Global stability and traveling wave solutions for a two-competing-prey and one-predator Lotka-Volterra model

Cuiping Wang

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GLOBAL STABILITY AND TRAVELING WAVE SOLUTIONS FOR A TWO-COMPETING-PREY AND ONE-PREDATOR LOTKA-VOLTERRA MODEL

by

CUIPING WANG

A DISSERTATION

Submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in The Department of Mathematical Sciences to The School of Graduate Studies of The University of Alabama in Huntsville

HUNTSVILLE, ALABAMA

2018
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Cuiping Wang
03/01/19
(date)
DISSERTATION APPROVAL FORM

Submitted by Cuiping Wang in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Applied Mathematics and accepted on behalf of the Faculty of the School of Graduate Studies by the dissertation committee.

We, the undersigned members of the Graduate Faculty of The University of Alabama in Huntsville, certify that we have advised and/or supervised the candidate of the work described in this dissertation. We further certify that we have reviewed the dissertation manuscript and approve it in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Applied Mathematics.

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ABSTRACT

School of Graduate Studies
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Degree Doctor of Philosophy College/Dept. Science/Mathematical Sciences
Name of Candidate Cuiping Wang
Title Global Stability and Traveling Wave Solutions for a Two-Competing-Prey and One-Predator Lotka-Volterra Model

This dissertation consists of four chapters. In Chapter 1, I give an introduction of background, models to be investigated, and main goal of this dissertation.

In Chapter 2, I study the space homogeneous model. The existence and local stability analysis of nonnegative equilibrium points in three cases, as well as the sufficient and necessary conditions for the existence of the positive interior equilibrium point are given in Section 2.1. In Section 2.2, I perform a careful analysis to show the global stability of the boundary equilibrium point at which two prey coexist with extinction of predator for case 1. In Section 2.3, I construct a Lyapunov function to prove that the positive interior equilibrium point is globally stable under some assumptions. In Section 2.4, I prove that the boundary equilibrium point at which predator and prey 1 coexist but prey 2 is extinct is globally stable under certain assumptions by using another Lyapunov function for case 3.

In Chapter 3, I consider a space variation or space diffusive predator-prey mode described by a reaction-diffusion system. The main purpose is to investigate the existence of the traveling wave solutions. My approach consists of two steps. First I show the existence of nonnegative semi-traveling wave solutions (i.e., nonnegative
wave solutions that converge to an unstable equilibrium point as time goes to \(-\infty\) for the models by a geometric method. Then I use the Lyapunov method to show that the traveling wave solutions converge to another equilibrium point as time goes to \(+\infty\).

Chapter 4 provides a couple of conclusion remarks that give brief descriptions of biological implication from the mathematical results obtained in this dissertation. Further research efforts, as a continuation of this dissertation has been addressed in Chapter 4.

Abstract Approval: Committee Chair

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ACKNOWLEDGMENTS

The work described in this dissertation could not been accomplished without the assistance of the numerous people who deserve special acknowledgement. First, I would like to sincerely thank my advisor Dr. Wenzhang Huang for his encouragements, great patience, constant help, professional guidance and understandings. Without his knowledge and support, it would not have been possible to complete my dissertation. So many events happened to me during my graduate studies, but he has always encouraged me to achieve my goal.

I would like to express my gratitude to my committee members, namely Dr. Shangbing Ai, Dr. David Halpern, Dr. Boris Kunin, and Dr. Rudi Weikard for their valuable time, suggestions, and comments for improving my dissertation.

Additionally I would also like to thank the Department of Mathematical Sciences and the School of Graduate Studies for the financial support. I want to thank all faculty, staff, and students in the math department for their assistance, especially, I want to thank Ms. Tamara Lang, the senior staff of the Department of Mathematical Sciences, who is always ready to help me.

Finally, I thank to my family: my husband, Dr. Huaming Zhang, my son, Lawrence Zhang, and my daughter, Rachel J. Zhang for their love, understanding and support throughout this endeavor.
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<td>$\Omega$</td>
<td>The omega limit set of the orbit of a system</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>The alpha limit set of the orbit of a system</td>
</tr>
<tr>
<td>$X'$</td>
<td>The derivative of the function $X$ with respect to time</td>
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<tr>
<td>$\frac{\partial N}{\partial t}$</td>
<td>The partial derivative of the function $N$ with respect to time</td>
</tr>
<tr>
<td>$\dot{u}$</td>
<td>The first order derivative of the function $u$ with respect to time</td>
</tr>
<tr>
<td>$\ddot{u}$</td>
<td>The second order derivative of the function $u$ with respect to time</td>
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<tr>
<td>$\Delta$</td>
<td>The Laplace operator</td>
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<tr>
<td>$M^s(E)$</td>
<td>The stable manifold of the equilibrium $E$</td>
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<tr>
<td>$M^u(E)$</td>
<td>The unstable manifold of the equilibrium $E$</td>
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<tr>
<td>$p(t,q)$</td>
<td>The solution of a system of differential equations through a point $q$</td>
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<tr>
<td>$\Phi_t(q)$</td>
<td>The orbit of a system of differential equations through a point $q$</td>
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To my family
CHAPTER 1

INTRODUCTION

The main goal of this dissertation is to apply dynamical system theory and to develop techniques to investigate the global dynamical structure of a class of predator-prey systems with two competing prey and one predator.

One of important issues in the studies of ecology is to understand the fundamental mechanism that governs the evolution of predator-prey interaction in an ecological system, which will provide us the insight to establish the scientific strategy for the management of ecological systems and the protection of environment. The first mathematical study of the interaction between predator and prey can go back to 1910’s when the American mathematician Alfred J. Lotka introduced a system of two nonlinear differential equations to describe the dynamical interaction of the predator and prey, or two species competing for the same resource [7]. In 1920’s, Lotka extended his work to modeling the dynamics of plant species and herbivorous animals, and made the mathematical analysis to understand the dynamics of predator-prey interactions. In 1926 an Italian mathematician, Vito Volterra, published the same equations [8,9] independently from Lotka and used them to explain the marine biologist Umberto D’Ancona’s observation. Due to their pioneer work, the models
introduced by Lotka and Volterra were named the Lotka-Volterra predator-prey system and Lotka-Volterra competition model. In 1936 [11], the Russian mathematician Andrey Nikolaevich Kolmogorov extended the systems to arbitrary dimension, so these kinds of systems are also called Kolmogorov systems.

Lotka-Volterra models on species competition and predator-prey interaction were relatively simple. It has been said that the Lotka-Volterra model is too simple to realistically reflect ecological processes. So for last several decades Lotka-Volterra systems have been intensively modified and generalized to take into account many factors that affect the dynamics of predator-prey interaction, such as hunting effect, selection effect, diffusion and aggregation effect, intraspecific and interspecific interference effect, intraspecific and interspecific competition, and so on. Many predator-prey or competition models have been formulated through the addition of non-linear terms, time-delays, and stochastic disturbances based on more realistic explicit and implicit biological assumptions. These models play a very important role in the theoretical studies of species interaction in ecology that greatly increases our understanding the evolution of ecology. The mechanism of model formulation has also been applied to model dynamical phenomena in different subjects, such as chemical reactions [12], hydrodynamics [13], etc. The models with one prey and one predator have been extensively investigated and well understood with many interesting phenomena observed after many researchers’ efforts in last few decades.

Partial research on Lotka-Volterra type of systems with the interaction of three or more species have been done by some authors. For example in [23, 24, 26, 30], the authors showed the presence of a predator may stabilize the species competition in
an ecological system which may otherwise be unstable. In [4, 5, 10, 22, 25, 27–29], the authors analyzed the persistence, local stability and some global dynamical properties of three species interaction systems under very strong conditions on parameters. Like in [10], the author analyzed the global properties of two prey and one predator system with the direct Lyapunov method under the assumptions that interspecies competition and intraspecies competition are absent. It is not common for the absence of the direct competition between two prey species having a common predator. However, in contrast to the research done on two species models, the progress on the research for three species interaction models is quite limited, in particular, the problem on global dynamical structure of models still remains largely open. The main difficulty one confronts in the analysis of three species interaction systems is the lack of efficient analytical or geometrical techniques, while in the analysis of two species interaction models, which are planar systems, the powerful Poincaré-Bendixson Theorem and the geometrical analysis of vector fields can be applied.

Even though the Lotka-Volterra model may be ecologically over simplified, it is worth to study it. I hope that through a comprehensive understanding of the behavior of this relative simple model it can help me to better understand the more general or complex ones. So in this dissertation, I attempt to study a two-competing-prey and one-predator Lotka-Volterra Model. Both the space homogeneous and space diffusive models will be considered. My goal is to use and develop some approaches, such as the ω and α-limit theorem, Lyapunov direct method, the shooting argument, etc., that enable us to handle the three dimensional systems, and that can be applied to
investigate the existence of important class of solutions, the traveling wave solutions, when the predator-prey models considered are reaction-diffusion type of systems.

1.1 Space Homogeneous Lotka-Volterra Model

The graphical description of the two-competing-prey and one-predator Lotka-Volterra Model is given by the figure 1.1.

![Figure 1.1: Schematic diagram for a two-competing-prey and one-predator interaction](image)

The Figure 1.1 illustrates that the populations of prey are affected by the death, the birth, the competitor and the predator. Two prey and the death and birth of predator also effect the population of the predator. Suppose we consider a relatively simple environment in which the populations of all species are independent
of space. Then based on the above diagram, the model is given by

\[
\begin{align*}
X'_1 &= X_1(\alpha_1 - \beta_1 X_1 - \gamma_1 X_2 - \delta_1 Y), \\
X'_2 &= X_2(\alpha_2 - \beta_2 X_1 - \gamma_2 X_2 - \delta_2 Y), \\
Y' &= Y(-s + \varepsilon_1 X_1 + \varepsilon_2 X_2),
\end{align*}
\]

(1.1)

with the initial condition

\[
X_{10} = X_1(0) \geq 0, \quad X_{20} = X_2(0) \geq 0, \quad Y_0 = Y(0) \geq 0.
\]

Where \( X'_i = \frac{dX_i}{dt} \), represents the instantaneous growth rates of the populations of prey \( i \) for \( i = 1, 2 \), \( Y' = \frac{dY}{dt} \), represents the instantaneous growth rate of the populations of predator, \( X_1(t) \) and \( X_2(t) \) denote the populations of the prey 1 and prey 2 at time \( t \) respectively, \( Y(t) \) denotes the populations of the predator feeding exclusively on prey with the system \( X_1(t) \) or \( X_2(t) \) or both), and hence will become extinct if prey does not survive. \( \alpha_i, \beta_i, \gamma_i, \delta_i, \varepsilon_i (i = 1, 2) \) and \( s \) are positive constants. \( s \) denotes the difference between the death and birth rates of the predators, \( \alpha_1 \) is the growth rate of the prey 1 and \( \alpha_2 \) is the growth rate of prey 2. Each prey has its own carrying capacity \( \frac{\alpha_1}{\beta_1} \) and \( \frac{\alpha_2}{\gamma_2} \). Because this is the competitive version of the model, all interactions must be harmful and therefore all \( \alpha_i, \beta_i, \gamma_i (i = 1, 2) \) are positive.

The model (1.1) was mentioned in [4], the author gave the persistence conditions for the model (1.1) and also pointed out if one prey wins the competition in the absence of a predator, and both prey persist with a predator, then the predator’s
adverse effect on the winner is greater than on other prey. This is observed in nature. For example in [33], the author Morin observed: “the outcome of competition among three species of anuran tadpoles in replicated artificial pond communities depends on the density of predatory salamanders present in the community. Predators differentially affect the survival of anuran species to metamorphosis and reverse the pattern of anuran relative abundance resulting from interspecific competition among tadpoles in the absence of predators.” So the model (1.1) is very realistic, but no one has conducted a comprehensive analysis of this model. In this dissertation, I would like to study the local and global stability of equilibrium points from three different aspects in Chapter 2.

Chapter 2 consists of 4 sections and is organized as follows. First by a scaling, I transform the system (1.1) to a dimensionless system (2.2) that is simpler than (1.1) and equivalent to (1.1). I analyze the existence and local stability of all nonnegative equilibrium points of the system (2.2) in three cases. Then I discuss the global stability of the system (2.2) in three cases in three sections. In Section 2, I study the case 1, the system (2.2) has no positive equilibrium point. I perform a careful analytical analysis to show the global stability of the boundary equilibrium at which the two prey coexist with extinction of predator. In Section 3, I study the case 2, the system (2.2) has a positive interior equilibrium point. I construct a Lyapunov function to prove that the positive interior equilibrium is globally stable under some assumptions. In Section 4, I consider the case 3, positive interior equilibrium point does not exist. I construct another Lyapunov function which is different from case
2 to prove the global stability of one boundary equilibrium point at which predator and one prey coexist but another prey is extinct under some assumptions.

