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MODELING A POWER-GENERATING PULSED NUCLEAR MAGNETIC NOZZLE

by

NATHAN SCHILLING

A DISSERTATION

Submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in The Department of Mechanical and Aerospace Engineering to The School of Graduate Studies of The University of Alabama in Huntsville

HUNTSVILLE, ALABAMA

2022
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Nathan Schilling  
06/23/2022 (date)
DISSERTATION APPROVAL FORM

Submitted by Nathan Schilling in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Aerospace Systems Engineering and accepted on behalf of the Faculty of the School of Graduate Studies by the dissertation committee.

We, the undersigned members of the Graduate Faculty of The University of Alabama in Huntsville, certify that we have advised and/or supervised the candidate of the work described in this dissertation. We further certify that we have reviewed the dissertation manuscript and approve it in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Aerospace Systems Engineering.

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ABSTRACT

The School of Graduate Studies
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Degree Doctor of Philosophy College/Dept. Engineering/Mechanical and Aerospace Engineering

Name of Candidate Nathan Schilling

Title Designing a Power-Generating Pulsed Nuclear Magnetic Nozzle

Pulsed nuclear (fusion, fission, and fission/fusion hybrid) propulsion systems provide the potential for 10,000 s specific impulse and specific powers of 1000 W of jet power per kg of propulsion system mass. This enables a new class of medium thrust, high specific impulse capability, resulting in 1/3 of the trip times and significantly higher payload mass fractions compared with chemical propulsion performing the same mission. One of the challenges for pulsed nuclear propulsion is the conversion of an isotropic explosion to directed thrust, another challenge is the generation of electrical power needed to run these systems.

To meet both of these challenges we model the design for a pulsed power-generating magnetic nozzle. This nozzle has two functions; 1) Generate power, and 2) Generate thrust. For the first function, we develop an archetype pulsed power generation system using MATLAB and determine performance scaling given in terms of two non-dimensional parameters: the ratio of initial plasma energy to initial pickup coil energy, and the ratio of initial pickup coil inductance to primary-side transformer inductance. We find that the first ratio should be 0.01, and the second ratio should
be 1 for idea system performance. We also find that for a system to provide 1.2 MJ of energy, its mass is 35 metric tons.

For the second function, we leverage the software SPFMax (Smoothed Particle Fluid with Maxwell equation solver). We compare SPFMax with results from the literature for two cases: a solenoidal case and an axial case. We find good quantitative and qualitative agreement for both cases. Next, we generated three nozzle designs (7.5 MA/strut, 15 MA/strut, 30 MA/strut) and analyzed their performance. The ideal design has 15 MA/strut and generates 2,400 sec specific impulse, 16 kilo-Newton-sec total impulse, and an efficiency of 0.34. The nozzle efficiency is low, but indicates a power-generating pulsed axial nozzle design is feasible. In conclusion, further development and sub-scale testing of a power-generating magnetic nozzle would make pulsed nuclear propulsion systems more feasible, opening up an entire new class of interplanetary and interstellar missions and allowing humanity to safely explore vast new reaches of space.

Abstract Approval: Committee Chair

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I would like to thank Dr. Hideki Nakashima for being a pioneer and trailblazer in the magnetic nozzle work, providing data for me to compare SPFMax against, answering any and all questions I had regarding the data and magnetic nozzle models, and being generally helpful and enjoyable to work with.

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I would like to thank my committee members Dr. Robert Frederick, Dr. Gabe Xu, and Dr. Babak Shotorban for challenging me, and in whose classes I gained the skills I needed to succeed in performing this work.

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Without it I do not think I would have pursed this passion project of mine to a PhD dissertation.

I would like to thank Anthony Edmonston, Jessica Rodgers, and Gabriele Cromartie for supporting me administratively, and ensuring I could be properly funded for my work.

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<tr>
<td>A</td>
<td>Area, $m^2$</td>
</tr>
<tr>
<td>B</td>
<td>Magnetic field, T</td>
</tr>
<tr>
<td>C</td>
<td>Capacitance, F</td>
</tr>
<tr>
<td>d</td>
<td>Separation distance (between plasma and nozzle), m</td>
</tr>
<tr>
<td>D</td>
<td>Wire, diameter, m</td>
</tr>
<tr>
<td>E</td>
<td>Energy, J</td>
</tr>
<tr>
<td>F</td>
<td>Force, N</td>
</tr>
<tr>
<td>G</td>
<td>Any property of a fluid (pressure, density, temperature)</td>
</tr>
<tr>
<td>h</td>
<td>Compact support distance, m</td>
</tr>
<tr>
<td>H</td>
<td>Ratio of particle length scale to particle inductance times particle area, m</td>
</tr>
<tr>
<td>I</td>
<td>Current, A</td>
</tr>
<tr>
<td>J</td>
<td>Pulse frequency, Hz</td>
</tr>
<tr>
<td>j</td>
<td>Current density, $A/m^2$</td>
</tr>
<tr>
<td>k</td>
<td>Thermal conductivity, $W/(m K)$</td>
</tr>
<tr>
<td>K</td>
<td>Transformer coupling constant</td>
</tr>
<tr>
<td>Symbol</td>
<td>Definition</td>
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<td>------------</td>
</tr>
<tr>
<td>L</td>
<td>Inductance, H</td>
</tr>
<tr>
<td>l</td>
<td>Loss inductance, H</td>
</tr>
<tr>
<td>m</td>
<td>Mass, kg</td>
</tr>
<tr>
<td>MW</td>
<td>Plasma molecular weight, g/mol</td>
</tr>
<tr>
<td>M</td>
<td>Mutual inductance, H</td>
</tr>
<tr>
<td>$\mathcal{M}$</td>
<td>Magnetic Moment, $A \times m^2$</td>
</tr>
<tr>
<td>N</td>
<td>Number of (turns, particles)</td>
</tr>
<tr>
<td>P</td>
<td>Gas pressure, Pa</td>
</tr>
<tr>
<td>$P_B$</td>
<td>Magnetic field pressure, Pa</td>
</tr>
<tr>
<td>Q</td>
<td>Charge, C</td>
</tr>
<tr>
<td>r</td>
<td>Radius (from electromagnetic field source, of plasma, etc.), m</td>
</tr>
<tr>
<td>R</td>
<td>Resistance, $\Omega$</td>
</tr>
<tr>
<td>t</td>
<td>Time, s</td>
</tr>
<tr>
<td>T</td>
<td>Temperature, K</td>
</tr>
<tr>
<td>u</td>
<td>Specific internal energy, $J/kg$</td>
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<td>Velocity, m/s</td>
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<tr>
<td>V</td>
<td>Voltage, V</td>
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<tr>
<td>$\mathcal{V}$</td>
<td>Volume, $m^3$</td>
</tr>
<tr>
<td>w</td>
<td>Width, m</td>
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</table>
W  Interpolating kernel function, $1/m^3$

XC  Wire cross-sectional area, $m^2$

Z  Ionization state

$\beta$  Steering angle, degrees

$\gamma$  Ratio of specific heats

$\epsilon_b$  Non-dimensional ratio of ion larmor radius to the initial plasma radius

$\varepsilon$  Induced Electro-motive force, V

$\eta$  Efficiency

$\theta$  Azimuthal angle in cylindrical and spherical coordinate systems, radians

$\kappa$  Non-dimensional plasma/nozzle energy

$\lambda$  Length, m

$\mu$  Material permeability

$\pi_1$  Power-generation circuit non-dimensional parameter 1: inductance ratio

$\pi_2$  Power-generation circuit non-dimensional parameter 2: energy ratio

$\pi_3$  Power-generation circuit non-dimensional parameter 3: timing ratio

$\rho$  Mass density, kg$/m^3$

$\sigma$  Electrical Conductivity, S/m
\[ \tau \] Burnout time, s

\[ \omega \tau \] Viscous stress tensor, J/kg

\[ \Phi_B \] Magnetic flux, T/m²

\[ \chi_{Plank} \] Single-group Plank emission opacity

\[ ln(\Lambda) \] Coulomb logarithm

Constants

\[ e \] Fundamental charge, \(1.60 \times 10^{-19}\) C

\[ k_b \] Boltzmann’s constant, \(1.38 \times 10^{-23}\) J/K

\[ m_e \] Mass of an electron, \(9.91 \times 10^{-31}\) kg

\[ \epsilon_0 \] Permittivity of free space, \(8.85 \times 10^{-12}\) F/m

\[ \mu_0 \] Permeability of free space, \(4\pi \times 10^{-7}\) H/m

\[ \sigma_{sb} \] Stefan-Boltzmann constant, \(5.67 \times 10^{-8}\) W/(m² K⁴)
CHAPTER 1

INTRODUCTION

Human piloted missions to Mars and deep space robotic missions are challenging because of the long trip times. For the Mars journey, current mission architectures for either conjunction or opposition class missions will take 2-3 years [10]. Robotic rendezvous missions to the gas giant planets are 5-10 years; for the outer ice giants (Uranus and Neptune) it is 10-15 years [10]. Long term exposure to microgravity and galactic cosmic rays pose threats to the physical and mental health of the astronauts. Deep space robotic missions require a significant portion of a Principal Investigator’s lifespan in order to collect data, and sample returns from the gas giant planets and beyond may not be possible at all with current technology.

In spite of the difficulties, there is measurable interest from the public for space. For example, prominent members of the private space industry, such as Elon Musk, as well as public figures within NASA are seriously considering and planning human piloted missions to Mars [11]. Further, there was the success at the box office with the theatrical release of movies like The Martian.

To make these ambitious human piloted and robotic missions more routine, trip times must be reduced dramatically. Such advances can only come from dra-
matic improvements in the propulsion technology. It has been known since at least the 1960’s that fission and fusion propulsion systems offer high specific impulse and moderate thrust sufficient to reduce trip times. For the crewed Mars mission, travel times are as little as 2 months each way [10]. For outer planetary missions, one way rendezvous times are, best-case, 1 year for Jupiter and 4 years to Neptune [10]. Round-trip sample return missions from these destinations can be accomplished in as little as 2-10 years [10]. These travel times are simply not practical for chemical or electric propulsion systems. Chemical propulsion is limited by the energy released per unit mass of the propellant, putting its maximum specific impulse around 1,000 sec. This specific impulse necessitates a low payload mass fraction (<0.01) to hit these low trip time missions, meaning the vehicle mass must be prohibitively large [10]. Electric propulsion systems have high specific impulse, but electric propulsion without a solar power source cannot be used effectively beyond Jupiter [12], using current technology. This necessitates using a nuclear reactor, reducing thrust-to-weight of the propulsion system, thus increasing trip time. Therefore, a nuclear fission, fusion, or hybrid system is highly advantageous for reducing trip times for interplanetary missions; this significantly reduces radiation exposure and health hazard for crewed missions, and greatly increases the return on investment for ambitious robotic missions.

1.1 Statement of the Challenges

Using Figure 1.1, one can classify nuclear space propulsion systems. Of these, some operate in an steady-state 'always-on’ mode, continuously producing thrust (such as Nuclear Thermal Propulsion or NTP), and some operate in a pulsed mode,
producing impulse bits as opposed to continuous thrust. Steady-state nuclear systems are good for shorter-distance missions (crewed Mars), but are less attractive for longer-distance missions [10]. This is because nuclear systems (steady-state and pulsed) must achieve high exhaust temperatures to realize high performance, and therefore enable long-distance missions. These high exhaust temperatures create high heat loads; Steady state systems must be constantly managing these heat loads with an active system, which can get quite massive. From a thermal management perspective, it is more attractive to dissipate this heat over a long period of time than a short one. Pulsed nuclear systems can operate in a pulsed mode to take advantage of this principle, operating for a short period of time before having a long period of time to dissipate heat. Thus, these systems can achieve higher performance than steady-state systems.

Of the pulsed nuclear systems, some use pure fission (such as the Pulsed Plasma Rocket or PPR [13]), some use fusion (such as the Vehicle for Interplanetary Space Transport or VISTA [14]), and some use a hybrid approach (such as the

**Figure 1.1**: Nuclear Space Propulsion System Taxonomy.
Pulsed Fission-Fusion or PuFF [15] concept, as well as Orion [16]). To reduce the amount of time necessary to generate the required energy, and increase the frequency a pulsed system can operate at, current systems get their input energy by extracting it from the exhaust stream [2, 14, 17–21]. Traditional schemes, like photovoltaics or heat cycles, are either too brittle or too massive to provide the input energy required. So, a direct power generation scheme is used. Unfortunately, little detail is found in the concepts on how to design such a system [2, 14, 17–20, 22]. Additionally, little is known about how design of these systems impacts their performance or their mass. Therefore, it is necessary to investigate how the design of a pulsed nuclear power generation system affects its performance and mass. This is the first half of the research objective.

In addition to the power generation system, another challenge facing pulsed nuclear propulsion is the efficient conversion of the exhaust stream into impulse. This challenge is addressed with a device called a magnetic nozzle - which is similar to a regular nozzle but is composed of magnetic field instead of metal (as magnetic fields cannot melt). Design of pulsed magnetic nozzles, which differ in function from their steady-state counterparts, has proceeded fairly steadily over the past couple of years [5, 6, 9, 23–47]. Authors have considered the effect of plasma initial conditions, magnetic nozzle initial conditions, and magnetic nozzle topologies [5, 6, 9, 23–29, 31–44, 46, 47] on magnetic nozzle performance. However, authors have primarily considered only solenoidal magnetic nozzle topologies, wherein the magnetic field is primarily in the axial direction. Recent work [46] has suggested an alternate nozzle configuration, called the axial configuration, that features a field primarily
in the azimuthal direction. This is theorized to enhance nozzle performance [46]. We would like to determine the physical mechanisms by which this occurs, as well as look at the effect of changing the axial nozzle magnetic field topology on performance (impulse bit, specific impulse, nozzle efficiency). This is the second half of the research objective.

