = -40.907\text{dB} \implies \omega_m = 9.15\text{rad/sec} \); Step 5. Pole = 15.23 and Zero = 5.533. Therefore, the phase-lead compensator to Q-HOSM control is obtained as

\[
W_c(s) = \frac{15.23}{5.533} \times \frac{s + 5.533}{s + 15.23}.
\]

Step 6. Then, \( P_{SPM_{new}} = 35.04^\circ \) is obtained. Note that the \( P_{SPM_{new}} \) satisfies the required specification on \( P_{SPM} \), i.e., \( P_{SPM} \geq 35^\circ \).

6.3 Robustness study of C-HOSM control to unmodeled dynamics

In this section, the robustness of system (4.1) controlled by C-HOSM control (2.8) to cascade unmodeled dynamics (4.4) is studied.

Similar to Example 2, the sliding variable \( \sigma \) in the dynamically unperturbed system (4.1) is driven to zero from \( \sigma(0) = 0.5 \) by applying C-HOSM control \( (r = 3) \) in eq. (2.9) with \( \lambda = \chi = 1 \) as \( u_1 \). By defining the same C-HOSM control to \( u \), the dynamically perturbed system (4.1), (4.4) with \( \eta = 0.5 \) exhibits chattering.
Figure 6.3: a) $\sigma$ and b) $u_1$ in the C-HOSM controlled dynamically unperturbed system
Figure 6.4: a) $\sigma$ and b) $u_1$ in the C-HOSM controlled dynamically perturbed system

The evolution of $\sigma$ and $u_1$ over time for dynamically unperturbed (4.1) and dynamically perturbed (4.1), (4.4) systems controlled by C-HOSM control are shown in Figs. 6.3 and 6.4, respectively. The robustness metrics for C-HOSM control to cascade unmodeled dynamics are obtained, while analyzing the chattering, as follows.
6.3.1 Computing parameters of limit cycle

DF is numerically obtained for C-HOSM (r=3) in eq. (2.9) with $\lambda = \chi = 1$ based on the proposed procedure in the Section 5.2.2.1. Note that the DF method is applied by assuming $\zeta(t)$ in eq. (4.3) has exactly been compensated by the designed N-HOSM control and the assumptions A1 and A2 are hold.

Then, the transfer function for dynamically perturbed system (4.1), (4.4) is obtained as eq. (6.2), where $\eta = 0.5$.

The chattering parameters, amplitude $A$ and frequency $\omega$ of main harmonic, are predicted by solving HB eq. (3.3) via NR method, as shown in the Section 5.2.2.2, for the dynamically perturbed system (4.1), (4.4) controlled by N-HOSM control. Table 6.3 shows the comparison of $\sigma$-limit cycle parameters that are obtained numerically using NR method with the simulations and the accuracy of the proposed method. Note that the predicted limit cycle using DF-HB method is stable in accordance with the Loeb’s criterion in Proposition 3.

6.3.2 Practical stability margins

The Tolerance Limits ($TL$), reasonable amplitude and frequency of chattering, for the system (4.1) controlled by C-HOSM control are defined as $A_c = 0.02$ and $\omega_c = 5$rad/sec. Then, the proposed robustness metrics ($PSPM$ and $PSGM$) are identified for the C-HOSM controlled dynamically perturbed system (4.1), (4.4) by following the steps presented in the Section 5.2.2.4.
To compute $PSPM$: Step 1. The solution of eq. (5.27) for $A_c = 0.02$ is obtained as: $\omega_{A_c} = 7.371\text{rad/sec} > \omega_c$. Therefore,

$$PSPM = 1.015 - 0.7946 = 0.22\text{rad} = 12.61^\circ.$$ 

To compute $PSGM$: Step 1. The solution of eq. (5.33) for $A_c = 0.02$ is obtained as: $\omega_{A_c} = 10.418\text{rad/sec} > \omega_c$. Therefore,

$$PSGM = \frac{1.691 \times 10^{-3}}{8.353 \times 10^{-4}} = 2.024 = 6.124\text{dB}.$$ 

The prescribed values of practical stability margins for the system (4.1) are defined as: $PSPM \geq 35^\circ$ and $PSGM \geq 6\text{dB}$. Note that the computed $PSPM$ has not satisfied the required value. The prescribed value on $PSPM$ is achieved by cascading a phase-lead compensator with the designed C-HOSM controller.

6.3.3 Cascade compensator for C-HOSM controlled system

The cascaded phase-lead compensator is designed by following the steps presented in the Section 5.2.3 as follows.

Step 1. $PSPM_{un} = 12.61^\circ$; Step 2. $\phi_m = 35^\circ - 12.61^\circ + 8^\circ = 30.39^\circ$; Step 3. $\mu = 3.05$; Step 4. $|W(j\omega_m)| = -(4.839 + 46.56) = -51.399\text{dB} \implies \omega_m = 7.42\text{rad/sec}$. Step 5. Pole = 12.95 and Zero = 4.25. Then, the phase-lead compensator
to C-HOSM control is obtained as

\[ W_c(s) = \frac{12.95}{4.25} \times \frac{s + 4.25}{s + 12.95}, \]

Step 6. Then, \( P_{SPM_{new}} = 35.82^\circ \) is obtained. Note that the \( P_{SPM_{new}} \) satisfies the required specification on \( P_{SPM} \), i.e., \( P_{SPM} \geq 35^\circ \).

**Figure 6.5:** a) \( \sigma \) and b) \( u_1 \) in the fixed-time convergent 2-SM controlled dynamically unperturbed system.
6.4 Robustness study of fixed-time convergent 2-SM control to unmodeled dynamics

In this section, the robustness of fixed-time convergent 2-SM control (5.1) to cascade unmodeled dynamics is studied.

![Graph](image)

**Figure 6.6:** a) $\sigma$ and b) $u_1$ in the fixed-time convergent 2-SM controlled dynamically perturbed system
Example 4. A scalar system is considered as

\[
\dot{x} = -10x + 0.7\cos(x) + u_1, 
\]

\[
\sigma = x, 
\]

where \( x \) is a system state, \( u_1 \) is an actuator control input, and \( \sigma \) is a system output.

The cascade unmodeled dynamics are considered as

\[
\dot{u}_1 = -\frac{1}{\eta^2}u_1 + \frac{1}{\eta^2}u, \quad \eta = 0.2. 
\]

Then, the input-output dynamics for the system (6.3) are derived as

\[
\dot{\sigma} + 10\sigma = u_1 + \zeta, 
\]

where \( \zeta = 0.7\cos(\sigma) \), and \(|\zeta| \leq 0.7, |\dot{\zeta}| \leq 0.7|\).

Next, the system (6.3) is simulated with \( \sigma(0) = 0.5 \) by applying the fixed-time convergent 2-SM control in eq. (5.1) as \( u_1 \) with \( \lambda_1 = \lambda_2 = \chi = 1 \) and \( \kappa = 1.5 \).

