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DISCRETE-TIME STRUCTURED MODELS AND THEIR
DYNAMICS FOR INTERACTIVE WILD AND STERILE
MOSQUITOES AND MALARIA TRANSMISSIONS

by

YANG LI

A DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy
in
The Department of Mathematical Sciences
to
The School of Graduate Studies
of
The University of Alabama in Huntsville

HUNTSVILLE, ALABAMA

2017
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Yang Li

3/29/2017
(date)
DISSERTATION APPROVAL FORM

Submitted by Yang Li in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Graduate Program of Mathematical Science and accepted on behalf of the Faculty of the School of Graduate Studies by the dissertation committee.

We, the undersigned members of the Graduate Faculty of The University of Alabama in Huntsville, certify that we have advised and/or supervised the candidate of the work described in this dissertation. We further certify that we have reviewed the dissertation manuscript and approve it in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Graduate Program of Mathematical Science.

Dr. Jia Li  3/29/2017  (Date)  Committee Chair

Dr. Dongcheng Wu  3/29/2017  (Date)

Dr. Ellen Weisskopf  3/29/2017  (Date)

Dr. David Halpern  3/29/2017  (Date)

Dr. John Mayer  4/3/2017  (Date)

Dr. Boris Kunin  5/29/17  (Date)  Department Chair

Dr. Sundar Christopher  4/6/17  (Date)  College Dean

Dr. David Berkowitz  (Date)  Graduate Dean
Mosquito-borne diseases, including malaria, transmitted between human beings by blood-feeding mosquitoes, have been big concerns for the public health. No vaccines are available. An effective way to prevent such diseases is to control the amount of mosquitoes. The Sterile Insect Technique (SIT) is indeed a method of biological control. Mathematical models have proven useful in gaining insights into challenging questions in population dynamics and epidemiology. The objectives in this dissertation is to formulate new models for interactive wild and sterile mosquitoes so that the dynamics are relatively simpler and mathematically more tractable, but the fundamental model features are snatched.

Instead of the Ricker-type of nonlinearity for the survival functions, we assume Beverton-Holt-type survival functions. We first formulate models with the assumption that there are no generation overlaps in the mosquito population. Then the models based on the assumption of overlapped generations are considered. We consider three different strategies for the releases of sterile mosquitoes and investigate the model dynamics. Threshold values for the releases of sterile mosquitoes are established for all of the models that determine whether the wild mosquitoes are wiped out or coexist with the sterile mosquitoes.
We also formulate stage-structured interactive models. Detailed analysis is carried out. Threshold values for the existence and stability of positive fixed points are derived, respectively. When the positive fixed point is unstable, a 2-cycle is bifurcated.

To incorporate the interactive mosquitoes into malaria transmissions, we formulate susceptible-exposed-infective-recovered (SEIR) compartmental discrete-time models for malaria, which are of high dimensions, and then include the interactive mosquito models into these disease models. We derive formulas for the reproductive number $R_0$ of infection for the malaria models with or without sterile mosquitoes and explore the existence of endemic fixed points as well. We then study the impact of sterile mosquitoes releases on the disease transmissions by investigating the effects of varying the releases of sterile mosquitoes. We use numerical simulations to verify our results for all cases and finally give brief discussions of our findings and future study.
There are several people who have provided me with their help, knowledge and advice, to whom I am indebted. First of all, I would like to express my sincere gratitude to my supervisor, Dr. Jia Li, for his instructive advice and useful suggestions with my Ph.D obtaining. I have been so lucky to have such an interested and approachable supervisor with a wide-ranging breadth of knowledge. In addition to his role as a supervisor, he is more like an approachable friend who cares about me in my life and is responsible for sparking my interest in Mathematical Biology. For that I am extremely thankful for Dr. Jia Li.

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<tr>
<td>$w_n$</td>
<td>Wild mosquitoes population size at generation $n$</td>
</tr>
<tr>
<td>$C(w_n)$</td>
<td>The number of matings per individual (mating rate)</td>
</tr>
<tr>
<td>$A(w_n)$</td>
<td>The number of offspring produced per mating</td>
</tr>
<tr>
<td>$s(w_n)$</td>
<td>The survival probability of the offspring produced by wild adult mosquitoes</td>
</tr>
<tr>
<td>$p$</td>
<td>Survival probability from current generation $n$ to next generation $n + 1$</td>
</tr>
<tr>
<td>$B_n$</td>
<td>The number of the sterile mosquitoes released at generation $n$</td>
</tr>
<tr>
<td>$\exp(-d)$</td>
<td>The density-independent survival probability</td>
</tr>
<tr>
<td>$k$</td>
<td>The maximum survival probability</td>
</tr>
<tr>
<td>$\eta$</td>
<td>The density-dependant factor</td>
</tr>
<tr>
<td>$r_0$</td>
<td>The population intrinsic growth rate</td>
</tr>
<tr>
<td>$x_n$</td>
<td>The number of mosquito larvae at generation $n$</td>
</tr>
<tr>
<td>$y_n$</td>
<td>The number of mosquito adults at generation $n$</td>
</tr>
<tr>
<td>$f$</td>
<td>The per-capita birth rate</td>
</tr>
<tr>
<td>$s_1$</td>
<td>The survival rate of larvae</td>
</tr>
<tr>
<td>$g$</td>
<td>The progression rate of larvae or the adults emergence rate</td>
</tr>
</tbody>
</table>
\(s_2\) The survival rate of adults

\(N^h\) The total human being population

\(S^h\) The susceptible human beings

\(E^h\) The exposed or incubating human beings

\(I^h\) The infectious human beings

\(R^h\) The recovered human beings

\(S^v\) The susceptible mosquitoes

\(E^v\) The exposed or incubating mosquitoes

\(I^v\) The infectious mosquitoes

\(\Lambda\) The input flow of the susceptible human beings including birth rate

\(\lambda^v\) The mosquito infection rate

\(\lambda^h\) The human infection rate

\(\gamma^v\) The rate of incubating individuals becoming infective

\(b^v\) The per capita birth rate for wild mosquitoes

\(\alpha^v_b\) The survival probabilities of the newborns

\(\alpha^v_1\) The survival probabilities of susceptible adults to stay in susceptible group

\(\alpha^v_2\) The survival probabilities of susceptible adults to be incubating adults

\(\alpha^v_3\) The survival probabilities of incubating adults to stay in incubating group
\( \alpha_4^v \) The survival probabilities of incubating adults to be infective adults

\( \alpha_5^v \) The survival probabilities of infective adults to stay in infective group

\( k_b^v \) The maximum survival probability

\( r_{b}^v \) The density-dependant factor

\( \alpha_k^h \) The survival probabilities of susceptible, exposed or incubating, infective and recovered humans, respectively for \( k = 1, 2, \ldots, 8 \)

\( \gamma_h^h \) The developing rate of incubating humans becoming infective

\( \eta_h^h \) The recovery rate for infective humans

\( \theta_h^h \) The rate of partial immunity loss

\( r \) The number of average bites by a single mosquito on all human hosts

\( \beta_h^h \) The transmission probability per bite to a susceptible mosquito from an infective human

\( r_1 \) The number of average bites on a human host by all mosquitoes

\( \beta^v \) The transmission probability per bite to a susceptible human from an infective mosquito

\( L(t) \) The factor to determine the infection rate for humans

\( G(t) \) A positive function of \( L \) to determine \( \lambda^h \)

\( R_0 \) The reproductive number without sterile mosquitoes

\( R_{0b} \) The reproductive number with sterile mosquitoes
To my parents and fiance
CHAPTER 1

INTRODUCTION

Research is what I’m doing
when I don’t know what I’m doing.

—Wernher von Braun

1.1 Biological Background

Mosquito-borne diseases, such as malaria, are a big concern for the public health worldwide. Malaria is a mosquito infectious disease caused by a parasite plasmodium, which infects red blood cells. Malaria begins as a flu-like illness, with symptoms first occurring 9 to 14 days after infection. Symptoms include fever (typical cycles of fever, shaking chills, and drenching sweats may develop), joint pain, headaches, frequent vomiting, convulsions, and coma. Because of destruction of red blood cells and liver cells by malaria, the patients’ skin and eyes may become white. Simple malaria can become severe malaria if not treated quickly, which may cause much more severe problems such as coma, shock, liver or kidney failure, bleeding problems, central nervous system problems, and even die from the infection [69].

Malaria is a leading cause of death in many developing countries, where young children and pregnant women are the groups most affected. According to the World Health Organizations World Malaria Report 2013 and the Global Malaria Action
Figure 1.1: Approximate geographic distribution of malaria

Plan, there are 3.2 billion people, half of the world's population, living in areas at risk of malaria transmission in 106 countries and territories, as shown in Figure 1.1 (Approximate geographic distribution of malaria) [24]. From a worldwide view, Malaria is the fifth cause of death from infectious diseases. In 2015, there were an estimated 214 million malaria cases and some 438,000 malaria deaths worldwide, with 92% of these deaths occurring in Africa. Between 1957 and 2015, 63 outbreaks of locally transmitted malaria have occurred, and about 1,500 cases of malaria are diagnosed each year in the United States [53].

Since the nineteenth century, the Nobel Prize in Physiology or Medicine has been awarded for work associated with malaria five times. They are Sir Ronald Ross (1902), Charles Louis Alphonse Laveran (1907), Julius Wagner-Jauregg (1927), Paul Hermann Mller (1948), and Youyou Tu (2015).
The life cycle of the disease is complicated. They are transmitted between humans by blood-feeding female mosquitoes, through blood transfusion among people living in malarious areas, through organ transplantation, or vertically by infected mothers. While there are about 3500 species of mosquitoes in the world, of which 430 are the genus Anopheles [9], the human malaria is mainly caused by infection with P.falciparum (tropics) and P.vivax (tropical and temperate zones). Human malaria cannot spread to animals, nor can animal malaria spread to humans [35]. Human malaria can only be transmitted by the bites of Anopheles mosquitoes.

There are many stimulants such as high concentrations of carbon dioxide, lactic acid, sweat, moisture, certain body odors, body warmth and movement, which can stimulate mosquitoes to target and identify human beings [57,59]. When a mosquito lies on the skin of a human being, it attempts to pierce a small blood vessel with its proboscis in order to suck blood. At the same time, mosquitoes inject some saliva together with anticoagulant to prevent blood from clotting and closing the wound, which makes it easier for them to drink it.

Malaria infection begins when an infected female Anopheles mosquito bites a person. In the form of sporozoites, Plasmodium parasites are injected into the human body (see Figure 1.2, step (1), infective stage). The sporozoites pass quickly into the human liver (see Figure 1.2, step (2) and (3)), and develop into schizonts which rupture and release merozoites (see Figure 1.2, step (4)). The merozoites invade red blood cells (erythrocytes) and multiply again until the cells burst. Then they invade more erythrocytes. This cycle is repeated, causing fever each time parasites break free and invade blood cells (see Figure 1.2 step (5) and (6)). Some of the infected blood
cells leave the cycle of asexual multiplication. Instead of replicating, the merozoites in these cells develop into sexual forms of the parasite, called gametocytes, that circulate in the blood stream (see Figure 1.2, step (7)). When a mosquito bites an infected human, it ingests the gametocytes, which develop further into mature sex cells called gametes (see Figure 1.2, step (8)). The fertilized female gametes develop into actively moving ookinetes (see Figure 1.2, step (9)) that burrow through the mosquito’s midgut wall and form oocysts (see Figure 1.2, step (10) and (11)). The oocyst grow and rupture and then release thousands of active sporozoites into the body cavity that can travel to the mosquitoes’ salivary glands (see Figure 1.2, step (12)). The cycle of human malaria infection begins again when the mosquito bites another person [6,8,23,51,52].

1.2 The Sterile Insect Technique (SIT) Control Measure

Mosquito-borne diseases such as malaria are transmitted indirectly, which is not from a human to a human, but through vectors like blood-feeding mosquitoes. Since there is no effective vaccine available to control malaria yet, all efforts are directed to avoid the proliferation of the mosquito population. Massive spraying of insecticides or eliminating breeding sites has greatly limited malaria in some area. But the number of mosquito-borne diseases cases still continues to climb. This is partially because of economic and social difficulty in some developing countries and effects of environmental changes such as global warming. The fact that mosquitoes developing resistance to the pesticides has also made the controlling more difficult. To
Figure 1.2: CDC illustration of the life cycles of malaria parasites

prevent and control the amount of mosquitoes in order to control such mosquito-borne diseases, biologically control measures provide an effective weapon [11].

The Sterile Insect Technique (SIT) is one of the biological control measures. The SIT was first conceived by Knipling [41], and used successfully in 1958 in Florida to control Screwworm fly (Cochliomya omnivorax) [42, 43]. From that time, the releases of sterile insects have been used with varying success. Examples are screwworm fly in USA, Mexico and Libya; Mediterranean Fruit Fly (Ceratitis capitata Wiedemann) in USA and Mexico; Melon Fly (Dacus cucurbitae Coquillett) in Japan and Taiwan; Pink Bollworm (Pectinophora gossypiella Saunders) in USA; Tsetse Fly (Glossina species) in Tanzania, Zimbabwe and Upper Volta; Boll Weevil (Anthono-
mus grandis Boheman) in Southeastern USA; Mexican Fruit Fly (Anastrepha ludens Loew) in USA and Mexico; Gypsy Moth (Lymantria dispar Linnaeus) in USA and Canada [15].

The SIT is indeed a method of biological control in which the natural reproductive process of the target population is disrupted. By chemical or physical methods, male mosquitoes are genetically modified to be sterile despite being sexually active. These sterile male mosquitoes are then released into the environment to mate with the wild female mosquitoes. A wild female mosquito that mates with a sterile male mosquito will either not reproduce, or produce eggs that do not hatch. Repeated releases of genetically modified mosquitoes or the releases of a significantly large number of sterile mosquitoes may eventually wipe out a wild mosquito population, although it is, in practice, often more useful to consider controlling the population rather than eradicating it [4,16,70].

In fact, in the Sterile Insect Technique (SIT) process applied to mosquitoes, male mosquito pupae are subjected to sterilizing gamma rays. The insects are raised in massive numbers and irradiated by gamma rays. Then gamma-irradiated male mosquitoes are released in equally massive hordes to compete with wild males in order to reduce the amount of whole wild mosquitoes. It’s believed that if there are enough sterile males mating with wild females, the chances of producing offspring are reduced, and the population crashes.

In the meantime, SIT can provide many attractive features. Sterile insects are environmentally benign, with no toxic residues and minimal nontarget impact. Resistance has very rarely been seen in the 50+ year history of large-scale SIT pro-
grams against agricultural pests. Further, SIT programs are far less intrusive than most other such methods although it needs high degree of organization for control programs, especially with a goal of eradication or long-term suppression [3].

SIT has shown promising results in laboratory studies, but predicting the impact of releasing sterile mosquitoes into the field of wild mosquito populations is still a challenging task. Moreover, sterile mosquitoes can be released into the field in various ways. How do we investigate and assess the impacts of different strategies for releases of sterile mosquitoes? Or can we determine and adopt optimal release strategies?

1.3 Mathematical Modeling Basis for the Research

Mathematical models have proven useful in gaining insights into challenging questions in population dynamics and epidemiology. There are mathematical models in the literature formulated to study the interactive dynamics of mosquito populations or the control of mosquitoes [10–12, 14, 20, 33, 34]. Models for vector-borne diseases, incorporating sterile mosquitoes, have also been formulated to investigate the disease transmission dynamics in [28, 32, 68].

Dynamics of the interactive wild and sterile mosquitoes with different strategies of releasing sterile mosquitoes have been explored in several studies such as [20, 49, 50]. The models in [49] are of discrete-time and based on the Ricker type of difference equations. It has shown complex dynamics such as period-doubling bifurcation in those models.
Notice the nature of the dynamical complexity in Ricker population models and that the complicated features in [49] may not necessarily be induced by the interactions between the wild and sterile mosquitoes. Our fundamental goal nevertheless is eventually incorporate the mosquito models into malaria transmission models. Hence the objectives in this dissertation is to formulate new models for interactive wild and sterile mosquitoes so that the dynamics are relatively simpler and mathematically more tractable. In particular, there seem no clear evidence in the literature or existing data supporting such complexity.

To establish a mathematical basis for investigating the dynamics among the wild mosquitoes without interaction with sterile mosquitoes, we assume that the wild mosquito population is homogeneous, which means there is not distinguishing of their metamorphosis stages.

Let $w_n$ be the wild mosquitoes population size at generation $n$, where we assume there is no generation overlaps. Then the population dynamics are governed by the following equation

$$w_{n+1} = C(w_n)A(w_n)w_ns(w_n), \quad (1.1)$$

where $C(w_n)$ is the number of matings per individual (mating rate), $A(w_n)$ the number of offspring produced per mating and $s(w_n)$ the survival probability of the offspring produced by wild adult mosquitoes [54–56].

If there exist overlapped generations with $p$ as the survival probability from the current generation $n$ to next generation $n+1$, then the dynamics are presented
as
\[ w_{n+1} = C(w_n)A(w_n)w_ns(w_n) + pw_n. \] (1.2)

Now suppose sterile mosquitoes are released into a wild mosquito population, and we let \( B_n \) be the number of the sterile mosquitoes released at generation \( n \). Because of the mechanism to produce sterile male mosquitoes, the sterile mosquitoes do not reproduce and there is no maturation process from larvae to adults for sterile mosquitoes. Hence, the number of sterile mosquitoes at generation \( n \) is just the number of releases, and the size of the total mosquitoes will be \( w_n + B_n \) at generation \( n \). After the sterile mosquitoes are released, the mating interaction between the wild and sterile mosquitoes takes place. Similarly as in the homogeneous population models in [44, 46], we assume harmonic means for matings, that is to say, the number of wild offspring produced per mating by wild mosquitoes is

\[ \frac{A(w_n)w_n}{w_n + B_n}. \]

Then, our model basis for interaction with sterile mosquitoes can be described as

\[ w_{n+1} = C(w_n)\frac{A(w_n)w_n}{w_n + B_n}w_ns(w_n) \] (1.3)

without generation overlaps, or

\[ w_{n+1} = C(w_n)\frac{A(w_n)w_n}{w_n + B_n}w_ns(w_n) + pw_n \] (1.4)

9
with generation overlaps.

In the models in [49] with discrete-time and non-overlapping generations, the survival probability was assumed to be of a Ricker-type of nonlinearity

\[ s(w_n) = e^{-d - kw_n}, \]

where \( \exp(-d) \) is the density-independent survival probability and \( k \) is the intraspecific competition coefficient or carrying capacity parameter [25, 38, 44, 54–56, 64]. Since there is no density limitation when adult mosquitoes reproduce offsprings, the number of offspring per mating was assumed to constant, i.e. \( A(w_n) = a. \)

For mosquitoes, the interspecific competition and predation are rather rare events and could be discounted as major causes of larval mortality. Then intraspecific competition could represent a major density-dependent source for the population dynamics, and hence the crowding could be an significant factor in the population dynamics of mosquitoes [29, 36, 63]. Note that the intraspecific competition mainly takes place within the aquatic stages of mosquitoes because of resource and space limitations among larvae and pupa. It was assumed that the density dependence is based on larvae not adult mosquitoes, which means the survival probability of offspring, \( s(w_n) \), depends only on the wild mosquitoes and is independent of the released sterile mosquitoes [13, 14, 46].

In the case where there is no difficulty in finding mates for both of wild and sterile male mosquitoes at low population densities, the number of matings per individual is counted as a constant \( C(w_n) = c. \) By re-labeling the product \( ca \exp(-d) \) as
$a$, which is the intrinsic population growth rate, the interactive dynamics of the wild and sterile mosquitoes satisfy the following equation

$$w_{n+1} = \frac{aw_n}{w_n + B_n} w_n e^{-kw_n}, \quad (1.5)$$

which is the case without the so-called Allee effects [2,27,58,66,67].

If there exists difficulty in finding mates in small population sizes, we assume Allee effects and that the number of matings per individual satisfies $C(N_n) = c_0 N_n / (1 + N_n)$, where $c_0$ is the maximum number of matings and $N_n = w_n + B_n$ is the total number of mosquitoes. Then the mating rate is

$$C = \frac{c_0 (w_n + B_n)}{1 + w_n + B_n}.$$

By combining the coefficients as before, the model equation becomes

$$w_{n+1} = \frac{aw_n}{1 + w_n + B_n} w_n e^{-kw_n}. \quad (1.6)$$

Three releasing strategies of sterile mosquitoes were considered in [49]. They are constant releases with $B_n := b$, releases proportional to the number of wild mosquitoes with $B_n := bw_n$, and proportional releases with saturation where the releasing rate is $B_n := bw_n / (1 + w_n)$. In all of the three cases, the parameter $b$ is a positive constant.

For all of the three strategies, it was determined that there is a threshold value $b_c$ for the existence of positive fixed points such that when the release parameter $b$
Figure 1.3: Bifurcation diagram for Constant release

exceeds this existence threshold, no positive fixed points exist and hence the wild mosquito population goes to extinct, which finally control mosquito-borne diseases. If the release parameter, on the other hand, is less than but near the threshold value, a positive fixed point exists. Then another threshold value \( b_s \) is derived for the stability of the positive fixed points such that if the release parameter is greater than but near this stability threshold \( (b > b_s) \), the positive fixed point is stable and if the release parameter is less than this stability threshold \( (b < b_s) \), the positive fixed point becomes unstable and period-doubling bifurcations occur as the release parameter decreases. The schematic diagram is given in Figure 1.3 for constant release case.

Notice that this is consistent with the dynamical complexity of discrete Ricker population models, but the complicated features in [49] may not necessarily be induced by the interactions of the wild and sterile mosquitoes rather from the nature
of the Ricker-type itself. More importantly, our fundamental goal nevertheless is eventually to incorporate the mosquito models into malaria transmission models [48]. Hence the objectives in this dissertation is to formulate new models for interactive wild and sterile mosquitoes so that the dynamics are relatively simpler and mathematically more tractable, but the fundamental model features are snatched. In particular, there seem no clear evidence in the literature or existing data supporting such complexity.

Thus, we first formulate new discrete-time models for the interactive wild and sterile mosquitoes and investigate their dynamics in this dissertation. We then incorporate them into disease transmissions models [48]. To this end, we assume that the nonlinear survival function is not exponential but fractional so that the models become of Beverton-Holt-type [18,19,25]. We also consider the cases without or with Allee effects.

This dissertation is organized as follows. We formulate our models with the assumption that there are no generation overlaps in the mosquito population in Chapter 2. We give complete analysis for the existence and stability of all fixed points. The models based on the assumption of overlapped generations are considered in Chapter 3. Detailed analysis is also carried out.

Since mosquitoes undergo complete metamorphosis, going through four distinct stages of development during a lifetime: egg, pupa, larva, and adults. To better understand the interactive dynamics, predict and control mosquito-borne diseases, inclusion of the metamorphic stages in our models and more detailed investigations on the mosquito dynamics based on those models are apparently needed. Thus we con-
sider stage-structured interactive models without generation overlaps in Chapter 4, and the models with generation overlaps in Chapter 5.

To incorporate the mosquito models into the models for the malaria transmission, which are of high dimensions, we consider to formulate a susceptible-exposed-infective-recovered (SEIR) compartmental discrete-time model for humans and a susceptible-exposed-infective (SEI) compartmental discrete-time model for interactive of wild and sterile mosquitoes based on [48]. We derive a formula for the reproductive number $R_0$ for the infection and study the existence of fixed points and their stability in Chapter 6. Numerical simulations to verify our theoretical results and brief discussions of our findings and future study are also provided.
CHAPTER 2

BEVERTON-HOLT MODELS WITH STERILE MOSQUITOES
WITHOUT OVERLAPS

2.1 Wild Mosquito Model without Allee Effects

We study, in this section, the interactive dynamics of wild and sterile mosquitoes where we assume no mating difficulty for mosquitoes so that no Allee effects are concerned.

We first formulate models for wild mosquitoes, in the absence of sterile mosquitoes, according to equation (1.1) in the section of Mathematical Modeling Basis in Chapter 1 such that the population dynamics are governed by the following equation

\[ w_{n+1} = C(w_n)A(w_n)w_n s(w_n), \]

where \( w_n \) is the size of the wild mosquito population at generation \( n \), \( C(w_n) \) the number of matings per individual (mating rate), \( A(w_n) \) the number of offspring produced per mating and \( s(w_n) \) the survival probability of the offspring produced by wild adult mosquitoes [54–56]. We further assume that the number of matings per individual and the number of offspring produced per mating are constants and combine
these two constants as \( C(w_n)A(w_n) := a \). The survival probability function \( s(w_n) \) of the offspring produced by wild adult mosquitoes \([54–56]\) is assumed to have the Beverton-Holt-type of nonlinearity as

\[
s(w_n) = \frac{k}{1 + \eta w_n},
\]

where \( 0 \leq k \leq 1 \) is the maximum survival probability, and \( \eta \geq 0 \) is the density-dependant factor \([18,19,25]\). Then the model becomes

\[
w_{n+1} = aw_n \frac{k}{1 + \eta w_n} = \frac{r_0 w_n}{1 + \eta w_n}, \tag{2.1.1}
\]

where \( r_0 := ak \) is the population intrinsic growth rate. Clearly, if \( w_0 = 0, w_n = 0 \) for all \( n \geq 1 \) such that \( w = 0 \) is a trivial fixed point, and if \( w_0 > 0, w_n > 0 \) for all \( n \geq 1 \).

We now investigate the existence of fixed points and stability analysis of these fixed points.

**Theorem 2.1** For model (2.1.1), the trivial fixed point \( w = 0 \) is globally asymptotically stable if \( r_0 \leq 1 \) and is unstable if \( r_0 > 1 \). There exists a unique positive fixed point given by \( w^* = \frac{r_0 - 1}{\eta} \) if \( r_0 > 1 \) and this unique positive fixed point is globally asymptotically stable. There exist no periodic cycles.

**Proof.** Define function

\[
F(w) := \frac{r_0 w}{1 + \eta w}.
\]

It follows from \( F'(0) = r_0 \) that \( w = 0 \) is locally asymptotically stable if \( r_0 < 1 \) and unstable if \( r_0 > 1 \). Moreover, for any solution of (2.1.1) with positive initial value
if \( w_0 > 0 \), it follows from

\[
w_{n+1} < r_0 w_n < r_0^2 w_{n-1} < ... < r_0^{n+1} w_0
\]

that the trivial fixed point \( w = 0 \) is globally asymptotically stable if \( r_0 < 1 \).