1.2 Summary of related work

All pulsed nuclear concepts can trace their heritage to Project Orion. The Project Orion vehicle concept was a large spacecraft propelled by thermonuclear bomb pulses [16]. The bombs would be ejected out of the vehicle in a pulse and the momentum of the explosions would be transferred to the craft using a circular pusher plate and a set of shock absorbers. Project Orion vehicles did not summarily address the power-generation challenge, but they did address the exhaust conversion challenge with their pusher plate design. While promising high specific impulse and medium thrust, Project Orion vehicles were outlawed as a consequence of the 1963 Partial Nuclear Test Ban Treaty [48].

Lured by the promise of high specific impulse and medium thrust, a team of scientists lead by Dr. Alan Bond created a fusion-only pulsed nuclear propulsion concept called Daedalus. Daedalus addressed the power-generation and exhaust direction challenges with a combined power-generating magnetic nozzle design [19]. This design featured a high-power electromagnetic nozzle for directing the exhaust stream, and a pickup coil for power generation. However, little detail was given on the specific designs of both sub-components, leaving a lot of technical risk to retire.
This risk was somewhat reduced by F. Winterburg [49] and R. Hyde [18, 50] in their vehicle designs, but the most of the risk on the magnetic nozzle side was reduced through studies by H. Nakashima and other co-authors [5, 6, 9, 23–45]. However, others, not affiliated with H. Nakashima, F. Winterburg, or R. Hyde, have contributed as well [2, 17, 46, 47, 51]. Together, these studies measured the effect of plasma initial conditions and magnetic nozzle topology on performance, but only for solenoidal nozzles and not axial ones (which are supposed to be higher-performing [46]). Most of the technical risk on the power-generation side has not been reduced as on the magnetic nozzle side, with authors focusing on high-level designs [2, 14, 17–20]. Detailed design of power-generation systems has also been left to future work.

1.3 Objectives of this Dissertation

The objective of this work is to explore the feasibility and scaling of a power-generating pulsed magnetic nozzle for nuclear space propulsion. To meet this objective, the effort is split into two parts. For the first part, we develop an archetype pulsed power generation system and illustrate the performance scaling of said system. The second part focuses on multidimensional axial magnetic nozzle modeling to determine the important physical mechanisms driving nozzle processes, and studies the effect of changing magnetic nozzle topology on specific impulse, impulse bit, and nozzle efficiency.
1.4 Summary of the Approach

As the research objective is split into two halves, the approach is similarly split. For the first half, we developed a power-generation system model in MATLAB that utilized the differential-equation integrator ODE45 [52]. We used this model to perform a trade study that varied design parameters (pick-up coil inductance, pick-up coil initial current, primary-side transformer inductance, capacitor size) to determine the effect on the energy generated. For the second half, we developed a multidimensional magnetic nozzle model in MATLAB called SPFMax (Smoothed Particle Fluid with Maxwell Equation Solver [53]). We compare results from this model with analytical models, experimental results, and prior computational results. We use the validated model to perform a trade study to meet the research objective; determining the important physical mechanisms driving nozzle processes, and studying the effect of changing magnetic nozzle topology on specific impulse, impulse bit, and nozzle efficiency.

1.5 Synopsis of the Dissertation

The following manuscript will present a review of relevant literature, a detailed methodology, results of the studies, and overall conclusions. All chapters will be bifurcated and organized according to the research objective; material pertinent to the power-generation system will be presented first, followed by material pertinent to the magnetic nozzle system. Chapter 2 will present a review of literature related to power-generating pulsed nuclear magnetic nozzles, including design of previous systems,
and elucidation underlying relevant physics. Chapter 3 will present the methodology, detailing the power-generation system model and magnetic nozzle model (SPFMax), and compare the results of an older version of SPFMax with analytical models and experiments. Chapter 4 will present the results of the power generation system trade study, compare SPFMax results to computational results in the literature, present the magnetic nozzle baseline test case, and look at the effect of changing nozzle current on performance. Chapter 5 will summarize all material presented thus far, and give this work’s implications for the design of pulsed magnetic nozzles generally.
CHAPTER 2

LITERATURE REVIEW

2.1 Pulsed Nuclear Power Generation Systems

2.1.1 Introduction

Pulsed nuclear propulsion concepts use power-generation systems because said concepts require MJ levels of input energy to ignite nuclear reactions [2, 14, 15, 17, 18, 20, 22]. Current concepts get this energy from the exhaust stream [2, 14, 17–21] to increase the frequency at which they can initiate these reactions. Extraction of this energy at the required power levels could be viable via photo-voltaic systems, thermionic systems, thermodynamic power cycles, and plasma flux compression systems.

While all of the aforementioned systems may be viable for providing the required power, we discuss our rationale for pursuing a flux compression approach as follows. Photo-voltaic or PV systems are not optimized for the light spectrum that will be emitted in nuclear processes [54]. The substantial development of PV systems for terrestrial and space applications is focused on efficiently absorbing visible light - not the high frequency photons from nuclear reactions. In contrast, thermionic and
thermodynamic systems can use energy transmitted over the entire electromagnetic spectrum. However, their efficiencies are limited by the Carnot cycle and current technology. Thermionic systems can reach 20% efficiency [55] and various thermodynamic cycles can reach 20%-60% efficiency [56]. Thus at least 40% of the captured thermal energy will become waste heat. When required power levels are on the order of MW, then heat rejection systems must also dissipate MW of power. Since radiator mass scales with the required heat power to be dissipated, the radiator mass becomes prohibitive. In addition to the efficiency and thermal management challenges, a pulsed nuclear reactor may need 10-100 Mega-Joules of electrical energy delivered on a time scale of 10-1000 microseconds (µs) to prepare the reactor for the next pulse. Flux compression circuits are designed to handle high instantaneous power loads, and may be one of the few technologies feasible for keeping up with the demands of the reactor. Therefore, we chose a plasma flux compression system for the power generation system of pulsed nuclear reactors for propulsion.

Previous work explores plasma flux compression power generation systems to supply the required power for pulsed nuclear systems [2, 14, 17, 19]. Orth finds using an plasma flux compression power generation system can reduce power system mass by 90% [14], and so these system are fairly widely used [2, 14, 17, 19, 20].

2.1.2 Magnetic flux compression

Magnetic flux compression systems draw intellectual heritage from devices called ‘flux compression generators’ (FCGs) in the U.S.A or ‘magneto-cumulative generators’ (MCGs) in the former Soviet Union [57–61]. These devices feature a
current that induces a magnetic flux through some cross-sectional area of a conductor [57–61]. Over the course of FCG operation, the initial or ‘seed’ magnetic flux and current are compressed and amplified due to flux conservation [57–61]. The flux is amplified in between a stationary set of coils, called the stator, and a moving set of coils, called the armature [57–61]. The more common version is the explosively driven FCG, which uses high explosives encased in a conductive metal to rapidly move the armature into the stator, compressing the magnetic flux [57–61].

A less common version is the plasma FCG, where the moving armature is a plasma, and is propelled by thermal expansion [62, 63]. A plasma FCG in operation is illustrated in Figure 2.1.

The plasma FCG consists of a conventional or nuclear explosive target (the armature [64]) confined inside of a set of solenoidal inductor coils (the stator [64]). The initial current in the stator $I_{10}$ induces an initial magnetic field $B_0$ inside the device. At time $t = 0$, the armature is ignited and converted into a plasma ball of radius $r_{p0}$; the armature is converted to plasma by some process not related to FCG operation. When this occurs, the system progresses from Figure 2.1a to Figure 2.1b. If the magnetic Reynolds’ number is greater than 1 [64–67], the plasma armature generates an internal magnetic field that counteracts the external magnetic field from the stator. Hence, there are no field lines in the armature, as shown in Figure 2.1c. In order for the magnetic Reynolds’ number to be greater than 1, the conductivity of the plasma must be at least 10 Siemens per meter, but this requirement is satisfied for pulsed nuclear plasmas [67]. This requirement is satisfied because pulsed nuclear plasma are very hot (1 keV), meaning their conductivity is high ($10^7 – 10^8$ Si/m).
Figure 2.1: Magnetic Flux Compression in a Plasma FCG. Device internal configuration a) Initially b) After plasma ignition c) With plasma partially expanded d) With fully expanded plasma and device destruction
As the armature expands it compresses the stator’s magnetic field (emphasized by the green circle in Figure 2.1c). This compression increases the magnetic field inside the device from $B_0$ to $B(t)$ and the stator current from $I_{10}$ to $I_1(t)$ (see Eq. (2.1) and surrounding discussion). In traditional FCGs the armature is not restrained from further expansion; the plasma impacts the device walls and destroys the device, as shown in Figure 2.1d. Thus plasma FCGs are one-shot systems, operating over a short period of time (10s of µs) before destruction [62].

The increase in current is modelled by taking the inductor-plasma system to be perfectly bounded — that is, no flux enters or leaves the system [62, 64]. Since no flux enters or leaves the system, flux is conserved, yielding Eq. (2.1).

\[ \Phi_B(t) = \Phi_{B_0} = B_0A_0 = B(t)A(t) \]  \hspace{1cm} (2.1)

As the cross-sectional area $A(t)$ decreases, the magnetic flux $B(t)$ increases. As magnetic field and current are proportional [68], the current rises as well (Figure 2.1). This is better illustrated by reformulating the flux in terms of current; the flux is reformulated as the current $I$, multiplied by the inductance, $L$ [64], resulting in Eq. (2.2).

\[ \Phi_B(t) = L(t)I(t) \]  \hspace{1cm} (2.2)

Also important is Faraday’s law of induction [68].

\[ \mathcal{E}(t) = -\frac{d}{dt}(\Phi_B(t)) \]  \hspace{1cm} (2.3)
Using Eq. (2.2), Faraday’s law (Eq. (2.3)), and the product rule results in Eq. (2.4) [64].

\[
\mathcal{E}(t) = - \left( \frac{dL(t)}{dt}I(t) + L(t)\frac{dI(t)}{dt} \right)
\]  

(2.4)

The first term on the right \( \frac{dL(t)}{dt}I(t) \) is the voltage source term [64], which depends on the change of inductance with time. This factors into FCG current output. The latter term \( L(t)\frac{dI(t)}{dt} \) is the conventional terms for a voltage drop across an inductor [64].

### 2.1.3 Plasma FCG Literature

Characterizing \( L(t) \) is required to evaluate the right hand side of Eq. (2.4). However, a review of the current literature produced no explicit expression for \( L(t) \) [62,63,67,69–71]. Modeling efforts describe the power output of the FCG as the result of the changing mutual inductance between the armature and stator [69], with more detailed efforts empirically calculating the current through the stator using 1D [71] and 2D MHD codes [63, 70]. Experimental efforts reported output current over time curves [62,65–67], or desired plasma FCG performance [69].

Work on conventional plasma FCGs (Figure 2.1), led to the development of the axial plasma FCG [62,65,66]. The axial plasma FCG is designed with sufficient field strength and clearance so the axially moving plasma does not physically interact with the stator [62,65,66]. Experimental work with these FCGs has focused on measuring plasma conductivity [65–67], and reporting voltage traces as the plasma shock travels across the stator [67].
Figure 2.2: FCG powering a capacitive load circuit diagram. Adaptation of Figure 1b in Ref. [1].

2.1.4 FCGs powering capacitive loads

Recent nuclear propulsion concepts [14,17] evolve the axial plasma FCG process to power capacitive loads. The following reviews, in order of publication, FCGs connected to capacitive loads. Firstly, C. Fowler and L. Altgilbers suggest using an impedance-matching transformer for powering capacitive loads (high-inductance loads) [64]. In this case, the resulting lumped-parameter circuit is given in Figure 2.2.

In Figure 2.2 the FCG is $L_g$, the primary and secondary side of the transformer are $L_{1t}$ and $L_{2t}$ respectively, loss inductances on the primary and secondary side are $L_l$ and $L_P$ respectively, loss resistances on the primary and secondary sides of the transformer are $R_1$ and $R_2$ respectively, and the capacitors are $C$.

After analyzing this circuit, taking no loss resistance ($R_1 = R_2 = 0$) and for times much greater than the LC time constant of the secondary circuit ($t >> \sqrt{L_2 C}$) A. Pavlovskii et al. find the solution is an oscillating solution composed of Bessel functions [1]. This is illustrated in Figure 2.3. The solution illustrates the current and voltage ringing from the transformer to the capacitor and vice versa. This ringing behavior is to be avoided.
2.1.5 Review of plasma FCG power generation system concepts

FCGs powering pulsed nuclear propulsion systems has heritage in several different concepts [2,14,17,19,20]. In all cases the plasma FCG is configured so that it is open at one end and nested inside a magnetic nozzle. The magnetic nozzle will force the plasma out before it can destroy the device (avoiding the situation in Figure 2.1d).
The schematic in Figure 2.4 illustrates the device configuration. A plasma FCG power generation system is composed of an FCG attached to a power-conditioning circuit; the FCG is deconstructed into the armature (nuclear target) and stator (black coils); the power conditioning circuit is broken into a transformer or other power-conditioning circuit elements, and an energy storage system. The storage system could be inductive, (see discussion on the SMES below), but for more contemporary pulsed nuclear propulsion concepts, is capacitive [17].