1.2 Traveling Wave Solutions for Diffusive Lotka-Volterra Model

Many biological, physical and chemical processes exhibit the phenomenon of wave propagation. The wave propagation can be described by the traveling wave solutions in many mathematical models that are formulated by using reaction-diffusion systems. There are a lot of applications of traveling wave solutions, for example, see [16–18]. The significant role of wave solutions is well recognized [19–21]. All of these inspire me to study traveling wave solutions for a reaction-diffusion two-competing-prey and one-predator model. A very important component of two-prey and one-predator interaction is spatial variation in the populations when predators are spatially distributed and are active in search for prey. To take into account the movement of predator species with the movement being assumed randomly, I add a diffusion term to the equation for predator so that the resulting Lotka-Volterra model is a three dimensional reaction-diffusion system. I want to see how the predator may influence the two competing species in space. To be specific, I want to investigate whether or not there will be a zone of transition from one equilibrium point to another one. Mathematically this transition can be illustrated by a traveling wave fronts connecting two equilibrium points. Hence my main purpose in Chapter 3 is to study the existence of traveling wave solutions connecting two equilibrium points for the corresponding reaction-diffusion model.
Chapter 3 contains three sections. A diffusive Lotka-Volterra model governed a system of reaction-diffusion equations is provided in Section 3.1. In Section 3.2 I transform this model into a higher dimensional system of ordinary differential equations for finding traveling wave solutions. In the last section, I show the existence of nonnegative semi-traveling waves (i.e. wave solutions that converge to one equilibrium point as time goes to $-\infty$) by a geometric approach. I then show the convergence of traveling wave solutions to another equilibrium point as time goes to $+\infty$ by the Lyapunov method.
CHAPTER 2

GLOBAL STABILITY FOR A TWO-COMPETING-PREY AND ONE-PREDATOR LOTKA-VOLTERRA MODEL

In this chapter, we study the global stability of the system (2.2) from three different situations. First by a scaling, we transform the system (1.1) to a dimensionless system (2.2) that is simpler than (1.1). Then analyze the existence and local stability of all nonnegative equilibrium points of the system (2.2) in three cases in Section 1, and discuss the global stability of the system (2.2) in these three cases in three sections respectively.

In order to study the system (1.1), we make the following changes of variables and scaling in (1.1) by

\[ x_1(t) = \frac{\varepsilon_1}{s} X_1\left(\frac{t}{s}\right), \]

\[ x_2(t) = \frac{\varepsilon_2}{s} X_2\left(\frac{t}{s}\right), \]

\[ y(t) = s Y\left(\frac{t}{s}\right). \]
Then
\[ x'_1 = x_1 \left( \frac{\alpha_1}{s} - \frac{\beta_1}{\varepsilon_1} x_1 - \frac{\gamma_1}{\varepsilon_2} x_2 - \frac{\delta_1}{s^2} y \right), \]
\[ x'_2 = x_2 \left( \frac{\alpha_2}{s} - \frac{\beta_2}{\varepsilon_1} x_1 - \frac{\gamma_2}{\varepsilon_2} x_2 - \frac{\delta_2}{s^2} y \right), \]
\[ y' = y(-1 + x_1 + x_2). \]

Then the system (2.1) becomes
\[ x'_1 = x_1 (a_1 - b_1 x_1 - c_1 x_2 - d_1 y), \]
\[ x'_2 = x_2 (a_2 - b_2 x_1 - c_2 x_2 - d_2 y), \]
\[ y' = y(-1 + x_1 + x_2). \]

With initial condition \( p_0 = (x_1, x_2, y) \) such that
\[ x_1 = x_{10} \geq 0, \quad x_2 = x_{20} \geq 0, \quad y = y_0 \geq 0. \]

Where \( a_i, b_i, c_i, \) and \( d_i(i = 1, 2), \) all are positive constants.

The system (1.1) and the system (2.2) are equivalent, but (2.2) looks simpler than (1.1), since we reduced the number of parameters. That is, the third equation in the system (2.2) has no parameter at all. For the rest of this chapter, we will study the system (2.2).
2.1 Existence and Local Stability of the Equilibrium Points

To find all possible equilibrium points of the system (2.2), we solve the following system

\[ x_1(a_1 - b_1x_1 - c_1x_2 - d_1y) = 0, \]
\[ x_2(a_2 - b_2x_1 - c_2x_2 - d_2y) = 0, \]
\[ y(-1 + x_1 + x_2) = 0. \]

we obtain several solutions which are the equilibrium points of the system (2.2) e.g.

\[ E_0 = (0, 0, 0), \quad E_1 = \left( \frac{a_1}{b_1}, 0, 0 \right), \quad E_2 = (0, \frac{a_2}{c_2}, 0), \quad E_3 = \left( \frac{a_1c_2 - a_2c_1}{b_1c_2 - b_2c_1}, \frac{b_1a_2 - b_2a_1}{b_1c_2 - b_2c_1}, 0 \right), \]
\[ E_4 = (0, 1, \frac{a_2 - c_2}{d_2}), \quad E_5 = \left( 1, 0, \frac{a_1 - b_1}{d_1} \right), \quad \text{and the interior equilibrium } E^0. \]

Consider the practical applications of the system (2.2), we study the system (2.2) in the positive octant and its boundaries. Whether the system (2.2) has a positive interior equilibrium depends on all parameters. Let

\[ F = a_1 - b_1x_1 - c_1x_2 - d_1y, \]
\[ G = a_2 - b_2x_1 - c_2x_2 - d_2y, \]
\[ H = -1 + x_1 + x_2. \]

From the geometrical point of view, there are three cases for the system of \( F = 0, G = 0, \) and \( H = 0. \)(see Figure 2.1, 2.2).
Figure 2.1: Graphs of $F = 0, G = 0$ and $H = 0$ for case 1 and 2
According above graphs, we give the following assumptions

**H1:** \( \frac{c_1}{c_2} < \frac{d_1}{d_2} \leq \frac{a_1}{a_2} < \frac{b_1}{b_2} \),

(H1) means that the predator does not have a stronger preference of any of the prey over the other, so this could be reasonable for some ecological system.

**H2:** \( \frac{a_2}{c_2} \leq 1 \) and \( \frac{a_2}{b_2} \leq 1 \),

**H3:** \( \frac{a_1}{b_1} \leq 1 \) and \( \frac{a_1}{c_1} \leq 1 \),

**H4:** \( \frac{a_1 d_2 - a_2 d_1}{b_1 d_2 - b_2 d_1} < 1 < \frac{a_1}{b_1} \) and \( 1 < \frac{a_1}{c_1} \),

**H5:** \( \frac{a_1 d_2 - a_2 d_1}{b_1 d_2 - b_2 d_1} < 1 < \frac{a_2}{b_2} \) and \( 1 < \frac{a_2}{c_2} \),

**H6:** \( \frac{a_1 d_2 - a_2 d_1}{b_1 d_2 - b_2 d_1} \geq 1 \).
Let us first compute eigenvalues associated to the boundary equilibrium points of the system (2.2). The Jacobian matrix of the system (2.2) at point \((x_1, x_2, y)\) is given by

\[
J(x_1, x_2, y) = \begin{bmatrix}
  a_1 - 2b_1x_1 - c_1x_2 - d_1y & -c_1x_1 & -d_1x_1 \\
  -b_2x_2 & a_2 - b_2x_1 - 2c_2x_2 - d_2y & -d_2x_2 \\
  y & y & -1 + x_1 + x_2
\end{bmatrix}.
\]

Substitute each point \(E_i (i = 0, \cdots, 5)\) into the above matrix \(J(x_1, x_2, y)\), and solve \(\det(J(E_i) - \lambda I) = 0\) for \(\lambda\), we obtain the corresponding eigenvalues \(\lambda_i (i = 1, 2, 3)\) at each equilibrium point. Each equilibrium point has eigenvalues as following

At \(E_0 = (0, 0, 0)\), \(\lambda_1 = 1, \lambda_2 = a_1, \lambda_3 = a_2\).

At \(E_1 = \left( \frac{a_1}{b_1}, 0, 0 \right)\), \(\lambda_1 = -a_1, \lambda_2 = \frac{a_2b_1 - a_1b_2}{b_1}, \lambda_3 = \frac{a_1}{b_1} - 1\).

At \(E_2 = (0, \frac{a_2}{c_2}, 0)\), \(\lambda_1 = -a_2, \lambda_2 = \frac{a_1c_2 - a_2c_1}{c_2}, \lambda_3 = \frac{a_2}{c_2} - 1\).

At \(E_3 = \left( \frac{a_1c_2 - a_2c_1}{b_1c_2 - b_2c_1}, \frac{b_1a_2 - b_2a_1}{b_1c_2 - b_2c_1}, 0 \right) = (x_1^*, x_2^*, 0)\), \(\lambda_1 = -1 + x_1^* + x_2^*\),

\[
\lambda_{2,3} = \frac{- (b_1x_1^* + c_2x_2^*) \pm \sqrt{(b_1x_1^* + c_2x_2^*)^2 - 4(b_1c_2 - b_2c_1)x_1^*x_2^*}}{2}.
\]

At \(E_4 = (0, 1, \frac{a_2 - c_2}{d_2})\), \(\lambda_1 = a_1 - c_1 - d_1 \frac{a_2 - c_2}{d_2}, \lambda_2, \lambda_3 = -\frac{c_2 \pm \sqrt{c_2^2 - 4(a_2 - c_2)}}{2}\).

At \(E_5 = (1, 0, \frac{a_1 - b_1}{d_1})\), \(\lambda_1 = a_2 - b_2 - d_2 \frac{a_1 - b_1}{d_1}, \lambda_2, \lambda_3 = -\frac{b_1 \pm \sqrt{b_1^2 - 4(a_1 - b_1)}}{2}\).
Remark 1 It is obvious the system (2.2) always has boundary equilibrium points $E_0, E_1,$ and $E_2$. If the system (2.2) satisfies (H1), then $E_0$ has three positive eigenvalues, $E_1$ has one positive eigenvalue, $E_2$ has one positive eigenvalue, therefore all of them are unstable. Moreover, by checking the vector field of the system (2.2) at these points, we know that $E_0$ is unstable both in the positive $x_1$-direction and positive $x_2$-direction and asymptotically stable in the $y$-direction, $E_1$ is unstable in the positive $x_2$-direction and asymptotically stable in the $x_1$-direction, $E_2$ is unstable in the positive $x_1$-direction and asymptotically stable in the $x_2$-direction.

Now we discuss the existence and local stability of nonnegative equilibrium points of the system (2.2) in three different cases, which are also relative to the existence of positive interior equilibrium point. To this end, let us consider the system

$$a_1 - b_1 x_1 - c_1 x_2 - d_1 y = 0,$$
$$a_2 - b_2 x_1 - c_2 x_2 - d_2 y = 0,$$
$$-1 + x_1 + x_2 = 0.$$

That is

$$b_1 x_1 + c_1 x_2 + d_1 y = a_1,$$
$$b_2 x_1 + c_2 x_2 + d_2 y = a_2,$$  \hspace{1cm} (2.3)
$$x_1 + x_2 = 1.$$
Let

\[
D = \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ 1 & 1 & 0 \end{vmatrix}, \quad D_1 = \begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ 1 & 1 & 0 \end{vmatrix}, \quad D_2 = \begin{vmatrix} b_1 & a_1 & d_1 \\ b_2 & a_2 & d_2 \\ 1 & 1 & 0 \end{vmatrix}, \quad D_y = \begin{vmatrix} b_1 & c_1 & a_1 \\ b_2 & c_2 & a_2 \\ 1 & 1 & 1 \end{vmatrix}.
\]

Then

\[
D = c_1d_2 - c_2d_1 + b_2d_1 - b_1d_2, \quad D_1 = c_1d_2 - c_2d_1 + a_2d_1 - a_1d_2,
\]

\[
D_2 = a_1d_2 - a_2d_1 + b_2d_1 - b_1d_2, \quad D_y = a_2c_1 - a_1c_2 + b_2a_1 - b_1a_2 + b_1c_2 - b_2c_1.
\]

**Case 1.** The system (2.2) satisfies the assumptions (H1) and at least one of the (H2) and (H3).

**CLAIM 2.1.1.** Under the assumptions given in case 1, the system (2.2) does not have positive interior equilibrium, but has nonnegative boundary equilibrium points \(E_0, E_1, E_2,\) and \(E_3,\) only \(E_3\) is locally asymptotically stable.