If \( r_0 = 1 \), then it follows from

\[
w_{n+1} - w_n = w_n \left( \frac{1}{1 + \eta w_n} - 1 \right) < 0
\]

that sequence \( \{w_n\} \) is monotone decreasing and thus \( \lim_{n \to \infty} w_n := u \) exists. Suppose \( u > 0 \). Letting \( n \to \infty \) in (2.1.1), we have

\[
u = \frac{u}{1 + \eta u},
\]

which leads to \( 1 + \eta u = 1 \), a contradiction. Thus \( u = 0 \) and the trivial fixed point \( w = 0 \) is globally asymptotically stable.

We then assume \( r_0 > 1 \) and \( w \) is a positive fixed point of (2.1.1). Then \( w \)

must satisfy the equation

\[
w = \frac{r_0 w}{1 + \eta w}.
\]

(2.1.2)

Clearly, equation (2.1.2) has a unique solution \( w^* = \frac{r_0 - 1}{\eta} \). It then follows from

\[
F'(w^*) = \frac{r_0}{(1 + \eta w^*)^2} = \frac{1}{r_0} < 1
\]
that the unique positive fixed point \( w^* \) is locally asymptotically stable.

Next we prove the global stability of \( w^* \).

Note that

\[
\begin{align*}
  w_{n+1} - w_n &= \frac{r_0 w_n}{1 + \eta w_n} - w_n = \frac{r_0 - 1 - \eta w_n}{1 + \eta w_n} \cdot \frac{w_n}{w_n} \\
  &= (w^* - w_n) \frac{\eta w_n}{1 + \eta w_n} = (w^* - w_n) g_n,
\end{align*}
\]

where \( g_n := \frac{\eta w_n}{1 + \eta w_n} < 1 \), for all \( n \geq 0 \). Then it follows that

\[
  w_{n+1} - w^* = w_{n+1} - w_n + w_n - w^* = (w_n - w^*) (1 - g_n). \tag{2.1.4}
\]

Hence, if \( w_n < w^* \), \( w_{n+1} < w^* \) and if \( w_n > w^* \), \( w_{n+1} > w^* \).

Furthermore, for initial value \( w_0 \) with \( w_0 < w^* \), it follows from (2.1.3) that \( w_1 > w_0 \). Repeating this process, we have \( w_{n+1} > w_n \) for all \( n \geq 0 \). Thus, we have an increasing sequence \( \{w_n\} \) satisfying \( w_{n+1} < w^* \) for all \( n \geq 1 \), if \( w_0 < w^* \).

Similarly, for initial value \( w_0 \) with \( w_0 > w^* \), we have a decreasing sequence \( \{w_n\} \) satisfying \( w_{n+1} > w^* \), for all \( n \geq 1 \). Thus \( \lim_{n \to \infty} w_n \) exists for any initial value \( w_0 \). By taking limits in (2.1.1), it follows that \( \lim_{n \to \infty} w_n = w^* \).

To investigate whether there exist periodic 2-cycles for \( r_0 > 1 \), we assume that there is a positive solution, \( \bar{w} \), satisfying \( F(F(\bar{w})) = \bar{w} \) and \( \bar{w} \neq w^* \), which implies

\[
\bar{w} = \frac{r_0 \bar{w}}{1 + \eta \bar{w}} = \frac{r_0^2 \bar{w}}{1 + \eta (1 + r_0) \bar{w}}. \tag{2.1.5}
\]
Solving equation (2.1.5), we have \( \bar{w} = \frac{r_0 - 1}{\eta} = w^* \). Thus there exist no periodic 2-cycles.

Moreover, in addition to (2.1.5), if \( w_{n+t} = \frac{r_0^t w_n}{1 + \eta(1 + r_0 + r_0^2 + \ldots + r_0^{t-1})w_n} \),
then
\[
w_{n+t+1} = \frac{r_0 w_{n+t}}{1 + \eta w_{n+t}} = \frac{r_0^{t+1} w_n}{1 + \eta(1 + r_0 + r_0^2 + \ldots + r_0^{t})w_n}.
\]

Thus, by mathematical induction, for all \( n \geq 1 \), we have
\[
w_{n+k} = \frac{r_0^k w_n}{1 + \eta(1 + r_0 + r_0^2 + \ldots + r_0^{k-1})w_n}.
\] (2.1.6)

Hence if there exists a periodic \( k \)-cycle \( w_{n+k} = w_n \), then it follows from (2.1.6) that
\[
w_n = \frac{r_0^k - 1}{\eta(1 + r_0 + r_0^2 + \ldots + r_0^{k-1})} = \frac{r_0 - 1}{\eta} = \bar{w}.
\]

Therefore, there exist no any periodic \( k \)-cycles with \( k \geq 2 \), except the unique positive fixed point \( w^* = \frac{r_0 - 1}{\eta} \), which is globally asymptotically stable. Then Theorem 2.1 is proven.

### 2.2 Wild Mosquito Model with Allee Effects

Considering possible difficulties for mosquitoes finding mates, we assume Allee effects \([2,27,67]\) and thus the following model equation
\[
w_{n+1} = \frac{aw_n}{1 + w_n} \left( 1 + \eta w_n \right)^{k} = \frac{r_0 w_n^2}{(1 + w_n)(1 + \eta w_n)}. \] (2.2.1)
Clearly, if $w_0 = 0$, $w_n = 0$ and if $w_0 > 0$, $w_n > 0$ for all $n \geq 1$. Thus $w = 0$ is a fixed point.

We then determine the existence of positive fixed points $w$ of equation (2.2.1), which corresponds to the positive roots of the following equation

$$P(w) := (1 + w)(1 + \eta w) - r_0 w = \eta w^2 + (1 + \eta - r_0)w + 1 = 0. \quad (2.2.2)$$

If $r_0 \leq 1 + \eta$, $P(w) > 0$ for all $w \geq 0$, and thus $P(w) = 0$ has no positive roots. Assume $r_0 > 1 + \eta$. The roots of $P(w) = 0$ are

$$w^\pm = \frac{r_0 - (1 + \eta) \pm \sqrt{(r_0 - (1 + \eta))^2 - 4\eta}}{2\eta}. \quad (2.2.3)$$

Let

$$\Delta := (r_0 - (1 + \eta))^2 - 4\eta = r_0^2 - 2r_0(1 + \eta) + (1 - \eta)^2$$

and define

$$r_c := (1 + \sqrt{\eta})^2. \quad (2.2.4)$$

Then

$$\Delta = (r_0 - (1 + \eta - 2\sqrt{\eta}))(r_0 - (1 + \eta + 2\sqrt{\eta}))$$

$$= (r_0 - (1 - \sqrt{\eta})^2)(r_0 - (1 + \sqrt{\eta})^2).$$

Since

$$(1 - \sqrt{\eta})^2 < 1 + \eta < (1 + \sqrt{\eta})^2,$$
equation $P(w)$ has no positive solution, one positive solution

$$\bar{w} = \frac{r_0 - (1 + \eta)}{2\eta}, \quad (2.2.5)$$

or two positive solutions given in (2.2.3), if $r_0 < r_c$, $r_0 = r_c$, $r_0 > r_c$, respectively.

We next investigate the stability of the fixed points.

**Theorem 2.2** For model (2.2.1) with Allee effects, the trivial fixed point $w = 0$ is globally asymptotically stable if $r_0 \leq r_c$ and locally asymptotically stable with basin of attraction $(0, w^-)$ if $r_0 > r_c$, where $w^-$ is the smaller positive root of $P(w) = 0$ given in (2.2.2). There exists a unique positive fixed point given by $\bar{w} = \frac{r_0 - (1 + \eta)}{2\eta}$ if $r_0 = r_c$ and this unique fixed point is unstable. If $r_0 > r_c$, there exist two positive fixed points $w^\pm$ given in (2.2.3). Fixed point $w^-$ is unstable and $w^+$ is locally stable with basin of attraction $(w^-, \infty)$.

**Proof.** Define function $G(w) := \frac{r_0w^2}{(1 + w)(1 + \eta w)}$. Then

$$G'(w) = \frac{r_0w((1 + \eta)w + 2)}{(1 + w)^2(1 + \eta w)^2}.$$ 

Clearly, the trivial fixed point $w = 0$ is always locally asymptotically stable.

We then consider the case of $r_0 < r_c$ such that there exist no positive fixed points, and $r_0 < 1 + \eta < r_c$ or $1 + \eta \leq r_0 < r_c$, respectively.

For $r_0 < 1 + \eta < r_c$, it follows from

$$w_{n+1} = \frac{r_0w_n}{(1 + w_n)(1 + \eta w_n)}w_n = \frac{r_0w_n}{\eta w_n^2 + (1 + \eta)w_n + 1}w_n$$
that for any \( w_0 > 0 \),

\[
w_{n+1} < \frac{r_0}{1 + \eta} w_n < ... < \left( \frac{r_0}{1 + \eta} \right)^{n+1} w_0.
\]

(2.2.6)

Thus the trivial fixed point is globally asymptotically stable.

For \( 1 + \eta \leq r_0 < r_c \), The quadratic function \( P(w_n) \) in (2.2.2) is positive for all \( w_n \). Then it follows from

\[
w_{n+1} - w_n = \left( \frac{r_0 w_n}{\eta w_n^2 + (1 + \eta) w_n + 1} - 1 \right) w_n
\]

(2.2.7)

that \( w_{n+1} < w_n \) for all \( n \geq 0 \), which implies that \( \{w_n\} \) is a decreasing sequence. Thus \( \lim_{n \to \infty} w_n \) exists for any initial value \( w_0 \). By taking limits in (2.2.1), it follows that \( \lim_{n \to \infty} w_n = 0 \); that is, the trivial fixed point \( w = 0 \) is globally asymptotically stable.

For \( r_0 = r_c \), there exists a unique positive fixed point \( \bar{w} := \frac{r_0 - (1 + \eta)}{2\eta} \).

Simple calculation yields

\[
G''(w) = \frac{2r_0((\eta + 1)w + 1)(1 + w)(1 + \eta w) - 2r_0 w((\eta + 1)w + 2)(2\eta w + \eta + 1)}{(1 + w)^3(1 + \eta w)^3}.
\]

(2.2.8)

At this unique positive fixed point \( \bar{w} \), it follows from \( r_0 \bar{w} = (1 + \bar{w})(1 + \eta \bar{w}) \) that

\[
G''(\bar{w}) = \frac{2r_0^2 \bar{w}((\eta + 1)\bar{w} + 1) - 2r_0^2 \bar{w}((\eta + 1)\bar{w} + 2)}{(r_0 \bar{w})^3}
\]

\[
= \frac{-2}{r_0 \bar{w}^2} < 0.
\]

(2.2.9)
Hence \( \tilde{w} \) is unstable [30,31].

Meanwhile, since \( P(w_n) > 0 \) for all \( w_n \neq \tilde{w} \), it follows from (2.2.7) that \( \{w_n\} \) is a decreasing sequence. By taking limits in (2.2.1), \( \lim_{n \to \infty} w_n \) exists. However, since \( \tilde{w} \) is the only positive fixed point and is unstable, the limit must be zero. Thus the trivial fixed point \( w = 0 \) is globally asymptotically stable.

We next assume \( r_0 > r_c > 1 + \eta \) such that there exist two positive fixed points given in (2.2.3).

At a positive fixed point \( w \), it follows from \( \frac{r_0 w}{(1 + w)(1 + \eta w)} = 1 \) that

\[
G'(w) = \frac{(1 + \eta)w + 2}{(1 + w)(1 + \eta w)} = \frac{(1 + \eta)w + 2}{r_0 w}.
\]

Hence, a positive fixed point \( w \) is locally asymptotically stable if \( (r_0 - (1+\eta))w - 2 > 0 \), and is unstable if \( (r_0 - (1+\eta))w - 2 < 0 \). Moreover, it follows from (2.2.2) that, at a fixed point, \((r_0 - (1 + \eta))w - 2 = \eta w^2 - 1\). Thus, a positive fixed point \( w \) is locally asymptotically stable if \( \eta w^2 - 1 > 0 \), and is unstable if \( \eta w^2 - 1 < 0 \).

Consider

\[
4\eta(\eta w^2 - 1) = 4\eta^2 w^2 - 4\eta = (2\eta w + 2\sqrt{\eta})(2\eta w - 2\sqrt{\eta})
\]

and let \( H(w) := 2\eta w - 2\sqrt{\eta} \). Then, equivalently, the positive fixed point \( w \) is locally asymptotically stable if \( H(w) > 0 \), and is unstable if \( H(w) < 0 \).
Write $\Sigma := r_0 - (1 + \eta)$. The two positive fixed points in (2.2.3) can be expressed as
\[
 w^\pm = \frac{\Sigma \pm \sqrt{\Sigma^2 - 4\eta}}{2\eta},
\] (2.2.10)
and thus
\[
 H(w^+) = \Sigma \pm \sqrt{\Sigma^2 - 4\eta} - 2\sqrt{\eta} = \Sigma - 2\sqrt{\eta} \pm \sqrt{(\Sigma + 2\sqrt{\eta})(\Sigma - 2\sqrt{\eta})}
 = \sqrt{\Sigma - 2\sqrt{\eta}} \left( \sqrt{\Sigma - 2\sqrt{\eta}} \pm \sqrt{\Sigma + 2\sqrt{\eta}} \right).
\] (2.2.11)

Then
\[
 \begin{cases}
 H(w^+) > 0, \\
 H(w^-) < 0,
\end{cases}
\]
and hence fixed point $w^+$ is locally asymptotically stable and $w^-$ is unstable.

Furthermore, it follows from
\[
 w_{n+1} - w_n = -\frac{P(w_n)}{P(w_n) + r_0 w_n} w_n = -\frac{P(w_n)}{\eta w_n^2 + (1 + \eta)w_n + 1} w_n,
\]
where $\eta w_n^2 + (1 + \eta)w_n + 1 > 0$ for all $w_n > 0$, that
\[
 \begin{cases}
 P(w_n) > 0, & \text{if } w_n \in (0, w^-), \\
 P(w_n) < 0, & \text{if } w_n \in (w^-, w^+), \\
 P(w_n) > 0, & \text{if } w_n \in (w^+, \infty),
\end{cases}
\]
which implies that

\[
\begin{cases}
\{w_n\} \text{ is decreasing, if } w_n \in (0, w^-), \\
\{w_n\} \text{ is increasing, if } w_n \in (w^-, w^+), \\
\{w_n\} \text{ is decreasing, if } w_n \in (w^+, \infty).
\end{cases}
\]

Then, by taking limits in (2.2.1), it follows that

\[
\begin{cases}
\lim_{n \to \infty} w_n = 0, \quad \text{if } w_n \in (0, w^-), \\
\lim_{n \to \infty} w_n = w^+, \quad \text{if } w_n \in (w^-, w^+), \\
\lim_{n \to \infty} w_n = w^+, \quad \text{if } w_n \in (w^+, \infty).
\end{cases}
\]

The theorem 2.2 is proven.

We further show that there exist no periodic positive cycles for equation (2.2.1) as follows.

We first show that there exist no periodic 2-cycles satisfying \( F(F(w_n)) = w_{n+2} = w_n \) and \( w_n \neq w_{n+1} \).

It follows from (2.2.1) that

\[
w_n = w_{n+2} = \frac{w_n^2}{\frac{r_0 w_n^2}{(1 + w_n)(1 + \eta w_n)} \left(1 + \eta \frac{r_0 w_n^2}{(1 + w_n)(1 + \eta w_n)}\right)} = \frac{r_0^2 w_n^4}{((1 + w_n)(1 + \eta w_n) + r_0 w_n^2)((1 + w_n)(1 + \eta w_n) + \eta r_0 w_n^2)}.
\]
Thus, by simple calculation, any positive 2-cycle equivalently satisfies

\[ H(w) := \eta(\eta + r_0)w^4 + Aw^3 + Bw^2 + 2(1 + \eta)w + 1 = 0. \] (2.2.13)

where \( A = (1 + \eta)(2\eta + r_0 + \eta r_0) - r_0^2 \), \( B = \eta^2 + 4\eta + \eta r_0 + r_0 + 1 > 0 \). If \( H(w) = 0 \) has a positive solution satisfying \( P(w) \neq 0 \), then (2.2.1) has a 2-cycle.

It is clear that if \( A > 0 \), all the coefficients of (2.2.13) are positive, and thus \( H(w) > 0 \) for all \( w \geq 0 \). If \( A < 0 \), from Descartes’ rule of signs, it follows that equation (2.2.13) has at most 2 positive solutions. However, since \( H(w^\pm) = 0 \) and \( w^\pm > 0 \), there are no other positive solutions satisfying \( H(w) = 0 \). That is, there exist no periodic 2-cycles for (2.2.13).

We next show that there exist no other periodic \( k \)-cycles with \( k > 2 \). Assume \( w_{n+k} = w_n := w^* \) and \( P(w^*) \neq 0 \), that is, \( G^{(k)}(w^*) = w^* \) and \( w^* \neq w^\pm \). Since \( G'(w) > 0 \) for all \( w > 0 \), if \( G(w^*) > w^* \), then \( G(G(w^*)) > G(w^*) > w^* \). By iteration, \( G^{(k)}(w^*) > G^{(k-1)}(w^*) > \cdots > w^* \). On the other hand, if \( G(w^*) < w^* \), then \( G(G(w^*)) < G(w^*) < w^* \), and again by iteration, \( G^{(k)}(w^*) < G^{(k-1)}(w^*) < \cdots < w^* \). Therefore, \( G^{(k)}(w^*) \neq w^* \), which is a contradiction.

Therefore, there exist no periodic \( k \)-cycles for \( k \geq 2 \).

### 2.3 Interactive Model with Wild and Sterile Mosquitoes

After building the model basis for our study, we suppose sterile mosquitoes are released into a field with wild mosquitoes and we let \( B_n \) be the number of the sterile mosquitoes released at generation \( n \). Since sterile mosquitoes do not reproduce,
there is no maturation process from larvae to adults for sterile mosquitoes. Hence the
number of sterile mosquitoes at $n$ is just the number of released sterile mosquitoes, and
the size of total mosquitoes is $w_n + B_n$ at generation $n$. After the sterile mosquitoes
are released, the mating interaction between the wild and sterile mosquitoes takes
place. Similarly as in [44, 47, 49], we assume harmonic means for matings, that is to
say, the number of wild offspring produced per mating by wild mosquitoes is

$$\frac{aw_n}{w_n + B_n}.$$ 

We assume that the complete life cycle of the mosquito occurs within one
time unit of our model. While interspecific competition and predation are rather
rare events and could be discounted as major causes of larval mortality, intraspecific
competition could represent a major density dependent source for the population
dynamics, and hence the effect of crowding could be an important factor in the popu-
lation dynamics of mosquitoes [29,36,63]. We note that the intraspecific competition
mainly takes place within the aquatic stages of mosquitoes and is due to resource
and space limitations among larvae and pupa. For this reason we assume that the
density dependence is based only on larvae not adult numbers. Thus, we assume
in our model that the probability of survival probability depends only on the wild
mosquitoes and is independent of the released sterile mosquitoes [13,14,46]. Instead
of the Ricker-type of nonlinearity for mosquitoes survivability assumed in [47, 49],
we let the mosquitoes survivability be of Beverton-Holt-type. Thus, if no difficulty
in finding mates at low population densities exists, based on (2.1.1) the interactive
dynamics of the wild and sterile mosquitoes satisfy the equation

\[ w_{n+1} = \frac{aw_n}{w_n + B_n \frac{k}{1 + \eta w_n}}. \]  

(2.3.1)

If there exists difficulty in finding mates, based on (2.2.1) the interactive dynamics are described by the equation

\[ w_{n+1} = \frac{aw_n}{1 + w_n + B_n \frac{k}{1 + \eta w_n}}. \]  

(2.3.2)

We then consider three different strategies of sterile mosquitoes releases.

2.4 Interactive Model with Constant Release

Assume sterile mosquitoes are constantly released in each generation so that

\[ B_n := b > 0 \] is a positive constant. Then the model equation becomes

\[ w_{n+1} = \frac{aw_n}{w_n + b \frac{k}{1 + \eta w_n}} = \frac{r_0 w_n^2}{(b + w_n)(1 + \eta w_n)}, \]  

(2.4.1)

where \( r_0 = ak \).

Let \( w_n := by_n \) and \( \eta_1 = b\eta \). Equation (2.4.1) can be written as

\[ y_{n+1} = \frac{ay_n}{1 + y_n \frac{k}{1 + \eta_1 y_n}} = \frac{r_0 y_n^2}{(1 + y_n)(1 + \eta_1 y_n)}. \]
It follows from Theorem 2.2 that there exists an existence threshold for positive fixed points as
\[ \hat{r}_c := (1 + \sqrt{\eta_1})^2 = \left(1 + \sqrt{b\eta}ight)^2 \]
such that there exist no, one, or two positive fixed points if \( r_0 < \hat{r}_c \), \( r_0 = \hat{r}_c \), \( r_0 > \hat{r}_c \), respectively.

For \( r_0 > \hat{r}_c \), we have \( r_0 > (1 + \sqrt{b\eta})^2 \), that is,
\[ b < \frac{(\sqrt{r_0} - 1)^2}{\eta}. \]

For given \( r_0 \) and \( \eta \), we define the threshold value for releases of sterile mosquitoes
\[ b_c := \frac{(\sqrt{r_0} - 1)^2}{\eta}. \quad (2.4.2) \]

Then model (2.4.1) has no positive solution, one positive solution
\[ w_c = \frac{r_0 - (1 + b\eta)}{2\eta}, \quad (2.4.3) \]
or two positive fixed points
\[ w_c^\pm = \frac{r_0 - (1 + b\eta) \pm \sqrt{(r_0 - (1 + b\eta))^2 - 4b\eta}}{2\eta}, \quad (2.4.4) \]
if \( b > b_c \), \( b = b_c \), or \( b < b_c \), respectively.
For the stability of the fixed points of Equation (2.4.1), it follows from Theorem 2.2 that if \( r_0 \leq \hat{r}_c \), that is, \( b \geq b_c \), the trivial fixed point \( w = 0 \) is globally asymptotically stable; if \( r_0 > \hat{r}_c \), that is, \( b < b_c \), there are two positive fixed points and \( w_c^- \) is unstable while \( w_c^+ \) is locally asymptotically stable with basin of attraction \((w^-, \infty)\).

These results can be summarized as follows.

**Theorem 2.4** For given \( r_0 \) and \( \eta \), there is a threshold value for releases of sterile mosquitoes \( b_c \) defined in (2.4.2). If the number of releases is greater than the threshold, that is, \( b > b_c \), there exists no positive fixed point and \( w = 0 \) is the only fixed point which is globally asymptotically stable. Thus the wild mosquito population goes extinct regardless of the initial population size. If \( b = b_c \), there exists a unique positive fixed point \( w_c \) given in (2.4.3) which is unstable, and the trivial fixed point \( w = 0 \) is globally asymptotically stable. Thus the wild mosquito population again goes extinct regardless of the initial population size. If \( b < b_c \), there exist two positive fixed points \( w_c^\pm \) given in (2.4.4), where \( w_c^- \) is unstable and \( w_c^+ \) is locally asymptotically stable with a basin of attraction \((w_c^-, \infty)\) while the trivial fixed point \( w = 0 \) is locally asymptotically stable with basin of attraction \((0, w_c^-)\). Solutions approach either \( w = 0 \) or \( w = w_c^+ \) depending on their initial values. That is to say that the two types of mosquitoes can either go extinct or coexisting, depending on their initial sizes.

We give an example to demonstrate the results for model (2.4.1) as follows.

**Example 2.3.1** In this example, we have parameters given as

\[
a = 6, \quad k = 0.5, \quad \eta = 0.4
\]  

(2.4.5)
Figure 2.1: For $b = 2 > b_c = 1.339745$, there exists no positive fixed point and the trivial fixed point $w = 0$ is globally asymptotically stable.

such that $r_0 = ak = 3$ and the release threshold is

$$b_c = \frac{(\sqrt{r_0} - 1)^2}{\eta} = 1.339745.$$  

For $b = 2 > b_c$, there exists no positive fixed point and the trivial fixed point $w = 0$ is globally asymptotically stable as shown in Figure 2.1. For $b = 1 < b_c$, there exist two positive fixed point

$$w_c^- = 0.7752556, \quad w_c^+ = 3.22474.$$  

Positive fixed point $w^-$ is unstable; the trivial fixed point $w = 0$ and $w^+$ are both locally asymptotically stable. Solutions approach either $w = 0$ or $w^+$ depending on their initial values as shown in Figure 2.2.
**Figure 2.2:** For $b = 1 < b_c$, there are two positive fixed point $w^-_c = 0.7752556$, and $w^+_c = 3.22474$. Fixed point $w^-_c$ is unstable and $w^+$ is locally asymptotically stable. Solutions approach $w = 0$ or $w^+_c$ depending on their initial values.

Furthermore, by straight calculation, we have

$$
\frac{\partial w^\pm}{\partial b} = \frac{1}{2\eta} \left( -\eta \pm \frac{-2\eta(r_0 - (1 + b\eta)) - 4\eta}{2\sqrt{(r_0 - (1 + b\eta))^2 - 4b\eta}} \right)
$$

$$
= -\frac{1}{2\sqrt{(r_0 - (1 + b\eta))^2 - 4b\eta}} \left( \sqrt{(r_0 - (1 + b\eta))^2 - 4b\eta \mp (r_0 + 1 - b\eta)} \right).
$$

Since $b < b_c < \frac{r_0 + 1}{\eta}$, $\frac{\partial w^+}{\partial b} < 0$, and

$$
\frac{\partial w^-}{\partial b} = -\frac{((r_0 - (1 + b\eta))^2 - 4b\eta - (r_0 - (1 + b\eta))^2)}{2\sqrt{(r_0 - (1 + b\eta))^2 - 4b\eta} \left( \sqrt{(r_0 - (1 + b\eta))^2 - 4b\eta + (r_0 + 1 - b\eta)} \right)}
$$

$$
= -\frac{2b\eta}{\sqrt{(r_0 - (1 + b\eta))^2 - 4b\eta} \left( \sqrt{(r_0 - (1 + b\eta))^2 - 4b\eta + (r_0 + 1 - b\eta)} \right)} > 0.
$$

As $b$ increases, the basins of traction for $w^+_c$ and $w^-_c$ are decreasing and increasing for $0 < b < b_c$, respectively, until $b = b_c$ when $w^+_c$ and $w^-_c$ collide to become a unique
unstable fixed point. That is, as the amount of the releases of sterile mosquitoes is small, the basin of traction for $w_c^-$ is relatively small and the basin of traction for $w_c^+$ is relatively large, which means only a small range of initial sizes of wild mosquitoes can be wiped out whereas most of mosquito populations with initial sizes greater than the small $w_c^-$ survive. As the amount of the releases of sterile mosquitoes increases, nevertheless, the basin of traction for $w_c^-$ becomes larger and the range of initial sizes of wild mosquitoes leading to their extinction becomes bigger.