The idea to use a plasma FCG, but open it at one end so the device is not destroyed, dates at least back to the 1970s [19], with the Laser Fusion Rocket (LFR) Daedalus concept (see Section 2.2.2). Daedalus has ”an induction loop situated at the reaction chamber exit” [19] for power production. This induction loop at the exit behaves very similarly to a one-turn stator of a plasma FCG - making the Daedalus concept one forerunners of using plasma FCG for power production. However, the Daedalus concept does not go into much detail about the FCG, the power conditioning circuit, or energy storage system [19].

Similar to the Daedalus concept is VISTA (Vehicle for Interplanetary Space Transport Application), which also is a LFR [14]. The designers describe something that is akin to a plasma FCG which they call the ”inductive power conversion system”. A sheet of beryllium sits between the expanding plasma and the coils that create the magnetic nozzle. As the coils are powered and “the magnetic-field lines are shoved ahead of the [expanding] plasma and compressed and then decompressed” [14] the conducting beryllium sheet develops a voltage across it and this voltage rise is used to power the lasers. The designers of VISTA specifically mention that the plasma-
assisted magnetic field compression induces a voltage in the conducting beryllium sheet. This indicates they are describing a plasma FCG. The designers estimate the system takes 2.5 GW of power from the plasma as it expands, of the 225 GW available after the target ignites and the plasma forms. The plasma FCG is able to capture “≤5%” [14] of the available power, which is well within the usual range for FCGs [64]. Of the 2.5 GW of power input into the system, the designers estimate half is lost in the power conditioning circuit [14], and half is provided to run the lasers. Thus, 1.25 GW is the output power of the VISTA FCG. The power conditioning circuit drains into a set of capacitors for temporary power storage before the lasers are fired again. The capacitors take up the majority of the estimated total system mass of 115 tons, resulting in a specific power of 0.092 kg/kW. The designers also do not go into a particularly detailed design of their system, nor do they discuss scaling extensively.

The HOPE (Human Outer Planet Exploration) group began working on a MTF (Magnetized Target Fusion) concept in 2002 [2]. The HOPE MTF concept uses a series of high-speed plasma guns to compress a nuclear fusion target. The guns are integrated with the plasma FCG power generation system; instead of a single-turn coil stator in the FCG, the FCG is broken into an 8-turn stator, with each turn powering a different bank of plasma guns. Additionally, the design of the plasma FCG is similar to that of Ref. [62,63], where a set of superconducting coils provide the magnetic field inside the FCG, and the stator is placed outside these coils. The outer coils that ‘seed’ the chamber with magnetic field are called ‘seed coils’, and the inner coils that form the stator are called the thrust coils [2]. Together, the seed and thrust coils provide input power to the power generation system of the MTF HOPE
vehicle [2], and also form the magnetic nozzle that pushes the plasma out of the FCG in between shots of the plasma guns [2].

**Figure 2.5**: Reproduction of Fig. 36 in Ref. [2]. *Seed coils are placed concentrically inside the outer thrust coils.*

This results in an overall nozzle that looks like Figure 2.5. The thrust coils in the nozzle drain into a superconducting power conditioning circuit that drains into the SMES (Super Conductive Magnetic Energy Storage) which is a large superconducting inductor, as shown in Fig. 39 in Ref. [2]. It is composed of two superconducting lines that form a coaxial superconductive transmission line to match circuit impedance, as shown in Fig. 40 of Ref. [2]. The SMES has low specific energy compared to capacitive storage systems due to the high structural loads, and is not considered for this work, despite its heritage with previous work [62,63,67]. We consider capacitive storage instead. Capacitive storage also has heritage with prior work [17].

R. Adams et al. estimate the mass of the system with all associated elements (FCG, circuit, and energy storage system) to be 38064 kg, including the mass of
neutron shielding for the FCG [2]. Neutron shielding for the nozzle was 16036kg, so
presuming that the mass of neutron shielding for the FCG is the same, the system
mass is 22028kg [2]. For output energy/power, we could not find specific numbers
within Ref. [2], so instead we used an earlier version of the HOPE MIF concept from
Ref. [72]. This version has an output energy of 28.6 MJ per pulse [72], but had a
jet power of 25 GW [72], compared to the 2.038 GW jet power of HOPE MTF [2].
Assuming that output energy scales linearly with jet power, the HOPE MTF FCG
had an output energy of 2.36 MJ per pulse. This gives a specific energy of 107 J/kg,
and given the 20 Hz pulse rate, results in a specific power of 0.47 kg/kWe, which is
higher than for the VISTA vehicle.

Working at the same time as the designers of HOPE MTF, the designers
of Mini-MagOrion were also working on a similar MTF pulsed nuclear propulsion
concept [20]. Instead of a series of plasma guns, the designers of Mini-MagOrion
used a Z-pinch machine to initiate fission in a plasma target [20]. A Z-pinch machine
drives a terrawatt (high-current, high-voltage) electrical pulse through a target to
initiate nuclear reactions and drive the target to a plasma state [20]. R. Lenard and
D. Andrews also use a plasma FCG for power conversion, and they estimate that the
energy yield from the plasma FCG to be "≤ 1%" [20] of the 89.1 TJ total energy
yield from the fission reactions [20]. This results in the power system depositing 0.89
TJ of energy into the power storage system. However, R. Lenard and D. Andrews do
not give details on the mass of the power conversion system in their publication [20].

U. Shumlak et al. further developed the Z-pinch fusion concept incorporating a
shear-flow stabilized Z-pinch for increased burn stability. In their concept, the authors
mention a power system using direct energy conversion, which is similar in process to a plasma FCGs. The Z-pinch machine requires between 1-9 TW [73] of power, but it is unclear how this translates to requirements for the power conversion system. Also, U. Shumlak et al. do not present a mass for this system in this work [73].

Building on the 2003 HOPE study, the Mini-MagOrion, and the work of U. Shumlak et al., T. Polsgrove et al. create a new concept that uses a Z-pinch to ignite fusion targets. For this concept the plasma FCG forms a parabolic shape, and creates thrust in addition to generating power [17]. It assumes that the z-pinch originates at the center of the FCG and then spectrally reflects off the magnetic field produced by the stator of the plasma FCG. The use of a plasma FCG for thrust (as a magnetic nozzle), is outside the scope of this work, but due to the spectral reflection assumption, T. Polsgrove et al. can calculate the trajectory of the plasma armature as it moves through the FCG.

Using the data provided by T. Polsgrove et al. the power system needs to provide 833 MJ of energy to the energy storage system, 667 MJ of which ends up in the storage system [17]. Therefore, the output energy is 667 MJ. For the masses of system components the circuit has an estimated mass of 4187 kg, the nozzle coils themselves have a mass of 18000 kg, and the radiators just for the power generation system have a mass of 6083 kg. This gives a total mass of 22184 kg without radiators, and a mass of 28270 kg with radiators. Both of these masses yield a specific energy of 30000 J/kg and 23000 J/kg per pulse, and with a pulse rate of 10 Hz [17], a specific power of 0.0033 kg/kW_e and 0.0042 kg/kW_e respectively. Both specific energy and
specific power are two orders of magnitude higher than the VISTA or the 2003 HOPE vehicle.

2.1.6 Summary

So, while current efforts focus on modeling/characterizing these systems, little work has been done to understand how system performance changes as system design changes. This leads us to a research question that will address the first half of our objective - how does performance of a pulsed nuclear power generation system scale with system design?

2.2 Pulsed Nuclear Magnetic Nozzles

Before discussing pulsed nuclear magnetic nozzles, it must be acknowledged that pulsed nozzles have important similarities with steady-state nozzles [3, 4, 74, 74]. These similarities will be discussed first.

2.2.1 Steady-state magnetic nozzles

Pulsed magnetic nozzles differ markedly in shape from their steady-state counterparts. Steady-state nozzles use the familiar de Laval converging-diverging magnetic field topology whereas pulsed nozzles use a parabolic or hemispherical coil struts to effectively reflect exhaust. However, one can glean important insights into underlying physical processes from experimental steady-state nozzle efforts.

York et al. first experimentally prove the feasibility of a steady-state magnetic nozzle in 1992 [3]. They use a 23 kG field to confine a plasma with $n_e = 10^{24}$
Figure 2.6: Reproduction of Figure 2 in Ref. [3] showing nozzle and experimental setup

#/m³ and a temperature of 20 eV. This solenoidal field demonstrates choked flow in a throat section, and accelerating flow downstream from the throat, proving the magnetic nozzle operates similarly to the conventional nozzle. Their magnetic theta-pinch (azimuthal) magnetic field topology results in the plasma traveling along the familiar de Laval converging-diverging shape. This is illustrated in Figure 2.6.

Building off the work of York et al., Hoyt et al. use MACH2, a 2D-axisymmetric MHD code, to perform a parametric study of steady-state magnetic nozzles and to investigate using magnetic nozzle topologies for thin-film material deposition [4]. They benchmark their code based off the results from York et al. and find that magnetic drag is the primary driver of magnetic nozzle efficiency (driving design) [4]. Magnetic drag occurs when, as the plasma exits the nozzle, it does not cleanly detach from the magnetic field. It ‘sticks’ to the field lines, following them as they diverge radially. This retards plasma motion, producing off-axis motion, reducing thrust and
nozzle efficiency. Magnetic drag most likely affects pulsed nozzles as well as steady-state nozzles, and so knowledge of how to reduce it can theoretically be applied to pulsed nozzle design as well.

Host et al. find that, by moving the nozzle throat away from the nozzle exit, magnetic drag is reduced and efficiency increases to 103% [4]. Note that this value is greater than 100% because some of the increase is due to increased conversion of plasma rotational kinetic energy to directed kinetic energy. Initial rotational kinetic energy is not considered as part of the useful initial energy in the nozzle; only thermal energy is. However, the nozzle design that resulted in drastically increased efficiency also increases erosion; the new topology is shown in Figure 2.7. To increase nozzle efficiency without increasing erosion, Hoyt et al. try placing trim coils behind the nozzle to change the nozzle’s end behavior; this results in a modest efficiency increase of 25% [4].

Mikellides et al. perform another study of magnetic nozzles using MACH2, again benchmarked with the York et al. results, but this time with a focus on higher-energy plasma [74, 75]. Initially, the authors considered a plasma of comparable energy level to a mid-power MPD thruster [74], but later they consider a plasma of comparable energy level to a fusion plasma [75] (1 keV ion temperature). In their publications, the authors focus on uncovering some of the primary physics processes that drive acceleration. The authors find that the primary acceleration mechanism was not rotation of plasma flow as previously thought by Hoyt et al., but rather conversion of rotational energy in flow into thermal energy through viscous heating [74]. This thermal energy is converted into linear kinetic energy with a sufficiently efficient
magnetic nozzle topology. The authors also determined that an increased magnetic field strength increases the conversion of rotational kinetic energy to thermal energy, thereby increasing efficiency, but this has the adverse effect of increasing magnetic drag [75]. Since magnetic drag is what tends to drive magnetic nozzle design, increased magnetic field strength results in a decreased efficiency of 24%. Therefore, instead of simply increasing magnetic field strength, the field topology must be carefully designed to convert rotational energy into thermal energy to maximize efficiency [74,75].

Lastly, J. Cassibry and S. Wu report what is occurring in nozzles with high magnetic drag [76]. They compared plasmas in nozzles with high and low magnetic drag and found that in nozzles with high magnetic drag, the flow does not transition from sub-Alvenic to super-Alvenic. This transition makes the plasma detach from the
field lines, reducing or in some cases eliminating drag. These results clearly suggest, to reduce magnetic drag, the flow must transition from sub-Alvenic to super-Alvenic [76].

Taken together, these references suggest the pulsed nozzle topology must be carefully constructed to maximize conversion of rotational kinetic energy to thermal energy (and therefore directed kinetic energy) [74], allow for the transition to super-alvenic flow [76], and reduce magnetic drag [3, 4, 75].

2.2.2 Pulsed Magnetic Nozzles

Most works regarding pulsed nuclear magnetic nozzles, especially since the 1970s, [19], use the nozzle as part of a Laser-Fusion Rocket (LFR). Referring back to Figure 1.1, and starting from the top, pulsed nuclear propulsion systems can be divided further by the kind of nuclear energy release: pure fission, pure fusion, or a hybrid of both. Pure fusion concepts are further subdivided by the kind of fusion: MTF (Magnetized Target Fusion) and ICF (Inertial Confinement Fusion). ICF can be further divided by the method by which the target is confined; through plasma guns or lasers. The LFR falls under this latter branch. The LFR uses the same mechanisms for fusion as terrestrial fusion sites like the National Ignition Facility [77]. The earliest LFR-derived concept here is Daedalus [19]; more modern concepts like VISTA [14] and the Nakashima Laser-Fusion Rocket (LFR) [29] draw their intellectual heritage from this concept. Our work aims to extend the LFR magnetic nozzle work to the MTF and hybrid areas.

Since Daedalus is the oldest concept, it is instructive to go over it first. Daedalus is a two-stage ICF-laser pulsed nuclear vehicle [19]. The lasers are ar-
ranged in such a way that they ignite the target without hitting the electromagnet coils that form the magnetic nozzle. Daedalus also uses an induction loop for power generation. Daedalus runs extremely high current (Mega-amp) through the electromagnet coils, and are therefore composed of superconductor to minimize power losses. The coils produce a magnetic field that directs the plasma out the back of the vehicle, producing a change in the vehicle’s momentum (impulse bit) [19].

Winterberg was the next person to consider the magnetic nozzle. In 1971 he publishes a paper for a pulsed fusion vehicle, where, after ignition, the fusion pellet is directed out the back of the vehicle by ”a concave magnetic mirror produced by superconducting field coils. The magnetic pressure of the field reflects the fireball generated by the explosion the ignition” [49]. Here, Winterberg identifies a mechanism that makes the magnetic nozzle work - the intense magnetic pressure generated in the nozzle.