Proof. First we show positive interior equilibrium does not exist. Consider the system (2.3), by (H1), we have \(D < 0\) and \(D_1 < 0,\) by (H2) or (H3), we have \(D_2 < 0,\)

\[
D_y = a_2c_1 - a_1c_2 + b_2a_1 - b_1a_2 + b_1c_2 - b_2c_1
\]

\[
\geq \frac{b_1}{a_1}(a_2c_1 - a_1c_2) + \frac{c_1}{a_1}(b_2a_1 - b_1a_2) + b_1c_2 - b_2c_1 = 0 \quad \text{by (H1, H3),}
\]

or

\[
\geq \frac{b_2}{a_2}(a_2c_1 - a_1c_2) + \frac{c_2}{a_2}(b_2a_1 - b_1a_2) + b_1c_2 - b_2c_1 = 0 \quad \text{by (H1, H2).}
\]
From the Cramer’s Rule, it follows that the system $(2.3)$ has a unique solution $(x_1, x_2, y)$ with $x_1 = \frac{D_1}{D} > 0, x_2 = \frac{D_2}{D} > 0, y = \frac{D_y}{D} \leq 0$, which is not positive. Hence $(2.2)$ has no positive interior equilibrium point.

From above Remark 1, $E_0, E_1,$ and $E_2$ are not stable, but at $E_3$, the corresponding eigenvalues $\lambda_1 < 0$ by H2 or H3, $\lambda_2 < 0$ and $\lambda_3 < 0$ by H1, so $E_3$ is locally asymptotically stable. \(\Box\)

**Remark 2.** $E_4$ and $E_5$ are not nonnegative in this case, so they are not in our consideration.

**Case 2.** The system $(2.2)$ satisfies the assumptions (H1) and at least one of the (H4) and (H5).

**THEOREM 2.1.1.** Under the assumptions given in Case 2, the system $(2.2)$ has one and only one positive interior equilibrium point $E^0 = (x_1^0, x_2^0, y^0)$. Moreover $E^0$ is locally asymptotically stable.

Proof. First we want to prove the existence of $E^0$. Consider the system $(2.3)$, we have $D < 0$ and $D_1 < 0$ by (H1), $D_2 < 0$ By (H4) or (H5),

\[
D_y = a_2c_1 - a_1c_2 + b_2a_1 - b_1a_2 + b_1c_2 - b_2c_1 < \frac{b_1}{a_1}(a_2c_1 - a_1c_2) + \frac{c_1}{a_1}(b_2a_1 - b_1a_2) + b_1c_2 - b_2c_1 = 0 \quad \text{by (H1, H4),}
\]

or

\[
< \frac{b_2}{a_2}(a_2c_1 - a_1c_2) + \frac{c_2}{a_2}(b_2a_1 - b_1a_2) + b_1c_2 - b_2c_1 = 0 \quad \text{by (H1,H5).}
\]
According to the Cramer’s Rule, the system (2.3) has a unique solution \((x_0^1, x_0^2, y_0)\) with \(x_0^1 = \frac{D_1}{D} > 0\), \(x_0^2 = \frac{D_2}{D} > 0\), \(y_0 = \frac{D_y}{D} > 0\). Hence (2.2) has a unique positive interior equilibrium point \(E^0 = (x_0^1, x_0^2, y_0)\).

Next we want to show that the above \(E^0\) is asymptotically stable. A direct computation yields that the Jacobian matrix of the system (2.2) associated with the equilibrium \(E^0 = (x_0^1, x_0^2, y_0)\) is

\[
J(E^0) = \begin{bmatrix}
-b_1 x_0^1 & -c_1 x_0^1 & -d_1 x_0^1 \\
-b_2 x_0^2 & -c_2 x_0^2 & -d_2 x_0^2 \\
y_0 & y_0 & 0
\end{bmatrix},
\]

The characteristic polynomial of the linearized system (2.2) at \(E^0\) is

\[
P(\lambda) = \lambda^3 + \left(b_1 x_0^1 + c_2 x_0^2\right)\lambda^2 + \left(b_1 c_2 x_0^1 x_0^2 + b_2 c_1 x_0^1 x_0^2 - b_2 c_1 x_0^1 x_0^2 + d_2 x_0^2 y_0 + d_1 x_0^1 y \right)\lambda
\]

\[
+(c_2 d_1 - c_1 d_2 + b_1 d_2 - b_2 d_1)x_0^1 x_0^2 y_0
\]

\[
= \lambda^3 + \mu_2 \lambda^2 + \mu_1 \lambda + \mu_0.
\]

where

\[
\mu_2 = b_1 x_0^1 + c_2 x_0^2,
\]

\[
\mu_1 = (b_1 c_2 - b_2 c_1)x_0^1 x_0^2 + d_2 x_0^2 y_0 + d_1 x_0^1 y_0,
\]

\[
\mu_0 = (c_2 d_1 - c_1 d_2 + b_1 d_2 - b_2 d_1)x_0^1 x_0^2 y_0.
\]
By the assumptions of the system, we have

\[ \mu_i > 0, \quad i = 0, 1, 2. \]

and

\[ \mu_1 \mu_2 - \mu_0 = (c_1 d_2 + b_2 d_1) x_0^0 x_2^0 y_0^0 + (b_1 c_2^2 - c_1 c_2 b_2) x_1^0 (x_2^0)^2 \]

\[ + (b_2^2 c_2 - b_1 c_1 b_2) (x_1^0)^2 x_2^0 + c_2 d_2 (x_2^0)^2 y_0^0 + b_1 d_1 (x_1^0)^2 y_0^0 > 0. \]

So that \( p(\lambda) \) is a Hurwitz polynomial, and by the Routh-Hurwitz stability criterion, \( p(\lambda) = 0 \) has three roots with negative real part. Therefore \( E^0 \) is locally asymptotically stable. \( \square \)

Notice that under the assumptions of Case 2, the boundary equilibrium points \( E_0, E_1, E_2 \) and \( E_3 \) are all nonnegative. moreover, \( E_4 \) is nonnegative if \( \frac{a_2}{c_2} > 1 \), \( E_5 \) is nonnegative if \( \frac{a_1}{b_1} > 1 \).

**CLAIM 2.1.2.** Under the assumptions given in Case 2, all the boundary equilibrium points are unstable if they exist.

Proof. From above Remark 1, \( E_0, E_1, \) and \( E_2 \) are not stable, consider the eigenvalues of other equilibrium points, we have

At \( E_3 \), eigenvalue \( -1 + x_1^* + x_2^* \) is positive by H4 or H5, so \( E_3 \) is unstable.

At \( E_4 \) eigenvalue \( a_1 - c_1 - d_1 \frac{a_2 - c_2}{d_2} \) is positive by H1, so \( E_4 \) is unstable.

At \( E_5 \), eigenvalue \( a_2 - b_2 - d_2 \frac{a_1 - b_1}{d_1} \) is positive by H4 or H5, so \( E_5 \) is unstable. \( \square \)
Remark 3. By checking the vector field at each $E_i(i = 3, 4, 5)$ in this case, we know that $E_i(i = 3, 4, 5)$ is unstable in the positive direction orthogonal to the plane in which it is located if it exists.

Case 3. The system (2.2) satisfies the assumptions (H1) and (H6).

CLAIM 2.1.3. With the assumptions given Case 3, the system (2.2) does not have positive interior equilibrium point, but has unstable nonnegative boundary equilibrium points $E_0, E_1, E_3, E_4$ (if $\frac{a_2}{c_2} > 1$), and stable nonnegative boundary equilibrium $E_5$.

Proof. Now consider the system (2.3) with the same argument in the proof of Theorem 2.1.1, the only solution to the system (2.3) is $x_1 = \frac{D_1}{D}, x_2 = \frac{D_2}{D}, y = \frac{D_y}{D}$, and $D < 0, D_1 < 0, D_y < 0$, but $D_2 > 0$ by (H6), so $x_2 < 0$. Therefore the system (2.2) has no positive interior equilibrium.

Consider the system (2.2), with similarly argument of the Claim 2.1.2, we can confirm that $E_0, E_1, E_2$ and $E_4$ are unstable, but at $E_5$, all of its eigenvalues

$$ a_2 - b_2 - d_2 \frac{a_1 - b_1}{d_1} < 0 \quad \text{by H6}, \quad -b_1 \pm \sqrt{b_1^2 - 4(a_1 - b_1)} < 0 \quad \text{by H1}. $$

Hence $E_5$ is stable. \(\square\)

Remark 4. By checking the vector field at point $E_4$ in this case, we know that $E_4$ is unstable in the positive direction orthogonal to the $x_2 - y$ plane.
2.2 Case 1 — Global Stability of the Boundary Equilibrium Point where Two Prey Coexistence in the Absence of Predator

Now we study the global stability of the equilibrium $E_3$ under the conditions given in Case 1. Before we study it, we have the following basic results

a. Each coordinate axis and each plane are invariant sets.

b. All solutions of the system (2.2) are bounded in forward time.

c. The omega limit set of any nonnegative solution of the system (2.2) is bounded.

In this section, We use $p(t, p_0)$ to denote the solution of the system (2.2) through a point $p_0$; $\Omega(p_0)$ denotes the omega limit set of the orbit $p(t, p_0)$, where $p_0 = (x_{10}, x_{20}, y_0)$ is a positive initial condition; $\alpha(p_0)$ denotes the alpha limit set of the orbit $p(t, p_0)$; $M^s(E)$ denotes the stable manifold of an equilibrium $E$ and $M^u(E)$ denotes the unstable manifold of the equilibrium $E$.

**Lemma 2.2.1.** Let $E$ be an isolated hyperbolic equilibrium in the omega limit set $\Omega(p_0)$, then either $\Omega(p_0) = E$ or there exist point $Q_1, Q_2$ in $\Omega(p_0)$ with $Q_1 \in M^s(E)$ and $Q_2 \in M^u(E)$. (see [4])

**Lemma 2.2.2.** Let $p(t, p_0) = (x_1(t), x_2(t), y(t))$ be a positive solution of the system (2.2) and $\Omega(p_0)$ be the omega limit set of $p(t, p_0)$, then

$$\Omega(p_0) \cap \{(0, x_2, y) : x_2 \geq 0, y \geq 0\} = \emptyset,$$  

(2.4)

$$\Omega(p_0) \cap \{(x_1, 0, y) : x_1 \geq 0, y \geq 0\} = \emptyset.$$  

(2.5)
Proof. First we want to show that (2.4) is true. Assume it is false. Let $A \in \Omega(p_0) \cap \{(0, x_2, y) : x_2 \geq 0, y \geq 0\}$. We claim that $A \neq \Omega(p_0)$. If $A = \Omega(p_0)$, then $\Omega(p_0)$ is a single point. So that $A$ is an equilibrium and $p(t, p_0) \to A$ as $t \to \infty$. It follows that $p(t, p_0)$ must be in the stable manifold $M^s(A)$ of $A$. This leads to a contradiction since $M^s(A) \cap R^+_3 = \emptyset$, where $R^+_3 = \{(x_1, x_2, y) : x_1 > 0, x_2 > 0, y > 0\}$.

If $A$ is in the inner of $x_2-y$ plane, or on the positive $y-axis$ or on the $x_2-axis$ with $x_2 > \frac{a_2}{c_2}$, then the orbit $\{p(t, A) : t \in \mathbb{R}\}$ through the point $A$ is contained in $\Omega(p_0)$. On the other hand it is easy to see that the solution $p(t, A)$ is unbounded as $t \to -\infty$, which contradicts the boundedness of $\Omega(p_0)$.

If $A = E_0 = (0, 0, 0)$, since $E_0$ is a hyperbolic saddle equilibrium, it cannot be the only point in $\Omega(p_0)$. By Lemma 2.2.1, there exists at least one point $B \in \Omega(p_0) \cap M^s(E_0)$. Note that the positive point of $M^s(E_0)$ is the positive $y-axis$, so the unbounded orbit $p(t, B)$ through the point $B$ is contained in bounded $\Omega(p_0)$, a contradiction.

If $A = E_2 = (0, \frac{a_2}{c_2}, 0)$, noting that $E_2$ is also a hyperbolic saddle equilibrium. It is easy to see that $M^s(E_2)$ is the positive of $x_2-axis$. By Lemma 2.2.1, there exists $C \in \Omega(p_0) \cap M^s(E_2)$. The orbit $p(t, C)$ through point $C$ is contained in $\Omega(p_0)$. If $C$ is the left side of $E_2$, then $\alpha(C) = E_0 \in \Omega(p_0)$, a contradiction the fact that $E_0 \notin \Omega(p_0)$ as showed in above. If $C$ is the right side of $E_2$, then $p(t, C)$ is unbounded, a contradiction to the boundedness of $\Omega(p_0)$.