With the same parameters given in (2.4.5), it is shown in Figure 2.3 that, for $0 < b < b_c$, the curves for $w_c^+$ and $w_c^-$ are decreasing and increasing functions of $b$, respectively. The two curves collide at $b = b_c$ and positive fixed points no longer exist.
The trivial fixed point \( w = 0 \) becomes globally asymptotically stable for \( b > b_c \), and all wild mosquitoes are wiped out.

### 2.5 Interactive Model with Proportional Release

The constant release strategy studied in Section 2.4 seems appropriate in a field with a large wild mosquito population. However, if the wild mosquito population is relatively small, constantly releasing apparently wastes resources. To have a more economically effective strategy for releasing sterile mosquitoes in such a situation, instead of releasing sterile mosquitoes constantly in each generation, we keep the number of releases of sterile mosquitoes proportional to the population size of the wild mosquitoes [20, 49]; that is, \( B_n = bw_n \) where \( b > 0 \) is a constant. Since we include the case of a possible small mosquito population, Allee effects are assumed and the model becomes

\[
      w_{n+1} = \frac{aw_n}{1+(1+b)w_n} \frac{k}{1+\eta w_n} = \frac{r_0 w_n^2}{(1+(1+b)w_n)(1+\eta w_n)}. \tag{2.5.1}
\]

Here we assume, in the absence of sterile mosquitoes,

\[
r_0 > (1 + \sqrt{\eta})^2 \tag{2.5.2}
\]

such that the wild mosquito population goes to a steady state as long as their initial size satisfies \( w_0 > w^- \) where \( w^- \) is given in (2.2.3).
Write \( z_n := (1 + b)w_n \), \( R := \frac{r_0}{1 + b} \), and \( \eta_2 := \frac{\eta}{1 + b} \). Equation (2.5.1) is translated into

\[
z_{n+1} = \frac{R z_n}{1 + z_n} \cdot \frac{1}{1 + \eta_1 z_n} = \frac{R z_n^2}{(1 + z_n)(1 + \eta_2 z_n)}.
\]

It follows from Theorem 2.2 that there exists a threshold value for the existence of positive fixed points as

\[
R_c := (1 + \sqrt{\eta_2})^2 = \left(1 + \sqrt{\frac{\eta}{1 + b}}\right)^2
\]

such that there exist no, one, or two positive fixed points if \( R < R_c \), \( R = R_c \), or \( R > R_c \), respectively.

By simple algebra, it shows that \( R > R_c \), that is

\[
\frac{r_0}{1 + b} > \left(1 + \sqrt{\frac{\eta}{1 + b}}\right)^2,
\]

which is equivalent to

\[
 r_0 > (\sqrt{1 + b} + \sqrt{\eta})^2,
\]

or

\[
 b < (\sqrt{r_0} - \sqrt{\eta})^2 - 1.
\]

Define the threshold for the releases of sterile mosquitoes for model (2.5.1) as

\[
 b_p := (\sqrt{r_0} - \sqrt{\eta})^2 - 1,
\]  

(2.5.3)
which is positive based on assumption (2.5.2). Then model (2.5.1) has no positive fixed point, one positive fixed point

$$\hat{w}_p = \frac{r_0}{1+b} - \left(1 + \frac{\eta}{1+b}\right) = \frac{r_0 - (1 + b + \eta)}{2\eta(1+b)}$$

(2.5.4)
or two positive fixed points $w^-_p < \hat{w}_p < w^+_p$ where

$$w^\pm_p = \frac{r_0}{1+b} - \left(1 + \frac{\eta}{1+b}\right) \pm \sqrt{\left(\frac{r_0}{1+b} - \left(1 + \frac{\eta}{1+b}\right)\right)^2 - 4 \frac{\eta}{1+b}}$$

(2.5.5)

$$= \frac{r_0 - (1 + b + \eta) \pm \sqrt{(r_0 - (1 + b + \eta))^2 - 4(1+b)\eta}}{2\eta(1+b)},$$

if $b > b_p$, $b = b_p$, or $b < b_p$, respectively.

The stability analysis for the fixed points of equation (2.5.1) is similar to that for equation (2.4.1). Thus we omit it here.

In summary, we have the following results.

**Theorem 2.5** Define the threshold for the releases of sterile mosquitoes $b_p := (\sqrt{r_0} - \sqrt{\eta})^2 - 1$ for model (2.5.1). Then if $b > b_p$, there exists no positive point and the trivial fixed point $w = 0$ is globally asymptotically stable. If $b = b_p$ there exists a unique positive fixed point $\hat{w}$ given in (2.5.4) which is unstable and the trivial fixed point $w = 0$ is also globally asymptotically stable. Thus all mosquitoes are also eventually wiped out. If $b < b_p$, there exist two positive fixed points $w^\pm_p$ given in (2.5.5). Fixed point $w^-_p$ is unstable and $w^+_p$ is locally asymptotically stable with basin of attraction $(w^-_p, \infty)$ while the trivial fixed point is locally asymptotically stable with basin of attraction $(0, w^-_p)$. Solutions approach either $w = 0$ or $w = w^+_p$ depending on their initial
values. The two types of mosquitoes can either go extinct or coexisting, depending on their initial sizes.

2.6 Interactive Model with Proportional Release with Saturation

The strategy of releases proportional to the wild population size, compared to the constant releases, may have an advantage when the size of the wild mosquito population is small since the size of releases is also small. However, when the wild mosquito population size is large, the size of releases should presumably also be large, which may exceed the availability of limiting resources in many situations.

We then propose a new strategy in which the number of sterile mosquito releases is proportional to the wild mosquito population size when it is small, but saturates and approaches a constant when the wild mosquito population size increases. This is clearly a compromise between the two strategies we discussed in Sections 2.4 and 2.5. To this end, we assume the number of releases to be of the Holling-II type with the form of \( B(w) = bw/(1 + w) \) \([20, 39, 49]\). Then the model becomes

\[
\begin{align*}
\frac{w_{n+1}}{w_n} &= \frac{aw_n}{1 + w_n + B(w_n)w_n} \frac{k}{1 + \eta w_n} = \frac{aw_n(1 + w_n)}{(1 + w_n)^2 + bw_n} \frac{k}{1 + \eta w_n}, \quad (2.6.1)
\end{align*}
\]

Here we, as in Section 2.5, focus on the case where \( r_0 > r_c = (1 + \sqrt{\eta})^2 \).

The origin \( w = 0 \) is a trivial fixed point and is locally asymptotically stable. Positive fixed points satisfy the following equation

\[
1 = \frac{r_0 w (1 + w)}{(1 + w)^2 + bw} \frac{1}{1 + \eta w}.
\]
or, equivalently,

\[ L(w) := \frac{(1 + w)^2 + bw}{w} = \frac{(1 + w)^2}{w} + b = \frac{r_0(1 + w)}{1 + \eta w} := h(w). \]  \tag{2.6.2}

To solve for positive fixed points, we first have

\[ h'(w) = \frac{r_0(1 - \eta)}{(1 + \eta w)^2} \begin{cases} > 0, & \text{if } \eta < 1, \\ < 0, & \text{if } \eta > 1, \end{cases} \]  \tag{2.6.3}

and it follows from

\[ L'(w) = 1 - \frac{1}{w^2} \quad \text{and} \quad L''(w) = \frac{2}{w^3}, \]

for all \( w > 0 \), that \( L(w) \) has a minimum value at \( w = 1 \) and the curve is concave up.

Thus the two curves for \( L \) and \( h \), respectively, have two intersections, one intersection, or no intersection as \( b \) gradually increases.

We now determine the threshold value \( b_s \) for the releases of sterile mosquitoes as follows. With parameter \( b = b_s \), the two curves are tangent to each other at the critical point, denoted by \( w_s \). Thus they satisfy the two equations \( L = h \) and \( L' = h' \), that is,

\[ \frac{(1 + w_s)^2}{w_s} + b_s = \frac{r_0(1 + w_s)}{1 + \eta w_s}, \]  \tag{2.6.4a}

\[ 1 - \frac{1}{w_s^2} = \frac{r_0(1 - \eta)}{(1 + \eta w_s)^2}. \]  \tag{2.6.4b}
Solving for $w_s$ from (2.6.4b) and then substituting it into (2.6.4a), we can completely determine $b_s$. 

The critical value $w_s$ can be solved in (2.6.4b) implicitly. To ensure its existence, we first show equation (2.6.4b) has a unique positive solution. Set

$$K(w) := \frac{r_0(1 - \eta)}{(1 + \eta w_s)^2} + \frac{1}{w_s^2} - 1,$$

Then since $r_0 > (1 + \sqrt{\eta})^2$, simple calculation yields $K'(w) < 0$ if $\eta < 1$, or $K'(w) > 0$ if $\eta > 1$, for all $w > 0$. Thus $K(w)$ is a monotone decreasing or increasing function if $\eta < 1$ or $\eta > 1$, respectively, with $\lim_{w \to 0^+} K(w) = +\infty$, $\lim_{w \to \infty} K(w) = -1$, and equation (2.6.4b) has a unique positive solution $w_s$. With this positive solution $w_s$ from (2.6.4b), we obtain threshold value

$$b_s := \frac{r_0(1 + w_s)}{1 + \eta w_s} - \frac{(1 + w_s)^2}{w_s} = \frac{(w_s^2 - 1)(1 + \eta w_s)}{(1 - \eta)w_s^2} - \frac{(1 + w_s)^2}{w_s}$$

$$= \frac{(1 + w_s)^2}{w_s} \left( \frac{(w_s - 1)(1 + \eta w_s)}{(1 - \eta)w_s} - 1 \right)$$

$$= \frac{(1 + w_s)^2}{(1 - \eta)w_s^2} \left( (w_s - 1)(1 + \eta w_s) - (1 - \eta)w_s \right)$$

$$= \frac{(1 + w_s)^2}{(1 - \eta)w_s^2} \left( \eta w_s^2 - 1 \right).$$  \hspace{1cm} (2.6.5)

A schematic diagram to demonstrate the existence results is shown in Figure 2.4 and Figure 2.5.

We next investigate the stability of positive fixed points.

**Theorem 2.6** The trivial fixed point $w = 0$ for model (2.6.1) is always locally asymptotically stable. Assume $r_0 > (1 + \sqrt{\eta})^2$, and define the threshold for the releases of
**Figure 2.4:** This is to show the determination of the threshold value $b_s$ for the releases of sterile mosquitoes for model (2.6.1) when $\eta < 1$. The vertical axis is for $b$. The middle curve corresponds to $L$ with $b = b_s$ such that the two curves for $L$ and $h$ touch and are tangent to each other. The only intersection is the unique positive fixed point $w_s$. The upper course corresponds to $b > b_s$ such that there exists no intersection between the two curves and thus no positive fixed point, whereas the lower curve corresponds to $b < b_s$ such that there exist two intersections between the two curves and thus two fixed points, for $\eta < 1$.

**Figure 2.5:** This is to show the determination of the threshold value $b_s$ for the releases of sterile mosquitoes for model (2.6.1) when $\eta > 1$. The vertical axis is for $b$. The middle curve corresponds to $L$ with $b = b_s$ such that the two curves for $L$ and $h$ touch and are tangent to each other. The only intersection is the unique positive fixed point $w_s$. The upper course corresponds to $b > b_s$ such that there exists no intersection between the two curves and thus no positive fixed point, whereas the lower curve corresponds to $b < b_s$ such that there exist two intersections between the two curves and thus two fixed points, for $\eta > 1$. 

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sterile mosquitoes $b_s$ for model (2.6.1) in (2.6.5) in which $w_s$ is the unique positive
solution given in (2.6.4a) and (2.6.4b). Then model (2.6.1) has no, unique, or two
positive fixed points if $b > b_s$, $b = b_s$, or $b < b_s$, respectively. If $b > b_s$, there exists
no positive point and the trivial fixed point $w = 0$ is globally asymptotically stable.
All mosquitoes are eventually wiped out. If $b = b_s$ there exists a unique positive fixed
point which is unstable and the trivial fixed point $w = 0$ is globally asymptotically
stable. All mosquitoes are also eventually wiped out. If $b < b_s$, there exist two positive
fixed points $w_s^{(1)} < w_s^{(2)}$. Then $w_s^{(1)}$ is unstable and $w_s^{(2)}$ is locally asymptotically
stable while the trivial fixed point is also locally asymptotically stable. Solutions ap-
proach either $w = 0$ or $w = w_s^{(2)}$ depending on their initial values. The two types of
mosquitoes can either go to extinct or coexist, depending on their initial sizes.

Proof. Assume $b < b_s$ such that there exist two positive fixed points. It follows from
(2.6.1) that these positive fixed points satisfy

$$1 = \frac{aw(1 + w)}{(1 + w)^2 + bw} \frac{k}{1 + \eta w}. $$

Set

$$ Q(w) := \eta w^3 + Aw^2 + Bw + 1 = 0 $$

(2.6.6)

where $A = 1 + 2\eta + b\eta - r_0$ and $B = 2 + \eta + b - r_0$. Then the positive fixed points
of (2.6.1) correspond to the positive roots of (2.6.6). We write the two positive fixed
points as $w_s^{(1)} < w_s^{(2)}$. It follows from $Q(0) = 1$ and $\lim \limits_{w \to \infty} = +\infty$ that $Q'(w_s^{(1)}) < 0$
and $Q'(w_s^{(2)}) > 0$. 

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Define
\[ T(w) := \frac{r_0w^2(1 + w)}{(1 + w)^2 + bw)(1 + \eta w)}. \]

Then
\[ T'(w) = \frac{r_0w(3w + 2)((1 + w)^2 + bw)(1 + \eta w) - r_0w^2(1 + w)(3\eta w^2 + 2Aw + 2r_0w + B + r_0)}{(1 + w)^2 + bw)^2(1 + \eta w)^2}, \]

and, evaluated at the fixed points,
\[
T'(w) = \frac{r_0w(3w + 2) - 3\eta w^2 + 2Aw + 2r_0w + B + r_0}{r_0(1 + w)}
\]
\[ = 1 + \frac{2r_0w + r_0 - (3\eta w^2 + 2Aw + 2r_0w + B + r_0)}{r_0(1 + w)} \]
\[ = 1 - \frac{3\eta w^2 + 2Aw + B}{r_0(1 + w)} \]
\[ = 1 - \frac{Q'(w)}{r_0(1 + w)}. \tag{2.6.7} \]

Thus, it immediately follows from \( Q'(w_s^{(1)}) < 0 \) that positive fixed point \( w_s^{(1)} \) is unstable.

It follows from \( Q'(w_s^{(2)}) > 0 \) that \( T'(w_s^{(2)}) < 1 \) and thus \( w_s^{(2)} \) is locally asymptotically locally stable if 
\[ \frac{Q'(w_s^{(2)})}{r_0(1 + w_s^{(2)})} < 2, \text{ that is,} \]
\[ Q'(w_s^{(2)}) < 2r_0 (1 + w_s^{(2)}). \tag{2.6.8} \]

Write \( w_s^{(2)} = w_s \) for convenience. Inequality (2.6.8) is equivalent to
\[ w_s (3\eta w^2 + 2Aw + B) < 2r_0w_s (1 + w_s) = 2r_0w_s + 2r_0w_s^2. \tag{2.6.9} \]
At a fixed point, the left hand side of (2.6.9) equals

\begin{equation}
3(\eta w_s^3 + Aw_s^2 + Bw_s + 1) - Aw_s^2 - 2Bw_s - 3 = -Aw_s^2 - 2Bw_s - 3
\end{equation}

\begin{equation}
= (r_0 - (1 + 2\eta + b\eta))w_s^2 + 2(r_0 - (2 + \eta + b))w_s - 3.
\end{equation}

By substituting (2.6.10) in (2.6.9), inequality (2.6.8) is equivalent to

\[-(1 + 2\eta + b\eta)w_s^2 - 2(2 + \eta + b)w_s - 3 < r_0w_s^2,\]

which holds for all $w_s > 0$. Thus, fixed point $w_s^{(2)}$ is locally asymptotically stable.

Theorem 2.6 is proven.

We provide Example 2.6.2 below to demonstrate the results for model (2.6.1).

Example 2.6.2 Given parameters

\begin{equation}
a = 6, \quad k = 0.5, \quad \eta = 0.4,
\end{equation}

we have $r_0 = 3$. The unique solution to (2.6.4a) is $w_s = 1.6775$ and thus the threshold value is $b_s = 0.5334$. We then choose $b = 1 > b_s$ and $b = 0.4 < b_s$, respectively. For $b = 1$, there exists no positive fixed point as shown in Figure 2.6. All solutions approach $w = 0$. For $b = 0.4$, $w = 0$ is locally asymptotically stable and there exist two positive fixed points $w_s^{(1)} = 1.1529$ and $w_s^{(2)} = 2.3642$. Fixed point $w_s^{(1)}$ is unstable and $w_s^{(2)}$ is locally asymptotically stable. Solutions approach either $w = 0$ or $w_s^{(2)}$, depending on their initial values, as shown in Figure 2.7.
Figure 2.6: For $b = 1 > b_s$, there exists no positive fixed point and the trivial fixed point $w = 0$ is globally asymptotically stable.

Figure 2.7: For $b = 0.4 < b_s$, there are two positive fixed point $w_s^{(1)} = 1.1529$ and $w_s^{(2)} = 2.3642$. Fixed point $w_s^{(1)}$ is unstable and $w_s^{(2)}$ is locally asymptotically stable. Solutions approach either $w = 0$ or $w_s^{(2)}$, depending on their initial values.
CHAPTER 3

BEVERTON-HOLT MODELS WITH STERILE MOSQUITOES WITH
GENERATION OVERLAPS

3.1 Model Basis in the Absence of Sterile Mosquitoes

We first formulate models for wild mosquitoes, in the absence of sterile mosquitoes, as our basis in this section. We consider the cases where either there are difficulties for mosquitoes to find their mates such that Allee effects are included or there are no difficulties such that no Allee effects are included in the models.

3.1.1 Without Allee effects

Similarly as in Chapter 2, we let \( w_n \) be the size of the wild mosquito population at generation \( n \). The population dynamics are governed by the equation

\[
w_{n+1} = aw_n \frac{k}{1 + \eta w_n} + pw_n = \frac{r_0 w_n}{1 + \eta w_n} + pw_n, \tag{3.1.1}
\]

where \( r_0 := ak \) is the population intrinsic growth rate. Clearly, if \( w_0 = 0, w_n = 0 \) for all \( n \geq 1 \) such that \( w = 0 \) is a trivial fixed point, and if \( w_0 > 0, w_n > 0 \) for all \( n \geq 1 \).
Define function

\[
F(w) := w = \frac{r_0 w}{1 + \eta w} + pw.
\]

It follows from \( F'(0) = r_0 + p \) that \( w = 0 \) is locally asymptotically stable if \( r_0 + p < 1 \) and unstable if \( r_0 + p > 1 \). Moreover, for any solution of (3.1.1) with positive initial value \( w_0 > 0 \), it follows from

\[
w_{n+1} < (r_0 + p)w_n < (r_0 + p)^2w_{n-1} < \ldots < (r_0 + p)^{n+1}w_0,
\]

that the trivial fixed point \( w = 0 \) is globally asymptotically stable if \( r_0 + p < 1 \).

If \( r_0 + p = 1 \), it follows from

\[
w_{n+1} - w_n = w_n\left(\frac{r_0}{1 + \eta w_n} + p - 1\right) < w_n(r_0 + p - 1) = 0,
\]

that sequence \( \{w_n\} \) is monotone decreasing and thus \( \lim_{n \to \infty} w_n := u \) exists. Let \( n \to \infty \) in (3.1.1). Then

\[
u = \frac{r_0 u}{1 + \eta u} + pu,
\]

which leads to \( u = \frac{r_0 + p - 1}{\eta(1 - p)} = 0 \). Thus \( u = 0 \) and the trivial fixed point \( w = 0 \) is globally asymptotically stable.

We then assume \( r_0 + p > 1 \) and \( w \) is a positive fixed point of (3.1.1). Then \( w \) must satisfy the equation

\[
w = \frac{r_0 w}{1 + \eta w} + pw,
\]
which has a unique solution \( w^* = \frac{r_0 + p - 1}{\eta(1 - p)} \). It then follows from

\[
F'(w^*) = \frac{r_0}{(1 + \eta w^*)^2} + p = \frac{(1 - p)^2 + pr_0}{r_0} < 1
\]

that since \((1 - p)^2 + pr_0 - r_0 = (1 - p)(1 - p - r_0) < 0\), the unique positive fixed point \( w^* \) is locally asymptotically stable.

Note that

\[
w_{n+1} - w_n = \frac{r_0 w_n}{1 + \eta w_n} + p w_n - w_n
= \frac{(r_0 + p - 1) - \eta w_n(1 - p)}{1 + \eta w_n} w_n = \frac{\eta(1 - p)w^* - \eta w_n(1 - p)w_n}{1 + \eta w_n}
= (w^* - w_n) \frac{\eta(1 - p)w_n}{1 + \eta w_n} = (w^* - w_n)g_n,
\]

where \( g_n := \frac{\eta(1 - p)w_n}{1 + \eta w_n} < 1 \) for all \( n \geq 0 \). Then it follows that

\[
w_{n+1} - w^* = w_{n+1} - w_n + w_n - w^* = (w_n - w^*)(1 - g_n). \tag{3.1.3}
\]

Hence, if \( w_n < w^* \), \( w_{n+1} < w^* \) and if \( w_n > w^* \), \( w_{n+1} > w^* \).

Furthermore, for initial value \( w_0 \) with \( w_0 < w^* \), it follows from (3.1.2) that \( w_1 > w_0 \). Repeating this process, we have \( w_{n+1} > w_n \) for all \( n \geq 0 \). Thus, we have an increasing sequence \( \{w_n\} \) satisfying \( w_{n+1} < w^* \) for all \( n \geq 1 \), if \( w_0 < w^* \).

Similarly, for initial value \( w_0 \) with \( w_0 > w^* \), we have a decreasing sequence \( \{w_n\} \) satisfying \( w_{n+1} < w^* \), for all \( n \geq 1 \). Thus \( \lim_{n \to \infty} w_n \) exists for any initial value \( w_0 \). By taking limits in (3.1.1), it follows that \( \lim_{n \to \infty} w_n = w^* \).
We next show that there exist no periodic $k$-cycles, where $k \geq 2$. Assume $w_{n+k} = w_n := w^*$, that is, $F^{(k)}(w^*) = w^*$ and $w^* \neq w^\pm$ where $F^{(k)}$ is the $k$th iteration of $F$. Since $F'(w) > 0$ for all $w > 0$, if $F(w^*) > w^*$, then $F(F(w^*)) > F(w^*) > w^*$. By iteration, $F^{(k)}(w^*) > F^{(k-1)}(w^*) > \cdots > w^*$. On the other hand, if $F(w^*) < w^*$, then $F(F(w^*)) < F(w^*) < w^*$, and again by iteration, $F^{(k)}(w^*) < F^{(k-1)}(w^*) < \cdots < w^*$. Therefore, $F^{(k)}(w^*) \neq w^*$, which is a contradiction, and thus there exist no periodic $k$-cycles for $k \geq 2$.

Together with the monotonicity of sequence $\{w_n\}$ for either $w_0 < w^*$ or $w_0 > w^*$, we then have shown that the positive fixed point $w^*$ is globally asymptotically stable. In summary, we have the following theorem.

**Theorem 3.1.1** For model (3.1.1) in the absence of sterile mosquitoes, the trivial fixed point $w = 0$ is globally stable if $r_0 + p \leq 1$ and is unstable if $r_0 + p > 1$. There exists a unique positive fixed point given by $w^* = \frac{r_0 + p - 1}{\eta(1-p)}$ if $r_0 + p > 1$ and this unique positive fixed point is globally asymptotically stable. There exist no any periodic cycles for model (3.1.1).

### 3.1.2 With Allee effects

Considering possible difficulty for mosquitoes finding mates, we assume Allee effects and the following equation:

$$w_{n+1} = \frac{aw_n}{1 + w_n} \frac{w_n}{1 + \eta w_n} + pw_n = \frac{r_0 w^2_n}{(1 + w_n)(1 + \eta w_n)} + pw_n.$$  

(3.1.4)
Clearly, if \( w_0 = 0, w_n = 0 \) and if \( w_0 > 0, w_n > 0 \) for all \( n \geq 1 \). Thus \( w = 0 \) is a fixed point.

We then determine the existence of positive fixed points \( w \) of equation (3.1.4), which corresponds to the positive roots of the following equation

\[
P(w) := (1 + w)(1 + \eta w) - \bar{r}_0 w = \eta w^2 + (1 + \eta - \bar{r}_0)w + 1 = 0,
\]

where \( \bar{r}_0 = \frac{r_0}{1 - p} \).

Setting

\[
r_c := (1 + \sqrt{\eta})^2,
\]

we have the following results from Chapter 2 that the system has no positive fixed point, one positive fixed point

\[
\bar{w} = \frac{\bar{r}_0 - (1 + \eta)}{2\eta},
\]

or two positive fixed points

\[
w^\pm = \frac{\bar{r}_0 - (1 + \eta) \pm \sqrt{\left(\bar{r}_0 - (1 + \eta)\right)^2 - 4\eta}}{2\eta},
\]

if \( \bar{r}_0 < r_c, \bar{r}_0 = r_c, \) or \( \bar{r}_0 > r_c \), respectively.

We next investigate the stability of the fixed points.
Define function \( G(w) = \frac{r_0w^2}{(1 + w)(1 + \eta w)} + pw \). Then

\[
G'(w) = \frac{r_0w((1 + \eta)w + 2)}{(1 + w)^2(1 + \eta w)^2} + p.
\]

Clearly, the trivial fixed point \( w = 0 \) is always locally asymptotically stable since \( G''(0) = p < 1 \).

For the case of \( \bar{r}_0 < r_c \) such that there exists no positive fixed points, the quadratic function \( P(w_n) \) in (3.1.5) is positive for all \( w_n \). Then it follows from

\[
w_{n+1} - w_n = \left( \frac{r_0w_n^2}{\eta w_n^2 + (1 + \eta)w_n + 1} + p - 1 \right) w_n
\]

\[
= - \frac{P(w_n)(1 - p)}{\eta w_n^2 + (1 + \eta)w_n + 1} w_n = - \frac{P(w_n)(1 - p)}{P(w_n) + \bar{r}_0w_n} w_n,
\]

that \( w_{n+1} < w_n \) for all \( n \geq 0 \), which implies that \( \{w_n\} \) is a decreasing sequence. Thus \( \lim_{n \to \infty} w_n \) exists for any initial value \( w_0 \). By taking limits in (2.2.1), it follows that \( \lim_{n \to \infty} w_n = 0 \), i.e. the trivial fixed point \( w = 0 \) is globally asymptotically stable.