Hyde theorizes that this pressure from the electromagnet is amplified by the plasma expansion process [18,50]. Essentially, as the plasma expands, it will compress the magnetic field outside of itself and increase it, potentially reducing the needs of the magnetic nozzle. Inside the plasma, the magnetic field is theorized to be 0 - the plasma has pushed the applied field outside of itself. In his papers, Hyde sizes the magnetic nozzle by ensuring the magnetic nozzle has 5 times the plasma energy. Using a 2D Magneto-hydrodynamic model to calculate the efficiency of this magnetic nozzle design, he finds the efficiency is 65%. He then considers the effects the magnetic nozzle has on the overall vehicle; sizing radiators, heat pipes, structural tie-bars, radiation shielding, and MMOD shielding based on the nozzle design.
Based on these system considerations, he suggests the ideal electromagnet will not use superconductor, contradicting Daedalus. Instead it will use liquid lithium as this material handles the radiation and heat loads better than the super conductor [18]. After the publication of Hyde’s concept in 1983, work in magnetic nozzle modeling spread outside the English-speaking world.

### 2.2.2.1 Analytical modeling: Zakharov group

Researchers at the institute of laser physics in Novibrisk Russia could have learned of Hyde’s concept in the 1990s. They were doing work in astrophysics, regarding the high-magnetic fields around stars, but they realized their work could have applications to magnetic nozzles. They were investigating the simplest magnetic nozzle: a magnetic dipole. A magnetic dipole is a magnetic field with just two poles (hence di-pole): a North Pole and a South Pole [68]. The two poles create a regular magnetic field that has a simple mathematical description, given by Eq. (2.5) [68].

\[
\vec{B} = \frac{\mu_0 M}{4\pi r^3} \left(2\cos(\theta)\vec{r} + \sin(\theta)\vec{\theta}\right)
\]

(2.5)

This elegant analytical formulation allows for the calculation of the magnetic field everywhere in the domain, thus simplifying the analysis.

S. Nikitin and A. Ponomarenko use Eq. (2.5) to analyze the behavior of a plasma in a dipole field [78]. Their analysis lead to the development of a non-
dimensional parameter they called $\kappa$, which is defined in Eq. (2.6).

$$\kappa = \frac{12\pi E_p r_0^3}{\mu_0 M}$$  \hspace{1cm} (2.6)

In Eq. (2.6) $E_p$ is the plasma energy, $r_0$ is the distance from the dipole to the center of the plasma, $\mu_0$ is the permeability of free space, and $M$ is the strength of the dipole. $\kappa$ represents the ratio of the energy in the plasma to the integrated magnetic field energy. Thus Hyde’s point design operated with a $\kappa$ of 0.2. They also define two modes of plasma expansion, a ‘quasi-capture’ mode and a ‘rupture mode,’ that correspond to different values of $\kappa$. The ‘quasi-capture’ mode corresponds to plasma fully deflected by the magnetic field – there is little leakage in this mode. The ‘rupture’ mode corresponds to plasma not well deflected – there is a lot of leakage. The ‘critical $\kappa’ or $\kappa_c$ is the value of $\kappa$ that signals the transition from quasi-capture to rupture modes. For a plasma expanding inside a solenoidal magnetic nozzle they find $\kappa_c = 0.4$, but for plasma expanding outside the nozzle, they find $\kappa_c = 0.1$. This latter fact is confirmed by later work [23], but the former is challenged by later work [25, 26, 40].
S. Nikitin and A. Ponomarenko were part of a larger research group lead by Y. P. Zakharov. In 1999, Y. P. Zakharov and the rest of his team published an overview of their work up to that point [79]. Included in the work is the development of $\epsilon_b$, another non-dimensional parameter defined as

$$\epsilon_b = \frac{r_h}{r_b}$$

(2.7)

where $r_h$ is the ion larmor radius and $r_b$ is the initial plasma radius.

Y. P. Zakharov et al. theorized that, as the plasma expands in the applied magnetic field, a diamagnetic current will be created that negates the magnetic field inside the plasma and enhances the field outside it - leading to magnetic flux compression as Hyde theorized. This is the same effect found in a plasma FCG. Inside the plasma, the area of zero field, caused by the diamagnetic drift, is called a 'diamagnetic cavity.' Zakharov created an apparatus capable of detecting diamagnetic cavities: an array of magnetic field probes positioned around a laser-ablation plasma. A laser-ablation plasma is a plasma created by hitting a solid target with a laser - the group uses $CO_2$ lasers. Using their array of magnetic field (Bdot) probes, they showed that the magnetic field inside the plasma reduced by as much as 50%, but in most experiments, it usually reduced by only 30% [79]. They plotted the maximum extent of the cavity for different magnetic field strengths, and they found that the higher the magnetic field, the smaller the cavity. This investigation suggests that the diamagnetic cavity does exist, and that increased magnetic field might result in increased thrust. This is because the smaller cavities might indicate more collima-
tion of the plasma exhaust, resulting in higher thrust. However, the opposite might be true - and higher magnetic field strengths reduce the cavity size and confine the plasma to the nozzle area more, reducing thrust. In order to determine which of these explanations is correct, instead of investigating a plasma expanding in a simple dipole field, researchers needed to examine a plasma expanding in a full magnetic nozzle. The next set of works aimed to investigate this problem.

2.2.2.2 Computational modeling: Nakashima group

In 1999, just before the turn of the century, Y. Nagamine and H. Nakashima conduct one of the first magnetic nozzle parametric trade studies [9]. They use a 3D hybrid Particle In Cell (PIC) code developed by E. Horowitz et al. called QN3D [80]. E. Horowitz et al. needed to use a PIC method over conventional MHD methods, because conventional methods didn’t give results that matched with experiments. A PIC method is different from conventional MHD methods in that conventional methods approximate the fluid as a continuum; the fluid is thought to vary continuously in space and time. In reality, the fluid is composed of a set of many, many particles ($> 10^{23}$). A particle method works by treating groups of particles as
one big ‘macroparticle’ (reducing from $>10^{23}$ particles to $10^5 - 10^6$ [5,6,9,23–45]), then tracks the position and velocity of each macroparticle, and handling collisions when macroparticles collide. However, the particle method does not consider field quantities; that is, quantities that do vary continuously in space and time (such as electromagnetic fields). For the fields, the PIC method computes the fields only at specified grid points, and interpolates between these points to get the field values at specific particle locations.

E. Horowitz validates QN3D against known analytical solutions for single-particle motion, the normal modes, and the rigid rotor problem. It shows good agreement against all solutions [80].

In their work, Y. Nagamine and H. Nakashima add a term to QN3D to account for the ion current, and in addition to a trade study, they use the code to determine the effect the Rayleigh-Taylor instability will have on the magnetic nozzle [9]. Y. Nagamine and H. Nakashima simulate a ICF laser-fusion plasma (4 MJ, 0.11 g) that is cold (0 eV), made of gold (197 amu), and 1 m away from a single-turn solenoid magnetic nozzle. The nozzle has a radius of 1 m and a current of 3.57 MA. They assume the electrons are cold (electron temperature is 0) inertia-less fluid. They use a fairly coarse grid (60 x 60 x 70) but a fairly high number of particles (100,000). They find that while the Rayleigh-Taylor instability does grow appreciably in the nozzle plasma, the instability amplitude is small, so the instability does not need to be considered in pulsed magnetic nozzle design. For the magnetic nozzle trade study, they consider two parameters; the separation distance between the plasma and the nozzle, and the magnetic energy in the coil (in the simulation they vary just
the coil current). They consider 4 values of distance and 3 values of magnetic coil energy. They find that the thrust efficiency is maximized for a distance of 0.83 m. They also find that increasing the magnetic field energy in the coil increases thrust efficiency, but this increase is not appreciable. Taking into account the fact that generating high-energy magnetic fields is more difficult, their optimal magnetic field is the lowest considered (corresponds to a current of 3.57 MA – 5 times the initial plasma energy). They find that momentum efficiency maximizes around 65%, but they get an efficiency of 56% when they use the same parameters as R. Hyde (from Ref. [18]). This seems to overturn R. Hyde’s results, but R. Hyde’s model was lower fidelity (2D as opposed to 3D).

The group lead by H. Nakashima uses this QN3D-derived 3D hybrid PIC code to test more complicated nozzle/target designs than just a spherical plasma in a magnetic dipole. They try to match their simulations with the analytical theory the Russians developed, as well as their own experiments. Originally, they couched their results in parameters developed by the Russian group \( (\epsilon_b \text{ and } \kappa) \) - see Ref. [23,25,26]. But later they start to publicize their results using their own parameters (\( \eta \) - plume efficiency, steering angle \( \beta \), and the ratio of magnetic field energy to plasma energy) [24]. This transition first appears in the literature in 2005, when N. Sakaguchi et al. considers a two-coil magnetic nozzle design. After this point, they focus on plume efficiency \( (\eta) \) and steering angle \( (\beta) \), which are defined over a set of \( p \) super-particles.
in Eq. (2.8)-(2.9) \([5, 24, 27, 28, 31]\).

\[
\eta = \frac{\sum p m v_x}{\sum p m |v_0|} \tag{2.8}
\]

\[
\beta = \cos^{-1} \left( \frac{\sum p m v_x}{\sqrt{\sum p m v_x^2 + \sum p m v_y^2 + \sum p m v_z^2}} \right) \tag{2.9}
\]

The steering angle is the angle between the velocity of the center of mass of the plasma and the z-axis. A higher steering angle means the magnetic nozzle can be more easily used for Thrust Vector Control (TVC).

H. Nakashima et al. considered multiple nozzle and plasma configurations including: two-coil nozzles \([24, 27]\), and multi-coil \([27–29]\) nozzles, and shaped targets \([5, 31]\). For an example of the latter see Figure 2.9. The researchers did not vary target composition within any of their studies, instead opting to use the same cold (electron temperature 0 eV), gold (193 amu) \([24, 27–29]\), or hydrogen (3 amu) \([5, 31]\) targets in each case. The researchers initially used fairly coarse grid \((60 \times 60 \times 70)\)
and a fairly low number of macroparticles (100,000) [24], but they gradually increased domain fidelity to a finer grid (100 x 100 x 100) with a fairly high number of superparticles (1,000,000) [42]. They find that, to maximize plume efficiency, one should use the 2 coil configuration. The 2 coil configuration gives 78-75% [24, 27], multi coil gives 75-53% [27–29, 31], and single coil gives 78%-54% [5, 9, 25, 27, 31, 35]). One could use the single coil configuration as well to maximize plume efficiency [5], but this requires using a shaped target, which is hard to manufacture, and injecting it precisely, which is hard to do.

The groups also finds that, for maximizing steering angle, one should use the multi-coil configuration ($\beta_{\text{max}} = 5.2^\circ$ [28]) provided one does not use a shaped target. One can use the single coil configuration and eject the target very off-axis - this results in the highest steering angle of $37^\circ$ [27], but also has a lower plume efficiency (51%) [27] than in the multi-coil case (60% [29]). The single coil configuration with a shaped target has comparable maximal steering angle and nozzle efficiency to the multi-coil configuration (5° and 66% respectively [5]), but it is difficult to inject the shaped target precisely off-axis to maximize the steering angle. Simply turning coils on and off, as in the multi-coil configuration, is much easier [29]. Rather than investigate steering angle, Kajimura et al. investigates plasma leakage back to vehicle, in addition to plume efficiency, and find a multi-coil configuration to be superior here as well; resulting in 0% leakage back to the vehicle, compared with the two-coil case which has 3% leakage [28].

Around the same time, Deng et al. tried to investigate magnetic nozzle behavior also using a different 3D hybrid PIC code, but they were unable to run simulations
at the same resolutions as H. Nakashima’s group [51]. This might have been due to a lack of computational resources available to them (Y. Nagamine and H. Nakashima had access to a supercomputer for their publication).

The computational work by the H. Nakashima group also investigated plasma detachment in a pulsed magnetic nozzle concept, as this is a major problem for steady-state nozzles (see Section 2.2.1). To this end, they perform a series of 2D PIC simulations, with both the electrons and ions considered as particles. They use the TRISTAN [30] and EPOCH [45] codes and find that the plasma detaches sufficiently from the magnetic field [30,45]. This suggests plasma detachment is less of a problem for pulsed magnetic nozzles.

2.2.2.3 Experiments: Nakashima group

In 2010, the Japanese group lead by H. Nakashima built and tested, to the author’s knowledge, the first pulsed magnetic nozzle. Unfortunately, the ICF method of plasma generation was unavailable to them, so they used the laser ablation method instead. Maeno et al. discus their experimental setup in Ref. [32], and estimate thrust using images from the experiment and some theory (unrelated to the work of Y. P. Zakharov et al.). Maeno et al. experimentally prove the feasibility of a pulsed magnetic nozzle in Ref. [33]. In this text, the authors detail an experiment where they ablate a polystyrene target with a 0.7J pulse of a 1064 nm Nd:YAG laser, and record the impulse the plasma imparts to a permanent magnet. They record 1.5-2.3 \( \mu \) Ns of impulse per shot, depending on the length of time they ablate the target. This time is the same as the laser pulse width. They find that higher ablation times (\( \geq 9 \) ns)
correspond to higher impulses. They also try to match their results with some theory unrelated to the 3D hybrid PIC simulations without much success. Later, Yasunga et al. use the same theory, but try to calculate impulse from integrating signal from a Bdot probe [36]. The data demonstrate fair dissimilarity with theory [36]. In this work, rather than the permanent magnet used by Maeno et al. in 2011, Yasunga et al. use an 96-turn (8 axial, 12 radial) solenoid electromagnetic (coil) magnetic nozzle, with an inner coil radius of 13 mm and an outer coil radius of 25 mm [36].