If $A$ is on $x_2-axis$ between $E_0$ to $E_2$, then $\alpha(A) = E_0 \in \Omega(p_0)$, a contradiction.

From above arguments, we know (2.4) is true. By similar arguments, (2.5) is true. □
THEOREM 2.2.1. Under the same assumptions given in Case 1 in Section 2.1, the equilibrium $E_3 = \left( \frac{a_1c_2 - a_2c_1}{b_1c_2 - b_2c_1}, \frac{b_1a_2 - b_2a_1}{b_1c_2 - b_2c_1}, 0 \right)$ is globally stable. i.e., for any $p_0 \in \mathbb{R}_3^+$, the solution $p(t, p_0)$ of the system (2.2) converges to $E_3$ as $t$ goes to $\infty$.

Proof. Let $p_0 \in \mathbb{R}_3^+$, we first claim that there exists a point $q = (q_1, q_2, 0) \in \Omega(p_0)$ with $q_1 > 0$, $q_2 > 0$. To this end, let $p(t, p_0) = (x_1(t), x_2(t), y(t))$ be a solution of the system (2.2) through the point $p_0$, rewrite the system (2.2) as

\begin{align}
    x_1' &= b_1x_1\left(\frac{a_1}{b_1} - x_1 - \frac{c_1}{b_1}x_2 - \frac{d_1}{b_1}y\right), \\
    x_2' &= c_2x_2\left(\frac{a_2}{c_2} - \frac{b_2}{c_2}x_1 - x_2 - \frac{d_2}{c_2}y\right). \tag{2.6}
\end{align}

Let

\begin{align*}
    m_1 &= \frac{a_1}{b_1}, \quad n_1 = \frac{c_1}{b_1}, \quad e_1 = \frac{d_1}{b_1}, \\
    m_2 &= \frac{a_2}{c_2}, \quad n_2 = \frac{b_2}{c_2}, \quad e_2 = \frac{d_2}{c_2}.
\end{align*}

Then (2.6) becomes

\begin{align}
    x_1' &= b_1x_1(m_1 - x_1 - n_1x_2 - e_1y), \\
    x_2' &= c_2x_2(m_2 - n_2x_1 - x_2 - e_2y), \tag{2.7}
    y' &= y(-1 + x_1 + x_2).
\end{align}

Note

\begin{align*}
    m_1 \leq 1, \quad m_2 \leq 1.
\end{align*}
First consider the case \( n_1 \geq 1 \) or \( n_2 \geq 1 \).

If \( n_1 \geq 1 \), then

\[
\frac{d}{dt} \left( \frac{1}{b_1} \ln x_1 + \ln y \right) = \frac{x_1'}{b_1 x_1} + \frac{y'}{y}
\]

\[= -(1 - m_1) - (n_1 - 1)x_2 - e_1 y < 0.\]

So

\[
\frac{1}{b_1} \ln x_1 + \ln y = \frac{1}{b_1} \ln x_1(0) + \ln y(0) + \int_0^t \left\{ -(1 - m_1) - (n_1 - 1)x_2(s) - e_1 y(s) \right\} ds.
\]

Hence

\[
\frac{1}{b_1} \ln x_1 + \ln y \to -\infty, \quad \text{as } t \to \infty. \quad (2.8)
\]

Now pick a sequence \( t_n \to \infty \) as \( n \to \infty \) such that

\[
\lim_{t \to \infty} (x_1(t), x_2(t), y(t)) = q = (q_1, q_2, \bar{y}) \in \Omega(p_0).
\]

Then, by (2.8), either \( x_1(t_n) \to 0 \) or \( y(t_n) \to 0 \) as \( n \to \infty \). i.e, either \( q \in x_2 - y \) plane or \( q \in x_1 - x_2 \) plane. By (2.4), \( q \in x_1 - x_2 \) plane. Also it is clear that \( q_1 > 0, q_2 > 0 \), since if \( q_1 = 0 \), then \( q \in x_2 - y \) plane, if \( q_2 = 0 \), then \( q \in x_1 - y \) plane, which leads to a contradiction to (2.4) or (2.5). This confirms the claim for the case of \( n_1 \geq 1 \).

For \( n_2 \geq 1 \), by using a similar argument, we can have

\[
\frac{1}{c_2} \ln x_2 + \ln y \to -\infty, \quad \text{as } t \to \infty, \quad (2.9)
\]
and conclude that \( q \in x_1 - y \) plane or \( q \in x_1 - x_2 \) plane. By (2.4), \( q \in x_1 - x_2 \) plane with \( q_1 > 0, q_2 > 0 \).

Next consider the case \( n_1 < 1 \) and \( n_2 < 1 \). Recall that \( E_3 = (x_1^*, x_2^*, 0) \), we can write the system (2.7) as

\[
\begin{align*}
x_1' &= b_1x_1 \left[ -(x_1 - x_1^*) - n_1(x_2 - x_2^*) - e_1y \right], \quad (2.10) \\
x_2' &= c_2x_2 \left[ -n_2(x_1 - x_1^*) - (x_2 - x_2^*) - e_2y \right], \quad (2.11) \\
y' &= y \left[ -r + (x_1 - x_1^*) + (x_2 - x_2^*) \right]. \quad (2.12)
\end{align*}
\]

Where \( r = 1 - (x_1^* + x_2^*) > 0 \).

From (2.12), one has

\[
\ln \frac{y(t)}{y(0)} = - \int_0^t \{ r - [(x_1(s) - x_1^*) + (x_2(s) - x_2^*)] \} ds.
\]

The above expression yields that either there exists a sequence \( t_n^* \to \infty \) as \( n \to \infty \) such that \( y(t_n^*) \to 0 \) as \( n \to \infty \), or there exists an \( M > 0 \) such that

\[
\int_0^t \{ r - [(x_1(s) - x_1^*) + (x_2(s) - x_2^*)] \} ds \leq M, \quad \text{for all } t > 0. \quad (2.13)
\]

If \( y(t_n^*) \to 0 \), with the same argument as above, we can say the claim is true.

If (2.13) happens, then it yields that

\[
\int_0^t [(x_1(s) - x_1^*) + (x_2(s) - x_2^*)] ds \to \infty, \quad \text{as } t \to \infty.
\]
It follows that there exists a sequence $\tau_n \to \infty$ as $n \to \infty$ with

$$\int_0^{\tau_n} [(x_1(s) - x_1^*)]ds \to \infty, \quad \text{as } n \to \infty,$$

or

$$\int_0^{\tau_n} (x_2(s) - x_2^*)ds \to \infty, \quad \text{as } n \to \infty.$$

Without loss of generality, suppose

$$\int_0^{\tau_n} (x_1(s) - x_1^*)ds \to \infty, \quad \text{as } n \to \infty. \quad (2.14)$$

From (2.10) and (2.12), we have

$$\frac{x_1'}{b_1 x_1} + \frac{n_1 y'}{y} = -(1 - n_1)(x_1 - x_1^*) - e_1 y - n_1 r. \quad (2.15)$$

So

$$\frac{1}{b_1} \ln \frac{x_1(\tau_n)}{x_1(0)} + n_1 \ln \frac{y(\tau_n)}{y(0)} = - \int_0^{\tau_n} [(1 - n_1)(x_1(s) - x_1^*) + e_1 y(s) + n_1 r]ds. \quad (2.16)$$

From (2.14), (2.16), and the assumption $n_1 < 1$, we have

$$\frac{1}{b_1} \ln \frac{x_1(\tau_n)}{x_1(0)} + n_1 \ln \frac{y(\tau_n)}{y(0)} \to -\infty, \quad \text{as } n \to \infty.$$
Which tells us,
\[ x_1(\tau_n) \to 0, \quad \text{as } n \to \infty, \]
or
\[ y(\tau_n) \to 0, \quad \text{as } n \to \infty. \]

Arguing in the same way as done for the case of \( n_1 \geq 1 \), we see that the claim is true.

Finally we want to prove
\[ \lim_{t \to \infty} p(t, p_0) = E_3, \]
where \( p(t, p_0) \) is the solution of the system (2.2) with \( p_0 \in \mathbb{R}_3^+ \).

Since \( E_3 \) is asymptotically stable, there exists a neighborhood \( N(E_3) \) of \( E_3 \) such that, for any point \( p_1 \in N(E_3) \), the solution \( p(t, p_1) \) of (2.2) through \( p_1 \) will have limit and
\[ \lim_{t \to \infty} p(t, p_1) = E_3. \] (2.17)

By the claim, we can choose a point \( q = (q_1, q_2, 0) \in \Omega(p_0) \) with \( q_1 > 0, q_2 > 0 \). and let \( p(t, q) \) be the solution of the system (2.2) through the point \( q \), then \( p(t, q) = (x_1(t), x_2(t), 0) \) for all \( t \geq 0 \), since the set \( \{(x_1, x_2, 0) : x_1 > 0, x_2 > 0\} \) is invariant. So that \( x_1(t) \), and \( x_2(t) \) satisfy the competing system
\[ \begin{align*}
    x'_1 &= x_1(a_1 - b_1 x_1 - c_1 x_2), \\
    x'_2 &= x_2(a_2 - b_2 x_1 - c_2 x_2). 
\end{align*} \] (2.18)
It is known that, under Assumption (H1) and one of Assumptions (H2) and (H3), the positive interior equilibrium point \((x^*_1, x^*_2)\) is globally stable with respect to all positive solutions of the system (2.18). Hence there exists a large enough \(T > 0\) such that \(p(T, q) \in N(E_3)\). Note \(p(T, q) \in \Omega(p_0)\), so there exists a sequence \(T_n \to \infty\) as \(n \to \infty\) such that

\[
p(T_n, p_0) \to p(T, q), \quad \text{as } n \to \infty.
\]

(2.19)

Hence there exists a large \(k > 0\) such that \(p(T_k, p_0) \in N(E_3)\),

By (2.17),

\[
\lim_{t \to \infty} p(t, p_0) = E_3.
\]

Therefore \(E_3\) is globally stable. \(\square\)

**Example.**

\[
x'_1 = x_1(1 - 2x_1 - x_2 - y),
\]
\[
x'_2 = x_2(1 - x_1 - 4x_2 - y),
\]
\[
y' = y(-1 + x_1 + x_2).
\]

(2.20) has equilibria \((0, 0, 0), (0.5, 0, 0), (0, 0.25, 0)\) and \((\frac{3}{7}, \frac{1}{7}, 0), (\frac{3}{7}, \frac{1}{7}, 0)\) is globally stable. The numerical simulations of (2.20) with positive initial conditions agreed with these results. (See Figure 2.3)

### 2.3 Case 2 —Global Stability of the Positive Interior Equilibrium Point

**THEOREM 2.3.1.** \(E^0 = (x^0_1, x^0_2, y^0)\) is globally stable if the system (2.2) satisfies H1, H4 or H5, and in addition \(2b_1c_2d_1d_2 > c_1^2d_2^2 + d_1^2b_2^2\).
Figure 2.3: The numerical simulations of (2.17) with initial condition (0.5,0.1,1.5) and (1,1,4).
Proof. Rewrite the system (2.2) as

\[
\begin{align*}
x_1' &= d_1x_1\left( -\frac{b_1}{d_1}(x_1 - x_1^0) - \frac{c_1}{d_1}(x_2 - x_2^0) - (y - y^0) \right), \\
x_2' &= d_2x_2\left( -\frac{b_2}{d_2}(x_1 - x_1^0) - \frac{c_2}{d_2}(x_2 - x_2^0) - (y - y^0) \right), \\
y' &= y\left( (x_1 - x_1^0) + (x_2 - x_2^0) \right). 
\end{align*}
\]

Define a function \( L: \mathbb{R}_+^3 \to \mathbb{R} \) such that

\[
L(x_1, x_2, y) = \frac{1}{d_1} \int_{x_1^0}^{x_1} \frac{s - x_1^0}{s} ds + \frac{1}{d_2} \int_{x_2^0}^{x_2} \frac{s - x_2^0}{s} ds + \int_{y^0}^{y} \frac{s - y^0}{s} ds.
\]

Then \( L \) is well defined and \( L \geq 0 \) on \( \mathbb{R}_+^3 \). Upon direct calculation, we obtain the derivative of the function \( L \) along a solution of the system (2.21) as

\[
\dot{L} = \frac{x_1 - x_1^0}{d_1x_1} x_1' + \frac{x_2 - x_2^0}{d_2x_2} x_2' + \frac{y - y^0}{y} y'
\]

\[
= \frac{b_1}{d_1} (x_1 - x_1^0)^2 - \left( \frac{c_1}{d_1} + \frac{b_2}{d_2} \right) (x_1 - x_1^0)(x_2 - x_2^0) - \frac{c_2}{d_2} (x_2 - x_2^0)^2.
\]

We have \( 2b_1c_2d_1d_2 > c_1^2d_2^2 + d_1^2b_2^2 \) by the assumption. Also (H1) implies that, \( c_2d_1 > c_1d_2, b_1d_2 > b_2d_1 \). Therefore \( 4b_1c_2d_1d_2 > (c_1d_2 + b_2d_1)^2 \), that is

\[
\frac{b_1c_2}{d_1d_2} > \frac{1}{4} \left( \frac{c_1}{d_1} + \frac{b_2}{d_2} \right)^2.
\]
So that \(-\dot{L}\) is positive definite quadratic form, that is \(\dot{L} \leq 0\).