For the case \( \bar{r}_0 = r_c \) such that there exists one positive fixed point \( \bar{w} = \frac{\bar{r}_0 - (1 + \eta)}{2\eta} \). Simple calculation yields

\[
G''(w) = \frac{2r_0((\eta + 1)w + 1)(1 + w)(1 + \eta w) - 2r_0w((\eta + 1)w + 2)(2\eta w + \eta + 1)}{(1 + w)^3(1 + \eta w)^3}.
\]

(3.1.8)
At the positive fixed point $\bar{w}$, it follows from $\bar{r}_0 \bar{w} = (1 + \bar{w})(1 + \eta \bar{w})$ that

$$G''(\bar{w}) = \frac{1}{1 - p} \frac{2 \bar{r}_0^2 \bar{w}((\eta + 1)\bar{w} + 1) - 2 \bar{r}_0^2 \bar{w}((\eta + 1)\bar{w} + 2)}{(r_0 \bar{w})^3}$$

$$= \frac{-2}{(1 - p)(r_0 \bar{w}^2)} < 0.$$  (3.1.9)

Hence the unique positive fixed point $\bar{w}$ is unstable.

Meanwhile, since $P(w_n) > 0$ for all $w_n \neq \bar{w}$, it follows from (3.1.7) that $\{w_n\}$ is also a decreasing sequence. By taking limits in (3.1.4), the limit $\lim_{n \to \infty} w_n$ exists. However, since $\bar{w}$ is the only positive fixed point and is unstable, the limit must be zero. Thus the trivial fixed point $w = 0$ is globally asymptotically stable.

For the case of $\bar{r}_0 > r_c$ such that there exist two positive fixed points given in (3.1.6), it follows from $(1 + w)(1 + \eta w) = \bar{r}_0 w = \frac{r_0 w}{1 - p}$ that

$$G'(w) = \frac{r_0 w((1 + \eta)w + 2)}{(1 + w)^2(1 + \eta w)^2} + p$$

$$= \frac{(1 - p)((1 + \eta)w + 2)}{(1 + w)(1 + \eta w)} + p$$

$$= \frac{(1 - p)((1 + \eta)w + 2) + pr_0 w}{r_0 w}.$$

Hence, a positive fixed point $w$ is locally asymptotically stable if $(1 - p)(r_0 w - (1 + \eta)w - 2) > 0$, and is unstable if $(1 - p)(r_0 w - (1 + \eta)w - 2) < 0$. Moreover, it follows from (3.1.5) that, at a positive fixed point, $(\bar{r}_0 - (1 + \eta)w) - 2 = \eta w^2 - 1$. Thus, a positive fixed point $w$ is locally asymptotically stable if $\eta w^2 - 1 > 0$, and is unstable if $\eta w^2 - 1 < 0$ since $1 - p > 0$. 

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Consider

\[ 4\eta(\eta w^2 - 1) = 4\eta^2w^2 - 4\eta = (2\eta w + 2\sqrt{\eta})(2\eta w - 2\sqrt{\eta}), \]

and let \( H(w) := 2\eta w - 2\sqrt{\eta} \). Then, equivalently, the positive fixed point \( w \) is locally asymptotically stable if \( H(w) > 0 \) and is unstable if \( H(w) < 0 \).

Write \( \Sigma := r_0 - (1 + \eta) \). The two positive fixed points in (3.1.6) can be expressed as

\[ w^\pm = \frac{\Sigma \pm \sqrt{\Sigma^2 - 4\eta}}{2\eta}, \tag{3.1.10} \]

and thus

\[ H(x^\pm) = \Sigma \pm \sqrt{\Sigma^2 - 4\eta} - 2\sqrt{\eta} = \Sigma - 2\sqrt{\eta} \pm \sqrt{(\Sigma + 2\sqrt{\eta})(\Sigma - 2\sqrt{\eta})} \]

\[ = \sqrt{\Sigma - 2\sqrt{\eta}} \left( \sqrt{\Sigma - 2\sqrt{\eta}} \pm \sqrt{\Sigma + 2\sqrt{\eta}} \right). \tag{3.1.11} \]

Then

\[ \begin{cases} H(x^+) > 0, \\ H(x^-) < 0, \end{cases} \]

and hence fixed point \( w^+ \) is locally asymptotically stable and \( w^- \) is unstable.

Furthermore, it follows from

\[ w_{n+1} - w_n = -\frac{P(w_n)(1-p)}{P(w_n) + r_0 w_n}w_n = -\frac{P(w_n)(1-p)}{\eta w_n^2 + (1 + \eta)w_n + 1}w_n \]

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where \( \eta w_n^2 + (1 + \eta)w_n + 1 > 0 \) for all \( w_n > 0 \), that

\[
\begin{align*}
P(w_n) &> 0, \quad \text{if } w_n \in (0, w^-), \\
P(w_n) &< 0, \quad \text{if } w_n \in (w^-, w^+), \\
P(w_n) &> 0, \quad \text{if } w_n \in (w^+, \infty),
\end{align*}
\]

which means

\[
\begin{align*}
\{w_n\} \text{ is decreasing, if } w_n \in (0, w^-), \\
\{w_n\} \text{ is increasing, if } w_n \in (w^-, w^+), \\
\{w_n\} \text{ is decreasing, if } w_n \in (w^+, \infty).
\end{align*}
\]

By taking limits in (2.2.1), it follows that

\[
\begin{align*}
\lim_{n \to \infty} w_n &= 0, \quad \text{if } w_n \in (0, w^-), \\
\lim_{n \to \infty} w_n &= w^+, \quad \text{if } w_n \in (w^-, w^+), \\
\lim_{n \to \infty} w_n &= w^+, \quad \text{if } w_n \in (w^+, \infty).
\end{align*}
\]

We next show that there exist no periodic \( k \)-cycles for \( k \geq 2 \). Assume \( w_{n+k} = w_n := w^* \) and \( P(w^*) \neq 0 \), that is, \( G^{(k)}(w^*) = w^* \) and \( w^* \neq w^\pm \) for \( n \geq 2 \). Since \( G'(w) > 0 \) for all \( w > 0 \), if \( G(w^*) > w^* \), then \( G(G(w^*)) > G(w^*) > w^* \). By iteration, \( G^{(k)}(w^*) > G^{(k-1)}(w^*) > \cdots > w^* \). On the other hand, if \( G(w^*) < w^* \), then \( G(G(w^*)) < G(w^*) < w^* \), and again by iteration, \( G^{(k)}(w^*) < G^{(k-1)}(w^*) < \cdots < w^* \). Therefore, we have \( G^{(k)}(w^*) \neq w^* \), which is a contradiction. Thus, there exist no periodic \( k \)-cycles for \( k \geq 2 \).

In summary, we have the following theorem.
Theorem 3.1.2  For model (3.1.4) with Allee effects, the trivial fixed point $w = 0$ is globally stable if $\bar{r}_0 \leq r_c$ and locally asymptotically stable with basin of attraction $(0, w^-)$ if $\bar{r}_0 > r_c$, where $w^-$ is the smaller positive root of $P(w) = 0$ given in (3.1.5). There exists a unique positive fixed point given by $\bar{w} = \frac{\bar{r}_0 - (1 + \eta)}{2\eta}$ if $\bar{r}_0 = r_c$ with $\bar{r}_0 = \frac{r_0}{1 - p}$, and this unique fixed point is unstable. If $\bar{r}_0 > r_c$, there exist two positive fixed points $w^\pm$ given in (3.1.5). Fixed point $w^-$ is unstable and $w^+$ is locally stable with basin of attraction $(w^-, \infty)$.

3.2 Interactive Model with Constant Release

Based on Chapter 2, when we release sterile mosquitoes constantly with $B_n := b > 0$, then the model becomes

$$w_{n+1} = \frac{aw_n}{w_n + b}w_n \frac{k}{1 + \eta w_n} = \frac{r_0w_n^2}{(b + w_n)(1 + \eta w_n)} + pw_n, \quad (3.2.1)$$

where $r_0 = ak$.

Let $w_n := by_n$ and $\eta_1 = b\eta$. Equation (3.2.1) can be written as

$$y_{n+1} = \frac{ay_n}{1 + y_n}y_n \frac{k}{1 + \eta_1 y_n} + p_1y_n = \frac{r_0y_n^2}{(1 + y_n)(1 + \eta_1 y_n)} + py_n.$$ 

It follows from Theorem 3.1.2 that there exists existence threshold for positive fixed points as

$$\hat{r}_c := (1 + \sqrt{\eta})^2 = (1 + \sqrt{b\eta}),$$

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such that there exist zero, one positive fixed point or two positive fixed points if $\bar{r}_0 < \hat{r}_c$, $\bar{r}_0 = \hat{r}_c$, $\bar{r}_0 > \hat{r}_c$, respectively.

For $\bar{r}_0 > \hat{r}_c$, we have $\bar{r}_0 > (1 + \sqrt{b\eta})^2$, and equivalently,

$$b < \frac{(\sqrt{\bar{r}_0} - 1)^2}{\eta}.$$ 

For given $\bar{r}_0$ and $\eta$, we define the threshold value for releases of sterile mosquitoes as

$$b_c := \frac{(\sqrt{\bar{r}_0} - 1)^2}{\eta}, \quad (3.2.2)$$

where $\bar{r}_0 = \frac{r_0}{1 - p} = \frac{ak}{1 - p}$.

The model (3.2.1) has no positive solution, one positive solution

$$w_c = \frac{\bar{r}_0 - (1 + b\eta)}{2\eta}, \quad (3.2.3)$$

or two positive fixed points

$$w_c^\pm = \frac{\bar{r}_0 - (1 + b\eta) \pm \sqrt{(\bar{r}_0 - (1 + b\eta))^2 - 4b\eta}}{2\eta}, \quad (3.2.4)$$

if $b > b_c$, $b = b_c$, or $b < b_c$, respectively.

For the stability of the fixed points of Equation (3.2.1), it follows from Theorem 3.1.2 that if $\bar{r}_0 \leq \hat{r}_c$, that is, $b \geq b_c$, the trivial fixed point $w = 0$ is globally asymptotically stable; if $\bar{r}_0 > \hat{r}_c$, that is, $b < b_c$, there are two positive fixed points $w^\pm$
where $w^-$ is unstable and $w^+$ is locally asymptotically stable with basin of attraction $(w^-, \infty)$.

The results can be summarized as follows.

**Theorem 3.2** For given $r_0$, $\eta$ and $p$, there is a threshold value for releases of sterile mosquitoes $b_c$ defined in (3.2.2). If the number of releases is greater than the threshold, that is, $b > b_c$, there exists no positive fixed point and $w = 0$ is the only fixed point which is globally asymptotically stable. Thus the wild mosquito population goes extinct regardless of the initial population size. If $b = b_c$, there exists a unique positive fixed point $w_c$ given in (3.2.3) which is unstable and thus the trivial fixed point $w = 0$ is globally asymptotically stable. If $b < b_c$, there exist two positive fixed points $w^\pm$ given in (3.2.4), where $w^-$ is unstable and $w^+$ is locally asymptotically stable with a basin of attraction $(w^-, \infty)$ while the trivial fixed point $w = 0$ is locally asymptotically stable with basin of attraction $(0, w^-)$. Solutions approach either $w = 0$ or $w = w^+$ depending on their initial values. That is to say that the two types of mosquitoes can either go extinct or coexisting, depending on their initial values.

We give an example to demonstrate the results for model (3.2.1) as follows.

**Example 3.2.** In this example, we have parameters given as

\[ a = 10, \quad k = 0.3, \quad \eta = 0.4, \quad p = 0.6 \quad (3.2.5) \]

such that $\bar{r}_0 = \frac{ak}{1 - p} = 7.5$ and the release threshold is

\[ b_c = \frac{(\sqrt{\bar{r}_0} - 1)^2}{\eta} = 7.556936. \]
Figure 3.1: For $b = 8 > b_c$, there exists no positive fixed point and the trivial fixed point $w = 0$ is globally asymptotically stable.

For $b = 8 > b_c$, there exists no positive fixed point and the trivial fixed point $w = 0$ is globally asymptotically stable as shown in Figure 3.1. For $b = 6 < b_c$, there exist two positive fixed points

$$w_c^- = 1.768570, \quad w_c^+ = 8.481430.$$  

Positive fixed point $w^-$ is unstable; the trivial fixed point $w = 0$ and $w^+$ are both locally asymptotically stable. Solutions approach either $w = 0$ or $w^+$ depending on their initial values as shown in Figure 3.2.
Figure 3.2: For $b = 6 < b_c$, there are two positive fixed point $w_1^- = 1.768570$, and $w_1^+ = 8.481430$. Fixed point $w_1^-$ is unstable and $w_1^+$ is locally asymptotically stable. Solutions approach $w = 0$ or $w_1^+$ depending on their initial values.

3.3 Interactive Model with Proportional Releases

Now we consider the releases of sterile mosquitoes proportional to the wild mosquito size with $B_n := bw_n$. Then the model becomes

$$w_{n+1} = \frac{aw_n}{1 + (1 + b)w_n} \frac{k}{1 + \eta w_n} + pw_n = \frac{r_0 w_n^2}{(1 + (1 + b)w_n)(1 + \eta w_n)} + pw_n. \tag{3.3.1}$$

Let $z_n = (1 + b)w_n$, $R_0 = \frac{r_0}{1 + b}$, $\eta_2 = \frac{\eta}{1 + b}$. Equation (3.3.1) is translated into

$$z_{n+1} = \frac{R_0 z_n}{1 + z_n} \frac{1}{1 + \eta_2 z_n} + pz_n = \frac{R_0 z_n^2}{(1 + z_n)(1 + \eta_2 z_n)} + pz_n.$$
It follows from Theorem 3.1.2 that there exists a threshold value for the existence of the positive fixed points as

\[ R_c := (1 + \sqrt{\eta_2})^2 = \left(1 + \sqrt{\frac{\eta}{1+b}}\right)^2, \]
\[ \tilde{R}_0 := \frac{R_0}{1-p} = \frac{\tilde{r}_0}{1+b}, \]

such that there exist zero, one, or two positive fixed points if \( \tilde{R}_0 < R_c, \tilde{R}_0 = R_c, \) or \( \tilde{R}_0 > R_c, \) respectively.

Consider \( \tilde{R}_0 > R_c. \) We have

\[ \frac{\tilde{r}_0}{1+b} > \left(1 + \sqrt{\frac{\eta}{1+b}}\right)^2, \]

which is equivalent to

\[ \tilde{r}_0 > (\sqrt{1+b} + \sqrt{\eta})^2, \]

or

\[ b < (\sqrt{\tilde{r}_0} - \sqrt{\eta})^2 - 1. \]

The threshold value for the releases of sterile mosquitoes for model (3.3.1) is defined as

\[ b_p := (\sqrt{\tilde{r}_0} - \sqrt{\eta})^2 - 1, \]
which is positive based on $\tilde{r}_0 > r_c = (1 + \sqrt{\eta})^2$. Then model (3.3.1) has no positive fixed point, one positive fixed point

$$\hat{w}_p = \frac{\tilde{r}_0}{1 + b} - \left(1 + \frac{\eta}{1 + b}\right) \frac{r_0 - (1 + b + \eta)(1 - p)}{2\eta(1 - p)(1 + b)},$$

(3.3.2)
or two positive fixed points

$$w_p^\pm = \frac{\tilde{r}_0}{1 + b} - \left(1 + \frac{\eta}{1 + b}\right) \pm \sqrt{\left(\frac{\tilde{r}_0}{1 + b} - \left(1 + \frac{\eta}{1 + b}\right)\right)^2 - 4 \frac{\eta}{1 + b}}$$

(3.3.3)

$$= \frac{r_0 - (1 + b + \eta)(1 - p) \pm (1 - p)\sqrt{(r_0 - (1 + b + \eta))^2 - 4(1 + b)\eta}}{2\eta(1 - p)(1 + b)},$$

if $b > b_p$, $b = b_p$, or $b < b_p$, respectively.

The stability analysis for the fixed points of equation (3.3.1) is similar to that for equation (3.1.4). We omit it here. Then we have the following theorem.

**Theorem 3.3** Define the threshold value for the releases of sterile mosquitoes $b_p := (\sqrt{\tilde{r}_0} - \sqrt{\eta})^2 - 1$ for model (3.3.1). Then if $b > b_p$, there exists no positive point and the trivial fixed point $w = 0$ is globally asymptotically stable. If $b = b_p$, there exists a unique positive fixed point $\hat{w}_p$ given in (3.3.2) which is unstable and the trivial fixed point $w = 0$ is also globally asymptotically stable. If $b < b_p$, there exist two positive fixed points $\hat{w}_p^\pm$ given in (3.3.3). Fixed point $\hat{w}_p^-$ is unstable and $\hat{w}_p^+$ is locally asymptotically stable with basin of attraction $(\hat{w}_p^-, \infty)$ while the trivial fixed point is locally asymptotically stable with basin of attraction $(0, \hat{w}_p^+)$. Solutions approach either $w = 0$ or $w = \hat{w}_p^+$ depending on their initial values. The two types of mosquitoes can either go extinct or coexisting, depending on their initial sizes.
3.4 Interactive Model with Proportional Releases with Saturation

We assume the number of releases to be the Holling-II type with the form of

\[ B(w) = \frac{bw}{1 + w} \], similarly as in Chapter 2. Then the model becomes:

\[
\begin{align*}
    w_{n+1} &= \frac{aw_n}{1 + w_n + B(w_n)} \frac{k}{1 + \eta w_n} + pw_n \\
    &= \frac{aw_n(1 + w_n)}{(1 + w_n)^2 + bw_n} \frac{k}{1 + \eta w_n} + pw_n.
\end{align*}
\] (3.4.1)

Here we, as in Section 3.2, focus on the case where \( r_0 > r_c = (1 + \sqrt{\eta})^2 \).

The trivial fixed point \( w = 0 \) is a trivial fixed point and is locally asymptotically stable. A positive fixed point satisfies the following equation:

\[
1 = \frac{r_0 w(1 + w)}{(1 + w)^2 + bw} \frac{1}{1 + \eta w} + p.
\] (3.4.2)

That is,

\[
1 = \frac{\bar{r}_0 w(1 + w)}{(1 + w)^2 + bw} \cdot \frac{1}{1 + \eta w}.
\]

Equivalently, we have

\[
b = \frac{1 + w}{w(1 + \eta w)} (\bar{r}_0 w - (1 + \eta w)(1 + w)) = \frac{\bar{r}_0 (1 + w)}{1 + \eta w} - \frac{(1 + w)^2}{w},
\] (3.4.3)

with \( \bar{r}_0 w \geq (1 + \eta w)(1 + w) \) since the parameter \( b \) cannot be negative.

Define

\[
H(w) := \frac{\bar{r}_0 (1 + w)}{1 + \eta w} - \frac{(1 + w)^2}{w},
\] (3.4.4)
which has the derivative as

\[ H'(w) = \frac{\bar{r}_0(1 - \eta)}{(1 + \eta w)^2} - \frac{w^2 - 1}{w^2} = L(w) - F(w), \]

where \( L(w) := \frac{\bar{r}_0(1 - \eta)}{(1 + \eta w)^2} \) and \( F(w) := \frac{w^2 - 1}{w^2} \).

If \( w \leq 1, \)

\[ F(w) \leq 0, \quad L(w) > 0, \quad \Rightarrow H'(w) > 0, \]

then, \( H(w) \) is increasing with respect to \( w \in (0, 1] \).

If \( w > 1, \)

\[ L'(w) = -\frac{2\bar{r}_0(1 - \eta)(1 + \eta w)}{(1 + \eta w)^4} < 0, \quad F'(w) = 2w^{-3} > 0, \]

then, \( L(w) \) is decreasing and \( F(w) \) is increasing with respect to \( w \in (1, \infty) \). It follows from

\[ L(1) > 0, \quad \lim_{w \to \infty} L(w) = 0, \]

\[ F(1) = 0, \quad \lim_{w \to \infty} F(w) = 1, \]

that there exists a unique intersection point between the curves of \( L(w) \) and \( F(w) \), denoted as \( w^* \) with \( w^* > 1 \). This unique fixed point, \( w^* \) satisfying \( L(w^*) = F(w^*) \), makes \( H'(w^*) = 0 \). Clearly, \( H'(w) > 0 \) if \( L(w) > F(w) \) and \( H'(w) < 0 \) if \( L(w) < F(w) \). Hence,

\[ \begin{cases} 
H'(w) > 0, & w < w^*, \\
H'(w) < 0, & w > w^*. 
\end{cases} \]
We define $b_s := H(w^*)$. Then $b_s$ determines the release threshold value of sterile mosquitoes such that system (3.4.1) has no positive fixed point, one positive fixed point, or two positive fixed points if $b > b_s$, $b = b_s$, or $b < b_s$, respectively.

We next investigate the stability of positive fixed points and have the following results.

**Theorem 3.4** The trivial fixed point $w = 0$ for model (3.4.1) is always locally asymptotically stable. Assume $\bar{r}_0 > (1 + \sqrt{\eta})^2$, and define the threshold for the releases of sterile mosquitoes as $b_s := H(w^*)$ where $H$ is given in (3.4.4) and $H'(w^*) = 0$.

Then the model (3.4.1) has no, unique, or two positive fixed points if $b > b_s$, $b = b_s$, or $b < b_s$, respectively. If $b > b_s$, there exists no positive fixed point and the trivial fixed point $w = 0$ is globally asymptotically stable. All wild mosquitoes are eventually wiped out. If $b = b_s$, there exists a unique positive fixed point which is unstable and $w = 0$ is still globally asymptotically stable. All wild mosquitoes are also eventually wiped out. If $b < b_s$, there exist two positive fixed points $w_s^- < w_s^+$. Fixed point $w_s^-$ is unstable and $w_s^+$ is locally asymptotically stable while the trivial fixed point is also locally asymptotically stable. Solutions approach either $w = 0$ or $w = w_s^+$ depending on their initial values. The two types of mosquitoes can either go extinct or coexisting, depending on their initial sizes.

**Proof.** Assume $b < b_s$, there exist two positive fixed points. It follows from (3.4.1) that these positive fixed points satisfy

$$1 = \frac{\bar{r}_0 w (1 + w)}{(1 + w)^2 + bw} \cdot \frac{1}{1 + \eta w}.$$
Set
\[ Q(w) := \eta w^3 + Aw^2 + Bw + 1 = 0, \quad (3.4.5) \]

where \( A = 1 + 2\eta + B \eta - \bar{r}_0, \) \( B = 2 + \eta + b - \bar{r}_0. \) Then the positive fixed points of (3.4.1) correspond to the positive roots of (3.4.5), which are denoted as \( w_s^- < w_s^+. \)

Thus \( Q'(w_s^-) < 0 \) and \( Q'(w_s^+) > 0. \)

Consider the function
\[
T(w) := \frac{r_0 w^2 (1 + w)}{(1 + w)^2 + bw(1 + \eta w)} + pw.
\]

Then, evaluated at the fixed points with \( \bar{r}_0 w(1 + w) = ((1 + w)^2 + bw)(1 + \eta w), \)

\[
T'(w) = \frac{r_0 w(3w + 2)((1 + w)^2 + bw)(1 + \eta w)}{((1 + w)^2 + bw)^2(1 + \eta w)^2} - \frac{r_0 w^2(1 + w)(3\eta w^2 + 2Aw + 2\bar{r}_0 w + B + \bar{r}_0)}{((1 + w)^2 + bw)^2(1 + \eta w)^2} + p \\
= (1 - p) \frac{r_0 w(3w + 2)}{r_0 w(1 + w)} - (1 - p) \frac{3\eta w^2 + 2Aw + 2\bar{r}_0 w + B + \bar{r}_0}{\bar{r}_0 (1 + w)} + p \\
= (1 - p) \left( 1 + \frac{2\bar{r}_0 w + \bar{r}_0 - (3\eta w^2 + 2Aw + 2\bar{r}_0 w + B + \bar{r}_0)}{\bar{r}_0 (1 + w)} \right) + p \quad (3.4.6) \\
= (1 - p) \left( 1 - \frac{3\eta w^2 + 2Aw + B}{\bar{r}_0 (1 + w)} \right) + p \\
= (1 - p) \left( 1 - \frac{Q'(w)}{\bar{r}_0 (1 + w)} \right) + p \\
= 1 - \frac{(1 - p)Q'(w)}{\bar{r}_0 (1 + w)}.
\]

Thus, positive fixed point \( w_s^- \) is unstable since \( Q'(w_s^-) < 0 \) leading to \( T'(w_s^-) > 1. \)
It follows from $Q'(w_s^+) > 0$ that $T'(w_s^+) < 1$ and thus $w_s^+$ is locally asymptotically stable if

$$\frac{(1 - p)Q'(w)}{\bar{r}_0(1 + w)} < 2.$$ 

Since

$$Q(w) := ((1 + w)^2 + bw)(1 + \eta w) - \bar{r}_0 w(1 + w)$$

$$= bw(1 + \eta w) - w(1 + \eta w) \cdot \frac{\bar{r}_0 w(1 + w) - (1 + w)^2(1 + \eta w)}{w(1 + \eta w)}$$

$$= bw(1 + \eta w) - w(1 + \eta w) \cdot \left(\frac{\bar{r}_0 (1 + w)}{1 + \eta w} - \frac{(1 + w)^2}{w}\right)$$

$$= w(1 + \eta w)(b - H(w)).$$

Then, taking the derivative with respect to $w$, we have

$$Q'(w) = -H'(w) w(1 + \eta w) + (b - H(w))(2w + 2\eta w).$$

Since the positive fixed point $w_s^+$ satisfies $b - H(w_s^+) = 0$, we have

$$Q'(w_s^+) = -w_s^+(1 + \eta w_s^+)H'(w_s^+).$$

Therefore,

$$\frac{(1 - p)Q'(w_s^+)}{\bar{r}_0(1 + w_s^+)} = (1 - p) \frac{-w_s^+(1 + \eta w_s^+)H'(w_s^+)}{\bar{r}_0(1 + w_s^+)}$$

$$< -H'(w_s^+) \cdot \frac{w_s^+}{1 + w_s^+} \cdot \frac{1 + \eta w_s^+}{\bar{r}_0},$$

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Figure 3.3: For $b = 9 > b_s$, there exists no positive fixed point and trivial fixed point $w = 0$ is globally asymptotically stable.

where

$$- H'(w_s^+) = \frac{w_s^{+2} - 1}{w_s^{+2}} - \frac{r_0(1 - \eta)}{(1 + \eta w_s^+)^2} < 1,$$

$$\frac{w_s^+}{1 + w_s^+} < 1,$$

$$\frac{1 + \eta w_s^+}{r_0} < \frac{1 + \eta w_s^+}{r_0} \frac{1 + w}{w} < 1,$$

which means

$$\frac{(1 - p)Q'(w)}{r_0(1 + w)} < 2.$$

Thus, fixed point $w_s^+$ is locally asymptotically stable.