As shown in Fig. 6.5, the fixed-time convergent control has driven \( \sigma \) to zero in fixed-time in the presence of bounded disturbance \( \zeta \) with bounded derivative. By applying the same control as \( u \) for the dynamically perturbed system (6.3), (6.4), \( \sigma \) converges to a limit cycle as shown in Fig. 6.6. The robustness metrics for fixed-time convergent 2-SM control to cascade unmodeled dynamics are obtained, while analyzing the chattering, as follows.
Table 6.4: Parameters of $\sigma$-limit cycle in fixed-time convergent 2-SM controlled system

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Simulations</th>
<th>Analytic solution</th>
<th>Numerical solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>$6.84 \times 10^{-4}$</td>
<td>$6.97 \times 10^{-4}$</td>
<td>$7 \times 10^{-4}$</td>
</tr>
<tr>
<td>$\omega$ (rad/sec)</td>
<td>35.69</td>
<td>36.11</td>
<td>36.029</td>
</tr>
</tbody>
</table>

6.4.1 Computing parameter of limit cycles

DF is analytically obtained for fixed-time convergent 2-SM controller (5.1) with $\lambda_1 = \lambda_2 = \chi = 1$ and $\kappa = 1.5$ based on the proposed procedure in the Section 5.2.2.1. Note that the DF method is applied by assuming $\zeta$ in eq. (6.5) has exactly been compensated by the designed fixed-time convergent control and the assumptions A1 and A2 are hold.

The transfer function of dynamically perturbed system (6.3), (6.4) is

$$W(s) = \frac{1}{0.04s + 1} \frac{1}{s + 10}.$$ (6.6)

The negative inverse of frequency characteristics for the system (6.6) are

$$W^{-1}(j\omega) = (0.04j\omega + 1)(j\omega + 10) = (10 - 0.04\omega^2) + j(1.4\omega).$$

Then, eq. (5.20) is obtained as

$$\frac{1.1128\lambda_1}{\sqrt{A}} + 0.9153\lambda_2 \sqrt{A} - \frac{1.2732\alpha}{A\omega} = -(10 - 0.04\omega^2) - j(1.4\omega).$$ (6.7)
From the eq. (6.7),

\[
\frac{1.1128\lambda_1}{\sqrt{A}} + 0.9153\lambda_2\sqrt{A} = 0.04\omega^2 - 10, \quad (6.8)
\]

\[
\frac{1.2732\alpha}{A\omega} = 1.4\omega. \quad (6.9)
\]

From the eq. (6.9),

\[
A = \frac{0.9094}{\omega^2} \quad \text{or} \quad \sqrt{A} = \frac{0.9536}{\omega}. \quad (6.10)
\]

The following eq. is obtained by substituting the eq. (6.10) in eq. (6.8):

\[
0.04\omega^3 - 1.1668\omega^2 - 10\omega - 0.8728 = 0. \quad (6.11)
\]

The frequency of predicted limit cycle is obtained as \(\omega = 36.11\text{rad/sec}\) by solving the eq. (6.11). Then, the amplitude of predicted limit cycle is obtained as \(A = 6.97 \times 10^{-4}\) by substituting \(\omega = 36.11\text{rad/sec}\) in eq. (6.10).

On the other hand, DF is numerically obtained for fixed-time convergent 2-SM controller \((5.1)\) with \(\lambda_1 = \lambda_2 = \chi = 1\) and \(\kappa = 1.5\) based on the proposed procedure in the Section 5.2.2.1. The chattering parameters, amplitude \(A\) and frequency \(\omega\) of main harmonic, are predicted by solving HB eq. \((3.3)\) via NR method, as shown in the Section 5.2.2.2, for the dynamically perturbed system \((6.3), (6.4)\) controlled by fixed-time convergent control. Table 6.4 shows the comparison of \(\sigma\)-limit cycle parameters that are obtained analytically, and numerically using NR method with the simulations and the accuracy of the proposed method. Note that the predicted
limit cycle using DF-HB method is stable in accordance with the Loeb’s criterion in Proposition 3.

### 6.4.2 Practical stability margins

The Tolerance Limits (\(TL\)), reasonable amplitude and frequency of chattering, for the system (6.3) controlled by fixed-time convergent control (5.1) are defined as \(A_c = 0.001\) and \(\omega_c = 10\text{rad/sec}\). Then, the proposed robustness metrics (\(PSPM\) and \(PSGM\)) are identified for the fixed-time convergent controlled dynamically perturbed system (6.3), (6.4) by following the steps presented in the Section 5.2.2.4.

To compute \(PSPM\): Step 1. The solution of the eq. (5.27) is obtained as: 
\[
\omega_{A_c} = 31.616\text{rad/sec} > \omega_c.
\]
Therefore,
\[
PSPM = -2.166 - (-2.289) = 0.1224 = 7.012^\circ.
\]

To compute \(PSGM\): Step 1. The solution of the eq. (5.33) is obtained as:
\[
\omega_{A_c} = 84.949\text{rad/sec} > \omega_c.
\]
Therefore,
\[
PSGM = \frac{0.0262}{0.0033} = 17.99\text{dB}.
\]

The prescribed values of practical stability margins for the system (6.3) are defined as: \(PSPM \geq 35^\circ\) and \(PSGM \geq 6\text{dB}\). Note that the computed \(PSPM\) has not satisfied the required value. The prescribed value on \(PSPM\) is achieved by cascading a phase-lead compensator with the designed fixed-time convergent controller.
6.4.3 Cascade compensator for fixed-time convergent 2-SM controlled system

The cascaded phase-lead compensator is designed by following the steps presented in the Section 5.2.3 as follows.

Step 1. $PSPM_{un} = 7.012^\circ$; Step 2. $\phi_m = 35^\circ - 7.012^\circ + 10^\circ = 37.988^\circ$; Step 3. $\mu = 4.2015$; Step 4. $|W(j\omega_m)| = -(6.23 + 42.42) = -48.65\text{dB} \implies \omega_m = 80\text{rad/sec}$; Step 5. Pole = 164 and Zero = 39.03. Therefore, the phase-lead compensator to fixed-time convergent control is obtained as

\[
W_c(s) = \frac{164}{39.03} \times \frac{s + 39.03}{s + 164}.
\]

Step 6. Then, $PSPM_{new} = 35.47^\circ$ is obtained. Note that the $PSPM_{new}$ satisfies the required specification on $PSPM$, i.e., $PSPM \geq 35^\circ$.

6.5 Summary

In this chapter, several simulation examples were presented to demonstrate the certification of designed HOSM controller for robustness to cascade unmodeled dynamics using the tools/algorithms developed in Chapter 5. In the next Chapter 7, a HOSM attitude F-16 aircraft system is certified for robustness to cascade unmodeled dynamics as a case study.
Consider the following nonlinear model of an F-16 aircraft [14] at Mach = 0.7, height $h = 10,000$ ft, $\alpha_{\text{trim}} = \theta_{\text{trim}} = 0.106803$ rad, $\delta_{\text{e}} = -0.0295$ rad, $\varphi_{\text{trim}} = \beta_{\text{trim}} = p_{\text{trim}} = q_{\text{trim}} = r_{\text{trim}} = \delta_{\alpha_{\text{trim}}} = \delta_{r_{\text{trim}}} = 0$:

\[
\dot{\theta} = q \cos(\varphi) - r \sin(\varphi),
\]

\[
\dot{\varphi} = p + q \sin(\varphi) \tan(\theta) + r \cos(\varphi) \tan(\theta),
\]