And notice that

\[
V = \{(x_1, x_2, y) : \dot{L}(x_1, x_2, y) = 0\} = \{(x_1, x_2, y) : x_1 = x_1^0, x_2 = x_2^0, y \geq 0\}.
\]

Which tells us the set \(V \setminus E^0\) does not contain any globally defined solution of (2.2).

Thus by LaSalle’s invariance principle, we conclude that \(E^0\) is globally stable. \(\square\)

**Remark.** The assumption \(2b_1c_2d_1d_2 > c_1^2d_2^2 + d_1^2b_2^2\) looks depending on \(d_1\) and \(d_2\).

But actually the impact of \(d_1\) and \(d_2\) on the stability of the equilibrium point \(E^0\) is negligible. We can use conditions \(b_1 > c_1, c_2 > b_2\) instead of the above inequality.

From \(b_1 > c_1, c_2 > b_2\), we have \(b_1d_2 > c_1d_2, c_2d_1 > b_2d_1\). By (H1), \(c_2d_1 > c_1d_2, d_1\), so that \(2b_1c_2d_1d_2 > (c_1d_2)^2 + (b_2d_1)^2\). Therefore \(4b_1c_2d_1d_2 > (c_1d_2 + b_2d_1)^2\), and hence \(\dot{L} \leq 0\).

**Example.**

\[
\begin{align*}
x'_1 &= x_1(4 - 3x_1 - x_2 - y), \\
x'_2 &= x_2(4 - 2x_1 - 4x_2 - \frac{3}{2}y), \\
y' &= y(-1 + x_1 + x_2).
\end{align*}
\]

(2.22) has equilibria \((0, 0, 0), \left(\frac{4}{3}, 0, 0\right), (0, 1, 0), \left(\frac{6}{7}, \frac{2}{5}, 0\right), (1, 0, 1)\) and \(\left(\frac{9}{10}, \frac{1}{10}, \frac{6}{5}\right), \left(\frac{9}{10}, \frac{1}{10}, \frac{6}{5}\right)\) is globally stable. The numerical simulations of (2.22) with different positive initial conditions agreed with these results as shown in Figure 2.4.
Figure 2.4: The numerical solutions of (2.19) with initial condition (0.25,0.2,0.5) and (1,2,3).
2.4 Case 3—Global Stability of the Boundary Equilibrium where Prey 2 is Absent

**THEOREM 2.4.1**. $E_5$ is globally stable if the system satisfies the assumptions (H1) and (H6), and the inequality $2b_1c_2d_1d_2 > c_1^2d_2^2 + d_1^2b_2^2$.

Proof. Recall that $E_5 = (1, 0, \frac{a_1 - b_1}{d_1})$. Denote $E_5$ by $E_5 = (x^*, 0, y^*)$. According to (H4), we have $\mu = -\frac{a_2}{d_2} + \frac{b_2}{d_2}x^* + y^* > 0$. So rewrite the system (2.2) as

$$
\begin{align*}
x_1' &= d_1 x_1 \left[ -\frac{b_1}{d_1} (x_1 - x^*) - \frac{c_1}{d_1} x_2 - (y - y^*) \right],
\end{align*}
$$

$$
\begin{align*}
x_2' &= d_2 x_2 \left[ -\frac{b_2}{d_2} (x_1 - x^*) - \frac{c_2}{d_2} x_2 - (y - y^*) - \mu \right],
\end{align*}
$$

$$
\begin{align*}
y' &= y \left[ x_1 - x^* \right] + \frac{x_2}{\mu}.
\end{align*}
$$

(2.23)

Let $L$ be a function $L : \mathbb{R}^3_+ \to \mathbb{R}$ such that

$$
L(x_1, x_2, y) = \frac{1}{d_1} \int_{x^*}^{x_1} \frac{s - x^*}{s} ds + \frac{1}{d_2} x_2 + \int_{y^*}^{y} \frac{s - y^*}{s} ds.
$$

Then $L$ is well defined and $L \geq 0$ on $\mathbb{R}^3_+$. The derivative of the function $L$ along a solution of the dynamical system (2.23) is

$$
\dot{L} = \frac{x_1 - x^*}{d_1 x_1} x_1' + \frac{1}{d_2} x_2' + \frac{y - y^*}{y} y' - \frac{b_1}{d_1} (x_1 - x^*)^2 - \left( \frac{c_1}{d_1} + \frac{b_2}{d_2} \right) (x_1 - x^*) x_2 - \frac{c_2}{d_2} x_2^2 - \mu x_2.
$$
Since \( 2b_1c_2d_1d_2 > c_1^2d_2^2 + d_1^2b_2^2 \), and \( c_2d_1 > c_1d_2, b_1d_2 > b_2d_1 \) by (H1), we have \( 4b_1c_2d_1d_2 > (c_1d_2 + b_2d_1)^2 \), that is

\[
\frac{b_1c_2}{d_1d_2} > \frac{1}{4} \left( \frac{c_1}{d_1} + \frac{b_2}{d_2} \right)^2.
\]

Which tells us

\[
\frac{b_1}{d_1}(x_1 - x^*)^2 + \left( \frac{c_1}{d_1} + \frac{b_2}{d_2} \right)(x_1 - x^*)x_2 + \frac{c_2}{d_2}x_2^2 \geq 0.
\]

Hence \( \dot{L} \leq 0 \). Note that the set

\[
V = \{(x_1, x_2, y) : \dot{L}(x_1, x_2, y) = 0\} = \{(x_1, x_2, y) : x_1 = x^*, x_2 = 0, y \geq 0\}.
\]

Which tells us the set \( V \setminus \mathcal{E}_5 \) does not contain any orbit of the system (2.2). Thus by LaSalle’s invariance principle, \( \mathcal{E}_5 \) is globally stable. \( \square \)

Example.

\[
\begin{align*}
x'_1 &= x_1(3 - 2x_1 - x_2 - y), \\
x'_2 &= x_2(2 - x_1 - 4x_2 - 2y), \\
y' &= y(-1 + x_1 + x_2). 
\end{align*}
\]

(2.24) has equilibria \((0, 0, 0), (1.5, 0, 0), (0, 0.5, 0)\) and \((1, 0, 1), (1, 0, 1)\) is globally stable. The numerical simulations of (2.24) with positive initial conditions agreed with these results. (See Figure 2.5)
**Figure 2.5**: The numerical simulations of (2.21) with initial condition (0.2,0.4,0.6) and (2,3,1).
CHAPTER 3

TRAVELING WAVE SOLUTIONS FOR A REACTION-DIFFUSION
ONE PREDATOR-TWO PREY MODEL

There are three sections in this chapter. In first section, we introduce a model with reaction diffusion which is partial differential equations. In second section, we transform the model into ordinary differential equations for finding traveling wave solutions. In the last section, we show the existence of traveling waves by a geometric approach and the Lyapunov method.

3.1 The Model with Reaction-Diffusion System

There are numerous applications of traveling wave solutions to many areas of sciences and the significant role of wave solution is well recognized. These inspire me to study traveling wave solutions for the diffusive two-competing-prey and one-predator model. A very important component of two-prey and one-predator interaction is spatial variation in the populations when predators are spatially distributed caused by the movement of species in searching for food. We want to see the influence of the predator on the two competing species. In this chapter, we consider a spatial extension of the model (2.2) by taking consideration of the predator move-
ment. Assume the movement of the predator is random, which can be modeled as a diffusion process. We also suppose that the movement of prey species are negligible. For example the prey species are plants. Then the model (2.2) can be extended to the following reaction-diffusion system

\[
\begin{align*}
\frac{\partial N_1}{\partial t} &= N_1(a_1 - b_1 N_1 - c_1 N_2 - d_1 Y), \\
\frac{\partial N_2}{\partial t} &= N_2(a_2 - b_2 N_1 - c_2 N_2 - d_2 Y), \\
\frac{\partial Y}{\partial t} &= d \Delta Y + Y(-1 + N_1 + N_2).
\end{align*}
\] (3.1)

where \( N_1(t, x) \) and \( N_2(t, x) \) are the population densities of the first and second prey at time \( t \) and location \( x \in \mathbb{R}^n \) respectively; \( Y(t, x) \) denotes the population densities of predator at time \( t \) and location \( x \in \mathbb{R}^n \); \( \Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \); \( d \) denotes the diffusivity of predator; all \( a_i, b_i, c_i \) and \( d_i \) \( (i = 1, 2) \) are positive constants defined as in the model (2.2).

In the chapter 2, we have studied all possible nonnegative equilibria \( E_0, E_1, E_2, E_3, E_4, E_5 \) and \( E^0 \) of the model (2.2) and its stability in three cases. However, in this chapter, we only focus on the case 3 which is discussed in section 2.1 and the case 2 that the system (2.2) has positive equilibrium \( E^0 \).

First we study the case 3 and consider

\[ E_2 = (0, \frac{a_2}{c_2}, 0), E_5 = (1, 0, \frac{a_1 - b_1}{d_1}) \]

where \( E_2 \) is the equilibrium at which only prey 2 is present, whereas at \( E_5 \), the first prey and the predator coexist with the second prey being absent. The reason we
consider these two equilibria is because we are interested in how the predator may influence the prey in space. We wonder whether there will be a zone of transition from $E_2$ to $E_5$. Mathematically this transition is a particular type of solution, a traveling wave solution. Existence of traveling waves is very important (see [14], [17] and [18]). So our purpose is to study the existence of the traveling wave solutions of the system (3.1) connecting $E_2$ to $E_5$. Then we study the case that the system (2.2) has a positive interior equilibrium $E_0$. Similarly we shall show that there is a traveling solutions of the system (3.1) connecting $E_2$ and $E^0$.

3.2 An ODE System for Traveling Wave Solutions

To investigate the existence of traveling wave solutions, we need consider and have certain assumptions on the corresponding reaction equation of (3.1), which is given by (2.2). Suppose for the system (3.1), (H1), (H6), $\frac{a_2}{c_2} > 1$, and $2b_1c_2d_1d_2 > c_1^2d_2^2 + d_1^2b_2^2$ hold, Our interest is to investigate the existence of solutions of (3.1) of the form

$$N_i(t,x) = U_i(k \cdot x + ct), \quad i = 1, 2,$$

$$Y(t,x) = V(k \cdot x + ct).$$

satisfying the boundary condition at infinite:

$$(U_1(-\infty), U_2(-\infty), V(-\infty)) = E_2 = (0, \frac{a_2}{c_2}, 0),$$

$$(U_1(\infty), U_2(\infty), V(\infty)) = E_5 = (1, 0, \frac{a_1 - b_1}{d_1}).$$

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Here $k \in \mathbb{R}^n$ is a unit vector denoting the direction of wave propagation and $c \in \mathbb{R}$ is a wave speed. The existence of solutions of form (3.2) and (3.3) indicates the existence of a transition zone between the equilibrium points $E_2$ and $E_5$. (3.1) has a traveling wave solution of form (3.2) connecting $E_2$ and $E_5$ if and only if the functions $U_1(\xi), U_2(\xi)$ and $V(\xi)$ with $\xi = x \cdot k + ct$ is a solution of the system

$$
c\dot{U}_1 = U_1(a_1 - b_1 U_1 - c_1 U_2 - d_1 V),
$$
c\dot{U}_2 = U_2(a_2 - b_2 U_1 - c_2 U_2 - d_2 V), \tag{3.4}
$$
c\ddot{V} = d \ddot{V} + V (-1 + U_1 + U_2).
$$

where $\dot{V}$ and $\ddot{V}$ denote the first and second order derivative of a function $V$ respectively. Instead of using a standard transformation, for a constant $c > 0$, we make the following changes of variables and scaling

$$
u(t) = V(ct),
$$
w(t) = V(ct) - \frac{d}{c} \dot{V}(ct). \tag{3.5}$$
Then, by a straightforward calculation, (3.4) is transformed to a four dimensional system

\begin{align*}
\dot{u}_1 &= u_1(a_1 - b_1 u_1 - c_1 u_2 - d_1 v), \\
\dot{u}_2 &= u_2(a_2 - b_2 u_1 - c_2 u_2 - d_2 v), \\
\dot{v} &= \frac{c^2}{d}(v-w), \\
\dot{w} &= v(-1 + u_1 + u_2).
\end{align*}

(3.6)

It is obvious that \((U_1(\xi), U_2(\xi), V(\xi))\) is a nonnegative solution of (3.4) connecting the equilibria \(E_2\) and \(E_5\) if and only if \((u_1(t), u_2(t), v(t), w(t))\) with \(u_1(t) \geq 0, u_2(t) \geq 0, v(t) \geq 0, w(t) \geq 0\), is a solution of (3.6) satisfying the boundary condition

\begin{align*}
(u_1(-\infty), u_2(-\infty), v(-\infty), w(-\infty)) &= E_2 = (0, \frac{a_2}{c_2}, 0, 0, 0), \\
(u_1(\infty), u_2(\infty), v(\infty), w(\infty)) &= E_5 = (1, 0, \frac{a_1 - b_1}{d_1}, \frac{a_1 - b_1}{d_1}).
\end{align*}

(3.7)

Here, for convenience, we still use \(E_2\) and \(E_5\) to denote the equilibrium points \((0, \frac{a_2}{c_2}, 0, 0)\) and \((1, 0, \frac{a_1 - b_1}{d_1}, \frac{a_1 - b_1}{d_1})\) of the system (3.6).