Given the dependence Maeno et al. show in 2011 on pulse width, the group next considers the effect of laser wavelength and energy on performance. They keep the same experimental setup as their 2011 experiment but vary the wavelength (1053nm, 527 nm, and 351 nm) and energy (30-900 J) independently [33]. They find that higher energies and lower laser wavelengths result in higher impulses, but their results are not internally consistent; their data show impulses in the mNs range, but the detection limit of their thrust stand is 17 µNs [37]. This suggests further investigation, but this investigation is outside the scope of this work.

After this, in 2016, Morita et al. try to determine what is occurring physically in their nozzle by comparing plasma trajectories in a solenoidal magnetic nozzle at two different energy levels: 60J, and 600J. They use the same experimental setup as Yasunga et al. in 2012 and vary the magnetic field strength such that, for both energy levels, the ratio $E_B/E_p$ (where $E_B = \frac{1}{2}LI^2$ and $E_p = \eta_{laser}E_{laser}$) is the same. They find that the plasma trajectories agree well for plasma inside the nozzle, but outside the nozzle, the trajectories diverge. Therefore, Morita et al. in 2016 suggest
that $E_B/E_p$ is a helpful non-dimensional parameter, but only for quantifying plasma behavior inside the nozzle.

Figuring that $\kappa$ might be better at quantifying plasma behavior than $E_B/E_p$, Kawashima et al. attempt to get further physical insight by looking at line emission from plasma inside the nozzle. They test plasmas at different values of $\kappa$ using an ICCD camera and the same experimental setup as Yasunaga et al [36, 40]. After comparing their results with Nikitin & Ponomarenko, they find disagreement; they find that $\kappa_c = 0.1$ for all cases, not just plasma expansion outside the nozzle [40]. These results suggest that a different theory should be applied.

Therefore, Saito et al. try to develop their own theory. They use the same experimental setup as Yasunaga et al. in 2012, and position charge collectors at different locations around the vacuum chamber [43]. Results show that the nozzle reduces cross-field ion motion but the nozzle does not seem as effective as Saito et al. presume; charge collectors near electromagnet show increasing current for increasing magnetic field strength [43]. These results indicate that, with increasing magnetic field strength, more plasma particles are scattering back to the nozzle, not fewer. More plasma particle back-scattering means fewer particles leaving the nozzle, leading to reduced thrust and performance.

Concurrently with the the charge collector experiment, Itadani et al. use a 0.36 J Nd:YAG laser for a Thompson scattering diagnostic, and are yet again able to show that the magnetic field deflects the bulk plasma [44]. They use $\beta$ (ratio of magnetic field pressure to plasma pressure) instead of $\kappa$ or $E_B/E_p$, and find that an
applied magnetic field pressure of 10 times the plasma pressure results in optimized
deflection [44].

While members of H. Nakashima’s research group are running experiments,
they also are concurrently trying to match these results with computational results.
They show good agreement (with a factor of two), for the 3 ns laser pulse width
experiment (Ref. [33]) [34], but they are unable to get good agreement with other
laser energies/pulse widths, such as the 4 J, 9 ns pulse (see Ref. [38]). To remedy this,
the group augments their numerical model. They surmise that the best path forward
would be to develop a code that is able to model the plasma creation process as well
as the plasma expansion process. To model the plasma creation process, they need
to model the laser-ablation portion of plasma generation. So, they used a 1D radio-
hydrodynamic model [81] to model how the solid polystyrene target absorbs the laser
light. This code is called Star1D and outputs the initial positions and velocities of
the plasma particles [42]. After running a test simulation with the integrated Star1D
model and the 3D hybrid PIC model [35], in 2014 Maeno et al. are able to match
results from the model with experiment, and show agreement to within a factor of 5
for the 4J 9 ns pulse case [38]. This is better than the 3D hybrid PIC model alone,
as in that case, the model too big by a factor of 35 [38].

Returning to the idea of using TVC in a magnetic nozzle, the H. Nakashima’s
group designs and builds a magnetic nozzle that they think will allow them to change
the direction of the plasma exhaust. In 2016, Edamoto et al. numerically validate
this design using 3D hybrid PIC code and the Star 1D code extended to 2 dimensions
(called Star 2D), and show that by turning off the bottom coil in the box they get
a high steering angle of 10 degrees [6]. However, their results show the code is not quite verified, because instead of the expected 0 degree steering angle for the base case, they get 2 degrees [6]. Later efforts by Morita et al. in 2017 attempt to refine their numerical models with experimental validation, which does agree somewhat in the plasma position, but the agreement is not very strong [41]. In 2017, Edamoto et al. attempt to validate the combination 3D hyrbid PIC + Star 2D code further with another experiment [42], this time comparing values on charge collectors over time. They find that their computational results are 40x too big [42]. For future work, they propose using a method called 'inside irradiation' which seems very similar to using a hohlraum.

### 2.2.2.4 Computational modeling: Other groups

Concurrently with the group lead by H. Nakashima, groups in the US and Europe were attempting to design a pulsed magnetic nozzle. Instead of basing their
efforts on a laser-ICF fusion vehicle, they first based them off an plasma gun-ICF vehicle, then an MTF (Z-pinch) vehicle. The plasma-gun ICF vehicle is detailed in Ref. [2], and the Z-pinch vehicle is detailed in Ref. [17].

Taking the former vehicle, Adams et al. develop a 2D parabolic reflection code to estimate nozzle performance. T. Polsgrove et al. detail how the code works and show trajectories in Ref. [17]. The authors break the expanding plasma ball into a series of shells that are assumed to spectrally reflect off the magnetic field. While good for quick calculations, the code does not incorporate magnetohydrodynamic effects, giving it a lower fidelity than prior work by Y. P. Zakharov et al. and H. Nakashima et al.

G. Romanelli et al. take the work done for the 2003 and 2010 HOPE vehicles and VISTA and build on it by using a 2D Ideal MHD (cylindrical r-z) code called PLUTO [47,82]. Ref. [47] has a good overview of the performance numbers obtained (73,000 sec Isp and 320 kN thrust for the 2003 HOPE vehicle and 19,000 Isp and 32 kN thrust for the 2010 HOPE vehicle), and Ref. [82] goes into detail on the methodology (how PLUTO works). Romanelli et al. show a lot of leakage in their analysis; they suggest the results be confirmed with a resistive MHD code [82]. SPFMax is one such code, and Romanelli et al. directly suggest to use it for further investigations [47].

Cassibry et al. put forth a different magnetic nozzle design than the usual solenoid one - they suggest using one that looks like an old TV diverter yoke with coils running axially as opposed to azimuthally (see Figure 2.11). After testing this design against two variations of the conventional solenoidal one, they find this design to be the only one that strongly accelerates the plasma. This contrasts earlier authors
Cassibry et al. use an earlier version of SPFMax that time integrated 2nd order wave equations of the potentials to solve for electromagnetic fields. However, this version of SPFMax crashes 250 ns, which is before the plasma is able to fully leave the nozzle. However, during this time, they are about to record a peak Isp of 9,000 sec, a 4,000 sec improvement over a bare pusher plate.

In summary, while authors have considered 3D effects and simulated different nozzle topologies and energies, they have not considered a bare axial nozzle without a pusher plate. This is important as using a pusher plate can increase heat loads on the vehicle to unacceptable levels. Also, authors have not compared these results to results from other computational groups, and have not detailed the important physical mechanisms that characterize these systems. Additionally, while nozzles with fusion plasmas have been considered, hybrid fission-fusion plasmas have not been considered as rigorously. Therefore, a survey of the literature leads to the second half of our research objective, which has three components: 1) compare the results of our model against previous authors, 2) illustrate important magnetic nozzle processes in a purely
axial magnetic nozzle, and 3) determine the effect of axial magnetic nozzle topology on performance (specific impulse, impulse bit, and nozzle efficiency).

2.2.2.5 Summary

We can summarize the current state of the art of pulsed magnetic nozzle development using Table 2.1. In the table, $d$ is the initial separation distance between the nozzle and the plasma, $E_0$ is the target initial energy, $E_{las}$ is the laser energy, $\eta$ is the plume efficiency (see Eq. (2.8)), $\beta$ is the steering angle (see Eq. (2.9)), and $\eta_{th}$ is the nozzle efficiency.

$$\eta_{th} = \frac{\Delta m v_z}{\sqrt{2mE_0}}$$  

(2.10)

<table>
<thead>
<tr>
<th>Nozzle Type</th>
<th>Simulation and/or Experiment Information</th>
<th>Propellant and Nozzle Information</th>
<th>Independent Variable(s)</th>
<th>$\eta_{max}$</th>
<th>Other Results</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dipole</td>
<td>Math model</td>
<td>Propellant: 0.11 g, 0 eV, Au</td>
<td>$d$</td>
<td>0.85</td>
<td>—</td>
<td>[26]</td>
</tr>
<tr>
<td>Coils</td>
<td>Simulation Type</td>
<td>Propellant</td>
<td>Nozzle</td>
<td>Parameter</td>
<td>Value 1</td>
<td>Parameter</td>
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<td>------------</td>
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<tr>
<td>1 coil</td>
<td>2D MHD simulation</td>
<td>8.3 g, 1.3 GJ</td>
<td>I=22 MA, r=6.5 m</td>
<td>—</td>
<td>0.65</td>
<td>—</td>
</tr>
<tr>
<td>1 coil</td>
<td>3D hybrid PIC simulation, with or without SPH helper code</td>
<td>0.1 g, 0 eV, Au or H+DT</td>
<td>r=1 m or r=9 mm</td>
<td>d, E₀, I, Target shape, Star1D</td>
<td>0.78</td>
<td>β₀ max : [5, 37°, 9.25, 27, 31, 35]</td>
</tr>
<tr>
<td>1 coil</td>
<td>3D hybrid PIC simulation</td>
<td>0.11 g, 4 MJ, 0 eV, Au</td>
<td>I₁=3.57 MA, r₁=1 m, r₂=2 m</td>
<td>d₂, I₂, γ₂</td>
<td>0.78</td>
<td>β₀ max : [24, 4.9°, 27, 28]</td>
</tr>
<tr>
<td>Multi-coil ring of 12 squares</td>
<td>3D hybrid PIC simulation, with or without SPH helper code</td>
<td>0.11 g, 4 MJ, 0 eV, Au, d=-1m</td>
<td>Iₚ₀₆=14 kA or Iₚ₀₆=32</td>
<td>Iₚ₀₆, Target shape</td>
<td>0.75</td>
<td>β₀ max : [28, 5.2°, 31]</td>
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<tr>
<td>2-3 coil</td>
<td>3D hybrid PIC with or without SPH helper code</td>
<td>4 MJ, d=-1m</td>
<td>I=3.57 MA, r=1 m (each)</td>
<td>Coil configuration, Target shape</td>
<td>0.74</td>
<td>No leakage with 3 coil configuration</td>
</tr>
<tr>
<td>Perm- anent magnet</td>
<td>1064 nm Nd:YAG laser experiment</td>
<td>Propellant: $50 \mu g$, $r=0.1$ mm, $d=10$ mm or $15$ mm</td>
<td>Pulse width, $E_{laser}$, Laser wavelength</td>
<td>$0.5$</td>
<td>$I_{bat}$:</td>
<td>0.25 - 33, 10000 - 36- $\mu Ns$ 38</td>
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</tr>
<tr>
<td>96-turn solen- oid</td>
<td>3D Hybrid PIC simulation and 6 J 1064 nm Nd:YAG laser, pulse width 10 ns experiment</td>
<td>Propellant: $r=0.25$ mm CH, $d=13$ mm</td>
<td>$I$</td>
<td>$\kappa$:0.1-0.2, sim 40x small</td>
<td>$0.1-0.2$, sim 40x small</td>
<td>39, 40, 43</td>
</tr>
<tr>
<td>Figure 2.10</td>
<td>3D hybrid PIC with Star2D simulation</td>
<td>Propellant: $r=0.25$ mm CH, $d=10$ mm</td>
<td>$I$</td>
<td>$\beta_{max} = 56.7^\circ$</td>
<td>$0.4$</td>
<td>39, 40, 43</td>
</tr>
<tr>
<td>8 coil solen- oid</td>
<td>2D MHD simulation (PLUTO)</td>
<td>Propellant: $20-2$ g, $r_p=0.3$ m</td>
<td>$m$, $E_0$, $I$</td>
<td>$\eta_l = 0.6$</td>
<td>0.4</td>
<td>47, 82</td>
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<tr>
<td>Figure 2.11</td>
<td>3D simulation (SPFMax) with radiative cooling</td>
<td>Propellant: DT, $T_0 = 1$ keV, $\rho_0 = 80$ kg/$m^3$, $2.5$ mg</td>
<td>Coil configuration</td>
<td>$\eta_l = 0.2$</td>
<td>0.4</td>
<td>46, 82</td>
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</table>
CHAPTER 3

METHODOLOGY

3.1 Power Generation System Model

To address the first half of our research objective, we develop a model for the power-generation system. The model is shown graphically in Figure 3.1. This model takes 26 inputs that are grouped into four main categories; the stator properties, the armature properties, the transformer properties, and the circuit properties. The stator properties include the number of turns, $N_{stat}$, the initial current $I_{10}$, and the radius, $r_{stat}$. The radius of the stator also serves as the length scale of the stator. The armature properties include the mass, $m$, the molecular weight $MW$, the ratio of specific heats, $\gamma$, the initial temperature, $T_0$, and the starting plasma radius, $r_{po}$. The transformer properties include the relative permeability of the core $\mu_r$, the coupling constant of the transformer, $K$, the radius of the core, $r_T$, the length of the primary side, $\lambda_{T1}$, the length of the secondary side, $\lambda_{T2}$, the number of turns on the primary side, $N_{T1}$, and the number of turns on the secondary side $N_{T2}$. The circuit properties include the resistive losses on the primary side, $R_1$, the resistive losses on the secondary side, $R_2$, the inductive losses on the primary side, $l_1$, the inductive losses on the secondary side, $l_2$, and the capacitance of the energy storage system $C$. Subscripts
Figure 3.1: Power generation system schematic with model inputs.