\[
\dot{\alpha} = -\beta p + 0.0427 \cos(\theta) \cos(\varphi) + 0.083589 + \tilde{Z}_a \alpha + \tilde{Z}_q q + \tilde{Z}_\delta \delta_e,
\]

\[
\dot{\beta} = -0.9973 r + \alpha p + 0.0427 \cos(\theta) \sin(\varphi) + \tilde{Y}_\beta \beta + \tilde{Y}_p p + \tilde{Y}_\delta \delta_r + \tilde{Y}_\delta \delta_a,
\]

\[
\dot{p} = -0.1345 pq - 0.8225 qr + \tilde{L}_\beta \beta + \tilde{L}_p p + \tilde{L}_r r - 50.933 \delta_a + \tilde{L}_\delta \delta_r + \zeta_p(t),
\]

\[
\dot{q} = 0.9586 pr - 0.0833(r^2 - p^2) - 1.94166 + \tilde{M}_a \alpha + \tilde{M}_q q + \tilde{M}_\delta \delta_e + \zeta_q(t),
\]

\[
\dot{r} = -0.7256 pq + 0.1345 qr + \tilde{N}_\beta \beta + \tilde{N}_p p + \tilde{N}_r r + 4.125 \delta_a + \tilde{N}_\delta \delta_r + \zeta_r(t),
\]

where $\theta, \varphi, \alpha, \beta$ are pitch, roll, attack, and sideslip angles, respectively; $p, q,$ and $r$ are roll, attack, and sideslip angular rates, respectively; $\delta_a, \delta_e,$ and $\delta_r$ are aileron, elevator, and rudder deflections, respectively; the terms $\zeta_p(t), \zeta_q(t),$ and $\zeta_r(t)$ represent
disturbance terms, which are defined as: $\zeta_p(t) = 0.005 \sin(t)$, $\zeta_q(t) = 0.005 \cos(t)$, 
$\zeta_r(t) = 0.005(\sin(t) + \cos(t))$, and $|\zeta_p(t)| \leq 0.005$, $|\dot{\zeta}_p(t)| \leq 0.005$, $|\zeta_q(t)| \leq 0.005$, 
$|\dot{\zeta}_q(t)| \leq 0.005$, $|\zeta_r(t)| \leq 0.01$, $|\dot{\zeta}_r(t)| \leq 0.01$. Note that these disturbance terms are
used for the simulation purposes only, and are not included in the controller functions;

The other coefficients in eq. (7.1) are given as
$\tilde{Z}_a = -1.15$, $\tilde{Z}_q = 0.9937$, $\tilde{Y}_\beta = -0.297$, 
$\tilde{Y}_p = 0.00085$, $\tilde{L}_\beta = -53.48$, $\tilde{L}_p = -4.324$, $\tilde{L}_r = -0.224$, $\tilde{M}_a = 10.177$, $\tilde{M}_q = 3.724$, 
$\tilde{M}_q = -1.26$, $\tilde{M}_\delta = -19.5$, $\tilde{N}_\beta = 17.67$, $\tilde{N}_p = 0.234$, $\tilde{N}_r = -0.649$, $\tilde{N}_\delta = -6.155$.
Note that the terms $\tilde{Z}_\delta, \tilde{Y}_\delta$, and $\tilde{Y}_a$, which contain small control input perturbations,
are ignored and thus forming a square cascade structure of the aircraft model.

Actuator deflection and rate limits are given as

$$
|\delta_\alpha| \leq 0.37\text{rad}, \quad |\dot{\delta}_\alpha| \leq 1\text{rad/s} \quad \forall \alpha = a, e, r. \quad (7.2)
$$

The dynamics of the actuators are given by

$$
\eta_a \dot{\delta}_a = -(\delta_a - u_a), \quad \eta_e \dot{\delta}_e = -(\delta_e - u_e), \quad \eta_r \dot{\delta}_r = -(\delta_r - u_r), \quad (7.3)
$$

where $\eta_a = \eta_e = \eta_r = 0.02$.

The following filters are used to obtain the desired command profiles $\alpha_c$, $\beta_c$, 
and $\varphi_c$ by applying $\alpha_{ref}$, $\beta_{ref}$, and $\varphi_{ref}$, respectively:

$$
\frac{\alpha_c}{\alpha_{ref}} = \frac{4}{s^2 + 3s + 4}, \quad \frac{\beta_c}{\beta_{ref}} = \frac{16.98}{s^2 + 6.18s + 16.98}, \quad \frac{\varphi_c}{\varphi_{ref}} = \frac{2}{s^2 + 2.2s + 2}. \quad (7.4)
$$

The problem formulation for this case study is as follows:
• To design the continuous attitude single-loop Q-HOSM (2.7) controllers $\delta_\varphi \\forall \varphi = a, e, r$ that drive $\Lambda \rightarrow \Lambda_\varphi \\forall \Lambda = \varphi, \alpha, \beta$ asymptotically with respect to (7.1) in presence of bounded perturbations, where $\Lambda_\varphi \in \mathbb{R}^3$ is a smooth attitude vector-command profile.

• To introduce the robustness metrics, $PSPM$ and $PSGM$, to unmodeled dynamics for aircraft attitude HOSM-controlled system.

7.1 HOSM attitude control design: relative degree approach

The F-16 aircraft model (7.1) is represented in a relative degree format as

$$\ddot{\varphi} = -53.48\beta - 4.324p - 0.224r - 0.1345pq - 0.8225qr$$
$$-50.9333\delta_a + 10.177\delta_r + F_\varphi(x, t),$$
$$\ddot{\alpha} = -1.15\dot{\alpha} + 3.7\alpha - 1.252q + 0.9526pr$$
$$-0.0828(r^2 - p^2) - 1.9294 - 19.3772\delta_e + F_\alpha(x, t),$$
$$\ddot{\beta} = -0.297\dot{\beta} - 17.66775\beta - 0.237p + 0.6471r$$
$$+0.7235pq - 0.1348qr - 4.1572\delta_a + 6.147\delta_r + F_\beta(x, t),$$

where $x = [\varphi, \alpha, \beta, p, q, r]^T$, and the perturbation terms are defined

$$F_\varphi(x, t) = \zeta_\varphi(t) + \frac{df_\varphi(x, t)}{dt}, \quad F_\alpha(x, t) = 0.9937\zeta_q(t) + \frac{df_\alpha(x, t)}{dt},$$
$$F_\beta(x, t) = 0.00085\zeta_p(t) - 0.9973\zeta_r(t) + \frac{df_\beta(x, t)}{dt},$$
$$f_\varphi(x, t) = q \sin(\varphi) \tan(\theta) + r \cos(\varphi) \tan(\theta),$$
\[ f_\alpha(x, t) = -\beta p + 0.0427 \cos(\theta) \cos(\varphi) + 0.083589, \]
\[ f_\beta(x, t) = \alpha p + 0.0427 \cos(\theta) \sin(\varphi). \]

The tracking errors are defined as
\[ e_\varphi = \varphi_c - \varphi, \quad e_\alpha = \alpha_c - \alpha, \quad e_\beta = \beta_c - \beta. \] (7.6)

Then, the sliding variables are selected as
\[ \sigma_\varphi = \dot{e}_\varphi + c_\varphi e_\varphi, \quad \sigma_\alpha = \dot{e}_\alpha + c_\alpha e_\alpha, \quad \sigma_\beta = \dot{e}_\beta + c_\beta e_\beta, \] (7.7)

where \( c_\varphi, c_\alpha, c_\beta > 0 \) guarantee asymptotic convergence of sliding variables to zero.