**3.3 Existence of Traveling Wave Solution**

In this section we will use a similar technique developed in [2], [3] and [6] to show the existence of solutions of (3.6) satisfying the boundary conditions (3.7).

For

\[0 \leq u_1 \leq \frac{a_1}{b_1}, \quad 0 \leq u_2 \leq \frac{a_2}{c_2}.
\]
let

\[ M = \max\{u_1 + u_2\}. \] (3.8)

Let \( \sigma > 1 \) be a constant defined by

\[ \sigma = \frac{c^2 + \sqrt{c^4 + 4dc^2}}{2c^2}. \] (3.9)

For \( c > 2\sqrt{d(M - 1)} \), we construct a wedged region \( \Sigma \subset \mathbb{R}^4 \) as follows (see Figure 3.1).

![Figure 3.1: The region \( \Sigma \).](image)

In the graph, the axis \( u \) represents \( u_1 - u_2 \) plane, \( E \) stands for \((\frac{a_1}{b_1}, 0, 0, 0)\) and \((0, \frac{a_2}{c_2}, 0, 0)\). A solution of the system (3.6) through a point in the interior of \( \Omega \) can only exits region \( \Sigma \) from either a point in the face \( P_1 \) or a point in the face \( P_2 \) when time increases.
\[ \Sigma = \left\{ (u_1, u_2, v, w) : 0 \leq u_1 \leq \frac{a_1}{b_1}, \ 0 \leq u_2 \leq \frac{a_2}{c_2}, \ \frac{1}{2}v \leq w \leq \sigma v \right\} \] (3.10)

The boundary of \( \Sigma \) consists of the faces \( P_i \) for \( i = 1, \cdots, 7 \), with

\[
\begin{align*}
P_1 & = \left\{ w = \frac{1}{2}v, \ 0 < u_1 < \frac{a_1}{b_1}, \ 0 < u_2 < \frac{a_2}{c_2} \right\}, \\
P_2 & = \left\{ w = \sigma v, \ 0 < u_1 < \frac{a_1}{b_1}, \ 0 < u_2 < \frac{a_2}{c_2} \right\}, \\
P_3 & = \left\{ u_1 = 0, \ 0 < u_2 < \frac{a_2}{c_2}, \ \frac{1}{2}v \leq w \leq \sigma v \right\}, \\
P_4 & = \left\{ u_2 = 0, \ 0 < u_1 < \frac{a_1}{b_1}, \ \frac{1}{2}v \leq w \leq \sigma v \right\}, \\
P_5 & = \left\{ u_1 = \frac{a_1}{b_1}, \ 0 < u_2 < \frac{a_2}{c_2}, \ \frac{1}{2}v \leq w \leq \sigma v \right\}, \\
P_6 & = \left\{ u_2 = \frac{a_2}{c_2}, \ 0 < u_1 < \frac{a_1}{b_1}, \ \frac{1}{2}v \leq w \leq \sigma v \right\}, \\
P_7 & = \left\{ v = w = 0, \ 0 < u_1 < \frac{a_1}{b_1}, \ 0 < u_2 < \frac{a_2}{c_2} \right\}.
\] (3.11)

Now we study the vector field of (3.6) in the boundary of \( \Sigma \) which can be characterized by the following two lemmas.

**Lemma 3.3.1.** Let \( \Phi_t(p) \) be the flow of (3.6), i.e. \( \Phi_t(p) \) is a solution of (3.6) satisfying the initial condition \( \Phi_0(p) = p \in \mathbb{R}^4 \), then for any \( p \in \text{interior of } \Sigma \), \( \Phi_t(p) \) can not leave \( \Sigma \) from a point in the boundary \( \bigcup_{i=3}^{7} P_i \) of \( \Sigma \) at any positive time.

Proof. It is obvious that the plan \( \{u_1 = 0\}, \{u_2 = 0\} \) and \( \{v = w = 0\} \) are invariant sets of (3.6). Hence any solution of (3.6) through a point in the interior of
Σ can not leave Σ from a point in the set $P_3 \cup P_4 \cup P_7$ at a positive time $t$.

For any point $p = (u_1, u_2, v, w) \in P_5$, where $u_1 = \frac{a_1}{b_1}$ and $0 < u_2 < \frac{a_2}{c_2}$, then

$$u_1 = u_1(-c_1 u_2 - d_1 v) < 0. \quad (3.12)$$

Which tells us the vector field of (3.6) at each point $p$ in the set $P_5$ points into the left-side of the plane $u_1 = \frac{a_1}{b_1}$, that is the region in which the $u_1$-coordinate is less than $\frac{a_1}{b_1}$, also the set Σ is in this region, therefore we conclude that any solution $\Phi_t(p)$ of (3.6) through a point $p$ in the interior of Σ can not leave Σ from a point in the set $P_5$ at a positive time.

For each point $p = (u_1, u_2, v, w) \in P_6$, where $u_2 = \frac{a_2}{c_2}$ and $0 < u_1 < \frac{a_1}{b_1}$, then

$$u_2 = u_2(-b_2 u_1 - d_2 v) < 0, \quad (3.13)$$

which tells us the vector field of (3.6) at each point $p$ in the set $P_6$ points into the left-side of the plane $u_2 = \frac{a_2}{c_2}$, that is the region in which the $u_2$-coordinate is less than $\frac{a_2}{c_2}$, also the set Σ is in this region, therefore we conclude that any solution $\Phi_t(p)$ of (3.6) through a point $p$ in the interior of Σ can not leave Σ from a point in the set $P_6$ at a positive time. □

**Lemma 3.3.2.** If $c > 2\sqrt{d(M - 1)}$, then the vector field of (3.6) at any point $p \in P_1 \cap P_2$ points the exterior of Σ.
Proof. The out normal vector of the face $P_1$ is $(0,0,1,-1)^T$, at each point $p = (u_1, u_2, v, w) \in P_1$, the vector field of (3.6) is $(\dot{u}_1, \dot{u}_2, \dot{v}, \dot{w})^T$, then

\[
(0,0,1,-1) \cdot (\dot{u}_1, \dot{u}_2, \dot{v}, \dot{w})^T = \frac{1}{2} \dot{v} - \dot{w} \\
= \frac{c^2}{2d}(v - w) - v(-1 + u_1 + u_2) \\
= \frac{c^2}{4d}v - v(-1 + u_1 + u_2) \\
> (M - 1)v - v(-1 + u_1 + u_2) \\
= M - (u_1 + u_2) \geq 0.
\]

Which implies that the vector field at a point $p$ in the face $P_1$ points toward the outside of the region $\Sigma$.

Next, consider a point $p = (u_1, u_2, v, w) \in P_2$, then $w = \sigma v$ and $0 < u_1 < \frac{a_1}{b_1}$.

At point $p$, with the use of (3.6) and (3.9), we have

\[
\dot{v} = \frac{c^2}{d}(v - w) \\
= \frac{c^2}{d}(1 - \sigma)v \\
= \frac{c^2 - \sqrt{c^4 + 4dc^2}}{2d}v < 0, \\
\dot{w} = \frac{2d(1 - u_1 - u_2)}{\sqrt{c^4 + 4dc^2} - c^2} \\
< \frac{2d}{\sqrt{c^4 + 4dc^2} - c^2} = \sigma.
\]

(3.15) implies the vector field of (3.6) at the point $p$ points to the exterior of $\Sigma$. \qed
Now we would like to study the unstable manifold of the equilibrium point \( E_2 \) of the system (3.6). First linearize the system (3.6) at \( E_2 \), a direct computation yields that the Jacobian matrix of (3.6) associated with the equilibrium \( E_2 = (0, \frac{a_2}{c_2}, 0, 0) \) is

\[
J(E_2) = \begin{bmatrix}
    a_1 - c_1 \frac{a_2}{c_2} & 0 & 0 & 0 \\
    -b_2 \frac{a_2}{c_2} & -a_2 & -d_2 \frac{a_2}{c_2} & 0 \\
    0 & 0 & c^2 & -\frac{c^2}{d} \\
    0 & 0 & \frac{a_2 - c_2}{c_2} & 0
\end{bmatrix}
\] (3.16)

The Characteristic equation of \( J(E_2) \) is

\[
P(\lambda) = \det(J - \lambda I) = (\lambda + a_2)(\lambda - \frac{a_1 c_2 - a_2 c_1}{c_2})(\lambda^2 - \frac{c^2}{d} \lambda + \frac{(a_2 - c_2)^2 c^2}{c_2 d}) = 0.
\]

So

\[
\lambda_0 = -a_2,
\]

\[
\lambda_1 = \frac{a_1 c_2 - a_2 c_1}{c_2},
\]

\[
\lambda_2 = \frac{c^2 + \sqrt{c^4 - 4d c^2 (\frac{a_2}{c_2} - 1)}}{2d},
\]

\[
\lambda_3 = \frac{c^2 - \sqrt{c^4 - 4d c^2 (\frac{a_2}{c_2} - 1)}}{2d}.
\] (3.17)

Recall that \( c^2 > 4d(M - 1) \geq 4d(\frac{a_2}{c_2} - 1) > 0 \), so \( E_2 \) has only one negative eigenvalue \( \lambda_0 \) and three positive eigenvalues \( \lambda_1, \lambda_2, \lambda_3 \). Hence, the unstable manifold of the equilibrium \( E_2 \) is three dimensional. By more computing, the eigenvectors \( h_i \)
corresponding to eigenvectors \( \lambda_i \), for \( i = 1, 2, 3 \), are respectively given by

\[
\begin{align*}
  h_1 &= \left( 1, -\frac{a_2 b_2}{c_2(a_2 + \lambda_1)}, 0, 0 \right)^T, \\
  h_2 &= \left( 0, -\frac{a_2 d_2}{c_2(a_2 + \lambda_2)}, 1, \frac{c^2 - \sqrt{c^4 - 4dc^2\left(\frac{a_2}{c_2} - 1\right)}}{2c^2} \right)^T, \\
  h_3 &= \left( 0, -\frac{a_2 d_2}{c_2(a_2 + \lambda_3)}, 1, \frac{c^2 + \sqrt{c^4 - 4dc^2\left(\frac{a_2}{c_2} - 1\right)}}{2c^2} \right)^T.
\end{align*}
\] (3.18)