1 and 2 refer circuit elements or properties on the primary and secondary sides of the transformer, respectively. Subscript $T$ refers to the transformer. Subscript 0 denotes the initial value of a quantity. Lastly, bolded quantities in Figure 3.1: $r_{stat}$, $N_{stat}$, $I_{10}$, $N_{T1}$, $N_{T2}$, and $C$, denote parameters for parametric investigation.

We analyzed the schematic in Figure 3.1 to derive a set of ordinary differential equations. Derivation will proceed right to left in the schematic; beginning with the circuit in general, proceeding to the transformer, and concluding with the FCG. Section 3.1.1 concerns the circuit in general, Section 3.1.2 concerns the transformer, and Section 3.1.3 concerns the FCG.
3.1.1 Circuit analysis

Focusing on the circuit model in Figure 3.1 (the right half of the graphic), and using the T-equivalent transformer decomposition [83] results in the circuit in Figure 3.2. Using the annotated graphic Figure 3.3, and applying Kirchhoff’s voltage and current laws to the left loop and right loop results in Eq. (3.1)-(3.3).

\[
I_1(t) \frac{dL}{dt} - (L_1 - M + l_1 + L(t)) \frac{dI_1(t)}{dt} - R_1 I_1(t) - M \frac{dI_3(t)}{dt} = 0 \quad (3.1)
\]

\[
-(L_2 - M + l_2) \frac{dI_2(t)}{dt} - R_2 I_2(t) - V_{cap}(t) + M \frac{dI_3(t)}{dt} = 0 \quad (3.2)
\]

Figure 3.2: Power generation circuit lumped-parameter model with T-equivalent transformer decomposition.

Figure 3.3: Power generation circuit lumped-parameter model.
\[ I_1(t) = I_2(t) + I_3(t) \]  

(3.3)

Differentiating with respect to time results in

\[ \frac{dI_1}{dt} = \frac{dI_2}{dt} + \frac{dI_3}{dt} \]  

(3.4)

Solving for \( \frac{dI_3}{dt} \) gives

\[ \frac{dI_3}{dt} = \frac{dI_1}{dt} - \frac{dI_2}{dt} \]  

(3.5)

Plugging in Eq. (3.5) into Eq. (3.1) and Eq. (3.2)) results in

\[ I_1(t) \frac{dL}{dt} - (L_1 + l_1 + L(t)) \frac{dI_1(t)}{dt} - R_1 I_1(t) + M \frac{dI_2(t)}{dt} = 0 \]  

(3.6)

\[ -(L_2 + l_2) \frac{dI_2(t)}{dt} - R_2 I_2(t) - V_{cap}(t) + M \frac{dI_1(t)}{dt} = 0 \]  

(3.7)

In both equations, voltage source terms here are positive, and sink terms are negative. In Eq. (3.6), \( I_1(t) \frac{dL}{dt} \) is the voltage pulse coming off the stator, \( L_1 \) is the inductance on the primary side of the transformer, \( I_1 \) is the current through the loop on the primary side of the transformer, \( M \) is the mutual inductance, and \( I_2 \) is the current through the loop on the secondary side of the transformer. \( l_1 \) and \( R_1 \) are the same as in Figure 3.1.

In Eq. (3.7), \( L_2 \) is the inductance on the secondary side of the transformer and \( V_{cap} \) is the voltage on the capacitor (see Eq. (3.10)) and surrounding discussion). \( l_2 \) and \( R_2 \) are the same as in Figure 3.1. \( I_1 \), \( I_2 \), and \( M \) are the same as in Eq. (3.9).
Since $I_1(t) \frac{d}{dt}$ term is the source voltage for the rest of the circuit [64], it is captured as $\mathcal{E}(t)$.

$$\mathcal{E}(t) = \frac{dL(t)}{dt} I(t)$$  \hspace{1cm} (3.8)

Plugging the above into Eq. (3.6) results in

$$\mathcal{E}(t) - (L_1 + l_1 + L(t)) \frac{dI_1(t)}{dt} - R_1 I_1(t) + M \frac{dI_2(t)}{dt} = 0$$  \hspace{1cm} (3.9)

Returning to Eq. (3.7), $V_{\text{cap}}$ does not have an associated equation. To this end, beginning with the definition of capacitance,

$$Q(t) = V_{\text{cap}}(t) C$$  \hspace{1cm} (3.10)

solving for $V_{\text{cap}}$,

$$V_{\text{cap}}(t) = \frac{1}{C} Q(t)$$  \hspace{1cm} (3.11)

and differentiating results in Eq. (3.12).

$$\frac{dV_{\text{cap}}(t)}{dt} = \frac{1}{C} I_2(t)$$  \hspace{1cm} (3.12)

This closes the circuit equation set.
3.1.2 Transformer analysis

The transformer mutual inductance term, $M$, is given by

$$M = K \sqrt{L_1 L_2}$$ (3.13)

The transformer is assumed to couple efficiently, so $K = 0.9$. We model the inductance of the transformer primary and secondary windings as an ideal solenoids. This is shown in the following

$$L_i = \frac{\mu_0 \mu_r N^2_{T_i} A_T}{\lambda_{T_i}}$$ (3.14)

The transformer is taken to be air-core, so $\mu_r = 1$. Taking Eq. (3.14) and substituting in $\mu_r = 1$ results in the following

$$L_i = \frac{\mu_0 N^2_{T_i} A_T}{\lambda_{T_i}}$$ (3.15)

which gives the inductance on the $i$th side of the transformer. $i = 1$ for the primary side of the transformer and $i = 2$ for the secondary side. In Eq. (3.15), $N_{T_i}$ is the number of turns on the $i$th side, $A_T$ is the cross-sectional area of the core (calculated via $\pi r_T^2$), and $\lambda_{T_i}$ is the length on the $i$th side. The transformer is assumed to have a cylindrical cross-sectional area with a major radius and length of 0.1m, meaning $r_T = 0.1$ m, and $\lambda_{T_1} = \lambda_{T_2} = 0.1$ m.
3.1.3 FCG analysis and determination of $L(t)$

It is instructive to approximate $L(t)$ by assuming the inductance of the armature and stator set is the same as the inductance of a set of parallel plate transmission lines. Here, the stator coil forms one of the plates, and the section of armature the stator coil interacts with forms the other transmission plate. As the plasma expands the distance between the plates decreases, increasing the source voltage $\mathcal{E}(t)$. The inductance of a set of $N$ parallel conducting plates is given as

$$L(t) = N\frac{\mu_0\lambda}{w} d(t)$$  \hspace{1cm} (3.16)

where $N$ is the number of turns, $\mu_0$ is the permeability of free space, $\lambda$ is plate length, $w$ is the plate width, and $d(t)$ is the separation distance between the plates.

For a hemispherical nozzle, a representative length $\lambda$ is the number of turns $N_{stat}$ times the mean coil length, which is $\frac{\pi}{2} r_{stat}$. For a similar nozzle with radius $r_{stat}$, a representative plate width $w$ is $\pi r_{stat}$. Plugging these into Eq. (3.16) results in the following:

$$L(t) = N_{stat} \frac{\mu_0 N_{stat} \frac{\pi}{2} r_{stat}}{\pi r_{stat}} d(t) = \frac{\mu_0 N_{stat}^2}{2} d(t)$$  \hspace{1cm} (3.17)

The separation distance between the plasma and coils $d(t)$ is given by the mean coil radius minus the radius of the expanding armature. Mathematically, this is expressed as

$$d(t) = \frac{r_{stat}}{2} - r_p(t)$$  \hspace{1cm} (3.18)
where $r_{stat}$ is the radius of the stator, and $r_p(t)$ is the radius of the plasma. To determine $r_p(t)$, we elected to use a 1D 'snowplow’ plasma model and force balance. While using a full 3D model that considers second order effects, such as transport, offers much higher accuracy \[70\], for this work we thought it sufficient to use a 1D model and add more detail to the circuit model (see previous section). However, 3D effects and transport will be important to consider as plasma FCG modeling matures.

See Figure 3.4 for the 1D force balance. The only force assumed to act on the armature is assumed to be entirely in the radial direction, and is given by $F_{P\theta}$, the magnetic pressure force, given in below

$$F_{P\theta} = \frac{B(t)^2}{2\mu_0} A(t)$$  \hspace{1cm} (3.19)
where \( B(t) \) is the magnetic field, which is given in Eq. (3.21), \( \mu_0 \) is again the permeability of free space, and \( A(t) \) is a representative area given as follows

\[
A(t) = A_{\text{stat}} - 2\pi r_p(t)^2 = \pi r_{\text{stat}}^2 - 2\pi r_p(t)^2
\]  

(3.20)

Here, \( A_{\text{stat}} \) is the cross-sectional area of the stator, and \( r_p(t) \) and \( r_{\text{stat}} \) are the same as before. Since flux is conserved inside an FCG, the magnetic field is proportional to the area \( A(t) \), as given by

\[
B(t) = \frac{\Phi_{\text{ref}}(t)}{A(t)} = \frac{B_{\text{ref}}(t) A_{\text{stat}}}{A(t)}
\]  

(3.21)

where \( \Phi_{\text{ref}}(t) \) is the reference magnetic flux, \( B_{\text{ref}}(t) \) is the reference magnetic field, and \( A_{\text{stat}} \) and \( A(t) \) are the same as before. \( B_{\text{ref}}(t) \) is assumed to be the same as the magnetic field inside an ideal solenoid

\[
B_{\text{ref}}(t) = \frac{\mu_0 N_{\text{stat}} I_1(t)}{r_{\text{stat}}}
\]  

(3.22)

Here, \( \mu_0 \), \( N_{\text{stat}} \), \( I_1(t) \), and \( r_{\text{stat}} \) are the same as before.

Putting Eq. (3.20)-(3.22) together results in the following equation for the magnetic field inside the stator.

\[
B(t) = \frac{\mu_0 N_{\text{stat}} I_1(t)}{r_{\text{stat}}} \frac{\pi r_{\text{stat}}^2}{A_0 - 2\pi r_p(t)^2} = \frac{\mu_0 N_{\text{stat}} I_1(t)}{r_{\text{stat}} \left(1 - 2 \left(\frac{r_p(t)}{r_{\text{stat}}}ight)^2\right)}
\]  

(3.23)
Using Figure 3.4, we find the equation of motion for the expanding plasma ball

\[- \frac{B(t)^2}{2\mu_0} A(t) = m \frac{d^2 r_p(t)}{dt^2} \]  

(3.24)

Here, \( m \) is the pass of the armature and \( t \) is the simulation time, but all other variables are the same as in other equations. The pressure force acts in the direction opposite to the direction the plasma expands, hence the negative sign. Plugging in Eq. (3.20) and Eq. (3.23) into the Eq. (3.24) results in

\[- \frac{\mu_0 N_{stat} I_1(t)}{r_{stat} \left( 1 - 2 \left( \frac{r_p(t)}{r_{stat}} \right)^2 \right)} \left( \pi r_{stat}^2 - 2 \pi r_p(t)^2 \right) = m \frac{d^2 r_p(t)}{dt^2} \]  

(3.25)

which is simplified to yield

\[- \frac{\mu_0 N_{stat}^2 I_1(t)^2 \pi}{2 \left( 1 - 2 \left( \frac{r_p(t)}{r_{stat}} \right)^2 \right)} = m \frac{d^2 r_p(t)}{dt^2} \]  

(3.26)

The plasma expansion speed is related back to the plasma radius as follows

\[ v_p(t) = \frac{dr_p(t)}{dt} \]  

(3.27)

Together, Eq. (3.26) and the above equation characterize the plasma motion. However, we also use them to determine \( E(t) \).
The plasma radius factors into $E(t)$ through the derivative of $L(t)$, given below

$$E(t) = \frac{dL(t)}{dt} = \frac{\mu_0 N^2_{stat}}{2} \frac{d}{dt}(d(t)) = \frac{\mu_0 N^2_{stat}}{2} \frac{dr_p(t)}{dt} = \frac{\mu_0 N^2_{stat}}{2} v_p(t)$$

(3.28)

The negative sign is dropped here to be consistent with the sign convention that voltage sources are positive and voltage sinks are negative.