For mathematical simplicity, the angular rates in eq. (7.1) are rewritten as
\[ \dot{p} = \Upsilon_p - 53.48\beta + \zeta_p(t), \quad \dot{q} = \Upsilon_q + 3.724\alpha + \zeta_q(t), \quad \dot{r} = \Upsilon_r + 17.67\beta + \zeta_r(t), \] (7.8)

where
\[ \Upsilon_p = -0.1345pq - 0.8225qr - 4.324p - 0.224r - 50.933\delta_a + 10.177\delta_r, \]
\[ \Upsilon_q = 0.9586pr - 0.0833(r^2 - p^2) - 1.94166 - 1.26q - 19.5\delta_e, \]
\[ \Upsilon_r = -0.7256pq + 0.1345qr + 0.234p - 0.649r + 4.125\delta_a - 6.155\delta_r. \]
In this case study, Q-HOSM control (2.6) is used to control the attitude of F-16 aircraft (7.1). Since the Q-HOSM control is a discontinuous control, the HOSM control is designed in terms of angular deflection rates $\dot{\delta}_a, \dot{\delta}_e, \dot{\delta}_r$.

The system (7.5) is rewritten as

$$\sigma_\varphi^{(2)} = \Upsilon_{11} + \Upsilon_{12} + 50.93\dot{\delta}_a - 10.17\dot{\delta}_r,$$

$$\sigma_\alpha^{(2)} = \Upsilon_{21} + \Upsilon_{22} - 1.15\dot{\sigma}_\alpha + 3.7\sigma_\alpha + 19.37\dot{\delta}_e,$$

$$\sigma_\beta^{(2)} = \Upsilon_{31} + \Upsilon_{32} - 0.297\dot{\sigma}_\beta - 17.66\dot{\sigma}_\beta + 4.15\dot{\delta}_a - 6.14\dot{\delta}_r,$$

where $\Upsilon_{11}, \Upsilon_{21}, \Upsilon_{31}$ are known terms; $\Upsilon_{12}, \Upsilon_{22}, \Upsilon_{32}$ are unknown terms and

$$\Upsilon_{11} = \varphi_c^{(3)} + c_\varphi \varphi_c + (4.324 + 0.1345q)\Upsilon_p$$
$$+(0.1345p + 0.8225r)\Upsilon_q + (0.224 + 0.8225q)\Upsilon_r$$
$$+c_\varphi (4.324p + 0.224 + 0.1345pq + 0.8225qr + 50.93\dot{\delta}_a - 10.17\dot{\delta}_r),$$

$$\Upsilon_{12} = 53.48\dot{\beta} + 53.48c_\varphi \beta - \dot{F}_\varphi(x, t) - c_\varphi F_\varphi(x, t)$$
$$+(4.324 + 0.1345q) \times (-53.48\beta + \zeta_p(t))$$
$$+(0.1345p + 0.8225r) \times (3.724\alpha + \zeta_a(t))$$
$$+(0.224 + 0.8225q) \times (17.67\beta + \zeta_r(t)),$$

$$\Upsilon_{21} = \alpha_c^{(3)} + (1.15 + c_\alpha)\alpha_c + (1.15c_\alpha - 3.7)\dot{\alpha}_c - 3.7c_\alpha\alpha_c$$
$$-(0.9526r + 0.1656p)\Upsilon_p + 1.252\Upsilon_q + (0.1656r - 0.9526p)\Upsilon_r$$
$$+c_\alpha (1.251q - 0.9526pr + 0.0828(r^2 - p^2) + 1.9294 + 19.37\dot{\delta}_e),$$

$$\Upsilon_{22} = -\dot{F}_\alpha(x, t) - c_\alpha F_\alpha(x, t) - (0.9526r + 0.1656p) \times (-53.48\beta + \zeta_p(t))$$
\[ +1.252 \times (3.724\alpha + \zeta_q(t)) + (0.1656r - 0.9526p) \times (17.67\beta + \zeta_r(t)), \]

\[ Y_{31} = \beta_c^{(3)} + (0.297 + c_\beta)\ddot{\beta}_c + (17.6677 + 0.297c_\beta)\dot{\beta}_c + 17.6677c_\beta \beta_c \]

\[ + (0.237 - 0.7235q)Y_p + (0.1348r - 0.7235p)Y_q + (0.1348q - 0.6471)Y_r \]

\[ + c_\beta (0.237p - 0.6471r + 0.7233pq + 0.1348qr + 4.1572\delta_a - 6.147\delta_r), \]

\[ Y_{32} = -\dot{F}_\beta(x, t) - c_\beta F_\beta(x, t) + (0.237 - 0.7235q)Y_p \]

\[ + (0.1348r - 0.7235p)Y_q + (0.1348q - 0.6471)Y_r. \]

Note that the unknown terms \( Y_{12}, Y_{22}, Y_{32} \) are assumed to be bounded in a reasonable flight domain \( |Y_{12}| \leq \lambda_a, |Y_{22}| \leq \lambda_e, \) and \( |Y_{32}| \leq \lambda_r. \)

Then, eq. (7.9) is written in the form of input-output decouple as

\[ \sigma^{(2)}_\varphi = Y_{11} + Y_{12} + v_a, \]

\[ \sigma^{(2)}_\alpha = Y_{21} + Y_{22} - 1.15\dot{\sigma}_\alpha + 3.7\sigma_\alpha + v_e, \]

\[ \sigma^{(2)}_\beta = Y_{31} + Y_{32} - 0.297\dot{\sigma}_\beta - 17.6677\sigma_\beta + v_r, \]

where

\[
\begin{bmatrix}
v_a \\
v_e \\
v_r
\end{bmatrix} =
\begin{bmatrix}
50.933 & 0 & -10.177 \\
0 & 19.3772 & 0 \\
4.1572 & 0 & -6.147
\end{bmatrix}
\begin{bmatrix}
\dot{\delta}_a \\
\dot{\delta}_e \\
\dot{\delta}_r
\end{bmatrix}.
\]

In this case study, the Q-HOSM control in eq. (2.7) with \( r = 2 \) is considered and is defined as

\[ v_a = -Y_{11} + \hat{v}_a, \quad v_e = -Y_{21} + \hat{v}_e, \quad v_r = -Y_{31} + \hat{v}_r, \]
\[
\begin{align*}
\dot{\nu}_a &= -\lambda_a \left( \dot{\varphi} + |\varphi|^{1/2} \text{sign}(\varphi) \right) / \left( |\dot{\varphi}| + |\varphi|^{1/2} \right), \\
\dot{\nu}_e &= -\lambda_e \left( \dot{\alpha} + |\alpha|^{1/2} \text{sign}(\alpha) \right) / \left( |\dot{\alpha}| + |\alpha|^{1/2} \right), \\
\dot{\nu}_r &= -\lambda_r \left( \dot{\beta} + |\beta|^{1/2} \text{sign}(\beta) \right) / \left( |\dot{\beta}| + |\beta|^{1/2} \right). \\
\end{align*}
\]