From the unstable manifold theorem and (3.18), there is a smooth twice continuously differentiable function \( \Phi = (\Phi_1, \Phi_2, \Phi_3, \Phi_4) : N \to \mathbb{R}^4 \) in a small neighborhood \( N \) of the origin in \( \mathbb{R}^3 \) such that the unstable manifold \( M^u(E_2) \) of \( E_2 \) can be expressed as

\[
M^u(E_2) = \left\{ k_1 h_1 + k_2 h_2 + k_3 h_3 + E_1 + \Phi(k_1, k_2, k_3) : (k_1, k_2, k_3) \in N \right\}
\]

\[
= \left\{ u_1(k_1, k_2, k_3), u_2(k_1, k_2, k_3), v(k_1, k_2, k_3), w(k_1, k_2, k_3) : (k_1, k_2, k_3) \in N \right\},
\] (3.19)

with

\[
\begin{align*}
  u_1(k_1, k_2, k_3) &= k_1 + \Phi_1(k_1, k_2, k_3), \\
  u_2(k_1, k_2, k_3) &= -a_{11} k_1 - a_{21} k_2 - a_{31} k_3 + \frac{a_2}{c_2} + \Phi_2(k_1, k_2, k_3), \\
  v(k_1, k_2, k_3) &= k_2 + k_3 + \Phi_3(k_1, k_2, k_3), \\
  w(k_1, k_2, k_3) &= a_{24} k_2 + a_{34} k_3 + \Phi_4(k_1, k_2, k_3),
\end{align*}
\]
where

\[
\begin{align*}
a_{11} &= \frac{a_2 b_2}{c_2 (a_2 + \lambda_1)}, \\
a_{21} &= \frac{a_2 d_2}{c_2 (a_2 + \lambda_2)}, \\
a_{31} &= \frac{a_2 d_2}{c_2 (a_2 + \lambda_3)}, \\
a_{24} &= \frac{c^2 - \sqrt{c^4 - 4d^2 (\frac{a_2}{c_2} - 1)}}{2c^2}, \\
a_{34} &= \frac{c^2 + \sqrt{c^4 - 4d^2 (\frac{a_2}{c_2} - 1)}}{2c^2}.
\end{align*}
\] (3.20)

and notice that \(a_{11} > 0, \ a_{21} > 0, \ a_{31} > 0, \ 0 < a_{24} < \frac{1}{2}, \ \frac{1}{2} < a_{34} < \sigma\) and the function \(\Phi_i(k_1, k_2, k_3)\) satisfies

\[
\Phi_i(0, 0, 0) = \frac{\partial \Phi_i(0, 0, 0)}{\partial k_j} = 0, \quad i = 1, 2, 3, 4, j = 1, 2, 3.
\] (3.21)

**LEMMA 3.3.3.** There are an interval \([s_2, s_1]\) and a continuous function \(p : [s_2, s_1] \rightarrow \mathbb{R}^4\) such that \(p(s) \in M^u(E_2)\) for \(s \in [s_2, s_1]\), and \(p(s_i) \in P_i\) for \(i = 1, 2\), as well as \(p(s) \in \text{interior of } \Sigma\) for \(s \in (s_2, s_1)\).

**Proof.** First let \(\overline{s}_1 = 1, \overline{s}_2 = -\frac{\sigma - a_{34}}{\sigma - a_{24}}\), then

\[
\begin{align*}
\frac{a_{34} + a_{24} \overline{s}_2}{1 + \overline{s}_2} &= \sigma, \\
\frac{a_{34} + a_{24} \overline{s}_1}{1 + \overline{s}_1} &= \frac{1}{2}.
\end{align*}
\] (3.22)
and $-1 < \bar{s}_2 < 0$. Next, for $i = 1, 2, 3, 4$, define functions

$$
\Psi_i(s, \theta) = \frac{\Phi_i(s \theta, \theta)}{\theta}
$$

$$
= \int_0^1 \left[ \frac{\partial \Phi_i(\theta \tau, s \theta \tau, \theta \tau)}{\partial k_1} + \frac{\partial \Phi_i(\theta \tau, s \theta \tau, \theta \tau)}{\partial k_2} + \frac{\partial \Phi_i(\theta \tau, s \theta \tau, \theta \tau)}{\partial k_3} \right] d\tau.
$$

From (3.21) and (3.23) it follows that

$$
\Psi_i(s, \theta) = O(\theta), \quad \frac{\partial \Psi_i}{\partial s} = O(\theta), \quad \text{as} \quad \theta \to 0.
$$

uniformly for $s$ in a bounded interval. Hence $\Psi_i(s, \theta)$ is well defined and is differentiable with respect to $s$ for all sufficiently small $\theta$, including $\theta = 0$.

Now define a function

$$
X(s, \theta) = \frac{sa_{24} + a_{34} + \Psi_4(s, \theta)}{1 + s + \Psi_3(s, \theta)}
$$

$$
= \frac{sa_{24} + a_{34}}{1 + s} + \frac{(1 + s)\Psi_4(s, \theta) - (sa_{24} + a_{34})\Psi_3(s, \theta)}{(1 + s + \Psi_3(s, \theta))(1 + s)}.
$$

Then $X(\bar{s}_1, 0) = \frac{1}{2}$, $X(\bar{s}_2, 0) = \sigma$, and with the use of (3.20), (3.24), (3.25), and following a direct computation we have

$$
\frac{\partial X(s, \theta)}{\partial s} = \frac{a_{24} - a_{34}}{(1 + s)^2} + O(\theta)
$$

$$
= -\sqrt{\frac{c^4 - 4dc^2(a_1 - 1)}{c^2(1 + s)^2}} + O(\theta).
$$
This implies when \( \theta \) is sufficiently small,

\[
\frac{\partial X(s, \theta)}{\partial s} < 0. \tag{3.27}
\]

Recall \( X(s_1, 0) = \frac{1}{2}, \quad X(s_2, 0) = \sigma \), by the Implicit Function Theorem, for each very small \( \theta > 0 \), there are \( s_1(\theta) \) and \( s_2(\theta) \) such that \( s_1(\theta) \approx s_1, \quad s_2(\theta) \approx s_2 \), and \( X(s_1(\theta), \theta) = \frac{1}{2}, \quad X(s_2(\theta), \theta) = \sigma \). Since \( s_1 = 1, -1 < s_2 < 0 \), we can choose a sufficiently small \( \theta_0 \) such that \( s_1(\theta_0) \approx 1 \) and \( s_1(\theta_0) > -s_2(\theta_0) \). Now let \( s_1 = s_1(\theta_0), s_2 = s_2(\theta_0) \), then

\[
X(s_1, \theta_0) = \frac{1}{2}, \quad X(s_2, \theta_0) = \sigma.
\]

Notice (3.27) implies \( X(s, \theta_0) \) is strictly decreasing on \( s \) for \( s \in [s_2, s_1] \). So we have

\[
\sigma = X(s_2, \theta_0) > X(s, \theta_0) > X(s_1, \theta_0) = \frac{1}{2}, \quad s \in (s_1, s_2). \tag{3.28}
\]

Now let

\[
p(s) = (u_1(\theta_0, s\theta_0, \theta_0), u_2(\theta_0, s\theta_0, \theta_0), v(\theta_0, s\theta_0, \theta_0), w(\theta_0, s\theta_0, \theta_0)), \quad s \in [s_2, s_1]. \tag{3.29}
\]
where
\[ u_1(\theta_0, s\theta_0, \theta_0) = \theta_0(1 + \Psi_1(s, \theta_0)), \]
\[ u_2(\theta_0, s\theta_0, \theta_0) = \theta_0(-a_{11} - a_{21} - a_{31} + \Psi_2(s, \theta_0)) + \frac{a_2}{c_2}, \]
\[ v(\theta_0, s\theta_0, \theta_0) = \theta_0(1 + s + \Psi_3(s, \theta_0)), \]
\[ w(\theta_0, s\theta_0, \theta_0) = \theta_0(a_{24}s + a_{34} + \Psi_4(s, \theta_0)). \]
\[ (3.30) \]
Then by (3.19) we can say \( p(s) \in M^u(E_2) \). And from (3.23), (3.25), (3.28) and (3.30), for sufficiently small \( \theta_0 > 0 \) and \( s \in [s_2, s_1] \) we can obtain
\[ u_1(\theta_0, s\theta_0, \theta_0) > 0, \text{ but very small}, \]
\[ 0 < u_2(\theta_0, s\theta_0, \theta_0) < \frac{a_2}{c_2}, \]
\[ \frac{w(\theta_0, s\theta_0, \theta_0)}{v(\theta_0, s\theta_0, \theta_0)} = X(s, \theta_0), \]
and
\[ \frac{1}{2} = \frac{w(\theta_0, s_1\theta_0, \theta_0)}{v(\theta_0, s_1\theta_0, \theta_0)} < \frac{w(\theta_0, s\theta_0, \theta_0)}{v(\theta_0, s\theta_0, \theta_0)} < \frac{w(\theta_0, s_2\theta_0, \theta_0)}{v(\theta_0, s_2\theta_0, \theta_0)} = \sigma. \]

So we can conclude that \( p(s_1) \in P_1, \ p(s_2) \in P_2 \) and \( p(s) \in \text{interior of } \Sigma \) for \( s \in (s_2, s_1) \). \( \square \)

**THEOREM 3.3.1.** For \( c > 2\sqrt{d(M-1)} \), the system (3.6) has a nonnegative solution satisfying the boundary condition (3.7). Hence, the reaction-diffusion system (3.1) has a traveling wave solution connecting the points \( E_2 = (0, \frac{a_2}{c_2}, 0, 0) \) and \( E_5 = (1, 0, \frac{a_1 - b_1}{d_1}) \).
Proof. Let \( \gamma = \{ p(s) : s \in [s_2, s_1] \} \), where \( p(s) \) is defined as in Lemma 3.3.3., then \( \gamma \subset M^u(E_1) \), here \( M^u(E_1) \) is the unstable manifold of \( E_1 \) defined as in (3.19).

Let \( \Phi_t(p) \) be the solution of (3.6) through the point \( p \in \gamma \) with \( \Phi_0(p) = \Phi(0, p) = p \).

Then we define two subsets \( \gamma_1 \) and \( \gamma_2 \) of \( \gamma \) as follows

\[ \gamma_i = \{ p \in \gamma : \text{there is a time } t_p \geq 0 \text{ such that } \Phi_{t_p}(p) \in P_i \}, \quad i = 1, 2. \] (3.32)

Obviously \( \gamma_i \) is not an empty set, since by Lemma 3.3.3., \( p(s_i) \in P_i \cap \gamma \) for \( i = 1, 2 \).

Notice that from Lemma 3.3.2, we know the vector field of (3.6) at any point \( p \in P_1 \cup P_2 \) points to the exterior of \( \Sigma \). By using continuity of solutions on the initial condition, we can deduce that both \( \gamma_1 \) and \( \gamma_2 \) are open relative to the curve \( \gamma \). It is obvious that \( \gamma \) is a closed and connected set. Therefore

\[ \gamma \setminus (\gamma_1 \cup \gamma_2) \neq \emptyset. \]

So let \( p^* \in \gamma \setminus (\gamma_1 \cup \gamma_2) \), by the definitions of \( \gamma, \gamma_1, \gamma_2 \) and Lemma 3.3.1, we have

\[ \Phi_t(p^*) \in \text{interior of } \Sigma \text{ for all } t \geq 0. \] (3.33)

Let \( \Phi_t(p^*) = (u_1(t), u_2(t), v(t), w(t)) \) and \( E_5 = (1, 0, \frac{a_1 - b_1}{a_1}, \frac{a_1 - b_1}{a_1}) = (u_*, 0, v_*, w_*) \).

Since \( p^* \) is in the unstable manifold \( M^u(E_1) \) of \( E_1 \), we have

\[ \Phi_t(p^*) = (u_1(t), u_2(t), v(t), w(t)) \to \left( \frac{a_1}{b_1}, 0, 0, 0 \right) \text{ as } t \to -\infty. \]
Now we prove

$$\Phi_t(p^*) \to E_5$$ as $t \to \infty$.