Theorem 3.4 is proven.

We provide an example below to demonstrate the results for model (3.4.1).

**Example 3.4.** In this example, we have parameters given as

$$a = 10, \quad k = 0.3, \quad \eta = 0.4, \quad p = 0.6 \quad (3.4.7)$$
Figure 3.4: For $b = 7 < b_s$, there are two positive fixed points $w^- = 1.136752$, and $w^+ = 6.454005$. Fixed point $w^-$ is unstable and $w^+$ is locally asymptotically stable. Solutions approach $w = 0$ or $w^+$ depending on their initial values

such that $\bar{r}_0 = \frac{ak}{1 - p} = 7.5$ and the release threshold is

$$b_s = 8.305099.$$ 

For $b = 9 > b_s$, there exists no positive fixed point and the trivial fixed point $w = 0$ is globally asymptotically stable as shown in Figure 3.3. For $b = 7 < b_s$, there exist two positive fixed points

$$w^- = 1.136752, \quad w^+ = 6.454005.$$ 

Positive fixed point $w^-$ is unstable; the trivial fixed point $w = 0$ and $w^+$ are both locally asymptotically stable. Solutions approach either $w = 0$ or $w^+$ depending on their initial values as shown in Figure 3.4.
STAGE-STRUCTURED BASIC MODELS

4.1 Introduction

Mosquitoes undergo complete metamorphosis, going through four distinct stages of development during a lifetime, egg, pupae, larva, and adults. After a female mosquito drinking blood, she can lay from 100 to 300 eggs at a time in standing water or very slow-moving water. In her lifetime, she can produce from 1000 to 3000 eggs [59]. Within a week, the eggs hatch into larvae, which will use their tubes to breathe air by poking above the surface of the water. Larvae eat a bit of floating organic matter and each other. Larvae molt four times totally as they grow and after the fourth molt, they are called pupae. Pupae also live near the surface of water and breathe through two horn-like tubes (called siphons) on their back. But pupae do not eat. When the skin splits after a few days from a pupae, an adult mosquito emerges. The adults live for only a few weeks and a full life-cycle of a mosquito takes about a month [5,17], as shown in Figure 4.1.

To have more realistic modeling of mosquitoes, we need to include stage structure since the different stages have different response to environment and regulating factors to the population [65]. While the interspecific competition and predation
are rare events and could be discounted as major causes of larval mortality, the intraspecific competition could represent a major density dependent source. Thus the effect of crowding could be an important factor in the population dynamics of mosquitoes [29, 36, 63].

Since the first three stages in a mosquito’s life time are aquatic and the major density dependent source comes from the larval stage, we group the three aquatic stages of mosquitoes into one class, which means we divide the whole mosquito population into only two classes to keep our mathematical modeling as simple as possible. The class consisting of the first three stages is simply called larvae and the other class is called adult. We assume that the density dependence is based on larvae not the adults.
4.2 Discrete-time Stage-structured Model

Let $x_n$ and $y_n$ be the number of mosquito larvae and adults at time $n$, respectively, and assume that population dynamics of the mosquitoes are described by the system

$$
x_{n+1} = f(x_n, y_n)y_n s_1(x_n, y_n),
$$

$$
y_{n+1} = g(x_n, y_n)x_n s_2(x_n, y_n).
$$

where $f$ is the per-capita birth rate, $s_1$ is the survival rate of larvae, or the fraction of larvae who survive, $g$ is the progression rate of larvae or the adults emergence rate, and $s_2$ is the survival rate of adults.

We assume a constant birth rate with $f := a$. Since the intraspecific competition mainly takes place within the aquatic stages of mosquitoes, we assume that the death and the progression rates of larvae are density dependent of the larvae size, with the Beverton-Holt type of nonlinearity, such that $s_1(x_n, y_n) = \frac{k_1}{1 + \eta_1 x_n}$ and $g(x_n, y_n) = \frac{\gamma}{1 + \eta_2 x_n}$, where $k_1$ is the maximum survival probability, $\eta_1$ and $\eta_2$ are density-dependent factors, and $\gamma$ is the maximum progression rate.

We assume food is abundant for mosquito adults, so that the adults survival rate is constant, denoted $s_2(x_n, y_n) := s_2$. Then the model equations become

$$
x_{n+1} = \frac{ay_n}{1 + \eta_1 x_n},
$$

$$
y_{n+1} = \frac{\gamma x_n}{1 + \eta_2 x_n},
$$

(4.2.1)
where we merge $k_1$ and $s_2$ into $a$ and $\gamma$, respectively. We still use $a$ and $\gamma$ for those parameters without confusion.

### 4.3 Existence and Stability of Fixed Points

It is easy to see that the first quadrant of the $xy$ plane is positively invariant for system (4.2.1), and all the solutions of (4.2.1) are bounded.

Define the intrinsic growth rate of the stage-structured mosquito population as $r_0 := a \gamma$. The origin is a trivial fixed point. This trivial fixed point is locally asymptotically stable if $r_0 < 1$ and is unstable if $r_0 > 1$.

Suppose $r_0 > 1$. Then there exists a unique positive fixed point $\bar{E} := (\bar{x}, \bar{y})$ satisfying

$$
\eta_1 \eta_2 x^2 + (\eta_1 + \eta_2)\bar{x} + 1 - r_0 = 0,
$$

with

$$
\bar{x} = \frac{-(\eta_1 + \eta_2) + \sqrt{\Delta}}{2\eta_1 \eta_2},
$$

$$
\bar{y} = \frac{\gamma \bar{x}}{1 + \eta_2 \bar{x}} = \frac{\gamma(-\eta_1 + \eta_2 + \sqrt{\Delta})}{2\eta_1 \eta_2 + \eta_2(-\eta_1 + \eta_2) + \sqrt{\Delta}},
$$

(4.3.1)

where $\Delta = (\eta_1 - \eta_2)^2 + 4\eta_1 \eta_2 a \gamma$.

Then, we investigate the stability of this unique positive fixed point $\bar{E}$.

The Jacobian matrix of system (4.2.1) at this positive fixed point is

$$
J_1 := \begin{pmatrix}
\frac{-a \bar{y} \eta_1}{(1 + \eta_1 \bar{x})^2} & \frac{a}{1 + \eta_1 \bar{x}} \\
\frac{\gamma}{(1 + \eta_2 \bar{x})^2} & 0
\end{pmatrix} = \begin{pmatrix}
-\frac{\eta_1 \bar{x}^2}{\bar{y}} & \bar{x} \\
\frac{a \bar{y}}{\bar{y} \bar{x}^2} & 0
\end{pmatrix}.
$$
Since

\[ \text{tr} J_1 = -\frac{\eta_1 \bar{x}^2}{a \bar{y}} = -\frac{\eta_1 \bar{x}}{1 + \eta_1 \bar{x}}, \quad \det J_1 = -\frac{\bar{y}}{\gamma \bar{x}} = -\frac{1}{1 + \eta_2 \bar{x}}, \]

both eigenvalues of \( J_1 \) are inside the unit circle if and only if

\[ \eta_1 < \eta_2. \tag{4.3.2} \]

We summarize our results as follows.

**Theorem 4.3** System (4.2.1) has a trivial fixed point, which is locally asymptotically stable if \( r_0 < 1 \) and unstable if \( r_0 > 1 \). If \( r_0 > 1 \), there exists a unique positive fixed point, given by (4.3.1), which is locally asymptotically stable if \( \eta_1 < \eta_2 \) and unstable if \( \eta_1 > \eta_2 \).

### 4.4 Existence and Stability of Synchronous 2-Cycles

System (4.2.1) may have periodic cycles of different periods. We first consider 2-cycles with \( x_{n+2} = x_n \neq 0 \) and \( y_{n+2} = y_n \neq 0 \), for all \( n \geq 0 \).

#### 4.4.1 Existence of 2-Cycles

It follows from system (4.2.1) that

\[
\begin{align*}
x_{n+2} &= \frac{ay_{n+1}}{1 + \eta_1 x_{n+1}} = \frac{a \gamma x_n}{(1 + \eta_2 x_n)(1 + \eta_1 x_{n+1})}, \\
y_{n+2} &= \frac{\gamma x_{n+1}}{1 + \eta_2 x_{n+1}} = \frac{a \gamma y_n}{(1 + \eta_1 x_n)(1 + \eta_2 x_{n+1})}.
\end{align*}
\tag{4.4.1}
\]
Then there exists a positive nontrivial 2-cycle if and only if

\[(1 + \eta_2 x_n)(1 + \eta_1 x_{n+1}) = (1 + \eta_1 x_n)(1 + \eta_2 x_{n+1}) = a\gamma, \quad (4.4.2)\]

which implies

\[(\eta_2 - \eta_1)(x_n - x_{n+1}) = 0,
\]

for all \(n \geq 0\).

If \(\eta_1 \neq \eta_2\), there exists no positive 2-cycles. If \(\eta_1 = \eta_2 := \eta\), there exists positive nontrivial 2-cycles. Before we investigate their existence and dynamics, we consider synchronous 2-cycles which are not strictly positive, but are non-negative with alternating zero and positive components. In such a situation, the mosquito larvae and adults are synchronized in such a way as to appear and vanish alternately in one time unit.

Synchronous 2-cycles of equation (4.2.1) can be found by looking for nontrivial equilibria of the equations (4.4.1) which have one component equal to zero.

Now we consider the synchronous 2-cycles with two components \((x_*, y_*)\) satisfying \(x_{n+2} = x_n = x* \neq 0\), \(x_{n+1} = 0\) and \(y_{n+2} = y_n = 0\), \(y_{n+1} = y* \neq 0\), for all \(n > 0\). It follows from

\[x_{n+2} = \frac{a\gamma x_n}{(1 + \eta_2 x_n)(1 + \eta_1 x_{n+1})}\]

that

\[x* = \frac{a\gamma x*}{(1 + \eta_2 x*)(1 + \eta_1 \cdot 0)},\]
which yields
\[ x_* = \frac{a\gamma - 1}{\eta_2}. \]  
(4.4.3)

We also have
\[ x_{n+2} = \frac{ay_{n+1}}{1 + \eta_1 x_{n+1}}, \]
which leads to
\[ x_* = \frac{ay_*}{1 + \eta_1 \cdot 0}. \]

Thus
\[ y_* = \frac{x_*}{a} = \frac{a\gamma - 1}{a\eta_2}. \]  
(4.4.4)

Therefore, there exists a synchronous 2-cycle with positive component given in (4.4.3) and (4.4.4), which indicates the mosquito larvae and adults are synchronized in such a way as to appear and vanish alternately in one time unit.

4.4.2 Stability of Synchronous 2-Cycles

It follows from equations (4.4.1) that
\[ x_{n+2} = \frac{a\gamma x_n}{(1 + \eta_2 x_n)(1 + \eta_1 x_{n+1})}, \]
\[ y_{n+2} = \frac{a\gamma y_n}{(1 + \eta_1 x_{n+1})(1 + \eta_2 x_{n+1})}, \]  
(4.4.5)

where \( x_{n+1} = \frac{ay_n}{1 + \eta_1 x_n} \).
At the synchronous cycle, the Jacobian matrix becomes

\[
J_2 := \begin{pmatrix}
\frac{a\gamma}{(1 + \eta_2x_*)^2} & -\frac{a\eta_1x_*}{1 + \eta_1x_*} \\
0 & \frac{a\gamma}{1 + \eta_1x_*}
\end{pmatrix} = \begin{pmatrix}
\frac{1}{\eta_0} & -\frac{a\eta_1x_*}{1 + \eta_1x_*} \\
0 & \frac{1}{\eta_0}
\end{pmatrix},
\]

where \( x_* = \frac{a\gamma - 1}{\eta_2} \) with \( \eta_0 > 1 \).

The eigenvalues are

\[
\lambda_1 = \frac{1}{a\gamma} < 1, \quad \lambda_2 = \frac{a\gamma}{1 + \eta_1x_*},
\]

and both eigenvalues of \( J_2 \) are inside of the unit circle if and only if

\[
\eta_1 > \eta_2,
\]

which makes the synchronous 2-cycle locally asymptotically stable.

Therefore, when \( r_0 < 1 \), the trivial fixed point is globally stable since there exist no positive fixed points and no periodic cycles. When \( r_0 > 1 \) with \( \eta_1 > \eta_2 \), the trivial fixed point and positive fixed point are unstable and only the synchronous 2-cycle is locally asymptotically stable, which makes the synchronous 2-cycle globally asymptotically stable. If \( \eta_1 < \eta_2 \), the trivial fixed point and the synchronous 2-cycle are both unstable and only the positive fixed point is locally asymptotically stable, which makes the positive fixed point globally asymptotically stable. We have the following results.
**Theorem 4.4** System (4.2.1) has a trivial fixed point which is globally stable if $r_0 < 1$ and unstable if $r_0 > 1$. If $r_0 > 1$, system (4.2.1) has a unique positive fixed point given in (4.3.1), which is globally asymptotically stable if $\eta_1 < \eta_2$ and unstable if $\eta_1 > \eta_2$, and a unique synchronous 2-cycle, given in (4.4.3) and (4.4.4), which is globally asymptotically stable if $\eta_1 > \eta_2$ and unstable if $\eta_1 < \eta_2$.

### 4.5 Existence of Positive Cycles

As is shown in (4.4.2), if $\eta_1 \neq \eta_2$, there exists no periodic positive 2-cycle. We now assume $\eta_1 = \eta_2 := \eta$. Then the model (4.2.1) becomes

\[
x_{n+1} = \frac{ay_n}{1 + \eta x_n},
\]
\[
y_{n+1} = \frac{\gamma x_n}{1 + \eta x_n}.
\]

The unique nontrivial fixed point has the components as:

\[
x_0 = \frac{\sqrt{r_0} - 1}{\eta}, \quad y_0 = \frac{\sqrt{\gamma (\sqrt{r_0} - 1)}}{\sqrt{a \eta}}.
\]

The Jacobian matrix at this positive fixed point is:

\[
J_3 := \begin{pmatrix}
-\eta x & \sqrt{\frac{a}{\gamma}} \\
\frac{1}{1 + \eta x} & 0 \\
\frac{1}{a} & 0
\end{pmatrix},
\]
where \( x = \frac{\sqrt{a \gamma} - 1}{\eta} \) with \( r_0 = a \gamma > 1 \). Then the two eigenvalues of \( J_3 \) are

\[
\lambda_1 = -1, \quad \lambda_2 = \frac{1}{\sqrt{r_0}} < 1.
\]

with associated eigenvectors

\[
v_1 = (a, -1)^T, \quad v_2 = \left( \frac{1}{\sqrt{\gamma}}, 1 \right)^T.
\]

While the solutions with the initial \( x \) and \( y \) lying on the line which has the direction parallel to \( v_1 \) and goes through the unique positive fixed point \((x_0, y_0)\) given by (4.5.2), approach \((x_0, y_0)\) along the line, the stability of the positive fixed point is indeterminant by its linearization. On the other hand, however, the eigenvalue \( \lambda_1 = -1 \) suggests the possibility of bifurcating 2-cycles. We now explore the existence of such 2-cycles.

It follows from system (4.5.1) that

\[
x_{n+2} = \frac{a \gamma x_n}{(1 + \eta x_n)(1 + \eta x_{n+1})} = \frac{r_0 x_n}{1 + \eta x_n + a \eta y_n},
\]

\[
y_{n+2} = \frac{a \gamma y_n}{(1 + \eta x_n)(1 + \eta x_{n+1})} = \frac{r_0 y_n}{1 + \eta x_n + a \eta y_n},
\]

(4.5.4)

where \( x_{n+1} = \frac{a y_n}{1 + \eta x_n} \).

Let \((x, y)\) be a positive 2-cycle. Then it follows from (4.5.4) the two components satisfy the following equation

\[
1 + \eta x + a \eta y = r_0.
\]
Let a solution of 4Model2 have initial values \((x_0, y_0)\) satisfying

\[
0 < x_0 < \frac{r_0 - 1}{\eta}, \quad y_0 = \frac{r_0 - 1 - \eta x_0}{a\eta}. \tag{4.1}
\]

Then this solution is a positive 2-cycle.

We further explore whether there exist positive periodic k-cycles for system (4.2.1) or not for \(k > 2\) and consider \(k = 3\) first. The model (4.2.1) then becomes

\[
x_{n+3} = \frac{a^2 \gamma y_n}{1 + a\eta y_n + \eta(r_0 + 1)x_n},
\]

\[
y_{n+3} = \frac{a\gamma^2 x_n}{1 + a\eta y_n + \eta(r_0 + 1)x_n}.
\tag{4.5.6}
\]

Any points \((x, y)\) on a 3-cycle should satisfy \(x \neq \frac{\sqrt{r_0 - 1}}{\eta}, 1 + \eta x + a\eta y \neq r_0\) and \(y = \sqrt{\frac{\gamma}{a}}x\). According to the first equation of (4.5.6), we have

\[
1 + a\eta y + \eta(r_0 + 1)x = a^2 \gamma \frac{y}{x}.
\]

Plugging \(y = \sqrt{\frac{\gamma}{a}}x\) into it, we have

\[
x = \frac{\sqrt{r_0 - 1}}{\eta}
\]

which is exactly the fixed point of the system (4.5.1). Thus 3-cycles do not exist and as a consequence, any \((2n + 1)\)-cycles do not exist with integer \(n > 0\).
We next check for 4-cycles. If \( k = 4 \), the model (4.2.1) becomes

\[
\begin{align*}
x_{n+4} &= \frac{r_0^2 x_n}{1 + \eta(r_0 + 1)x_n + a\eta(r_0 + 1)y_n}, \\
y_{n+4} &= \frac{r_0^2 y_n}{1 + \eta(r_0 + 1)x_n + a\eta(r_0 + 1)y_n}.
\end{align*}
\] (4.5.7)

For a point \((x, y)\) on 4-cycle, it satisfies

\[ r_0^2 = 1 + \eta(r_0 + 1)x + a\eta(r_0 + 1)y. \]

Equivalently, we can write it in this form

\[ r_0^2 - 1 = (r_0 + 1)(\eta x + a\eta y). \]

Simplifying this a little bit, we get

\[ r_0 = 1 + \eta x + a\eta y, \]

which is exactly the condition for 2-cycles. Thus there exist no 4-cycles, and as a consequence, there exist no 2\(n\)-cycles for integer \( n > 1 \).

**Theorem 4.5** System (4.2.1) has positive 2-cycles with initial values with \( x_0 < \frac{r_0 - 1}{\eta} \) and \( y_0 = \frac{r_0 - 1 - \eta x_0}{a\eta} \) if and only if \( \eta_1 = \eta_2 = \eta \).
Figure 4.2: The trivial fixed point is globally asymptotically stable since $r_0 = 0.9 < 1$.

4.6 Numerical Simulations

We provide simple numerical simulations to demonstrate our results in Sections 4.4 and 4.5.

Example 4.6.1 In this example, we have parameters given as

$$a = 5, \quad \gamma = 0.18, \quad \eta_1 = 0.1, \quad \eta_2 = 0.3,$$

such that $r_0 = a\gamma = 0.9 < 1$, which makes the trivial fixed point globally asymptotically stable as shown in Figure 4.2.

Example 4.6.2 In this example, we have parameters given as

$$a = 5, \quad \gamma = 0.4, \quad \eta_1 = 0.2, \quad \eta_2 = 0.3,$$
Figure 4.3: The positive fixed point is globally asymptotically stable since \( r_0 = 2 > 1 \) and \( \eta_1 < \eta_2 \).

such that \( r_0 = a\gamma = 2 > 1 \) and \( \eta_1 < \eta_2 \), which makes the unique positive fixed point globally asymptotically stable as shown in Figure 4.3.

Example 4.6.3 In this example, we have parameters given as

\[
a = 5, \quad \gamma = 0.4, \quad \eta_1 = 0.7, \quad \eta_2 = 0.5,
\]

such that \( r_0 = a\gamma = 2 > 1 \) and \( \eta_1 > \eta_2 \), which makes the synchronous 2-cycle globally asymptotically stable as shown in Figure 4.4.

Example 4.6.4 In this example, we have parameters given as

\[
a = 5, \quad \gamma = 0.4, \quad \eta_1 = 0.3, \quad \eta_2 = 0.3,
\]

such that \( r_0 = a\gamma = 2 > 1 \) and \( \eta_1 = \eta_2 \). There exists a positive 2-cycle as shown in Figure 4.5.
**Figure 4.4:** The synchronous 2-cycle is globally asymptotically stable since $r_0 = 2 > 1$ and $\eta_1 > \eta_2$.

**Figure 4.5:** There exists a positive 2-cycle for $r_0 = 2 > 1$ and $\eta_1 = \eta_2$. 
CHAPTER 5

STAGE-STRUCTURED INTERACTIVE MODELS

Suppose sterile mosquitoes are released into the field of wild mosquitoes and we let $B_n$ be the number of sterile mosquitoes released at generation $n$. Since sterile mosquitoes do not reproduce, $B_n$ only depends on the size of the releases of the sterile mosquitoes. After the sterile mosquitoes are released, the mating interaction between wild and sterile mosquitoes take place. We assume harmonic means for matings such that the per capita birth rate is given by

$$C(N_n) \frac{ay_n}{y_n + B_n} = C(N_n) \frac{ay_n}{N_n},$$

where $C(N_n)$ is the number of matings per mosquito, per unit of time, and $N_n = y_n + B_n$; $a$ is the number of wild larvae produced per wild mosquito [22, 40, 44]. The interactive dynamics of wild and sterile mosquitoes are then described by the following system:

$$x_{n+1} = C(N_n) \frac{ay_n}{y_n + B_n} \frac{y_n}{1 + \eta_1 x_n},$$

$$y_{n+1} = \frac{\gamma x_n}{1 + \eta_2 x_n}. \tag{5.1.1}$$
5.1 Constant Release of Sterile Mosquitoes

We first consider the case where $B_n := b$ is a constant which means sterile mosquitoes are constantly released for each generation, and assume that the number of matings $C(N_n)$ is a constant which is merged into the birth rate $a$ with the same notation for convenience. Then system (5.1.1) becomes

\[
\begin{align*}
  x_{n+1} &= \frac{ay_n}{y_n + b} \frac{y_n}{1 + \eta_1 x_n} = \frac{ay_n^2}{(y_n + b)(1 + \eta_1 x_n)}, \\
  y_{n+1} &= \frac{\gamma x_n}{1 + \eta_2 x_n},
\end{align*}
\]  

(5.1.2)

Clearly, the origin $(0, 0)$ is a fixed point and is always locally asymptotically stable by its linearization. Let $(x, y)$ be a positive fixed point. It satisfies the following equations:

\[
\begin{align*}
  x &= \frac{ay^2}{(y + b)(1 + \eta_1 x)}, \\
  y &= \frac{\gamma x}{1 + \eta_2 x},
\end{align*}
\]

which leads to

\[
\frac{ay}{(y + b)(1 + \eta_1 x)} \frac{\gamma}{1 + \eta_2 x} = 1,
\]

and then

\[
b = \frac{a \gamma y}{(1 + \eta_1 x)(1 + \eta_2 x)} - y
\]

\[
= \frac{\gamma x}{(1 + \eta_1 x)(1 + \eta_2 x)^2} (a \gamma - (1 + \eta_1 x)(1 + \eta_2 x)) := \gamma H(x),
\]

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for \((1 + \eta_1 x)(1 + \eta_2 x) \leq r_0\), i.e. \(x \leq \bar{x}\).

Function \(H(x) > 0\) if \(r_0 > 1\). Hence if \(r_0 \leq 1\), there exists no positive fixed point.

Now we assume \(r_0 > 1\), and only consider \(x \in \Omega\) where set \(\Omega\) is defined by

\[
\Omega := \{ x : 0 < x < \bar{x} \},
\]

where \(\bar{x}\) is given in (4.3.1).

From

\[
H'(x) = \frac{1}{(1 + \eta_1 x)(1 + \eta_2 x)^3} \left( r_0 (1 - \eta_2 x - \frac{\eta_1 x (1 + \eta_2 x)}{1 + \eta_1 x}) - (1 + \eta_1 x)(1 + \eta_2 x) \right),
\]

we define \(L(x) := r_0 \left( 1 - \eta_2 x - \frac{\eta_1 x (1 + \eta_2 x)}{1 + \eta_1 x} \right)\) and \(F(x) := (1 + \eta_1 x)(1 + \eta_2 x)\). Then \(H'(x) = 0\) for \(x \geq 0\) if and only if \(L(x) = F(x)\) for \(x \geq 0\).

Since

\[
L'(x) = r_0 \left( -\eta_2 - \frac{\eta_1 + 2\eta_1 \eta_2 x + \eta_1^2 \eta_2 x^2}{(1 + \eta_2 x)^2} \right) < 0,
\]

\(L(x)\) is decreasing in set \(\Omega\). But for \(F(x)\), it is increasing in \(\Omega\) and \(F(0) = 1\). Then it follows from \(L(0) = r_0 > 1\) that there exists a unique intersection point between the curves of \(F(x)\) and \(L(x)\), denoted as \(x^*\) with \(0 < x^* \leq \bar{x}\) since \(F(x^*) = L(x^*) < r_0\).
With this unique $x^*$, we have $H'(x) = 0$. Clearly, $L(x) > F(x)$ for $0 < x < x^*$ and $L(x) < F(x)$ for $x^* < x < \bar{x}$. Hence

$$
\begin{cases}
  H'(x) > 0, & 0 < x < x^*, \\
  H'(x) < 0, & x^* < x < \bar{x}.
\end{cases}
$$

We define $b_c := \gamma H(x^*)$. Then $b_c$ determines the release threshold value of sterile mosquitoes such that system (5.1.2) has no fixed point, one positive fixed point $(x^*, y^*)$, or two positive fixed points $(x_i^*, y_i^*)$, $i = 1, 2$ with $x_1^* < x^* < x_2^*$, if $b > b_c$, $b = b_c$, or $b < b_c$, respectively.

We next investigate the stability of the positive fixed points.