Then eq. (7.10) becomes

\[
\begin{align*}
\sigma^{(2)}_\varphi &= \Upsilon_{12} + \dot{\nu}_a, \\
\sigma^{(2)}_\alpha &= \Upsilon_{22} - 1.15\dot{\alpha} + 3.7\sigma_\alpha + \dot{\nu}_e, \\
\sigma^{(2)}_\beta &= \Upsilon_{32} - 0.297\dot{\beta} - 17.6677\sigma_\beta + \dot{\nu}_r. \\
\end{align*}
\]

The unknown terms \(\Upsilon_{12}, \Upsilon_{22}, \Upsilon_{32}\) are assumed to be exactly compensated by the HOSM controllers \(\dot{\nu}_a, \dot{\nu}_e, \dot{\nu}_r\), and thus the dynamics in the sliding modes become

\[
\begin{align*}
\sigma^{(2)}_\varphi &= \dot{\nu}_a, \\
\sigma^{(2)}_\alpha &= -1.15\dot{\alpha} + 3.7\sigma_\alpha + \dot{\nu}_e, \\
\sigma^{(2)}_\beta &= -0.297\dot{\beta} - 17.6677\sigma_\beta + \dot{\nu}_r. \\
\end{align*}
\]

The HOSM control law with regards to actuator deflections that corresponds to eqs. (7.4) is obtained as

\[
\begin{bmatrix}
\delta_a \\
\delta_e \\
\delta_r
\end{bmatrix} = \begin{bmatrix}
50.933 & 0 & -10.177 \\
0 & 19.3772 & 0 \\
4.1572 & 0 & -6.147
\end{bmatrix}^{-1} \begin{bmatrix}
\int_0^t \nu_a d\tau \\
\int_0^t \nu_e d\tau \\
\int_0^t \nu_r d\tau
\end{bmatrix}. 
\]
Figure 7.1: Aircraft (dynamically unperturbed) tracking the angular commands a) $\varphi_c$, b) $\alpha_c$, and c) $\beta_c$
Figure 7.2: Sliding variables a) $\sigma_\phi$, b) $\sigma_\alpha$, and c) $\sigma_\beta$ in the dynamically unperturbed system
Figure 7.3: Aircraft (dynamically unperturbed) angular deflections

Remark 9. It is worth noting that the designed HOSM control in eq. (7.14) is not only robust to external perturbations, but also to model uncertainties.

7.2 Simulation Results

In this section, the system (7.1) is simulated with the designed HOSM controller (7.14) in terms of aerodynamics surface deflections, while analyzing the robustness of the HOSM control to cascade unmodeled actuator dynamics (7.3).

Note that in the following two scenarios, dynamically unperturbed and dynamically perturbed aircraft systems, the controller gains are taken as $\lambda_a = 7$, $\lambda_e = 16$, and $\lambda_r = 7$ to satisfy the angular rate limits in eq. (7.2), $c_\varphi = c_\alpha = c_\beta = 3$, and simulated with initial conditions $\varphi(0) = 0.2$, $\alpha(0) = 0.1$, and $\beta(0) = 0.1$. In other words, the robustness of the HOSM controlled aircraft system (7.1), (7.14) to cascade unmodeled actuator dynamics (7.3) is studied by having these parameters unchanged.
Figure 7.4: Aircraft (dynamically perturbed) tracking the angular commands a) $\varphi_c$, b) $\alpha_c$, and c) $\beta_c$
Figure 7.5: Sliding variables a) $\sigma_\phi$, b) $\sigma_\alpha$, and c) $\sigma_\beta$ in the dynamically perturbed system
7.2.1 Dynamically unperturbed HOSM attitude control system

Figure 7.1 shows the high tracking accuracy of aircraft controlled by the designed single-loop attitude HOSM controller in the presence of external perturbations. The continuous control functions $\delta_a$, $\delta_e$, and $\delta_r$ are shown in Fig. 7.3. Note that the limit on the deflections given in eq. (7.2) are satisfied.

The evolution of sliding variables $\sigma_\varphi$, $\sigma_\alpha$, and $\sigma_\beta$ are shown in Fig. 7.2. The designed HOSM control (7.14) is successful in driving the sliding variables to zero in finite-time in the presence of external perturbations.

7.2.2 Dynamically perturbed HOSM attitude control system

Here, the robustness of the aircraft system (7.1) controlled by the HOSM attitude control (7.14) to cascade unmodeled actuator dynamics (7.3) is studied.

Figure 7.4 shows the high tracking accuracy of aircraft controlled by the designed continuous single-loop attitude HOSM controller in the presence of unmodeled
actuator dynamics (7.3) and external perturbations. The angular deflections in Fig. 7.6 show the acceptable limit-cycles/chattering.

The evolution of sliding variables $\sigma_\varphi$, $\sigma_\alpha$, and $\sigma_\beta$ in the dynamically perturbed HOSM control system (7.1), (7.3) are shown in Fig. 7.5. The sliding variables are converged to limit cycles, which are acceptable.

7.3 Robustness study of HOSM attitude control to cascade unmodeled dynamics

In this section, the robustness of designed HOSM controller to cascade unmodeled actuator dynamics (7.3) is analyzed using the tools developed in the Chapter 5.

7.3.1 Identification of Describing Function for HOSM control

Using the step-by-step procedure proposed in Section 5.2.2.1, the DFs for Q-HOSM controllers $\hat{v}_\varphi (\varrho = a, e, r)$ have been obtained numerically, and the negative reciprocals of DFs are tabulated in the form of Table 5.1.

7.3.2 Computing parameters of limit cycles

Note that the perturbations are assumed to be compensated exactly by the HOSM control $\hat{v}_\varphi \forall \varrho = a, e, r$. Next, the dynamics of sliding variable (7.13) in the frequency-domain are written as,

\[
W_{\varphi}(j\omega) = \left. \frac{1}{s^2} \right|_{s=j\omega}, \\
W_{\alpha}(j\omega) = \left. \frac{1}{s^2 + 1.15s - 3.7} \right|_{s=j\omega},
\]

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\[ W_\beta(j\omega) = \frac{1}{s^2 + 0.297s + 17.66775} \bigg|_{s=j\omega}. \]

Then, the dynamically perturbed system is written as

\[ W_\varphi(j\omega) = \frac{1}{0.02s + 1} \cdot \frac{1}{s^2} \bigg|_{s=j\omega}, \]
\[ W_\alpha(j\omega) = \frac{1}{0.02s + 1} \cdot \frac{1}{s^2 + 1.15s - 3.7} \bigg|_{s=j\omega}, \tag{7.15} \]
\[ W_\beta(j\omega) = \frac{1}{0.02s + 1} \cdot \frac{1}{s^2 + 0.297s + 17.66775} \bigg|_{s=j\omega}. \]