Rewrite the system (3.6) as

\[
\begin{align*}
\dot{u}_1 &= d_1 u_1 \left[ -\frac{b_1}{d_1} (u_1 - u_*) - \frac{c_1}{d_1} u_2 - (v - v_*) \right], \\
\dot{u}_2 &= d_2 u_2 \left[ -\frac{b_2}{d_2} (u_1 - u_*) - \frac{c_2}{d_2} u_2 - (v - v_*) - s \right], \\
\dot{v} &= \frac{c_2}{d} \left[ (v - v_*) - (w - w_*) \right], \\
\dot{w} &= v \left[ u_1 - u_* + u_2 \right].
\end{align*}
\] (3.34)

Where $s = \frac{b_2}{d_2} u_* + v_* - \frac{a_2}{d_2} > 0$ by the assumption (H6). Define a function $L : [0, \infty) \to \mathbb{R}$ by

\[
L(t) = \frac{1}{d_1} \int_{u_*}^{u_1(t)} \frac{s - u_*}{s} ds + \frac{1}{d_2} u_2(t) + w(t) - v_* \frac{w(t)}{v(t)} - v_* \ln v(t). \quad (3.35)
\]

Then by (3.33), $L(t)$ is well defined and the derivative of $L(t)$ along (3.34) is

\[
\dot{L}(t) = \frac{u_1 - u_*}{d_1 u_1} \dot{u}_1 + \frac{1}{d_2} \dot{u}_2 + \dot{w} - \frac{v_*}{v^2} [\dot{w} v - w \dot{v} + v \dot{v}],
\]

\[
= -\frac{b_1}{d_1} (u_1 - u_*)^2 - \left( \frac{c_1}{d_1} + \frac{b_2}{d_2} \right) (u_1 - u_*) u_2 - \frac{c_2}{d_2} u_2^2 - s u_2 - \frac{v_* c^2}{v^2 d} (v - w)^2. \quad (3.36)
\]

By assumption

$$2b_1 c_2 d_1 d_2 > c_1^2 d_2^2 + d_1^2 b_2^2.$$
We can conclude

\[
\frac{b_1}{d_1} (u_1 - u_*)^2 + \left( \frac{c_1}{d_1} + \frac{b_2}{d_2} \right) (u_1 - u_*) u_2 + \frac{c_2}{d_2} u_2^2 > 0.
\]

So

\[
\dot{L}(t) \leq 0 \quad \text{for } t \geq 0. \tag{3.37}
\]

Hence \( L(t) \) is decreasing. Moreover \((u_1(t), u_2(t), v(t), w(t)) \in \Sigma, \) so

\[
\frac{1}{2} \leq \frac{w(t)}{v(t)} \leq \sigma. \tag{3.38}
\]

From (3.35),(3.36),(3.38), we can get

\[
L(0) \geq L(t) \geq \frac{v(t)}{2} - v_* \sigma - v_* \ln v(t).
\]

This yields that

\[
L(0) + v_* \sigma \geq \frac{v(t)}{2} - v_* \ln v(t). \tag{3.39}
\]

One can easily prove that

\[
\frac{v(t)}{2} - v_* \ln v(t) \to \infty \quad \text{if} \quad v(t) \to 0 \quad \text{or} \quad v(t) \to \infty. \tag{3.40}
\]

By (3.38),(3.39) and (3.40), we conclude that there are positive constant \( M_1 \) and \( M_2 \) such that

\[
M_1 \leq v(t) \leq M_2, \quad M_1 \leq w(t) \leq M_2. \tag{3.41}
\]
From (3.36) and (3.41), we conclude

(a). \((u_1(t), u_2(t), v(t), w(t))\) is bounded for \(t \geq 0\);

(b). The function \(L(t)\) is monotone decreasing and is bounded below as \(t \to \infty\);

(c). \(\dot{L}(t)\) is uniformly continuous.

Therefore from (b) and (c), we obtain \(\dot{L}(t) \to 0\) as \(t \to \infty\). It follows that

\[
-\frac{b_1}{d_1}(u_1(t) - u_*)^2 - \left(\frac{c_1}{d_1} + \frac{b_2}{d_2}\right)(u_1(t) - u_*)u_2(t) - \frac{c_2}{d_2}u_2(t)^2
\]

\[
- su_2(t) - \frac{v_*c^2}{v(t)^2d}(v(t) - w(t))^2 \to 0 \quad \text{as} \quad t \to \infty.
\]

Hence

\[u_1(t) \to u_, \quad u_2(t) \to 0 \quad \text{and} \quad v(t) - w(t) \to 0 \quad \text{as} \quad t \to \infty. \quad (3.42)\]

Notice that \(u_1(t)\) is uniformly continuous. Hence \(u_1(t) \to u_*\) yields that

\[\dot{u}_1(t) \to 0 \quad \text{as} \quad t \to \infty. \quad (3.43)\]

From (3.34), (3.42), and (3.43), we get

\[v(t) \to v_, \quad w(t) \to w_\ast \quad \text{as} \quad t \to \infty. \quad (3.44)\]
Therefore we conclude that

\[ \Phi_t(p^*) = \left( u_1(t), u_2(t), v(t), w(t) \right) \rightarrow (1, 0, \frac{a_1 - b_1}{a_1}, \frac{a_1 - b_1}{a_1}) = E_5 \text{ as } t \rightarrow \infty. \]

Now we turn to consider the case, the system (3.1) satisfies (H1) and (H4) or (H1) and (H5), in addition \(2b_1c_2d_1d_2 > c_1^2d_2^2 + d_1^2b_2^2\) and \(\frac{a_2}{c_2} > 1\). By Theorem 2.3.1, we know under these assumptions, the system (3.6) has the positive equilibrium \(E^0 = (u_1^0, u_2^0, v^0, v^0)\) where \((u_1^0, u_2^0, v^0)\) is the positive solution of the system (2.2) and the boundary equilibrium \(E_2 = (0, \frac{a_2}{c_2}, 0, 0)\). We shall show that the system (3.1) has a traveling wave solution connecting \(E_2\) and \(E^0\). To draw our conclusion, we set the boundary condition for the system (3.6) as following

\[
(u_1(-\infty), u_2(-\infty), v(-\infty), w(-\infty)) = E_2 = (0, \frac{a_2}{c_2}, 0, 0), \\
(u_1(\infty), u_2(\infty), v(\infty), w(\infty)) = E^0 = (u_1^0, u_2^0, v^0, v^0).
\]

THEOREM 3.3.2. For \(c > 2\sqrt{d(M - 1)}\), the system (3.6) has a nonnegative solution satisfying the boundary condition (3.45). Hence, the reaction-diffusion system (3.1) has a traveling wave solution connecting the equilibrium points \(E_2 = (0, \frac{a_2}{c_2}, 0)\) and \(E^0 = (u_1^0, u_2^0, v^0, v^0)\).

Proof. We can follow the same process as we did in the section 3.2, just replace \(E_5\) by \(E^0 = (u_1^0, u_2^0, v^0)\). Then we have the same results as stated in Lemma 3.3.1, Lemma 3.3.2 and Lemma 3.3.3. Just like the proof of Theorem 3.3.1, we are still able
to have the same curves $\gamma, \gamma_1, \gamma_2$ and

$$\gamma \setminus (\gamma_1 \cup \gamma_2) \neq \emptyset.$$ 

Now choose $p^0 \in \gamma \setminus (\gamma_1 \cup \gamma_2)$ and $\Phi_t(p^0) = (u_1(t), u_2(t), v(t), w(t))$ be the solution of (3.6) through the point $p^0$ with $\Phi_0(p^0) = \Phi(0, p^0) = p^0$, according the definitions of $\gamma, \gamma_1,$ and $\gamma_2$ and lemma 3.3.1, obtain

$$\Phi_t(p^0) \in \text{interior of } \Sigma \text{ for all } t \geq 0. \quad (3.46)$$

Now we prove

$$\Phi_t(p^0) \to E^0 = (u_1^0, u_2^0, v^0, v^0) \text{ as } t \to \infty,$$

$$\Phi_t(p^0) \to E_2 = (0, \frac{a_2}{c_2}, 0, 0) \text{ as } t \to -\infty.$$

Rewrite the system (3.6) as

\begin{align*}
\dot{u}_1 &= d_1 u_1 \left[ -\frac{b_1}{d_1} (u_1 - u_1^0) - \frac{c_1}{d_1} (u_2 - u_2^0) - (v - v^0) \right], \\
\dot{u}_2 &= d_2 u_2 \left[ -\frac{b_2}{d_2} (u_1 - u_1^0) - \frac{c_2}{d_2} (u_2 - u_2^0) - (v - v^0) \right], \\
\dot{v} &= \frac{c^2}{d} \left[ (v - v^0) - (w - v^0) \right], \\
\dot{w} &= v \left[ (u_1 - u_1^0) + (u_2 - u_2^0) \right].
\end{align*}

(3.47)
Define a function $L : [0, \infty) \to \mathbb{R}$ by

$$L(t) = \frac{1}{d_1} \int_{u_1^0}^{u_1(t)} \frac{s - u_1^0}{s} ds + \frac{1}{d_2} \int_{u_2^0}^{u_2(t)} \frac{s - u_2^0}{s} ds + w(t) - v^0 \frac{w(t)}{v(t)} - v^0 \ln v(t).$$  \hspace{1cm} (3.48)

Then by (3.46), $L(t)$ is well defined and the derivative of $L(t)$ along (3.47) is

$$\dot{L}(t) = \frac{u_1 - u_1^0}{d_1 u_1} u_1 + \frac{u_2 - u_2^0}{d_2 u_2} u_2 + \dot{w} - \frac{v^0}{v^2} [\dot{w} v - \dot{w}^* + \dot{v} v]$$

$$= -\frac{b_1}{d_1} (u_1 - u_1^0)^2 - \left( \frac{c_1}{d_1} + \frac{b_2}{d_2} \right) (u_1 - u_1^0)(u_2 - u_2^0) - \frac{c_2}{d_2} (u_2 - u_2^0)^2 - \frac{v^0 c^2}{v^2 d} (v - w)^2.$$ \hspace{1cm} (3.49)

By assumption

$$2b_1 c_2 d_1 d_2 > c_1^2 d_2^2 + d_1^2 b_2^2,$$

We can conclude

$$\frac{b_1}{d_1} (u_1 - u_1^0)^2 + \left( \frac{c_1}{d_1} + \frac{b_2}{d_2} \right) (u_1 - u_1^0)(u_2 - u_2^0) + \frac{c_2}{d_2} (u_2 - u_2^0)^2 > 0.$$  

So

$$\dot{L}(t) \leq 0 \quad \text{for} \quad t \geq 0. \hspace{1cm} (3.50)$$

Hence $L(t)$ is decreasing. Notice $(u_1(t), u_2(t), v(t), w(t)) \in \Sigma$, the same arguments as we did in the proof of above theorem 3.3.1, we conclude $\dot{L}(t)$ is uniformly continuous and

$$\dot{L}(t) \to 0 \quad \text{as} \quad t \to \infty.$$
By (3.49), we have

\[ u_1(t) \to u_1^0, \quad u_2(t) \to u_2^0 \quad \text{and} \quad v(t) - w(t) \to 0 \quad \text{as} \quad t \to \infty. \quad (3.51) \]

Notice that \( u_1(t) \) is uniformly continuous. Hence \( u_1(t) \to u_1^0 \) yields that

\[ u_1(t) \to 0 \quad \text{as} \quad t \to \infty. \quad (3.52) \]

By (3.47), (3.51) and (3.52) we can get

\[ v(t) \to v^0, \quad w(t) \to v^0 \quad \text{as} \quad t \to \infty. \quad (3.53) \]

Therefore we conclude that

\[ \Phi_t(p^0) = (u_1(t), u_2(t), v(t), w(t)) \to E^0 \quad \text{as} \quad t \to \infty. \]

Since \( p^0 \) is in the unstable manifold \( M^u(E_2) \) of \( E_2 \), we have

\[ \Phi_t(p^0) = (u_1(t), u_2(t), v(t), w(t)) \to (0, \frac{a_2}{c_2}, 0, 0) \quad \text{as} \quad t \to -\infty. \quad \square \]

**Remark.** If \( \frac{a_2}{c_2} < 1 \), notice that \( E_2 \) only has two positive eigenvalues which tells us the unstable manifold \( M^u(E_2) \) is two-dimension in stead of three-dimension, but one can follow the same procedure to prove both Theorem 3.3.1. and Theorem 3.3.2..
CHAPTER 4

CONCLUSIONS

In Chapter 2, we prove the global stability of the equilibrium points of the system (2.2) in three cases. Future studies may focus on generalizing the results by relaxing some assumptions, for example relaxing the condition $2b_1c_2d_1d_2 > c_1^2d_2^2 + d_1^2b_2^2$.

The main finding of Chapter 3 is the existence of traveling wave solutions of the reaction-diffusion system (3.1). Traveling wave solutions describe the distributions of interacting species and the speed of convergence from one equilibrium to another. The system (3.1) can give rise to traveling wave solutions connecting the equilibrium $E_2 = (0, \frac{a_2}{c_2}, 0)$, where only prey 2 is present, to the equilibrium $E_5 = (1, 0, \frac{a_1 - b_1}{d_1})$, where the prey 1 coexists with predator in absence of the prey 2 in case 3. In case 2 traveling wave solutions connect $E_2$ and $E^0$, where two prey and one predator coexist. These results are biologically interesting. In both case 2 and case 3, with small perturbations from the point $E_2$, there will be a solution converging to a stable steady state $E^0$ and $E_5$ respectively. In the future I may also consider the movements of two prey and modify the system (3.1) by adding diffusive term $d_1\Delta N_1$ and $d_2\Delta N_2$ to the first and second equations of the system (3.1) respectively. I expect that
the analysis of this modified system is mathematically challenging, but biologically important and interesting.
REFERENCES


[6] Zhilan Feng and Wenzhang Huang, Spatially Heterogeneous Invasion of Toxic Plant Mediated by Herbivory, Mathematical Biosciences and Engineering, Volume 10, Number 5&6, (October & December 2013)