The Jacobian matrix at a positive fixed point has the form of

$$
J := \begin{pmatrix}
\frac{\eta_1x}{1 + \eta_1x} & \frac{x y + 2b}{y + b} \\
\frac{y}{x} & 1 + \eta_2 x \\
\end{pmatrix}.
$$

Since

$$
\text{tr} J = -\frac{\eta_1 x}{1 + \eta_1 x}, \quad \det J = -\frac{y + 2b}{(y + b)(1 + \eta_2 x)},
$$

a positive fixed point $(x, y)$ is locally asymptotically stable if and only if

$$
\frac{\eta_1 x}{1 + \eta_1 x} + \frac{y + 2b}{(y + b)(1 + \eta_2 x)} < 1 \quad (5.1.4)
$$

and

$$
-\frac{\eta_1 x}{1 + \eta_1 x} + \frac{y + 2b}{(y + b)(1 + \eta_2 x)} < 1. \quad (5.1.5)
$$
We further assume condition \( \eta_1 < \eta_2 \) is satisfied such that the wild mosquitoes maintain a locally steady state before the sterile mosquitoes are released. It is easy to check that condition (5.1.5) is contained in condition (5.1.4).

Condition (5.1.4) is equivalent to

\[
b \cdot (1 + 2\eta_1 x - \eta_2 x) < \eta_2 xy - \eta_1 xy.
\]

Define

\[
b_s := \frac{\eta_2 xy - \eta_1 xy}{1 + 2\eta_1 x - \eta_2 x},
\]

where \( y = \frac{\gamma x}{1 + \eta_2 x} \). If \( 1 + 2\eta_1 x - \eta_2 x < 0 \), no matter what value of \( b \) is, the conditions (5.1.4) is satisfied, which means the positive fixed point is locally asymptotically stable. If \( 1 + 2\eta_1 x - \eta_2 x > 0, b < b_s \), which makes the positive fixed points locally asymptotically stable.

Then we have the following results.

**Theorem 5.1** For the system (5.1.2), the trivial fixed point \((0,0)\) is always locally asymptotically stable. Suppose the intrinsic growth rate of the stage-structured mosquito population \( r_0 := a\gamma > 1 \) and condition (4.3.2) is satisfied. We define the threshold value for the release sterile mosquitoes \( b_c := \gamma H(x^*) \) where \( x^* \) satisfies \( H'(x^*) = 0 \) with

\[
H(x) := \frac{x}{(1 + \eta_1 x)(1 + \eta_2 x)^2} (a\gamma - (1 + \eta_1 x)(1 + \eta_2 x)).
\]
Then system (5.1.2) has no positive fixed point, one positive fixed point \( E^* := (x^*, y^*) \), or two positive fixed points \( E_i^* := (x_i^*, y_i^*) \) with \( x_1^* < x^* < x_2^* \), if \( b > b_c \), \( b = b_c \), or \( b < b_c \), respectively. A positive fixed point \((x, y)\) is locally asymptotically stable if \( b < b_s \), where

\[
b_s := \frac{\eta_2 xy - \eta_1 xy}{1 + 2\eta_1 x - \eta_2 x}
\]

with \( y = \frac{\gamma x}{1 + \eta_2 x} \), and is unstable if \( b > b_s \).

We give the following example to demonstrate the results in Theorem 5.1.

**Example 5.1** Choose the following parameters

\[
a = 2.25, \quad \gamma = 0.8, \quad \eta_1 = 0.2, \quad \eta_2 = 0.3.
\]  

(5.1.6)

We have \( r_0 = a\gamma = 1.8 > 1 \) and condition (4.3.2)) is satisfied. The threshold value of release \( b_c = 0.5514 \). For \( b = 0.02 \), there exist two positive fixed points \( E_1^* = (0.0888, 0.0692) \) and \( E_2^* = (1.1905, 0.7018) \). For \( E_1^* \), we have \( 1 + 2\eta_1 x_1^* - \eta_2 x_1^* = 1.0089 > 0 \) and \( b_s = 0.0006088 \). Then \( E_1^* \) is unstable since \( b > b_s \). Similarly, at \( E_2^* \), we have \( b_s = 0.07466 \) and thus it is locally asymptotically stable since \( b < b_s \).

The origin \((0, 0)\) is always locally asymptotically stable. Solutions approach either the origin or \( E_2^* \) depending on their initial values. The solution with \( x_0 = 0.018 \) approaches the origin as shown in Figure 5.1, and the solution with \( x_0 = 2 \) approaches \( E_2^* \) as shown in Figure 5.2. We also notice that the attracting region for \( E_2^* \) is much larger than that for the origin.
**Figure 5.1:** The threshold value of release is $b_c = 0.5514$ and $b = 0.02$. The solution with initial value $x_0 = 0.01$ approaches the locally asymptotically stable trivial fixed point (the origin).

**Figure 5.2:** The threshold value of release is $b_c = 0.5514$ and $b = 0.02$. The solution with initial value $x_0 = 2$ approaches the locally asymptotically stable positive fixed point $E_2^*$. 
5.2 Release Proportional to the wild mosquito population size

To have a more optimal and economically effective strategy for releasing sterile mosquitoes in an area where the population size of the wild mosquitoes is relatively small, instead of releasing the sterile mosquitoes constantly in each generation, we may consider, by keeping a close eye on the abundance of the wild mosquitoes, to let the releases be proportional to the population size of the wild mosquitoes such that the number of releases is $B(\cdot) := by$ where $b$ is a constant.

We assume that there is no mating difficulty even as the mosquito population size is small. Then the model dynamics are described as follows:

$$
x_{n+1} = \frac{ay_n}{y_n + by_n} \cdot \frac{y_n}{1 + \eta_1 x_n} = \frac{ay_n}{(1 + b)(1 + \eta_1 x_n)},
$$

$$
y_{n+1} = \frac{\gamma x_n}{1 + \eta_2 x_n}.
$$

The system becomes, mathematically, the same system as (4.2.1). Define the sterile mosquito release threshold as

$$b_c := a\gamma - 1.$$

Then the trivial fixed point is locally asymptotically stable if $b > b_c$ and is unstable if $b < b_c$. 
There exists no positive fixed point if \( b > b_c \) and a unique positive fixed point
\[ E^* := (x^*, y^*) \]
exists if \( b < b_c \) with
\[
\begin{align*}
x^* &= \frac{-(\eta_1 + \eta_2) + \sqrt{\Delta}}{2\eta_1 \eta_2}, \\
y^* &= \frac{\gamma x^*}{1 + \eta_2 x^*} = \frac{\gamma(\eta_1 + \eta_2)}{2\eta_1 \eta_2 + \eta_2(\eta_1 + \eta_2) + \sqrt{\Delta}},
\end{align*}
\] (5.2.2)
where \( \Delta = (\eta_1 - \eta_2)^2 + \frac{4\eta_1 \eta_2 a \gamma}{1 + b} \).

For the stability of the positive fixed point, we know that \( E^* \) is locally asymptotically stable if \( \eta_1 < \eta_2 \) and unstable if \( \eta_1 > \eta_2 \), no matter what value of \( b \) is. In summary, we have the following.

**Theorem 5.2** The trivial fixed point \( (0, 0) \) for system (5.2.1) is globally asymptotically stable if \( b > b_c \) and is unstable if \( b < b_c \) with the sterile mosquito release threshold \( b_c := a \gamma - 1 = r_0 - 1 \). If \( b < b_c \), there exist a positive fixed point and a synchronous 2-cycle. The positive fixed point, given in (5.2.2), is globally stable if \( \eta_1 < \eta_2 \). The synchronous 2-cycle is globally stable if \( \eta_1 > \eta_2 \). Positive 2-cycles exist only if \( \eta_1 = \eta_2 \).

The stability condition remain the same even we have released some sterile mosquitoes. But from the (5.2.2), we know \( x^* \) becomes smaller when the size of the sterile mosquitoes released \( b \) is larger. Since \( y^* \) is a strictly increasing function related to \( x^* \), \( y^* \) becomes smaller with the smaller \( x^* \). That is to say, increasing the releases of sterile mosquitoes can reduce the population size of the wild mosquitoes. We demonstrate our findings in Example 5.2.
Example 5.2  We choose the following parameters

\[ a = 5, \quad \gamma = 0.8, \quad \eta_1 = 0.2, \quad \eta_2 = 0.3. \]  \hspace{1cm} (5.2.3)

Since \( b_c = a\gamma - 1 = 3 \), the trivial fixed point is locally asymptotically stable if we choose \( b = 4 \) as shown in Figure 5.3. When we choose \( b = 0.5 < b_c = 3 \), there exists a unique positive fixed point \( E^* = (2.5519, 1.1563) \) which is globally asymptotically stable as shown in Figure 5.4 since \( \eta_1 < \eta_2 \). Increase the releases and let \( b = 1.5 \). There exists a globally stable positive fixed point \( E^* = (1.0642, 0.6453) \) which has smaller magnitudes \( x \) and \( y \) compared to those for \( b = 0.5 \) as shown in Figure 5.5.

If we choose a different parameter \( \eta_1 = 0.4 \), which results in an unstable positive fixed point no matter what value we choose for \( b \). Let \( b = 1 \), the positive fixed point \( E^* = (1.1870, 0.7003) \) is unstable, which results in a globally asymptotically stable synchronous 2-cycle as shown in Figure 5.6.

5.3 Proportional Release with Saturation

The proportional releases introduced in Section 5.2, compared to the constant release in Section 5.1, may have an advantage when the size of the wild mosquito population is small since the size of releases is also small. However, if the wild mosquito population size is large, the release size should presumably also be large, which may exceed our affordability. Then we propose a new strategy such that the number of releases is proportional to the wild adult mosquito population size when it is small, but the number of releases is saturated and approaches a constant when
Figure 5.3: The threshold value of release is $b_c = 3$. For $b = 4$, the trivial fixed point (the origin) is globally asymptotically stable.

Figure 5.4: The threshold value of release is $b_c = 3$. For $b = 0.5$ the positive fixed point $E^* = (2.5519, 1.1563)$ is globally asymptotically stable.
**Figure 5.5:** The threshold value of release is $b_c = 3$. For $b = 1.5$ the globally asymptotically stable positive fixed point is $E^* = (1.0642, 0.6453)$, which has smaller magnitudes $x$ and $y$ compared to Figure 5.4.

**Figure 5.6:** The threshold value of release is $b_c = 3$. For $b = 1$ and $\eta_1 = 0.4$, the positive fixed point $E^* = (1.1870, 0.7003)$ is unstable. The synchronous 2-cycle is globally asymptotically stable.
the wild adult mosquito population size is sufficiently large. To this end, we let the
releases be of Holling-II type \[39\] such that \( B(\cdot) := \frac{by}{1+y} \). Then we consider the
following system of equations:

\[
x_{n+1} = \frac{ay_n}{y_n + \frac{by_n}{1+y_n}}, \quad y_n = \frac{ay_n(1+y_n)}{(1+b+y_n)(1+\eta_1x_n)}, \quad (5.3.1)
\]

\[
y_{n+1} = \frac{\gamma x_n}{1+\eta_2x_n}.
\]

We define an initial sterile mosquitoes release threshold \( b_0 := a\gamma - 1 = r_0 - 1 \)
such that the origin \((0, 0)\) is locally asymptotically stable if \( b > b_0 \) and is unstable if
\( b < b_0 \).

A positive fixed point \( E = (x, y) \) should satisfy

\[
\frac{a(1+y)}{(1+b+y)(1+\eta_1x)} \cdot \frac{\gamma}{1+\eta_2x} = 1,
\]

that is,

\[
b = \left( 1 + \frac{\gamma x}{1+\eta_2x} \right) \left( \frac{a\gamma}{(1+\eta_1x)(1+\eta_2x)} - 1 \right).
\]

Equivalently, we have:

\[
G(x) := Ax^3 + Bx^2 + Cx + b + 1 - a\gamma, \quad (5.3.2)
\]
where:

\[ A = \eta_1 \eta_2 (\eta_2 + \gamma + b \eta_2), \]

\[ B = \eta_1 \eta_2 (1 + b) + (\eta_1 + \eta_2)(\eta_2 + \gamma + b \eta_2), \]

\[ C = (\eta_1 + 2 \eta_2)b - (a \gamma \eta_2 + a \gamma^2 - 2 \eta_2 - \gamma - \eta_1). \]

Define

\[ b_1 := \frac{a \gamma \eta_2 + a \gamma^2 - 2 \eta_2 - \gamma - \eta_1}{\eta_1 + 2 \eta_2}. \]

Since \( b_0 = a \gamma - 1 \), under the condition \( b_1 < b_0 \), we know there exists a unique positive fixed point if \( b < b_0 \) and none positive fixed points if \( b \geq b_0 \), according to Descartess rule of sign [7,37].

Under the condition \( b_1 > b_0 \), there exists a unique positive fixed point if \( b < b_0 \) and no positive fixed points if \( b \geq b_1 \). But if \( b_0 < b < b_1 \), there exists some threshold value of releases \( b_c \in (b_0, b_1) \) such that if \( b > b_c \), \( b = b_c \) or \( b < b_c \), there exist no, one, or two positive solutions to equation \((5.3.2)\) and hence no, one, or two positive fixed points to system \((5.3.1)\), respectively. That is to say, there exist no, one, or two positive fixed points if \( b > b_c \), \( b = b_c \) or \( b < b_c \), respectively when \( b > b_0 \).

The threshold value \( b_c \) can be actually determined as follows. When \( b = b_c \), there is only one positive \( x \)-intercept in the graph, which means the local minimum
point should be on the $x$-axis, that is to say, $b_c$ should satisfy the following condition:

\[
\begin{cases}
G'(x^*) = 0, \\
G(x^*(b), b) = 0.
\end{cases}
\]

With the critical point $(x^*, y^*)$, there exist no, one, or two positive solutions if $G(x^*(b), b) > 0$, $G(x^*(b), b) = 0$, or $G(x^*(b), b) < 0$, respectively.

Let function $H(b)$ be defined as

\[H(b) := G(x^*(b), b) = A(x^*(b))^3 + B(x^*(b))^2 + C x^*(b) + b + 1 - a \gamma,\]

where $G'(x^*) = 0$. Since $A'(b) = \eta_1 \eta_2^2$, $B'(b) = \eta_1 \eta_2 + \eta_2 (\eta_1 + \eta_2)$, and $C'(b) = \eta_1 + 2 \eta_2$, it follows from

\[H'(b) = \partial_x G(x^*)'(b) + \partial_b G(x^*(b), b) = \partial_b G(x^*(b), b)\]

\[= 3A(x^*(b))^2(x^*)'(b) + 2B x^*(b)(x^*)'(b) + C(x^*)'(b)\]

\[+ A'(b)(x^*(b))^3 + B'(b)(x^*(b))^2 + C'(b)x^*(b) + 1\]

\[= A'(b)(x^*(b))^3 + B'(b)(x^*(b))^2 + C'(b)x^*(b) + 1 > 0\]

that $G(x^*(b), b)$ is an increasing function of $b$, which implies that there exists no, one, or two positive fixed points if $b > b_c$, $b = b_c$, or $b < b_c$, respectively.
We then investigate the stability of the positive fixed points. The Jacobian at a positive fixed point \( E = (x, y) \) has the form of

\[
\mathbf{J} := \begin{pmatrix}
-\frac{\eta_1 x}{1 + \eta_1 x} & \frac{(b + 2y) + (y + 1)^2}{\gamma(1 + y)(1 + y + b)} \\
\frac{\gamma}{(1 + \eta_2 x)^2} & 0
\end{pmatrix}.
\]

Since

\[
\text{tr} \mathbf{J} = -\frac{\eta_1 x}{1 + \eta_1 x}, \quad \text{det} \mathbf{J} = \frac{b(1 + 2y) + (y + 1)^2}{(1 + \eta_2 x)(1 + y + b)(1 + y)}.
\]

then \( E \) is locally asymptotically stable if and only if

\[
\frac{b(1 + 2y) + (y + 1)^2}{(1 + \eta_2 x)(1 + y + b)(1 + y)} < 1 - \frac{\eta_1 x}{1 + \eta_1 x},
\]

that is,

\[
[(1 + \eta_1 x)(1 + 2y) - (1 + \eta_2 x)(1 + y)] b < (\eta_2 - \eta_1)(1 + y)^2. \quad (5.3.3)
\]

Substituting \( y = \frac{\gamma x}{1 + \eta_2 x} \) for \( y \) in (5.3.3) leads to

\[
[(1 + \eta_1 x)(1 + \eta_2 x + 2\gamma x) - (1 + \eta_2 x)(1 + \eta_2 x + \gamma x)] \cdot b < \frac{(1 + \eta_2 x + \gamma x)^2(\eta_2 - \eta_1)}{1 + \eta_2 x}.
\]

Define

\[
b_s := \frac{(1 + \eta_2 x + \gamma x)^2(\eta_2 - \eta_1)}{(1 + \eta_2 x)(1 + \eta_1 x)(1 + \eta_2 x + 2\gamma x) - (1 + \eta_2 x)^2(1 + \eta_2 x + \gamma x)}. \quad (5.3.4)
\]
If the denominator is positive, then \( E \) is locally asymptotically stable if \( b < b_s \) and unstable if \( b > b_s \).

In summary, we have the following results.

**Theorem 5.3** Suppose \( \eta_1 < \eta_2 \). With the initial sterile mosquitoes release threshold defined as \( b_0 = a\gamma - 1 \), the trivial fixed point \((0,0)\) is locally asymptotically stable if \( b > b_0 \) and is unstable if \( b < b_0 \). If \( b < b_0 \), there exists a unique positive fixed point \( E^* \). For \( b > b_0 \), we define the releases threshold value of sterile mosquitoes from \( G'(x^*(b_c)) = 0 \) and \( G(x^*(b_c), b_c) = 0 \). Then there exist no, one, or two positive fixed points to system (5.3.1), if \( b > b_c \), \( b = b_c \) or \( b < b_c \), respectively. A positive fixed point \( E^* = (x^*, y^*) \) is locally asymptotically stable if \( b < b_s \), where \( b_s \) is given in (5.3.4), and is unstable if \( b > b_s \).

We give an example to demonstrate the existence and stability results for model system (5.3.1) as follows.

**Example 5.3** Choosing the following parameters

\[
a = 2.25, \quad \gamma = 0.8, \quad \eta_1 = 0.02, \quad \eta_2 = 0.03. \tag{5.3.5}
\]

We have \( b_0 = 0.8 \) and the threshold value of release \( b_c = 1.8987 \). If \( b = 0.05 < b_0 \), there exist a unique positive fixed point \( E^* = (13.5813, 7.7197) \), which has the corresponding \( b_s = 0.08808 > b = 0.05 \), so \( E^* \) is locally asymptotically stable as shown in Figure 5.7. If we choose \( b = 0.78 < b_0 \), the unique positive fixed point \( E^* = (11.1872, 6.7008) \) is unstable since \( b_s = 0.08081 < b = 0.78 \), which leads to appearance of a positive 2-cycle as shown in Figure 5.8. If \( b = 1 > b_0 \), there exist two positive
Figure 5.7: The threshold value $b_c = 1.8987$ and $b_s = 0.08808$. For $b = 0.05 < b_s$, the unique positive fixed point $E^* = (13.5813, 7.7197)$ is locally asymptotically stable.

Figure 5.8: The threshold value $b_c = 1.8987$ and $b_s = 0.08081$. For $b_s < b < 0.78 < b_0$, the positive fixed point $E^* = (11.1872, 6.7008)$ is unstable, which leads to a positive 2-cycle.
Figure 5.9: The threshold value $b_c = 1.8987$. For $b = 1$, the two positive fixed points $E^*_1 = (0.3870, 0.3060)$ and $E^*_2 = (10.3864, 6.3351)$ are unstable. Solution approaches the stable origin.

Figure 5.10: The threshold value $b_c = 1.8987$. For $b = 1$, the two positive fixed points $E^*_1 = (0.3870, 0.3060)$ and $E^*_2 = (10.3864, 6.3351)$ are unstable. A stable synchronous 2-cycle is bifurcated.
fixed points $E_1^* = (0.3870, 0.3060)$ and $E_2^* = (10.3864, 6.3351)$, and the trivial fixed point $(0, 0)$ is locally asymptotically stable. For $E_1^*$, the corresponding $b_s = 0.05623$, which implies $E_1^*$ is unstable. For $E_2^*$, the corresponding $b_s = 0.07810$, which implies $E_2^*$ is also unstable. A stable synchronous 2-cycle is bifurcated. Solutions either approach the origin or to this stable synchronous 2-cycle, as shown in Figure 5.9 and Figure 5.10, respectively.
CHAPTER 6

TRANSMISSION MODELS WITH CONSTANT RELEASES

We have studied dynamics of interactive wild and sterile mosquitoes with different release strategies in previous chapters. Now we are ready to incorporate the mosquito models into malaria transmission models and investigate the impact of releases of the sterile mosquitoes on the disease transmission dynamics. Our focus will be on the constant releases.

6.1 Baseline Model for Wild Mosquitoes

We first formulate general malaria models to study the transmission dynamics between mosquitoes and humans. Since the mosquitoes lifespan is usually shorter than their infective period, we assume that mosquitoes are unable to recover after infection. Then we divide the mosquito population into groups of susceptible, exposed or incubating [6], and infective individuals, and denote their numbers, at generation $n$, by $S_v(n)$, $E_v(n)$ and $I_v(n)$, respectively.

Let $\lambda_v(n)$ be the mosquito infection rate with $0 \leq \lambda_v(n) \leq 1$; $\gamma_v$ the rate of incubating individuals becoming infective, which is a constant with $0 \leq \gamma_v \leq 1$; $b_v$ the per capita birth rate; $\alpha_{b_v}, \alpha_{1_v}, \alpha_{2_v}, \alpha_{3_v}, \alpha_{4_v}$ and $\alpha_{5_v}$ the survival probabilities of the
newborns, susceptible adults, incubating adults and infective adults from generation $n$ to generation $n+1$, respectively. We assume that the infection has no effects on the birth and all newborns that the intraspecific competition only affects the survivability of the susceptible newborns, such that $\alpha_b^v = \alpha_b^v(N^v(n))$ with Beverton-Holt -II type of nonlinearity as

$$\alpha_b^v(N(n)) := \frac{k_b^v}{1 + \eta_b^v N^v(n)}; \quad (6.1.1)$$

where $N^v(n) = S^v(n) + E^v(n) + I^v(n)$ is the total wild mosquito population, $0 \leq k_b^v \leq 1$ is the maximum survival probability, and $\eta_b^v \geq 0$ is the density-dependant factor. The other survival probabilities $\alpha_k^v$, $k = 2,...,5$, $b^v$ and $\gamma^v$ are all constants and also less than 1.

Then, the model for wild mosquitoes can be illustrated as in Figure 6.1 and the equations are given by:

$$S^v(n+1) = b^v N^v(n) \frac{k_b^v}{1 + \eta_b^v N^v(n)} + (1 - \lambda^v(n))S^v(n)\alpha_1^v,$$

$$E^v(n+1) = \lambda^v(n)S^v(n)\alpha_2^v + (1 - \gamma^v)E^v(n)\alpha_3^v, \quad (6.1.2)$$

$$I^v(n+1) = \gamma^v E^v(n)\alpha_4^v + I^v(n)\alpha_5^v.$$

The model in (6.1.2) is much general. If the unit time step is larger than the incubating period, no exposed individuals at generation $n$ will stay in this group at generation $n+1$, which implies that $\alpha_3^v = 0$. But if the unit time step is shorter than the incubating period, a certain proportion of exposed or incubating individuals have not become infective so that they stay in $E^v$ group at generation $n+1$, which leads to $\alpha_3^v > 0$. 

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The mosquito population is divided into three groups, $S^v$, $E^v$ and $I^v$. The mosquito infection rate $\lambda^v(n)$ is given in (6.1.4); $\gamma^v$ is the rate of incubating individuals becoming infective; $b^v$ is the per capita birth rate; $\alpha^v_1$, $\alpha^v_2$, $\alpha^v_3$, $\alpha^v_4$ and $\alpha^v_5$ the survival probabilities of the newborns, susceptible adults, incubating adults and infective adults from generation $n$ to generation $n+1$, respectively.

For the human part in our malaria transmission model, we divide the human population into groups of susceptible, exposed or incubating, infective and recovered individuals. The infection dynamics among humans during the same period of time are relatively simpler than the dynamics among mosquitoes.

Then, we let $S^h(n)$ be the number of susceptible humans, $E^h(n)$ the number of exposed or incubating humans, who are infected but not infectious yet, $I^h(n)$ the number of infective humans, who are infected and also infectious, $R^h(n)$ the number of humans who are recovered from infection but partly lose their immunity [45,60–62], and $N^h(n) = S^h(n)+E^h(n)+I^h(n)+R^h(n)$ the total number of humans at generation $n$. We assume $\alpha^h_k$, $k = 1, ..., 8$, as the survival probabilities of susceptible, exposed or incubating, infective and recovered humans, respectively, which are all constants.
Figure 6.2: The human population is divided into four groups, $S^h$, $E^h$, $I^h$, and $R^h$. The human infection rate $\lambda^h(n)$ is given in (6.1.5); $\gamma^h$ is the rate of incubating individuals becoming infective; $\Lambda$ is the constant input flow of the susceptible humans; $\alpha_k^h$, $k = 1, \ldots, 8$, as the survival probabilities of susceptible, exposed or incubating, infective and recovered humans from generation $n$ to generation $n + 1$, respectively.

Then, the model for humans as shown in Figure 6.2 are given by

\begin{align}
S^h(n + 1) &= \Lambda + (1 - \lambda^h(n))S^h(n)\alpha_1^h + \theta^h R^h(n)\alpha_8^h, \\
E^h(n + 1) &= \lambda^h(n)S^h(n)\alpha_2^h + (1 - \gamma^h)E^h(n)\alpha_3^h, \\
I^h(n + 1) &= \gamma^hE^h(n)\alpha_4^h + (1 - \eta^h)I^h(n)\alpha_5^h, \\
R^h(n + 1) &= \eta^h I^h(n)\alpha_6^h + (1 - \theta^h)R^h(n)\alpha_7^h, \tag{6.1.3}
\end{align}

where $\Lambda$ is the constant input flow of the susceptible humans, $\gamma^h$ is the developing rate of incubating humans becoming infective, such that $1/\gamma^h$ is the incubating period, $\eta^h$ is the recovery rate for infective humans and $\theta^h$ is the rate of partial immunity loss.

To determine the infection rate, $\lambda^e(n)$ and $\lambda^h(n)$, we let $r$ be the number of average bites by a single mosquito on all human hosts, and $\beta^h$ the transmission probability per bite to a susceptible mosquito from an infective human. Then, the
infection rate for mosquitoes is given by

\[ \lambda^v(n) = \beta^h r \frac{I^h(n)}{N^h(n)}, \]  

(6.1.4)

where we assume \( \beta^h r < 1 \).

Similarly, the infection rate for humans is determined by the factor:

\[ L(n) := \beta^v r_1 \frac{I^v(n)}{N^v(n)}, \]

where \( r_1 \) is the number of average bites on a human host by all mosquitoes, and \( \beta^v \) is the transmission probability per bite to a susceptible human from an infective mosquito.