### Table 7.1: Parameters of limit cycles \( \sigma_\varphi, \sigma_\alpha, \) and \( \sigma_\beta \)

<table>
<thead>
<tr>
<th>Sliding variable</th>
<th>Simulations</th>
<th>Numerical solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma_\varphi )</td>
<td>( A_{\sigma_\varphi} = 1.094 \times 10^{-3} ) ( \omega_{\sigma_\varphi} = 69.81 \text{rad/sec} )</td>
<td>( A_{\sigma_\varphi} = 9.461 \times 10^{-4} ) ( \omega_{\sigma_\varphi} = 69.03 \text{rad/sec} )</td>
</tr>
<tr>
<td>( \sigma_\alpha )</td>
<td>( A_{\sigma_\alpha} = 1.169 \times 10^{-3} ) ( \omega_{\sigma_\alpha} = 89.75 \text{rad/sec} )</td>
<td>( A_{\sigma_\alpha} = 1.108 \times 10^{-3} ) ( \omega_{\sigma_\alpha} = 88.82 \text{rad/sec} )</td>
</tr>
<tr>
<td>( \sigma_\beta )</td>
<td>( A_{\sigma_\beta} = 9.735 \times 10^{-4} ) ( \omega_{\sigma_\beta} = 69.81 \text{rad/sec} )</td>
<td>( A_{\sigma_\beta} = 9.303 \times 10^{-4} ) ( \omega_{\sigma_\beta} = 69.55 \text{rad/sec} )</td>
</tr>
</tbody>
</table>

Using the proposed procedure in the Section 5.2.2.2, the parameter of predicted limit cycles for sliding variables are numerically obtained and are compared with the simulations as shown in Table 7.1.

### Table 7.2: Parameters of limit cycles \( e_\varphi, e_\alpha, \) and \( e_\beta \)

<table>
<thead>
<tr>
<th>Tracking error</th>
<th>Simulations</th>
<th>Numerical solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e_\varphi )</td>
<td>( A_{e_\varphi} = 1.565 \times 10^{-6} ) ( \omega_{e_\varphi} = 69.81 \text{rad/sec} )</td>
<td>( A_{e_\varphi} = 1.369 \times 10^{-6} ) ( \omega_{e_\varphi} = 69.03 \text{rad/sec} )</td>
</tr>
<tr>
<td>( e_\alpha )</td>
<td>( A_{e_\alpha} = 1.302 \times 10^{-6} ) ( \omega_{e_\alpha} = 89.75 \text{rad/sec} )</td>
<td>( A_{e_\alpha} = 1.246 \times 10^{-6} ) ( \omega_{e_\alpha} = 88.82 \text{rad/sec} )</td>
</tr>
<tr>
<td>( e_\beta )</td>
<td>( A_{e_\beta} = 1.393 \times 10^{-6} ) ( \omega_{e_\beta} = 69.81 \text{rad/sec} )</td>
<td>( A_{e_\beta} = 1.331 \times 10^{-6} ) ( \omega_{e_\beta} = 69.55 \text{rad/sec} )</td>
</tr>
</tbody>
</table>
7.3.3 Tolerance Limits for limit cycles

Note that the Tolerance Limits are somewhat easy to introduce as the acceptable limit cycle parameters, amplitude and frequency, that may occur due to cascade unmodeled dynamics for attitude angle tracking errors. In this case study, the admissible Tolerance Limits for the attitude angle tracking errors are defined as $TL$: $A_{e_{\Lambda}} = 4.5 \times 10^{-5}$ and $\omega_{e_{\Lambda}} = 45\text{rad/sec}$ $\forall \Lambda = \varphi, \alpha, \beta$. However, the proposed analysis estimates the parameters of predicted limit cycles for sliding variables $\sigma_{\varphi}$, $\sigma_{\alpha}$ and $\sigma_{\beta}$, but not for the attitude angle tracking errors.

Then, the frequency characteristics of the errors $e_{\Lambda} \forall \Lambda = \varphi, \alpha, \beta$ that correspond to inputs $\sigma_{\Lambda} = A_{\sigma_{\Lambda}} \sin(\omega_{\sigma_{\Lambda}} t)$ are written using eq. (7.7) as

$$\frac{e_{\Lambda}(j\omega_{\sigma_{\Lambda}})}{\sigma_{\Lambda}(j\omega_{\sigma_{\Lambda}})} = \frac{1}{(j\omega_{\sigma_{\Lambda}}) + c_{\Lambda}} \quad \forall \Lambda = \varphi, \alpha, \beta. \quad (7.16)$$

Therefore, the parameters of limit cycles for the attitude angle tracking errors are obtained from eq. (7.16) as $A_{e_{\Lambda}} = A_{\sigma_{\Lambda}}/(c_{\Lambda}^2 + \omega_{e_{\Lambda}}^2)^{1/2}$ and $\omega_{e_{\Lambda}} = \omega_{\sigma_{\Lambda}}$, where $A_{e_{\Lambda}}$ and $\omega_{e_{\Lambda}}$ are the amplitude and the frequency of limit cycle, respectively. Then, corresponding parameters are computed and tabulated in Table 7.2.

Next, $TL$ for sliding variables (7.7) are computed from the $TL$ for the attitude angle tracking errors using eq. (7.16) as

$$A_{\sigma_{\Lambda}} = A_{e_{\Lambda}}(c_{\Lambda}^2 + \omega_{e_{\Lambda}}^2)^{1/2} = 2 \times 10^{-3}, \quad (7.17)$$

$$\omega_{\sigma_{\Lambda}} = \omega_{e_{\Lambda}} = 45\text{rad/sec} \quad \forall \Lambda = \varphi, \alpha, \beta.$$

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Figure 7.7: Sliding variables a) $\sigma_\varphi$, b) $\sigma_\alpha$, and c) $\sigma_\beta$ in the dynamically perturbed aircraft system after cascading compensator with HOSM control
Figure 7.8: Aircraft angular deflections after cascading compensator with HOSM control

7.3.4 Practical stability margins

The robustness metrics are obtained for the aircraft system (7.1) controlled by HOSM control to cascade unmodeled actuator dynamics (7.3) using Section 5.2.2.4.

Table 7.3: Practical stability margins for $\sigma_\varphi$, $\sigma_\alpha$, and $\sigma_\beta$

<table>
<thead>
<tr>
<th>Sliding variable</th>
<th>PSPM</th>
<th>PSGM</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_\varphi$</td>
<td>9.669°</td>
<td>24.01dB</td>
</tr>
<tr>
<td>$\sigma_\alpha$</td>
<td>7.048°</td>
<td>18.66dB</td>
</tr>
<tr>
<td>$\sigma_\beta$</td>
<td>9.993°</td>
<td>24.44dB</td>
</tr>
</tbody>
</table>

Table 7.4: $PSPM$ of $\sigma_\varphi$, $\sigma_\alpha$, and $\sigma_\beta$ after cascading linear compensators with the designed HOSM controllers

<table>
<thead>
<tr>
<th>Sliding variable</th>
<th>$PSPM_{new}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_\varphi$</td>
<td>33.07°</td>
</tr>
<tr>
<td>$\sigma_\alpha$</td>
<td>31.75°</td>
</tr>
<tr>
<td>$\sigma_\beta$</td>
<td>32.92°</td>
</tr>
</tbody>
</table>
Table 7.3 presents the robustness metrics, \( PSPM \) and \( PSGM \), that are deduced using the \( TL \) for the sliding variables in eq. (7.17).

Note that the identified \( PSPMs \) do not satisfy the prescribed value [17] (see page 33 of [17]): \( PSPM_\Lambda \geq 30^\circ \Lambda = \varphi, \alpha, \beta \). Then, the required \( PSPMs \) are achieved (see Table 7.4) by cascading phase-lead compensators with the designed HOSM controllers.