### 3.1.4 Final model equation set

To get the final equation set, we plug Eq. (3.15), Eq. (3.28), Eq. (3.17), Eq. (3.18), and Eq. (3.13), into Eq. (3.9). This results in the following:

$$\mu_0 N^2_{stat} v_p(t) I_1(t) - \left( \frac{\mu_0 N^2_{T_2} A_T}{\lambda_{T_1}} + l_1 + \frac{\mu_0 N^2_{stat}}{2} \left( \frac{r_{stat}}{2} - r_p(t) \right) \right) \frac{dI_1(t)}{dt}$$

$$- R_1 I_1(t) + K \sqrt{\frac{\mu_0 N^2_{T_1} A_T}{\lambda_{T_1}} \frac{\mu_0 N^2_{T_2} A_T}{\lambda_{T_2}}} \frac{dI_2(t)}{dt} = 0$$

(3.29)

We also plug in Eq. (3.15), and Eq. (3.13) into Eq. (3.7) to result in the following equation.

$$- \left( \frac{\mu_0 N^2_{T_2} A_T}{\lambda_{T_2}} + l_2 \right) \frac{dI_2(t)}{dt} - R_2 I_2(t) - V_{cap}(t) + K \sqrt{\frac{\mu_0 N^2_{T_1} A_T}{\lambda_{T_1}} \frac{\mu_0 N^2_{T_2} A_T}{\lambda_{T_2}}}$$

$$+ \frac{dI_1(t)}{dt} = 0$$

(3.30)

Eq. (3.29)-(3.30), along with Eq. (3.27)-(3.26) and Eq. (3.12) form a closed set of differential equations that can be numerically integrated using MATLAB’s
ODE45 [52]. However, they must be put in vector form first. Defining \( \vec{x}(t) \) as

\[
\vec{x}(t) = \begin{bmatrix}
I_1(t) \\
I_2(t) \\
V_{cap}(t) \\
r_p(t) \\
v_p(t)
\end{bmatrix}
\]

and differentiating results in

\[
\frac{d\vec{x}(t)}{dt} = \begin{bmatrix}
\frac{dI_1(t)}{dt} \\
\frac{dI_2(t)}{dt} \\
\frac{dV_{cap}(t)}{dt} \\
\frac{dr_p(t)}{dt} \\
\frac{dv_p(t)}{dt}
\end{bmatrix}
\]

Substituting in Eq. (3.27)-(3.26), Eq. (3.29)-(3.30), and Eq. (3.12) in the above results in

\[
\frac{d\vec{x}(t)}{dt} = \begin{bmatrix}
\frac{\mu_0 N_{stat}^2 x_5(t)x_1(t) - x_1(t)R_1 - K \frac{dx_2}{dt}}{L_1 + \frac{\mu_0 N_{stat}^2}{\lambda T_1}(r_{stat} - x_4(t)) + \frac{\mu_0 N_{T}^2}{\lambda T_2}} \\
\frac{\mu_0 N_{stat}^2 x_5(t)x_1(t) - x_1(t)R_1 - K \frac{dx_2}{dt}}{L_1 + \frac{\mu_0 N_{stat}^2}{\lambda T_1}(r_{stat} - x_4(t)) + \frac{\mu_0 N_{T}^2}{\lambda T_2}} \\
\frac{\mu_0 N_{stat}^2 x_5(t)x_1(t) - x_1(t)R_1 - K \frac{dx_2}{dt}}{L_1 + \frac{\mu_0 N_{stat}^2}{\lambda T_1}(r_{stat} - x_4(t)) + \frac{\mu_0 N_{T}^2}{\lambda T_2}} \\
\frac{\mu_0 N_{stat}^2 x_5(t)x_1(t) - x_1(t)R_1 - K \frac{dx_2}{dt}}{L_1 + \frac{\mu_0 N_{stat}^2}{\lambda T_1}(r_{stat} - x_4(t)) + \frac{\mu_0 N_{T}^2}{\lambda T_2}} \\
\frac{\mu_0 N_{stat}^2 x_5(t)x_1(t) - x_1(t)R_1 - K \frac{dx_2}{dt}}{L_1 + \frac{\mu_0 N_{stat}^2}{\lambda T_1}(r_{stat} - x_4(t)) + \frac{\mu_0 N_{T}^2}{\lambda T_2}}
\end{bmatrix}
\]

\[
\frac{d\vec{x}(t)}{dt} = \begin{bmatrix}
\frac{1}{C} x_2(t) \\
x_5(t) \\
\frac{-1}{m} \frac{\mu_0 N_{stat}^2 x_1(t)^2 \pi}{2 \left(1 - 2 \left(\frac{x_4(t)}{r_{stat}}\right)^2\right)}
\end{bmatrix}
\]
Table 3.1: Initial conditions

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Initial Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_{c\text{apo}}$</td>
<td>0 V</td>
</tr>
<tr>
<td>$I_{2\text{o}}$</td>
<td>0 A</td>
</tr>
<tr>
<td>$r_{p\text{o}}$</td>
<td>1 cm</td>
</tr>
<tr>
<td>$v_{p\text{o}}$</td>
<td>$\sqrt{\gamma \frac{R}{M_W} 2T_0}$</td>
</tr>
</tbody>
</table>

The above equation is input into MATLAB’s ODE45 integrator [52].

3.1.5 Initial conditions and additional model information

The initial conditions for the power-generation system model are given in Table 3.1; these are also needed to numerically integrate Eq. (3.33). Most circuit parameters start at 0, except for the initial stator current $I_{1\text{o}}$, which is specified as an input, the initial plasma radius, which is specified as 1 cm, and the initial plasma velocity, which is specified as the initial speed of sound in the plasma. Mathematically, the initial conditions factor into the starting vector $\ddot{x}_0$ using Eq. (3.34)

$$\ddot{x}_0 = \begin{bmatrix} I_{1\text{o}} \\ 0 \\ 0 \\ 0.01 \\ \sqrt{\gamma \frac{R}{M_W} 2T_0} \end{bmatrix} \quad (3.34)$$

Numerical integration of the equations is performed until either the plasma leaves the nozzle, or until the stator current returns to its initial value. When the
plasma radius equals the nozzle radius, the plasma 'bounces' off the walls and the integration is continued with the plasma velocity multiplied by negative 1. If during the simulation the capacitor voltage stops changing \((I_2 = 0)\), the capacitor is taken to be fully charged and the simulation ends.

### 3.1.6 Non-dimensionalization

The power-generation circuit model uses the dimensional form of the equations. However, for subsequent analyses it is necessary to non-dimensionalize these equations. Using the non-dimensionalization in Eq. (3.35)-(3.41)

\[
r^* = \frac{r_p(t)}{r_{stat}} \quad (3.35)
\]

\[
v^* = \frac{v_p(t)}{v_{p0}} \quad (3.36)
\]

\[
t^* = \frac{t}{\tau} = \frac{t}{r_{stat}/v_{p0}} \quad (3.37)
\]

\[
L_{stat0} = \frac{\mu_0 N_{stat}^2 r_{stat}}{4} \quad (3.38)
\]

\[
I_1^* = \frac{I_1(t)}{I_{10}} \quad (3.39)
\]

\[
I_2^* = \frac{I_2(t)}{I_{10}} \quad (3.40)
\]

\[
V_{cap}^* = \frac{V_{cap}(t)}{V_{capf}} \quad (3.41)
\]
results in the following non-dimensional forms of Eq. (3.26)-(3.27), Eq. (3.29)-(3.30), and Eq. (3.12) respectively.

\[ v^* = \frac{dr^*}{dt^*} \]  
\[ - \left( \frac{\frac{1}{2} L_{\text{stat}} I_{10}^2}{\frac{1}{2} mv_{p0}^2} \right) \left( \frac{I_{\tau^*}}{1 - 2r^*} \right) = \frac{1}{2} \frac{dv^*}{dt^*} \]  
\[ 2v^* I_1^* - \left( \frac{L_1}{L_{\text{stat}}} + \frac{l_1}{L_{\text{stat}}} + (1 - 2r^*) \right) \frac{dI_1^*}{dt^*} - \frac{R_1\tau}{L_{\text{stat}}} I_1^* + K \sqrt{\frac{L_1}{L_{\text{stat}}}} \frac{L_2}{L_{\text{stat}}} \frac{dI_2^*}{dt^*} = 0 \]  
\[ - \left( \frac{L_2}{L_{\text{stat}}} + \frac{l_2}{L_{\text{stat}}} \right) \frac{dI_2^*}{dt^*} - \frac{R_2\tau}{L_{\text{stat}}} I_2^* - \left( \frac{I_{10}\tau}{V_{\text{cap}_f} C} \right) \left( \frac{\frac{1}{2} CV_{\text{cap}_f}^2}{\frac{1}{2} L_{\text{stat}} I_{10}^2} \right) V_{\text{cap}}^* + K \sqrt{\frac{L_1}{L_{\text{stat}}}} \frac{L_2}{L_{\text{stat}}} \frac{dI_1^*}{dt^*} = 0 \]  
\[ \frac{dV_{\text{cap}}^*}{dt^*} = \left( \frac{I_{10}\tau}{V_{\text{cap}_f} C} \right) I_2^* \]  

From these equations and the Buckingham Pi theorem [84], the following non-dimensional parameters emerge:

\[ \pi_1 = \frac{L_1}{L_{\text{stat}}} \]  
\[ \pi_2 = \frac{\frac{1}{2} L_{\text{stat}} I_{10}^2}{\frac{1}{2} mv_{p0}^2} \]  
\[ \pi_3 = \frac{I_{10}\tau}{V_{\text{cap}_f} C} \]  
\[ \eta = \frac{E_{\text{charged}}}{\frac{1}{2} L_{\text{stat}} I_{10}^2} \]  

These pi-groups (except for Eq. (3.49)) are used throughout Section 4.1.1.3.
3.1.7 Study design

For parametric study we considered the following input variables: parameters that characterize the stator design \(N_{\text{stat}}, r_{\text{stat}}, I_{l_0}\), parameters that characterize the transformer design \(N_{T_1}, N_{T_2}\), and parameters that characterize the energy storage design \((C)\). We used the PuFF target for our armature [15]. Table 3.2 contains values for variables not considered in the study. For the study, we used Table 3.1 in addition to Table 3.2 to run the model; they should be considered in conjunction.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(m)</td>
<td>600 grams</td>
<td>(\mu_r)</td>
<td>1</td>
<td>(l_1)</td>
<td>10 nH</td>
</tr>
<tr>
<td>MW</td>
<td>8 grams/mol</td>
<td>(K)</td>
<td>0.9</td>
<td>(l_2)</td>
<td>100 nH</td>
</tr>
<tr>
<td>(\gamma)</td>
<td>1.3</td>
<td>(\lambda_{T_1})</td>
<td>0.1 m</td>
<td>(R_1)</td>
<td>0 Ω</td>
</tr>
<tr>
<td>(r_{po})</td>
<td>1 cm</td>
<td>(\lambda_{T_2})</td>
<td>0.1 m</td>
<td>(R_2)</td>
<td>0 Ω</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(r_T)</td>
<td>0.1 m</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

PuFF uses a Z-pinch to implode a fission or fission/fusion target [15]. We assume an armature with a mass of 600 grams, a molecular weight of 8 grams per mole (this corresponds to a by-mass-mixture ratio of Lithium to Uranium to 5), a constant of specific heats (\(\gamma\)) of 1.3, and an initial plasma radius of 1 cm. We also assume a circuit with a loss inductance on the primary side of 10 nH, a loss inductance on the secondary side of 100 nH, and no loss resistances. For future work, we can increase the loss resistances in later analysis and determine the effect on the final capacitor energy. Lastly, we assume an air-core transformer \((\mu_r = 1)\), with a coupling
constant of 0.9, a core length of 0.1 m, and a core radius of 0.1 m. This is because we want the transformer to be fairly small.

For the trade study, the figure of merit is the energy in the capacitive storage system after the capacitors have fully charged, or \( E_{\text{charged}} \). Using the capacitive stored energy allows for comparison of energy and power against previous work: VISTA, the 2003 MTF HOPE vehicle, and the 2010 HOPE vehicle.

### 3.2 Pulsed Magnetic Nozzle Model

To address the second half of our research objective, we develop a model for the magnetic nozzle subsystem. For this system, to accurately capture the fully 3D nature of electromagnetic fields, we must use a fully 3D model. In contrast to a more computationally intensive Particle-in-Cell (PIC) model used by the H. Nakashima team, we use a Smoothed Particle Fluid (SPH) model.

#### 3.2.1 SPFMax

##### 3.2.1.1 SPH theory

The SPH model is augmented with a Maxwell equations solver to capture electromagnetic fields, hence the model’s name (Smoothed Particle Fluid with Maxwell equation solver or SPFMax). The version of SPFMax we plan to use is different from the one in Ref. [46]; these differences are explained in Section 3.2.1.4. Firstly,
SPFMax uses a kernel function to approximate properties in the following manner:

\[
G_a(r) = \int G(r')W(r - r', h)dr'
\] (3.51)

\(G\) is any property (such as pressure, temperature, etc), subscript \(a\) indicates point \(a\), \(r\) is the position of point \(a\) in space, \(h\) is the compact support distance and \(W\) is the interpolating kernel function [46]. The integral is then replaced with a summation over \(b\) neighboring particles as follows

\[
A_a = \sum_b A_b \mathcal{V}_b W_{ab}(r - r', h)
\] (3.52)

with \(\mathcal{V}_b\) being the volume of neighboring \(b\) number of particles [46]. \(W_{ab}\) is the cubic-spline function, which is as follows

\[
W_{ab}(r) = \begin{cases} 
\frac{1}{4\pi h_{ab}^3} [(2 - r)^3 - 4(1 - r)^3] & \text{for } 0 \leq r \leq 1 \\
\frac{1}{4\pi h_{ab}^3} (2 - r)^3 & \text{for } 1 \leq r \leq 2 \\
0 & \text{for } r \geq 2 
\end{cases}
\] (3.53)

where \(h_{ab}\) is again the compact support distance, which must be chosen according to the following equations

\[
\sum_b \mathcal{V}_b W_{ab} = 1
\] (3.54)

\[
\sum_b \mathcal{V}_b \nabla W_{ab} = 0
\] (3.55)
where $\nabla W_{ab}$ denotes the gradient of the kernel function [46]. Choosing $h$ is the most important, but also most difficult part of the model. In implementation, the SPFMax code considers $b=60$ (60 nearest neighbors), and for a full expansion of $h$, see Ref. [46].