Notice that \( r \) and \( r_1 \) need to satisfy the following balance constraint:

\[ rN^v(n) = r_1 N^h(n). \]

Thus, the factor to determine the infection rate for humans becomes

\[ L(n) = \beta^v r N^v(n) \frac{I^v(n)}{N^v(n)} = \beta^v r \frac{I^v(n)}{N^h(n)}. \]

Since the number of infected mosquitoes can be sufficiently larger than the total number of humans which may lead to the factor \( L(n) \) much greater than 1, we
assume that the infection rate for humans is given by:

\[ \chi^h(n) = G(L(n)), \quad (6.1.5) \]

where \( G \) is a positive function of \( L \), satisfying the following conditions [21]:

\[ G : [0, \infty) \to [0, 1], \quad G(0) = 0, \quad G'(L) > 0, \quad G''(L) < 0, \]
\[ \forall L \geq 0, \quad \lim_{L \to \infty} G(L) = 1. \quad (6.1.6) \]

### 6.1.1 The Reproductive Number

We derive a formula for the reproductive number by investigating the local stability of the infection-free equilibrium. The Jacobian matrix at the infection-free equilibrium

\[ (E^v, I^v, E^h, I^h, S^v, S^h, R^h) = (0, 0, 0, S^v_0, S^h_0, 0), \]

where

\[ S^v_0 = \frac{k^v b^v}{\eta^v (1 - \alpha^v_1)}; \quad S^h_0 = \frac{\Lambda}{1 - \alpha^h_1}, \quad (6.1.7) \]

has the form of

\[ J_1 := \begin{pmatrix} F + T & 0 \\ \cdot & C \end{pmatrix}. \quad (6.1.8) \]
where $F$ is the fertility matrix given by

$$F = \begin{pmatrix}
0 & 0 & 0 & r \beta^h \alpha^v_2 \frac{s^v}{s^h_0} \\
0 & 0 & 0 & 0 \\
0 & G'(0) r \beta^v \alpha^h_2 & 0 & 0 \\
0 & 0 & 0 & 0 
\end{pmatrix},$$

and $T$ is the transition matrix given by

$$T = \begin{pmatrix}
(1 - \gamma^v) \alpha^v_3 & 0 & 0 & 0 \\
\gamma^v \alpha^v_4 & \alpha^v_5 & 0 & 0 \\
0 & 0 & (1 - \gamma^h) \alpha^h_3 & 0 \\
0 & 0 & \gamma^h \alpha^h_4 & (1 - \eta^h) \alpha^h_5 
\end{pmatrix}.$$  

Matrix $C$ has the form

$$C = \begin{pmatrix}
c_{11} & 0 & 0 \\
0 & \alpha^h_1 & \theta^h \alpha^h_3 \\
0 & 0 & (1 - \theta^h) \alpha^h_7 
\end{pmatrix},$$

where

$$c_{11} = \frac{k^v b^v}{1 + \eta^v_0 S^v_0} - \frac{k^v b^v \eta^v_0 S^v}{(1 + \eta^v_0 S^v_0)^2} + \alpha^v_1 < 1,$$

since

$$c_{11} - 1 = (1 - \alpha^v_1) \left( \frac{1 - \alpha^v_1}{k^v b^v} - 1 \right) < 0.$$
Thus we have all eigenvalues of matrix $C$ that are inside the unit circle, and the local stability of the infection-free fixed point is determined by matrix $F + T$.

Simple algebra yields

$$
(I - T)^{-1} = \begin{pmatrix}
\frac{1}{1 - (1 - \gamma^v\alpha^v_3)} & 0 & 0 & 0 \\
\frac{\gamma^v\alpha^v_4}{(1 - \alpha^v_5)(1 - (1 - \gamma^v\alpha^v_3))} & \frac{1}{1 - \alpha^v_5} & 0 & 0 \\
0 & 0 & \frac{1}{1 - (1 - \gamma^h\alpha^h_3)} & 0 \\
0 & 0 & \frac{tt}{1 - (1 - \eta^h\alpha^h_5)} & 1
\end{pmatrix},
$$

where $tt = \frac{\gamma^h\alpha^h_4}{(1 - (1 - \gamma^h\alpha^h_3)(1 - (1 - \eta^h\alpha^h_5))}$. Then, the next generation matrix $[1, 22, 25, 26]$, is given by

$$
Q = F(I - T)^{-1} = \begin{pmatrix}
0 & 0 & q_{13} & q_{14} \\
0 & 0 & 0 & 0 \\
q_{31} & q_{32} & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
$$

where

$$
q_{13} := \frac{r\beta^h\alpha^h_2\alpha^v_5 S^v_0}{S^h_0(1 - (1 - \gamma^h\alpha^h_3)(1 - (1 - \eta^h\alpha^h_5))},
$$

$$
q_{14} := \frac{r\beta^h\alpha^v_2 S^v_0}{S^h_0(1 - (1 - \eta^h\alpha^h_5)},
$$

$$
q_{31} := \frac{G'(0)r\alpha^h_2\beta^v\gamma^v\alpha^v_4}{(1 - \alpha^v_5)(1 - (1 - \gamma^v\alpha^v_3))},
$$

$$
q_{32} := \frac{G'(0)r\alpha^h_2\beta^v}{1 - \alpha^v_5}.
$$
The characteristic polynomial of $Q$ is

$$\det(\rho I - Q) = \rho^2(\rho^2 - q_{13}q_{31}) = \rho^2(\rho^2 - R_0^2).$$

Then define the reproductive number of infection for system (6.1.2) and (6.1.3) as

$$R_0 := \sqrt{\frac{G'(0)r\beta_h \gamma_h \alpha_2^h \alpha_4^h}{(1 - (1 - \gamma^h)\alpha_3^h)(1 - (1 - \eta^h)\alpha_5^h)} \cdot \frac{r\beta^v \gamma^v \alpha_2^v \alpha_4^v}{(1 - \alpha_5^v)(1 - (1 - \gamma^v)\alpha_3^v)} \cdot \frac{S_0^v}{S_0^h}},$$

(6.1.9)

where $S_0^v = \frac{k^v b^v - (1 - \alpha_1^v)}{\eta^v (1 - \alpha_1^v)}$, $S_0^h = \frac{\Lambda}{1 - \alpha_4^h}$.

Thus, the infection-free fixed point is locally asymptotically stable if $R_0 < 1$, and is unstable if $R_0 > 1$.

6.1.2 The Endemic Equilibrium

We next explore the existence of endemic equilibrium of system (6.1.2) and (6.1.3) as $R_0 > 1$.

The components of an endemic fixed point satisfy the following equations:
\[ S^h = \Lambda + (1 - \lambda^h)S^h \alpha_1^h + \theta^h R^h \alpha_8^h, \]  
(6.1.10a)

\[ E^h = \lambda^h S^h \alpha_2^h + (1 - \gamma^h)E^h \alpha_3^h, \]  
(6.1.10b)

\[ I^h = \gamma^h E^h \alpha_4^h + (1 - \eta^h)I^h \alpha_5^h, \]  
(6.1.10c)

\[ R^h = \eta^h I^h \alpha_6^h + (1 - \theta^h)R^h \alpha_7^h, \]  
(6.1.10d)

\[ S^v = \frac{k_{b\nu} N^v}{1 + \eta_b^v N^v} + (1 - \lambda^v)S^v \alpha_1^v, \]  
(6.1.10e)

\[ E^v = \lambda^v S^v \alpha_2^v + (1 - \gamma^v)E^v \alpha_3^v, \]  
(6.1.10f)

\[ I^v = \gamma^v E^v \alpha_4^v + I^v \alpha_5^v, \]  
(6.1.10g)

Solving (6.1.10a) - (6.1.10d), we have

\[ E^h = \omega_1^h \lambda^h S^h, \]

\[ I^h = \omega_1^h \omega_2^h \lambda^h S^h, \]  
(6.1.11)

\[ R^h = \omega_1^h \omega_2^h \omega_3^h \lambda^h S^h, \]

\[ S^h = \frac{\Lambda}{1 - \alpha_1^h + \lambda_1^h \lambda^h}, \]
where,

\[
\omega_1^h := \frac{\alpha_2^h}{1 - (1 - \gamma^h)\alpha_3^h}, \\
\omega_2^h := \frac{\gamma^h\alpha_4^h}{1 - (1 - \eta^h)\alpha_5^h}, \\
\omega_3^h := \frac{\eta^h\alpha_6^h}{1 - (1 - \theta^h)\alpha_7^h}, \\
A_1^h := \alpha_1^h - \theta^h\alpha_8^h\omega_1^h\omega_2^h\omega_3^h.
\]

Thus, we have

\[
N^h = \frac{\Lambda(1 + A_2^h\lambda^h)}{1 - \alpha_1^h + A_1^h\lambda^h}, 
\]  
(6.1.12)

where \(A_2^h := \omega_1^h + \omega_1^h\omega_2^h + \omega_1^h\omega_2^h\omega_3^h\).

Solving (6.1.10e) - (6.1.10g), we have

\[
S^v = \frac{k_b^v b^v N^v}{(1 - (1 - \lambda^v)\alpha_1^v)(1 + \eta_b^v N^v),} \quad E^v = \omega_1^v \lambda^v S^v, \quad I^v = \omega_1^v \omega_2^v \lambda^v S^v, \quad (6.1.13)
\]

where

\[
\omega_1^v := \frac{\alpha_2^v}{1 - (1 - \gamma^v)\alpha_3^v}, \quad \omega_2^v := \frac{\gamma^v\alpha_4^v}{1 - \alpha_5^v}.
\]

Thus, it follows from \(S^v(n) + E^v(n) + I^v(n) = N^v(n)\) that

\[
N^v = \frac{k_b^v b^v(1 + \lambda^v B_1^v) - (1 - (1 - \lambda^v)\alpha_1^v)}{\eta_b^v(1 - (1 - \lambda^v)\alpha_1^v)}, 
\]  
(6.1.14)

where

\[
B_1^v := \omega_1^v + \omega_1^v\omega_2^v = \frac{\alpha_2^v}{1 - (1 - \gamma^v)\alpha_3^v} \left(1 + \frac{\gamma^v\alpha_4^v}{1 - \alpha_5^v}\right).
\]
Then,

\[ S^v = \frac{k_v b_v}{\eta_b^v (1 - (1 - \lambda^c) \alpha_1^v)} = \frac{1}{\eta_b^v (1 + \lambda^c B^v_1)}. \]  

(6.1.15)

Substituting (6.1.11), (6.1.12) and (6.1.13) into (6.1.4) and (6.1.5), respectively, we obtain

\[ \lambda^v = \beta_r \frac{I^h}{N^h} = \beta_r \frac{\omega_1^h \omega_2^h \lambda^h}{1 + A_2^h \lambda^h}, \]  

(6.1.16)

and

\[
L(\lambda^h) = \beta_r \frac{I^v}{N^v} = \frac{\beta_r B_2^v \alpha_2^v S^v (1 - \alpha_1^h + A_1^h \lambda^h)}{\Lambda (1 + A_2^h \lambda^h)} \lambda^v \\
= \frac{\beta_r B_2^h \alpha_2^h S^v (1 - \alpha_1^h + A_1^h \lambda^h)}{\Lambda (1 + A_2^h \lambda^h)} \frac{\beta_r \omega_1^h \omega_2^h \lambda^h}{1 + A_2^h \lambda^h} \\
= R_0^2 \frac{1 - \alpha_1^h + A_1^h \lambda^h}{G'(0)(1 - \alpha_1^h)(1 + A_2^h \lambda^h)^2} \frac{S^v(\lambda^h)}{S_0^v} \lambda^h. 
\]  

(6.1.17)

Hence, there exists an endemic fixed point if and only if there is a positive solution to equation \( \lambda^h = G(L(\lambda^h)) \), or equivalently, \( H(\lambda^h) := G(L(\lambda^h)) - \lambda^h \), for \( 0 \leq \lambda^h \leq 1 \).

Notice that \( L(0) = 0 \) and hence \( G(L(0)) = 0 \) by (6.1.6). Then \( H(0) = 0 \) and \( H(1) < 0 \) since \( G'(L) > 0 \) and \( \lim_{L \to \infty} G(L) = 1 \). Moreover, it follows from \( S^v(0) = S_0^v \) and \( L'(0) = R_0^2 / G'(0) \) that

\[ H'(0) = G'(0) L'(0) - 1 = R_0^2 - 1. \]

Thus, if \( R_0 > 1 \), there exists a positive solution to \( H(\lambda^h) = 0 \), that is, to \( \lambda^h = G(L(\lambda^h)) \). Therefore, when \( R_0 > 1 \), the infection-free equilibrium is unstable and there exists an endemic equilibrium.
6.2 The Interactive Transmission Model with Wild and Sterile Mosquitoes

We now suppose sterile mosquitoes are released into a wild mosquito population and we let $B(n)$ be the number of the sterile mosquitoes released at generation $n$. Since sterile mosquitoes do not reproduce, there is no maturation process from larvae to adults for sterile mosquitoes. Hence, the number of sterile mosquitoes at generation $n$ is just the number of released sterile mosquitoes, and the size of total mosquitoes is $N^v(n) + B(n)$ at generation $n$. We assume sterile mosquitoes are constantly released in each generation so that $B(n) := b > 0$ is a positive constant. After the sterile mosquitoes are released, the mating interaction between the wild and sterile mosquitoes takes place. The number of susceptible offspring produced per mating is

$$\frac{b^v N^v(n)}{N^v(n) + b}.$$ 

Then, the model for mosquitoes can be described as follows:

$$S^v(n + 1) = \frac{k^v}{1 + \eta^v N^v(n)} N^v(n) \frac{b^v N^v(n)}{N^v(n) + b} + (1 - \lambda^v(n)) S^v(n) \alpha^v_1,$$

$$E^v(n + 1) = \lambda^v(n) S^v(n) \alpha^v_2 + (1 - \gamma^v) E^v(n) \alpha^v_3,$$ \hspace{1cm} (6.2.1)

$$I^v(n + 1) = \gamma^v E^v(n) \alpha^v_4 + I^v(n) \alpha^v_5.$$

For the human part in our malaria transmission model, it remains the same as described in system (6.1.3).

Assume all the survival probability $\alpha^v_i$, $i = 1, 2, 3, 4, 5$ in system (6.2.1) are the same, denoted as $\alpha^v$. Then the interactive dynamics between the total wild
mosquitoes and constant sterile mosquitoes are governed by the following system

\[
N^v(n + 1) = \frac{k^v b^v (N^v(n))^2}{(1 + \eta^v_b N^v(n))(N^v(n) + b)} + \alpha^v N^v(n).
\] (6.2.2)

It follows from Chapter 3 that the results for system (6.2.2) can be stated as follows.

**Lemma 6.2** (Theorem 3.2 [Chapter 3]). For given \( r_0 = k^v b^v \), system (6.2.2) has no, one, or two positive fixed points if \( b > b_c \), \( b = b_c \), or \( b < b_c \), respectively, where the release threshold \( b_c \) is given by

\[
b_c := \frac{\sqrt{k^v b^v} - \sqrt{1 - \alpha^v}}{\eta^v_b (1 - \alpha^v)}.
\] (6.2.3)

If \( b > b_c \), there exists no positive fixed point and \( w = 0 \) is the only fixed point which is globally asymptotically stable. Thus the wild mosquito population goes extinct regardless of the initial population size. If \( b = b_c \), there exists a unique positive fixed point which is unstable and thus the trivial fixed point \( w = 0 \) is globally asymptotically stable. If \( b < b_c \), there exist two positive fixed points which are

\[
N^{vb(\pm)} = \frac{k^v b^v - (1 + b \eta^v_b)(1 - \alpha^v) \pm \sqrt{(k^v b^v - (1 + b \eta^v_b)(1 - \alpha^v))^2 - 4b \eta^v_b (1 - \alpha^v)}}{2 \eta^v_b (1 - \alpha^v)},
\] (6.2.4)

where \( N^{vb(-)} \) is unstable and \( N^{vb(+)} \) is locally asymptotically stable with a basin of attraction \((N^{vb(-)}, \infty)\) while the trivial fixed point \( w = 0 \) is locally asymptotically stable with basin of attraction \((0, N^{vb(-)})\).
Figure 6.3: Solutions approach $N^v = 0$ or $N^{vb(+)}$ depending on their initial values, $y = N^{vb(-)}$ is the dividing line of stable manifolds.

Notice that if $b > b_c$, there exist no positive fixed point and only one globally stable trivial fixed point for mosquitoes, which leads to all wild mosquitoes eventually going extinct. Then there will be no malaria transmission. Thus we only consider $b < b_c$, and for given $b < b_c$, we assume the initial sizes of wild mosquitoes is larger than $N^{vb(-)}$, as shown in Figure 6.3.

6.2.1 The Reproductive Number and Disease Spread

We derive a formula for the reproductive number of infection after the sterile mosquitoes are released into the wild mosquito population with $b < b_c$ and the initial size in $(N^{vb(-)}, \infty)$. 

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The infection-free equilibrium of system (6.2.1) and (6.1.3) has the following form:

\[(E^v, I^v, E^h, I^h, S^v, S^h, R^h) = (0, 0, 0, 0, S_0^{vb}, S_0^{hb}, 0).\]

It’s obvious to find \(S_0^{hb} = S_0^h\) satisfying

\[S_0^{hb} := \frac{\Lambda}{1 - \alpha_1^h} \tag{6.2.5}\]

For \(S_0^{vb}\), it follows from model (6.2.1) that

\[S_0^{vb} = \frac{k^v_0 S_0^{vb}}{1 + \eta^v_0 S_0^{vb}} \cdot \frac{b^v S_0^{vb}}{S_0^{vb} + b} + S_0^{vb} \alpha_1^v,\]

or equivalently,

\[(1 - \alpha_1^v)(1 + \eta^v_0 S_0^{vb})(b + S_0^{vb}) = b^v k^v_0 S_0^{vb},\]

which gives the form of \(b\) as

\[b = \frac{k^v_0 b^v S_0^{vb}}{(1 - \alpha_1^v)(1 + \eta^v_0 S_0^{vb})} - S_0^{vb}, \tag{6.2.6}\]

where \(k^v_0 b^v \geq (1 - \alpha_1^v)(1 + \eta^v_0 S_0^{vb})\) since the release value \(b \geq 0\).

We let the function \(P(b) := \Phi(S_0^{vb}(b), b)\) be defined as

\[\Phi(S_0^{vb}(b), b) := (1 - \alpha_1^v)\eta^v_0 S_0^{vb} + [(1 - \alpha_1^v)(1 + \eta^v_0 b) - k^v_0 b^v] S_0^{vb} + b(1 - \alpha_1^v), \tag{6.2.7}\]

where \(k^v_0 b^v \geq (1 - \alpha_1^v)(1 + \eta^v_0 S_0^{vb})\).
Set
\[ b_0 := \frac{k_v b - (1 - \alpha_1^v)}{\eta_b^v (1 - \alpha_1^v)}. \]

There’s no positive solution of (6.2.7) if \((1 - \alpha_1^v)(1 + b\eta_b^v) - k_v b^v \geq 0\), that is to say, there’s no positive solution of (6.2.7) if \(b \geq b_0\).

Now assume \(b < b_0\), there’s no positive solution, one positive solution, or two positive solutions if \(\Delta < 0, \Delta = 0, \Delta > 0\), respectively with

\[ \Delta := \left[ \eta_v^v (1 - \alpha_1^v) b - (1 - \alpha_1^v + k_v b^v) \right]^2 - 4 k_v^2 b^v (1 - \alpha_1^v). \]

From \(\Delta = 0\), we get two threshold values for releases \(b\):

\[ b_1 = \frac{k_v^v b^v + (1 - \alpha_1^v) - 2 \sqrt{k_v^v b^v (1 - \alpha_1^v)}}{\eta_b^v (1 - \alpha_1^v)}, \]
\[ b_2 = \frac{k_v^v b^v + (1 - \alpha_1^v) + 2 \sqrt{k_v^v b^v (1 - \alpha_1^v)}}{\eta_b^v (1 - \alpha_1^v)}, \]  
(6.2.8)

where \(b_c = b_1\) from (6.2.3).

The threshold values have the following relationship

\[ b_c = b_1 < b_0 < b_2. \]

Then, there’s no positive solution, one positive solution, or two positive solutions if \(b > b_c, b = b_c, \) or \(b < b_c\), respectively. Therefore, in the case \(b < b_c\), there exist two infection-free equilibria.

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The Jacobian matrix $J_b$ evaluated at the infection-free fixed point

$$(E^v, I^v, E^h, I^h, S^v, S^h, R^h) = (0, 0, 0, S^{vb}_0, S^{vh}_0, 0),$$

has the form of

$$J_b := \begin{pmatrix} F_b + T_b & 0 \\ \cdot & C_b \end{pmatrix},$$

where $F_b$ is the fertility matrix satisfied $F_b = F$ and $T_b$ is the transition matrix satisfied $T_b = T$. Matrices $F$ and $T$ are given in (6.1.8).

Matrix $C_b$ has the form

$$C_b = \begin{pmatrix} c_{11}^b & 0 & 0 \\ 0 & \alpha_4^h & 0 \\ 0 & 0 & (1 - \theta^h)\alpha_2^b \end{pmatrix},$$

where

$$c_{11}^b = \frac{k_b^v b^v S^{vb}_0 [2(1 + \eta_b^v S^{vb}_0)(b + S^{vb}_0) - S^{vb}_0(1 + 2\eta_b^v S^{vb}_0 + b\eta_b^v)]}{(1 + \eta_b^v S^{vb}_0)^2(b + S^{vb}_0)^2}$$

$$= 2 - \frac{S^{vb}_0(1 + 2\eta_b^v S^{vb}_0 + b\eta_b^v)}{(1 + \eta_b^v S^{vb}_0)(b + S^{vb}_0)}$$

$$= 2 - \frac{(1 - \alpha_1^v)(1 + 2\eta_b^v S^{vb}_0 + b\eta_b^v)}{k_b^v b^v}$$

since

$$(1 + \eta_b^v S^{vb}_0)(b + S^{vb}_0) = \frac{k_b^v b^v S^{vb}_0}{1 - \alpha_1^v}.$$
It follows from (6.2.7) that

\[
\partial_{s_{vb}} \Phi(s_{vb}(b), b) = 2(1 - \alpha_1^v) \eta_b^v s_{vb}^b + (1 - \alpha_1^v)(1 + b \eta_b^v) - k_b^v b^v
\]

\[
= (1 - \alpha_1^v)(1 + 2 \eta_b^v s_{vb}^b + b \eta_b^v) - k_b^v b^v,
\]

which means for the two positive solutions \(s_{vb}^b(-) < s_{vb}^b(+),\) we have

\[
\partial_{s_{vb}(-)} \Phi < 0, \quad \partial_{s_{vb}(+)} \Phi > 0;
\]

that is

\[
\frac{(1 - \alpha_1^v)(1 + 2 \eta_b^v s_{vb}^b(-) + b \eta_b^v)}{k_b^v b^v} < 1, \quad \frac{(1 - \alpha_1^v)(1 + 2 \eta_b^v s_{vb}^b(+) + b \eta_b^v)}{k_b^v b^v} > 1.
\]

Therefore, we have \(c_{11}^b > 1\) for \(s_{vb}^b(-),\) which indicates \(s_{vb}^b(-)\) is unstable since there exists one eigenvalue of matrix \(C\) outside of the unit circle. We have \(c_{11}^b < 1\) for \(s_{vb}^b(+)\), which implies all eigenvalues of matrix \(C\) are inside the unit circle, and the local stability of this infection-free fixed point is determined by matrix \(F_b + T_b.\)

Then, it follows from Section 6.1 that

\[
R_0^b := \sqrt{\frac{G'(0)r^b \gamma^h \alpha_2^h \alpha_4^h}{(1 - (1 - \gamma^h) \alpha_2^h)(1 - (1 - \eta^h) \alpha_3^h)} \cdot \frac{r^b \gamma^v \alpha_2^v \alpha_4^v}{(1 - \alpha_5^v)(1 - (1 - \gamma^v) \alpha_3^v)} \cdot \frac{s_{vb}^{b(+)}}{S_0^h}},
\]

(6.2.9)

where \(s_{vb}^{b(+)}/S_0^h\) is given in (6.2.4) with \(s_{vb}^{b(+)}/S_0^h = N^{vb(+)}.\)
Using $b$ as a variable, we note that $S_0^{v(b)}(b)$ is a function of $b$ and so is $R_0^b$.

When $b = 0$, it is clear that $S_0^{v(b)}(b) = S_0^v$ and $R_0^b = R_0$, given in (6.1.9). Then

$$R_0^b(b) = \sqrt{\frac{S_0^{v(b)}(b)}{S_0^v}}R_0.$$ 

Moreover, for $0 < b < b_c$, $0 < S_0^{v(b)}(b) < S_0^v$, and thus $R_0^b < R_0$.

It follows from (6.2.7) that

$$P'(b) = \partial_{S_0^v} \Phi(S_0^{v(b)}(b), b) \cdot S_0^{v(b)}(b) + (1 - \alpha_1^v)(1 + \eta_b S_0^{v(b)}) = 0,$$

and we have

$$S_0^{v(b)}(b) = \frac{(1 - \alpha_1^v)(1 + \eta_b S_0^{v(b)})}{\partial_{S_0^v} \Phi(S_0^{v(b)}(b), b)} < 0,$$ (6.2.10)

since $\partial_{S_0^{v(b)}} \Phi > 0$, which indicates $S_0^{v(b)}$ is a decreasing function with respect to the release value $b$. But we know $R_0^b$ is an increasing function with respect to $S_0^{v(b)}(b)$; that is to say, $R_0^b$ is a decreasing function with respect to $b$. Then, there is a unique threshold $\bar{b}$ such that

$$R_0^b(\bar{b}) = 1,$$

and

$$R_0^b(b) \begin{cases} > 1, & if \ b < \bar{b}, \\ < 1, & if \ b > \bar{b}. \end{cases}$$ (6.2.11)

Indeed, threshold $\bar{b}$ can be explicitly solved as follows.
Since $1 = \sqrt{\frac{S_{0}^{vb}(+) (\bar{b})}{S_{0}^{v}}} R_{0}$, we have

$$S_{0}^{vb}(+) (\bar{b}) = \frac{S_{0}^{v}}{R_{0}^{2}}.$$  \hfill (6.2.12)

According to (6.2.6), we obtain

$$\bar{b} = \frac{k_{b}^{v} b^{v} S_{0}^{vb}}{(1 - \alpha^{v}_{1})(1 + \eta_{b}^{v} S_{0}^{vb})} - S_{0}^{vb} = \frac{k_{b}^{v} b^{v} S_{0}^{v}}{(1 - \alpha^{v}_{1})(R_{0}^{2} + \eta_{b}^{v} S_{0})} - \frac{S_{0}^{v}}{R_{0}^{2}}. \hfill (6.2.13)$$

Thus, the infection-free equilibrium of system (6.2.1) and (6.1.3) is locally asymptotically stable if $b > \bar{b}$ and unstable if $b < \bar{b}$.