### 7.4 Cascade compensator for HOSM controlled F-16 aircraft

Phase-lead compensators that cascade with the HOSM controllers are designed for the roll angle \( \varphi \), angle of attack \( \alpha \), and sideslip angle \( \beta \), respectively as follows:

\[
W_{dc\varphi}(s) = \frac{97.94}{39.25} \times \frac{s + 39.25}{s + 97.94}, \quad W_{dc\alpha}(s) = \frac{118.10}{42.70} \times \frac{s + 42.70}{s + 118.10},
\]

\[
W_{dc\beta}(s) = \frac{95.76}{38.86} \times \frac{s + 38.86}{s + 95.76}.
\]

The obtained \( PSPM_{new} \) (see Table 7.4) are satisfied the prescribed value of \( PSPMs \) for the attitude angle control. Moreover, Fig. 7.7 shows that the chattering in the dynamically perturbed aircraft system (7.1), (7.3) is alleviated, while satisfying the prescribed values of \( PSPM \) and \( PSGM \), after cascading phase-lead compensator with corresponding HOSM control. The chattering of angular deflections is also alleviated as shown in Fig. 7.8.

**Remark 10.** Note that the parameters of predicted limit cycles for the sliding variables may be not admissible because of the significant effect of unmodeled dynamics. In this scenario, one of the following proposed way outs can be used.

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1. By designing a linear compensator, the negative practical stability margins can be transformed to prescribed positive values.

2. When the first order compensator failed to achieve the prescribed value of practical stability margins, the complexity of compensator can be increased.

3. The order of HOSM can be increased by one, instead of increasing the compensator complexity, while taking a first order unmodeled dynamics implicitly.

**Remark 11.** Note that the designed HOSM control is operational for multiplicative disturbances that may occur due to aircraft battle damage, in addition to additive disturbances.

The aircraft internal perturbations are modeled by considering a 50% loss of horizontal tail and rudder areas. The symbols of form $\hat{G}$ in eq. (7.1) are represented as the nominal terms $(.)_n$ and the additional deviations from nominal due to the damage and uncertainty $\Delta(.)$.

$$\hat{G} = G_n + \Delta G,$$

(7.18)

where

$$G_n = [Z_\alpha, Z_q, Z_\delta, Y_\beta, Y_p, Y_\delta_a, Y_\delta_r, L_\beta, L_p, L_r, L_\delta_r, M_\alpha, M_q, M_\delta, N_\beta, N_p, N_r, N_\delta_r]$$

$$= [-1.15, 0.9937, 0, -0.297, 0.00085, 0, 0, -53.48, -4.324, -0.224, 10.177, 3.724, -1.26, -19.5, 17.67, 0.234, -0.649, -6.155],$$

$$\Delta G = [\Delta Z_\alpha, \Delta Z_q, \Delta Z_\delta, \Delta Y_\beta, \Delta Y_p, \Delta Y_\delta_a, \Delta Y_\delta_r, \Delta L_\beta, \Delta L_p, \Delta L_r, \Delta L_\delta_r, \Delta M_\alpha, \Delta M_q, \Delta M_\delta, \Delta N_\beta, \Delta N_p, \Delta N_r, \Delta N_\delta_r].$$
\[
\begin{bmatrix}
0.04\bar{U}(t - 5), 0.0031\bar{U}(t - 5), -0.177 + 0.0885\bar{U}(t - 5), 0.0534\bar{U}(t - 1.5), \\
-0.0005\bar{U}(t - 5), 0.0372 - 0.0186\bar{U}(t - 1.5), 0.002466, 8.024\bar{U}(t - 1.5), \\
0.071\bar{U}(t - 1.5), 0.055\bar{U}(t - 1.5), -5.089\bar{U}(t - 1.5), 1.856\bar{U}(t - 5), \\
0.42\bar{U}(t - 5), 9.75\bar{U}(t - 5), -5.82\bar{U}(t - 1.5), 0.01\bar{U}(t - 1.5), \\
0.133\bar{U}(t - 1.5), 3.077\bar{U}(t - 1.5)
\end{bmatrix},
\]

where functions \(\bar{U}(t)\) define the change of the F-16 math model due to the aerodynamic surface damages. Note that the controller does not know about these damages. The corresponding dynamics of the model changes are described by

\[
\frac{\bar{U}(s)}{U(s)} = \frac{\omega_n^2}{s^2 + 2\epsilon\omega_n s + \omega_n^2},
\]

(7.19)

where \(U(s)\) is a unit step function, \(\epsilon = 0.707\) and \(\omega_n = 100\text{rad/sec}\).

Figure 7.9: Damaged aircraft angular deflections
Figure 7.10: Damaged aircraft (dynamically unperturbed) tracking the angular commands a) $\varphi_c$, b) $\alpha_c$, and c) $\beta_c$
Figure 7.11: Sliding variables a) $\sigma_\varphi$, b) $\sigma_\alpha$, and c) $\sigma_\beta$ in the damaged aircraft system
The accuracy and robustness of the designed HOSM controller to multiplicative disturbances are shown in Figs. 7.10-7.9.

7.5 Summary

In this chapter, a continuous single-loop HOSM controller for F-16 aircraft was designed for the first time, and the HOSM attitude controller was certified for robustness to cascade unmodeled dynamics using the introduced robustness metrics: $PSPM$ and $PSGM$. 
CHAPTER 8

CONCLUSIONS

The gap in the certification of Higher Order Sliding Mode (HOSM) control for robustness to unmodeled dynamics was closed for the first time in this dissertation.

- The robustness metrics, Practical Stability Phase Margin ($PSPM$) and Practical Stability Gain Margin ($PSGM$), for HOSM control to cascade unmodeled dynamics were introduced, presiding the tool for HOSM control certification.

- The tools/algorithms to identify these robustness metrics using Describing Function-Harmonic Balance (DF-HB) method were developed.

  - A numerical algorithm to compute DFs for HOSM controllers (specifically, Nested, Quasi-continuous, and Continuous HOSM controllers) was proposed, and the analytic DF for fixed-time convergent 2-SM controller was obtained. A database of DFs for these controllers was developed.

  - A numerical algorithm that solves HB equation using Newton Raphson method to obtain predicted chattering parameters was proposed.

  - The computational algorithms to identify the HOSM control robustness metrics ($PSPM$ and $PSGM$) were proposed.
• When the obtained values of $PSPM$ and $PSGM$ do not satisfy the prescribed values, a cascade linear compensator to HOSM controller was suggested to achieve the prescribed values.

• The proposed technique was applied to certify the HOSM attitude controller of an F-16 aircraft for robustness to cascade unmodeled dynamics as a case study.

  – A continuous single-loop HOSM attitude F-16 aircraft controller was designed for the first time in this dissertation.

  – The robustness of designed HOSM control was quantified/certified using $PSPM$ and $PSGM$.

Several other simulation examples were presented in the dissertation to validate the proposed method.
APPENDIX A

A.1 Proof of Proposition 3

Consider the Cartesian complex form of HB eq. (3.3),

\[
H = 1 + \text{Re}\{N(A, \omega)\} \text{Re}\{W(j\omega)\} - \text{Im}\{N(A, \omega)\} \text{Im}\{W(j\omega)\} + \text{Im}\{N(A, \omega)\} \text{Re}\{W(j\omega)\}.
\]  \hspace{1cm} (A.1)

From Loeb’s criterion [34], the analysis of eq. (A.1) in the presence of small amplitude perturbations gives the necessary condition for limit cycle stability

\[
\frac{\partial U}{\partial A} \frac{\partial V}{\partial \omega} - \frac{\partial U}{\partial \omega} \frac{\partial V}{\partial A} |_{(A_0, \omega_0)} > 0.
\]  \hspace{1cm} (A.2)

Then, the stability condition in eq. (5.25) is obtained by replacing \(U\) and \(V\) in inequality (A.2).

The Proposition 3 is proven.
A.2 Proof of Theorem 1

Consider that $|\sigma_0| > \epsilon$, where $\epsilon > 0$ is a given constant. Consider that $\text{sign}(y(t))$ remains opposite to $\text{sign}(\sigma(t))$ for $t > t_0$, while $\sigma(t)$ does not cross the axis $\sigma = 0$, and $\chi > L_1$, the first equation in (5.2) gives

$$\frac{d |\sigma(t)|}{dt} \leq -\lambda_2 |\sigma(t)|^\kappa.$$  \hfill (A.3)

Equation (A.3) is rewritten as

$$\frac{d |\sigma(t)|}{|\sigma(t)|^\kappa} = |\sigma(t)|^{-\kappa} \frac{d |\sigma(t)|}{-\lambda_2 dt}. \hfill (A.4)$$

By integrating the eq. (A.4) with the initial condition $\sigma(t_0) = \sigma_0$ yields

$$\frac{|\sigma(t)|^{1-\kappa}}{1 - \kappa} - \frac{|\sigma_0|^{1-\kappa}}{1 - \kappa} \leq -\lambda_2(t - t_0),$$

and taking into account that $1 - \kappa < 0$, the above expression leads to

$$\frac{|\sigma(t)|^{1-\kappa}}{1 - \kappa} \leq -\lambda_2(t - t_0) + \frac{|\sigma_0|^{1-\kappa}}{1 - \kappa} \leq -\lambda_2(t - t_0).$$

Then, by multiplying both parts with $\kappa - 1 > 0$ yields to

$$-|\sigma(t)|^{1-\kappa} \leq -\lambda_2(t - t_0)(\kappa - 1).$$
For positive $\sigma, y$, i.e., $\sigma > 0$ and $y > 0$, $-\sigma \leq -y \implies \frac{1}{\sigma} \leq \frac{1}{y}$, and by using this in the above expression, we obtain

$$\left| \sigma(t) \right|^\kappa - 1 \leq \frac{1}{\lambda_2(\kappa - 1)(t - t_0)}.$$  

Therefore, $| \sigma(t) |$ decreases and reaches the value $| \sigma(t) | = \epsilon$ for a time $T_1 \leq \frac{1}{\lambda_2(\kappa - 1)\epsilon^{\kappa - 1}}$, which corresponds to the first term in eq. (5.3). It is worth noting that the obtained expression is independent of an unknown initial condition $\sigma_0$.

The step A concludes with $| \sigma(T_1) | = \epsilon > 0$. Given that $\text{sign}(y(t))$ is opposite to $\text{sign}(\sigma(t))$ for $t \in [t_0, T_1]$ and $\chi > L_1$, $| y(t) |$ increases for $t \in [t_0, T_1]$. Note that if $| \sigma_0 | \leq \epsilon$, Step A is not executed and therefore, the term $\frac{1}{\lambda_2(\kappa - 1)\epsilon^{\kappa - 1}}$ would be absent in (5.3).

B. For $t > T_1$, $| \sigma(t) |$ continues to decrease until reaching zero at a certain time $T_2$. Considering that $\text{sign}(y(t))$ remains opposite to $\text{sign}(\sigma(t))$ for $t \in [T_1, T_2]$ until $\sigma(t)$ becomes equal to zero, the first equation in (5.2) gives

$$\frac{d}{dt} \left| \sigma(t) \right| \leq -\lambda_1 \left| \sigma(t) \right|^{1/2},$$  

(A.5)

By solving (A.5), we obtain

$$2 \left| \sigma(t) \right|^{1/2} \leq -\lambda_1(t - T_1) + 2 \left| \sigma(T_1) \right|^{1/2} = -\lambda_1(t - T_1) + 2\epsilon^{1/2}.$$

Therefore, $| \sigma(t) |$ decreases and reaches zero for a time $T_2 \leq 2\epsilon^{1/2}/\lambda_1$, which corresponds to the second term in (5.3).
The step B concludes with \( | \sigma(T_2) | = 0 \). Given that sign\((y(t))\) is opposite to sign\((\sigma(t))\) for \( t \in [t_0, T_2] \) and \( \chi > L_1 \), \( | y(t) | \) increases for \( t \in [t_0, T_1] \). The value \( | y(T_2) | \) is bounded by \( | y(T_2) | < M(T_2 - t_0) = M(\frac{1}{\lambda_2(\kappa-1)e^{\kappa-1}} + \frac{2^{1/2}}{\lambda_1}) \) in view of the second equation in (5.2).

C. Note that the trajectory of system (5.2) starting at \((0, y(T_2))\) is dominated by the trajectory of system (5.2) starting at \((0, y_2 = M(\frac{1}{\lambda_2(\kappa-1)e^{\kappa-1}} + \frac{2^{1/2}}{\lambda_1})\) and converges to the origin faster. Following the proof of Theorem 4.5 in [1], the finite-time convergence for the latter trajectory is estimated by \( T_{STW} \leq \sum_i \dot{\sigma}(t_i)/m \), where \( \dot{\sigma}(t_i), i = 1, 2, ..., \) are the derivatives of \( \sigma(t) \) at subsequent time moments \( t_i \), such that \( \sigma(t_i) = 0 \) and \( \dot{\sigma}(t_i) = y(t_i) \), if \( q_c = | \frac{\dot{\sigma}(t_2)}{\dot{\sigma}(t_1)} | < 1 \). Note that Step C begins at one of such time moments, \( T_2 \). Therefore \( T_{STW} \) is calculated as \( T_{STW} = y_2/(1 - q_c)m \), where the value of \( q_c \) is estimated using the formula \( q_c \leq Mk(\lambda_1)/\lambda_1 \) derived in [50]. The necessary condition \( q_c = Mk(\lambda_1)/\lambda_1 < 1 \) corresponds to the second condition for control gains in this theorem. Note that the first condition \( \chi > L_1 \) is a mandatory condition for convergence of the conventional super-twisting algorithm, as confirmed in [50]. Substituting the obtained estimates for \( y_2 = | y(T_2) | \) and \( q_c \) yields \( T_{STW} < M(\frac{1}{\lambda_2(\kappa-1)e^{\kappa-1}} + \frac{2^{1/2}}{\lambda_1})/(1 - Mk(\lambda_1)/\lambda_1)m \), which, after dividing both parts of the fraction by \( M \), corresponds to the last two terms in (5.3).

D. The optimal value of \( \epsilon \) corresponding to the minimum uniform convergence time \( T_f \) is obtained by minimizing the first two terms in (5.3) with respect to \( \epsilon \).

The Theorem 1 is proven.
REFERENCES