We also notice that if $b = b_{c}$, the infection-free equilibrium with the unique positive component $S_{0}^{vb}(+)$ is unstable and if $b > b_{c}$, there exists no such positive component $S_{0}^{vb}(+)$. That is to say, all wild mosquitoes will be wiped out if $b \geq b_{c}$ and hence $\bar{b} < b_{c}$. We summarize the results as follows.

**Theorem 6.2** Assume sterile mosquitoes are released into the wild mosquito population constantly with the rate of releases $b$. Define two threshold values of releases $b_{c}$ and $\bar{b}$ in (6.2.3) and (6.2.13), respectively. Then we have the following.

- If $b > b_{c}$, there exists no positive equilibrium of the interactive mosquitoes system (6.2.2) and the only trivial fixed point which has $S_{0}^{vb} = 0$ as a component for mosquitoes is globally asymptotically stable. All wild mosquitoes are wiped out and there will be no infection.
• If $b = b_c$, the unique positive fixed point of system (6.2.2) is unstable and the trivial fixed point is globally asymptotically stable. All wild mosquitoes are wiped out as well and hence there is no infection.

• If $\bar{b} < b < b_c$, the sterile mosquito and wild mosquitoes coexist, but the infection-free equilibrium of system (6.1.3) and (6.2.1) has the reproductive number $R^b_0 < 1$, which leads to this infection-free equilibrium locally asymptotically stable. Then the infection will eventually go extinct.

• If $b < \bar{b}$, $R^b_0 > 1$, which makes the infection-free equilibrium of system (6.2.1) and (6.1.3) unstable. The disease spreads when the initial size of wild mosquito larger than $N^{vb(-)}$, given in (6.2.4).

6.2.2 The Endemic equilibrium of Interactive Model

Similarly as in Section 6.1.2, we determine the existence of endemic equilibria of system (6.1.3) and (6.2.1) as follows.

The components of endemic equilibrium for human part are the same as those in Section 6.1.2. The components of wild mosquitoes at an endemic equilibrium satisfy the following system

\begin{align*}
S^v &= \frac{k^v_b}{1 + \eta^v_b N^v} N^v \frac{b^v N^v}{N^v + b} + (1 - \lambda^v) S^v \alpha^v_1, \quad (6.2.14a) \\
E^v &= \lambda^v S^v \alpha^v_2 + (1 - \gamma^v) E^v \alpha^v_3, \quad (6.2.14b) \\
I^v &= \gamma^v E^v \alpha^v_4 + I^v \alpha^v_5, \quad (6.2.14c)
\end{align*}
which leads to

\[ N^v = \frac{k_i^v b^v N^{v2}}{(1 + \eta_b^v N^v)(b + N^v)} + \alpha N^v, \]

with \( \alpha_i^v = \alpha \) for \( i = 1, \ldots, 5 \). Then \( N^v = N^{vb(+)\prime} \), given in (6.2.4).

Solving (6.2.14a) - (6.2.14c), we have

\[
S^v = \frac{k_i^v b^v N^{v2}}{(1 + \eta_b^v N^v)(b + N^v)(1 - (1 - \gamma^v)\alpha_1^v)},
\]

\[
E^v = \frac{\lambda^v \alpha_2^v}{1 - (1 - \gamma^v)\alpha_3^v} S^v = B_2^v \lambda^v S^v,
\]

\[
I^v = \frac{\gamma^v \alpha_4^v}{1 - \alpha_5^v}, \quad \frac{\lambda^v \alpha_2^v}{1 - (1 - \gamma^v)\alpha_3^v} S^v = B_3^v \lambda^v S^v,
\]

where

\[
B_2^v := \frac{\alpha_2^v}{1 - (1 - \gamma^v)\alpha_3^v},
\]

\[
B_3^v := \frac{\gamma^v \alpha_4^v \alpha_2^v}{(1 - \alpha_5^v)(1 - (1 - \gamma^v)\alpha_3^v)}.
\]

Then, we have

\[
S^v = \frac{N^v}{1 + \lambda^v (B_2^v + B_3^v)},
\]

Similarly,

\[
\lambda^v = \beta^h r \frac{I^h}{N^h} = \beta^h r \frac{\omega^h}{1 + A_2^h \lambda^h}.
\]
Substituting (6.2.15) into the factor $L(t)$ of (6.1.5) yields

$$L(\lambda^h) = \beta^v r \frac{I^v}{N^h} = \frac{\beta^v r B^v_2 \alpha^v_2 S^v(1 - \alpha^h_1 + A^h_1 \lambda^h)}{\Lambda(1 + A^h_2 \lambda^h)} \lambda^v,$$

$$= \frac{\beta^v r B^v_2 \alpha^v_2 S^v(1 - \alpha^h_1 + A^h_1 \lambda^h)}{\Lambda(1 + A^h_2 \lambda^h)} \cdot \frac{\beta^h r \omega^h_1 \omega^h_2 \lambda^h}{1 + A^h_2 \lambda^h},$$

$$= R_0^2 \frac{1 - \alpha^h_1 + A^h_1 \lambda^h}{G'(0)(1 - \alpha^h_1)(1 + A^h_2 \lambda^h)^2} S^v(\lambda^h) \lambda^h.$$  

(6.2.17)

Since $S_0^v = N^{vb(+)},$ we have

$$\frac{S^v(\lambda^h)}{S_0^{vb}} = \frac{1}{1 + \lambda^v(B^v_2 + B^v_3)},$$

Then,

$$L(\lambda^h) = R_0^2 \frac{1 - \alpha^h_1 + A^h_1 \lambda^h}{G'(0)(1 - \alpha^h_1)(1 + A^h_2 \lambda^h)^2} \frac{1}{1 + \lambda^v(B^v_2 + B^v_3)} \lambda^h,$$

$$= R_0^2 \frac{1 - \alpha^h_1 + A^h_1 \lambda^h}{G'(0)(1 - \alpha^h_1)(1 + A^h_2 \lambda^h)(1 + D \lambda^h)} \lambda^h,$$

(6.2.18)

where $D = A^h_2 + \beta^h r \omega^h_1 \omega^h_2 (B^v_2 + B^v_3).$

Hence, similarly as in (6.1.17), there exists an endemic fixed point if and only if there is a positive solution to the equation $\lambda^h = G(L(\lambda^h)),$ or equivalently,

$$H^b(\lambda^h) := G(L(\lambda^h)) - \lambda^h = 0,$$

(6.2.19)

for $0 \leq \lambda^h \leq 1.$

Notice that $L(0) = 0$ and hence $G(L(0)) = 0$ by (6.1.6). Then $H^b(0) = 0$ and $H^b(1) < 0,$ since $G'(L) > 0$ and $\lim_{L \to \infty} G(L) = 1.$ Moreover, it follows from
Let \( L'(0) = R_0^{h_2}/G'(0) \) that

\[
H'(0) = G'(0)L'(0) - 1 = R_0^{h_2} - 1.
\]

Thus, if \( R_0^{h_2} > 1 \), there exists a positive solution \( \lambda^* \) to \( H^h(\lambda^h) = 0 \), that is, to \( \lambda^* = G(L(\lambda^*)) \). Then, there exists an endemic equilibrium if \( R_0^{h_2} > 1 \).

### 6.3 Impact of releases of sterile mosquitoes

To study the impact of the releases of sterile mosquitoes on the malaria transmission dynamics, we consider the interval \((0, \bar{b})\). For each \( b \in (0, \bar{b}) \), the corresponding reproductive number \( R_0^h > 1 \) and there exists an endemic equilibrium associated with \( \lambda^* \) to the equation \( H^h(\lambda) = 0 \).

With this \( \lambda^*(b) \), we have

\[
\frac{\partial H^b}{\partial \lambda}_{|\lambda=\lambda^*(b)} = G'(\lambda^*)L'(\lambda^*) - 1 < 0,
\]

by taking the derivative with respect to \( \lambda \). Now regard the release of sterile mosquitoes \( b \) as the variable for the unique \( \lambda^* \), we take the derivative of equation \( H^b(\lambda(b)) = 0 \) with respect to \( b \). Then,

\[
\lambda''(b)(G'(\lambda^*(b))L'(\lambda^*(b)) - 1) = - \frac{2R_0^hG'(\lambda^*)\lambda^*(1 - \alpha_1^h + A_1^h\lambda^*)}{G''(0)(1 - \alpha_1^h)(1 + A_2^h\lambda^*)(1 + D\lambda^*)} R_0'. \tag{6.3.1}
\]

Solving for \( \lambda''(b) \) then yields

\[
\lambda''(b) < 0,
\]
since \( G'(\lambda^*(b))L'(\lambda^*(b)) - 1 < 0 \) and \( R_0^b < 0 \).

It follows from (6.1.13) that

\[
I^h'(b) = \frac{\omega_1^h \omega_2^h \Lambda (1 - \alpha_1^h)}{(1 - \alpha_1^h + A_1^h \lambda^*)^2} \lambda^{*'} < 0.
\]

Moreover, it follows from (6.2.15) that

\[
I^v'(b) = \frac{B_3^u N^{vb'} \lambda^{v'} + (1 + \lambda^v)(B_2^v + B_3^v)B_3^u \lambda^v N^{vb'+v'}}{(1 + (B_2^u + B_3^u)\lambda^v)^2} < 0,
\]

since \( \lambda^{v'} = \frac{\beta^h r \omega_1^h \omega_2^h \lambda^{*'}}{(1 + A_2^h \lambda^*)^2} < 0 \) and \( N^{vb'}(b) < 0 \) from (6.2.10).

Therefore, for \( b \in (0, \bar{b}) \), even though \( R_0^b(b) > 1 \) such that the disease spreads and goes to a positive steady state for \( n \to \infty \), we can reduce the components of the infected humans and mosquitoes by increasing the release of sterile mosquitoes to make the transmission under control.

We provide an example below to demonstrate our findings.

**Example 6.3** We use the following parameters for the transmission model.

\[
\begin{align*}
\Lambda &= 6, \quad k^v_b = 0.5, \quad \eta^v_b = 1.5, \quad b^v = 80, \quad \gamma^v = 0.8, \quad \beta^v = 0.1, \quad \beta^h = 0.15, \\
\alpha_1^h &= 0.3, \quad \alpha_2^h = 0.7, \quad \alpha_3^h = 0.5, \quad \alpha_4^h = 0.8, \quad \alpha_5^h = 0.8, \quad \alpha_6^h = 0.8, \quad \alpha_7^h = 0.8, \\
\alpha_8^h &= 0.8, \quad \gamma^h = 0.7, \quad \eta^h = 0.25, \quad \theta^h = 0.5, \quad r = 6,
\end{align*}
\]

and \( \alpha_1^v = \alpha_2^v = \alpha_3^v = \alpha_4^v = \alpha_5^v = 0.5 \).

Before the sterile mosquitoes are released, the reproductive number of infection for system (6.1.2) and (6.1.3) is \( R_0 = 1.3039 < 1 \) and hence the mosquito-borne
Figure 6.4: With parameters given in (6.3.2), the threshold value is \( b_c = 42.0743 \). If \( b = 45 > b_c \), the trivial fixed point is globally asymptotically stable, which leads to the infection eventually extinct.

Disease spreads. After the release of sterile mosquitoes, we have the existence threshold value \( b_c = 42.0743 \) such that if \( b > 42.0743 \), there exists no positive equilibria and thus the infection eventually dies out, as shown in figure in Figure 6.4. If \( b < 42.0743 \), there exist two positive equilibria with components

\[
N_{vb}^\pm(b) = 26.3333 - 0.5000b \pm 0.6667 \sqrt{(39.5 - 0.75b)^2 - 1.500b},
\]

where \( N_{vb}^- \) is unstable and \( N_{vb}^+ \) is locally asymptotically stable with initial value larger than \( N_{vb}^- \). \( N_{vb}^+ \) is a decreasing function with respect to release value \( b \).

However, the threshold value for the disease spread is \( \bar{b} = 21.2330 \) such that the reproductive number \( R_0^b(b) < 1 \) if \( b > 21.2330 \) and thus the disease dies out eventually, which is shown in Figure 6.5 with \( b = 30 \). If \( b < 21.2330 \), the reproductive
Figure 6.5: With parameters given in (6.3.2), the threshold values are $\bar{b} = 21.2330$ and $b_c = 42.0743$, respectively. For $b = 30$, the infection-free equilibrium is locally asymptotically stable since $R_0(b) < 1$, then the infected human and infected mosquitoes eventually dies out.

Figure 6.6: With parameters given in (6.3.2), the reproductive number is $R_0 = 1.3039 > 1$ and hence the disease spreads when there are no sterile mosquitoes released.
Figure 6.7: With parameters given in (6.3.2), the threshold values are $\bar{b} = 21.2330$ and $b_c = 42.0743$, respectively. After the sterile mosquitoes are released, even the released amount $b = 15 < \bar{b}$, we can still make the amount of infected humans reduced.

Figure 6.8: With parameters given in (6.3.2), the threshold values are $\bar{b} = 21.2330$ and $b_c = 42.0743$, respectively. Using $b$ as the variable, the horizontal axis is for $b$ and the vertical axis is for $R_0^*(b)$. $R_0^*$ is a decreasing function with respect to $b$. 
**Figure 6.9:** With parameters given in (6.3.2), the threshold values are $\bar{b} = 21.2330$ and $b_c = 42.0743$, respectively. This curve is for $\lambda^h(b)$ at the endemic equilibrium for each $b$.

**Figure 6.10:** With parameters given in (6.3.2), the threshold values are $\bar{b} = 21.2330$ and $b_c = 42.0743$, respectively. The upper curve and lower curve are for $I^h(b)$ and $I^v(b)$ at the endemic equilibrium for each $b$, respectively. Clearly, $\lambda^h(b)$, $I^h(b)$ and $I^v(b)$ are all negative if $b > \bar{b}$, which implies that no endemic equilibrium exists although positive $N^{vb(+)}$ exist for $\bar{b} < b < b_c$. 
number \( R_{0}^{b}(b) > 1 \), which makes the infection-free equilibrium become unstable and thus the disease will spread. \( R_{0}^{b}(b) \) is a decreasing function with respect to \( b \), as shown in Figure 6.8.

For \( b < 21.2330 \), \( R_{0}^{b}(b) > 1 \), and the corresponding \( \lambda^{h}(b) \) determined in (6.2.19) is a positive and decreasing function as shown in Figure 6.9. Corresponding to \( \lambda^{h}(b) \), there exists a unique endemic equilibrium whose components \( I^{v}(b) \) and \( I^{h}(b) \) are decreasing functions of \( b \) as shown in Figure 6.10, which indicates that the increasing of the releases of sterile mosquitoes reduces the disease spread.

To show the dynamical impact of the releases of the sterile mosquitoes, we also present the solutions of the disease transmission system. When no sterile mosquitoes are released, the reproductive number \( R_{0} = 1.3039 > 1 \) and the disease spreads as shown in Figure 6.6. For \( b = 15 < \bar{b} = 21.2330 \), the reproductive number \( R_{0}^{b} = 1.0988 > 1 \) such that the infection still exists but the number of infected humans and infected mosquitoes are reduced significantly thereafter, as shown in Figure 6.7.
CHAPTER 7

CONCLUSION

The spread of mosquito-borne diseases, such as malaria, transmitted between human beings by blood-feeding mosquitoes, has always been a big concern and a threat to public health in the world. It has caused severe problems for the survival of human beings and other species, and for economic and social development of human society. Malaria is a leading cause of death in many developing countries, where young children and pregnant women are the groups most affected. According to the World Health Organizations World Malaria Report 2013 and the Global Malaria Action Plan, there are 3.2 billion people, half of the worlds population, living in areas at risk of malaria transmission in 106 countries and territories. From a worldwide view, Malaria is the fifth cause of death from infectious diseases. In 2015, there were an estimated 214 million malaria cases and some 438,000 malaria deaths worldwide, with 92% of these deaths occurring in Africa.

The life cycle of the mosquito-borne diseases is complicated. Since there is no effective vaccine available to control malaria yet, all efforts are directed to avoid the proliferation of the mosquito population. Massive spaying of insecticides or eliminating breeding sites has greatly limited malaria in some area. But the number of
mosquito-borne diseases cases still continues to climb. This is partially because of economic and social difficulty in some developing countries and effects of environmental changes such as global warming. To prevent and control the amount of mosquitoes in order to control such mosquito-borne diseases, biologically control measures provide an effective weapon.

The Sterile Insect Technique (SIT) is indeed a method of biological control in which the natural reproductive process of the target population is disrupted. By chemical or physical methods, male mosquitoes are genetically modified to be sterile despite being sexually active. These sterile male mosquitoes are then released into the environment to mate with the wild female mosquitoes. A wild female mosquito that mates with a sterile male mosquito will either not reproduce, or produce eggs that do not hatch. Repeated releases of genetically modified mosquitoes or the releases of a significantly large number of sterile mosquitoes may eventually wipe out a wild mosquito population, although it is, in practice, often more useful to consider controlling the population rather than eradicating it.

SIT has shown promising results in laboratory studies, but predicting the impact of releasing sterile mosquitoes into the field of wild mosquito populations is still a challenging task.

Mathematical models have proven useful in gaining insights into challenging questions in population dynamics and epidemiology. There are mathematical models in the literature formulated to study the interactive dynamics of mosquito populations or the control of mosquitoes, among which, in particular, discrete-time models were formulated in [49]. Models in [49] exhibited complex dynamics where the Ricker-type
nonlinearity was used for the survival probabilities of mosquitoes. Notice the nature of the dynamical complexity in Ricker population models. The complicated features in [49] may not necessarily be induced by the interactions between the wild and sterile mosquitoes. Moreover, there seem no clear evidence in the literature or existing data supporting such complexity.

Our fundamental goal is eventually incorporate the mosquito models into malaria transmission models. The mathematical analysis for the high-dimensional malaria disease transmission models is difficult already. Adding another one or more equations to the disease models leads to even higher dimension models which could make the analysis untractable. Hence the objectives of our research in this dissertation are to formulate new models for interactive wild and sterile mosquitoes so that the dynamics are relatively simpler and mathematically more tractable, but basic and important model features are snatched.

To begin with, we formulated models for interactive wild and sterile mosquitoes with the assumption that there are no generation overlaps in the mosquito population in Chapter 2. The survival probabilities for mosquitoes are no longer of the Ricker-type but the Beverton-Holt-type. We first fully investigated the dynamics for wild mosquitoes in the absence of sterile mosquitoes, where the models with or without Allee effects were both considered. For the model without Allee effects in (2.1.1), we showed that if the population intrinsic growth rate $r_0$ is less than or equal to one, all populations approach zero, and if $r_0$ is greater than one, all populations go to a unique positive fixed point, regardless of their initial sizes. For the model with Allee effects in (2.2.1), on the other hand, we determined a threshold value for the
population intrinsic growth rate, $r_c$, such that if $r_0 \leq r_c$, all populations approach zero, and if $r_0 > r_c$, all populations go to a unique positive fixed point, regardless of their initial sizes. We also showed the nonexistence of periodic solutions for these models.

We then introduced sterile mosquitoes into wild mosquito populations and studied their interactive dynamics. As in [20,49], we considered three different strategies of releases of sterile mosquitoes. We investigated the existence and stability of all nonnegative fixed points and established a threshold value of releases for each of the interactive models with different strategies. Based on the release threshold values, the models may have no, one or two positive fixed points mathematically, and the wild mosquitoes can be completely wiped out or coexist with sterile mosquitoes determined by these threshold values.

Compared to the models with the Ricker-type of nonlinearity, the dynamics for the interactive models we studied in Chapter 2 are relatively simpler such as no period-doubling bifurcation occurs in our models. This is indeed what we need because we need to exclude those complex dynamical features induced not by the interactions between the wild and sterile mosquitoes but other factors.

We studied models based on the assumption of overlapped generations in Chapter 3. The analysis is similar to that in Chapter 2. For the model (3.1.1) without Allee effects, there exists a trivial fixed point which is globally asymptotically stable if $r_0 + p < 1$. This leads to extinction of all mosquitoes. There also exists a unique positive fixed point which is globally asymptotically stable if $r_0 + p > 1$. For the model (3.1.4) with Allee effects, we determined a threshold value for the
population intrinsic growth rate, \( r_c \), such that if \( \hat{r}_0 \leq r_c \), all populations approach zero, and if \( \hat{r}_0 > r_c \), all populations go to a unique positive fixed point, regardless of their initial sizes. We then introduced sterile mosquitoes into the wild mosquito populations and considered three different strategies of releases of sterile mosquitoes. The analysis results are similar to those in Chapter 2 except the expressions of threshold values. When the releases exceed the threshold value, the trivial fixed point is globally asymptotically stable and all wild mosquitoes will be wiped out.

Since mosquitoes undergo complete metamorphosis, going through four distinct stages of development during a lifetime: egg, pupa, larva, and adults. To better understand the interactive dynamics, predict and control mosquito-borne diseases, we included the metamorphic stages in our models but to make our mathematical analysis tractable, we combine the three aquatic metamorphic stages into one group as in [20,49], called larvae, and assume that the density dependence, due to intraspecific competition, is only in the stage of larvae. We then formulated our stage-structured models in Chapter 4 and Chapter 5.

We then considered stage-structured models in the absence of sterile mosquitoes in Chapter 4. For the model (4.2.1), there exists a trivial fixed point if \( r_0 < 1 \), which is globally asymptotically stable. Thus all wild larvae and adults die out if \( r_0 < 1 \) eventually. In the case of \( r_0 > 1 \), there exists a unique positive fixed point, which is globally asymptotically stable if \( \eta_1 < \eta_2 \) and unstable if \( \eta_1 > \eta_2 \). If \( \eta_1 > \eta_2 \), there exists a unique synchronous 2-cycle and it is globally asymptotically stable, which means the mosquito larvae and adults are synchronized in such a way as to appear
and vanish alternately in one time unit. If \( \eta_1 = \eta_2 = \eta \), there exist positive periodic 2-cycles.

In Chapter 5, we included sterile mosquitoes and considered stage-structured models with three different strategies for the releases of sterile mosquitoes in model systems (5.1.2), (5.2.1) and (5.3.1). We established the threshold values of the releases for existence of positive fixed points, \( b_c \), and for stability \( b_s \), for each of the model systems. If \( b > b_c \), there exists no positive fixed point for all of the three model systems, in which case the wild mosquitoes will be wiped out if the origin is stable, or oscillates. If \( b < b_c \), there exist two positive fixed points for system (5.1.2) and (5.3.1), and a unique positive fixed point for system (5.2.1). A positive fixed point is asymptotically stable if \( b < b_s \) and is unstable if \( b > b_s \). When the positive fixed point is unstable, a 2-cycle is bifurcated.

To study the impact of releasing sterile mosquitoes on malaria transmission, we then incorporated the interactive mosquito models into disease models. To this end, we first formulated a simple compartmental SEIR model for malaria transmission in (6.1.2) and (6.1.3) as our baseline model in Section 6.1. We derived the formula for the reproductive number of infection \( R_0 \), in (6.1.9) and showed that the infection-free equilibrium of the baseline model is asymptotically stable if \( R_0 < 1 \) and unstable if \( R_0 > 1 \). We also showed that if \( R_0 > 1 \), there exists an endemic equilibrium for the baseline model.

We then included sterile mosquitoes in Section 6.2. The human components are the same as those in (6.1.3) but the mosquito components are given in (6.2.1). We only consider the case of constant releases. We derived a formula for the reproductive
number $R^b_0$, presented in (6.2.9), for the model with sterile mosquitoes of (6.2.1) and (6.1.3). We showed that the infection-free equilibrium is asymptotically stable if $R^b_0 < 1$ and is unstable if $R^b_0 > 1$. Using the constant rate of releases $b$ as a variable, we determined threshold value $b_c$ for the existence of positive equilibrium for the interactive wild and sterile mosquitoes and threshold value $\bar{b}$ that determines whether $R^b_0 < 1$ or $R^b_0 > 1$; that is, whether the disease dies out or spreads. We also showed the existence of an endemic equilibrium when $R^b_0 > 1$.

We studied the impact of the releases of sterile mosquitoes on the transmission dynamics in Section 6.3 by investigating the variation of the reproductive number $R^b_0(b)$, and the infected components $I^h(b)$ and $I^v(b)$, induced by $\lambda^h(b)$, as $b$ varies. Based on the threshold values $b_c$ and $\bar{b}$, we provided an example to confirm and demonstrate our findings. If the rate of releases $b$ is greater than threshold $b_c$, the trivial fixed point for the interactive wild and sterile mosquitoes is the only equilibrium and is globally asymptotically stable. All wild mosquitoes are wiped out and thus there is no infection. On the other hand, if $b < b_c$, while the wild mosquitoes can not be wiped out and the interactive wild and sterile mosquitoes coexist, the disease can still go extinct when there are sufficient sterile mosquitoes with $\bar{b} < b < b_c$ which leads to $R^b_0 < 1$. Even if we are unable to release enough sterile mosquitoes with $b < \bar{b}$, which leads to $R^b_0 > 1$ and thus the disease spreads when the initial wild mosquitoes is larger than $N^{ub(-)}$ given in (6.2.4), the infected components $I^h(b)$ and $I^v(b)$ are decreased functions with respect to $b$; that is, the infection is reduced as we increase the rate of releases $b$. 

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There are many unsolved open problems from this dissertation. For the stage-structured interactive models in Chapter 5, we are unable to obtain conditions for the existence and stability of positive 2-cycles. For the transmission models in Chapter 6, we have not been able to show the uniqueness and the stability of the endemic equilibrium for $R_0 > 1$. Moreover, we have only considered the case of constantly release of sterile mosquitoes in the disease transmission models with sterile mosquitoes. What will happen if we release the sterile mosquitoes proportionally or proportionally with saturation? All of these will be projects for our future studies.
REFERENCES


[35] Ana Franco. Modeling a novel method to control human malaria: Insecticide Treated Liverstock, Disease Control and Vector Biology Unit, Department of Infectious and Tropical Disease, London School of Hygiene and Tropical Medicine, Keppel Street, London WCI 7HT, UK, August 2003.


[41] E. F. Knipling, Possibilities of insect control or eradication through the use of sexually sterile males, J. Econ. Entomol. 48 (1955), 459-462.