### 3.2.1.2 Fluid equations of motion

The equations of motion the code solves are the continuity (Eq. (3.56)), momentum (Eq. (3.57)), and energy (Eq. (3.58)) equations:

\[
\frac{D\rho}{Dt} = -\rho(\nabla \cdot \vec{v})
\]
\[
\frac{D\vec{v}}{Dt} = -\frac{1}{\rho} \nabla p + \frac{1}{\rho} \nabla \cdot \vec{\tau} + \frac{1}{\rho} \vec{j} \times \vec{B}
\]
\[
\frac{Du}{Dt} = -\frac{p}{\rho} \nabla \cdot \vec{v} + \frac{1}{\rho} \left( \vec{\tau} : \nabla \vec{v} \right) - \frac{1}{\rho} \nabla \cdot \left( k \nabla T \right) - 4\sigma_{sb} T^4 \chi_{Planck} + \frac{1}{\rho \sigma} j^2
\]

In the above equations, $\rho$ is the mass density, $\vec{v}$ is the velocity vector, $\vec{j}$ is the current density vector, $\vec{B}$ is the magnetic field, $t$ is time, $p$ is the static pressure, $\tau$ is the deviatoric viscous stress tensor, $u$ is the specific internal energy, $k$ is the thermal conductivity, $\sigma_{sb}$ is the Stefan-Boltzmann constant, $T$ is temperature, $\chi_{Planck}$ is the single group Planck emission opacity, and $\sigma$ is the electrical conductivity [46]. $\chi_{Planck}$ as well as specific internal energies and pressures are calculated using PROPACEOS [85]. Eq. (3.56)-(3.58) are numerically integrated in time according to the second-order Runge-Kutta scheme [86].
For calculating thermal conductivity, we use the model outlined in Appendix B of Ref. [87]. However, for the electrical conductivity, we use the method in the next section.

### 3.2.1.3 Electrical conductivity model

We calculate $\sigma$, using the reciprocal of the Spitzer resistivity

$$\sigma = \frac{3}{4\sqrt{2\pi}} \frac{(4\pi \epsilon_0)^2}{(k_b T)^2} Z e^2 m_e^{1/2} \ln(\Lambda)$$

(3.59)

In Eq. (3.59), $\epsilon_0$ is the permittivity of free space, $k_b$ is Boltzmann’s constant, $Z$ is the ionization state, $e$ is the fundamental charge, $m_e$ is the mass of an electron, and $\ln(\Lambda)$ is the Coulomb logarithm. Ionization state $Z$ and $\ln(\Lambda)$ are calculated using PROPACEOS [85].

### 3.2.1.4 Electromagnetic fields

The electromagnetic quantities that feed into the equations of motion are $\vec{J}$ and $\vec{B}$. In contrast to Ref. [46], these are calculated in the following manner. Firstly, the code calculates $\vec{B}$ using the Biot-Savart law, given by the following

$$\vec{B} = \frac{\mu_0}{4\pi} \int \int \int \frac{\vec{j} dV \times \vec{r}'}{r'^3}$$

(3.60)

where $N$ is the number of particles, $dV$ is the volume of a plasma particle, $\vec{r}'$ is the displacement vector from the plasma particle to the point in space where the value of $\vec{B}$ is needed, and all other variables are the same as before. To calculate $\vec{J}$ we assume
SPH particles act as a 3D network of transmission lines, with each particle having an inductance and resistance. Mathematically this is stated as

\[
\frac{d\vec{j}}{dt} = \frac{\lambda_p}{L_p A_p} \left( \vec{\nu} \times \vec{B} - \frac{1}{\sigma^2} \vec{j} \right)
\]

(3.61)

where \(L_p\) is the inductance of a SPH particle, \(\lambda_p\) is the length scale of a SPH particle, \(A_p\) is the cross-sectional area of a SPH particle, and all other quantities are the same as before. For a complete derivation of Eq. (3.61), see Appendix C. In Eq. (3.61), the first term \(\vec{\nu} \times \vec{B}\) is the inductive term and the second term \(\frac{1}{\sigma^2} \vec{j}\) is the resistive term. In Eq. (3.61), \(L_p\) is calculated by assuming the inductance of a plasma particle is equal to that of a single-turn ideal solenoidal, with \(\mu_r = 1\), yielding

\[
L_p = \frac{\mu_0 A_p}{\lambda_p}
\]

(3.62)

The last quantity of note is the energy in the electromagnetic field, which we denote as \(E_{\text{current}}\). This is calculated using

\[
\frac{dE_{\text{current}}}{dt} = \mathcal{V} \vec{j} \cdot \left( \vec{\nu} \times \vec{B} \right)
\]

(3.63)

where \(\mathcal{V}\) is the total volume of the plasma, which is given by

\[
\mathcal{V} = A_{\text{plasma}} \lambda_{\text{plasma}} = \pi \left( \frac{3}{4\pi} \sum_p \mathcal{V}_p \right)^{2/3} \star \left( \sum_p \mathcal{V}_p \right)^{1/4}
\]

(3.64)
where $V_p$ is the volume of an individual plasma SPH particle, and $N$ is the total number of plasma particles in a simulation (varies based on the initial value of $h_{ab}$).

Both Eq. (3.61) and Eq. (3.63) are numerically integrated in time, using different time steps than Eq. (3.56)-(3.58), with the second-order Runge-Kutta scheme. $E_{current}$ is used to ensure energy conservation.

3.2.1.5 Limiting current

Over the course of code development, we noticed that the currents developing in the plasma due to $\vec{v} \times \vec{B}$, were too high, and were in some cases higher than the applied currents from the magnetic nozzle. We realized we needed to minimize the current developed by $\vec{v} \times \vec{B}$. Physically, the way this term is prevented from increasing without bound is collisions; as $\vec{v} \times \vec{B}$ is increased particle collide with other plasma particles more, reducing their motion and increasing heating. Therefore, this effect is captured with the resistive term $-\frac{1}{\sigma} \vec{j}$ in Eq. (3.61). In earlier versions of the code this term was not as large as it should have been, or was too low and violating energy conservation or something, so we put a current limiter in SPFMax, where currents are limited using the following equation

$$j_{\text{max}} = 1000 \frac{B}{\mu_0 \lambda_{\text{plasma}}}$$  \hspace{1cm} (3.65)

where $\lambda_{\text{plasma}}$ is defined as in Eq. (3.64). Additionally, once $j_{\text{max}}$ is calculated, $\vec{j}$ was limited so each component does not exceed $j_{\text{max}}$. This ensures the current does not increase without bound and the plasma self-field does not surpass the applied field.
This method of limiting $\tilde{j}$ is included in the current version of the code, but does not really affect results; this is because the resistive term was fixed and $\tilde{j}$ is limited using energy conservation.

### 3.2.1.6 Implementation of nozzles

For a given nozzle, SPFMax calculates the geometry of the current windings and the current at a given time step. Currents in nozzles can vary over time according to a user-specified function. However, SPFMax breaks the nozzle geometry into a series of segments, each of length $\Delta \lambda$ (for a current winding of total length $\lambda$). The magnetic field at a point away from the nozzle is calculated using Biot-Savart.

$$
\tilde{B} = \frac{\mu_0}{4\pi} \sum_l \frac{I d\lambda \times \tilde{r}}{r^3}
$$

(3.66)

However, this means that the nozzle windings must be broken into a sufficient number of segments of length $d\lambda$. We have found that, for most problems, using 20-30 segments is sufficient.

### 3.2.1.7 Summary

The fluid equation and electromagnetic field equations solvers of SPFMax are combined to solve for fluid properties and electromagnetic field properties at each time-step. SPFMax is run on a Windows 10 Enterprise machine with an Intel(R)Xeon(R) E5-1630 v4 CPU running at 3.70 GHz, and an NVIDIA Quatro M5000 graphics card. SPFMax is implemented in MATLAB with the GPU computing tool-
Using GPU computing allows for simulations to complete much faster than CPU computing.

3.2.2 Comparison of model with prior work

For the magnetic nozzle subsystem design, it is necessary to validate SPFMax through comparison with prior work. To this end, we compare SPFMax results primarily with two computational test cases; a solenoidal nozzle test case and a axial nozzle test case. We also compare an earlier version SPFMax with an analytical model and an experiment to ensure we were on the right track with model development.

3.2.2.1 Solenoidal nozzle comparison

To validate SPFMax against computational data, we attempted to reproduce the results in Ref. [9] for their introductory case. The case has a current of 3.57 MA, and is separated from the plasma 1 m. Other relevant simulation parameters are given in Table 3.3.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Plasma energy ($E_p$)</td>
<td>4 MJ</td>
</tr>
<tr>
<td>Plasma mass ($m$)</td>
<td>110 mg</td>
</tr>
<tr>
<td>Plasma molecular weight (MW)</td>
<td>197 amu</td>
</tr>
<tr>
<td>Plasma initial radius ($r_{p0}$)</td>
<td>0.3 m</td>
</tr>
<tr>
<td>Plasma composition</td>
<td>Gold (Au)</td>
</tr>
<tr>
<td>Electron temperature ($T_e$)</td>
<td>0 eV</td>
</tr>
<tr>
<td>Coil radius</td>
<td>1.0 m</td>
</tr>
<tr>
<td>Coil current ($I$)</td>
<td>3.57 MA</td>
</tr>
<tr>
<td>Coil axial position</td>
<td>$z=-1.0$ m</td>
</tr>
<tr>
<td>Number of macro-particles</td>
<td>100,000</td>
</tr>
</tbody>
</table>

Table 3.3: Plasma Input Parameters from Ref. [9]
The magnetic nozzle is a 1 turn solenoidal coil, wound along the +z axis. The plasma starts at (0,0,0) in the simulation domain. For a schematic representation of the plasma and nozzle initial positions, see Fig. 1 of Ref. [9].

To match the simulation results, we had to slightly modify the input conditions. Firstly, SPFMax gave more consistent results when we specified an initial temperature for the plasma than an initial energy. Using PROPACEOS for Gold, an initial plasma energy of 4 MJ corresponds to a temperature of about 100 million Kelvin (8.6 keV). Secondly, due to hardware constraints, using 100,000 macroparticles is infeasible. Instead, we vary the number of SPH macroparticles between 3,000-40,000 and compare the results with Ref. [9]. Lastly, SPFMax takes the electrons and ions to be at the same temperature, so instead of an electron temperature of 0 eV, we used an electron temperature of 100 million K.

For the nozzle, SPFMax can not have the coil loop back in on itself perfectly like in Ref. [9], so we used a coil thickness of 1 cm, to make the thickness negligible compared to the radius. Additionally, we broke the coil into 30 segments for magnetic field calculation. Results from this comparison are presented in Section 4.2.1.2.

### 3.2.2.2 Axial nozzle comparison

For the axial nozzle test case, we decided to use the nozzle from Fig. 3 of Ref. [46] (see Figure 2.11). This nozzle featured 20 turns (gold loops in figure), had a major radius of 0.1 m, a height of 0.04 m [46], and each turn of the nozzle had 10 MA. We only modeled inclined part of the strut (that went in the +r direction) in
Figure 3.5: Axial nozzle test case with plasma (pink) and pusher plate (green). Reproduction of Fig. 3d from Ref. [46]

SPFMax, and specified currents run in the +r direction as well. Cassibry et al. also modeled the nozzle this way.

The nozzle also has a pusher plate (outlined in green in Figure 3.5) along with the plasma (in pink). The pusher plate is 0.02 m in radius and approximately 1/3 mm thick. Both the plasma and the plate are represented by SPH particles - when an SPH particle gets within the simulation specified 'wall distance' of a pusher plate particle, the z-component of its velocity is reversed (it bounces off). This wall distance is usually set to the initial plasma particle compact support distance $h$. The plasma starts off 0.001 mm in radius, 0.01 m in length, cylindrical in shape, with a mass of 2.5 mg and an initial temperature of 1 keV temperature [46]. The plasma is a 50%-50% mixture of Deuterium and Tritium, giving it an effective molecular weight of 2.5 g/mol [46]. $\gamma$ and $\sigma$ for the plasma are both assumed to be constant, at 1.3 and $10^5$ Si/m respectively.
We used the data provided by Cassibry et al. when available, but when not we asked the authors directly. Results from this SPFMax simulation are compared with some results from Ref. [46], but also results obtained by T. Morita who used an updated version of the 3D Hybrid PIC code from Ref. [5,6,9,23–29,31–44]. Their code implements the pusher plate differently; when SPH particles enter the plate region, their $z$ component of velocity ($v_z$) is reversed. They do not use specific wall particles.

The original input file, used to generate the results in Ref. [46], used a relatively few number of particles in both the plasma and the pusher plate (about 1,000 for each). To increase accuracy, we elected to use 1,271 particles in the plasma and 13,025 in the pusher plate. More particles in the pusher plate reduces the chance a plasma particle fails to get within the wall distance and sneaks through the pusher plate (which is non-physical). But, more particles in the plate is also computationally expensive. We found 1,271 plasma particles and 13,025 pusher plate particles to offer a good trade; we tried other combinations but these did not not work as well.

### 3.2.2.3 Analytical model

For the analytical test case, we used an earlier version of SPFMax that featured an Ohmic heating term ($\frac{1}{\sigma_p} j^2$) in the energy conservation equation (Eq. (3.58)). In later tests, we found that inclusion of this term resulted in insufficient plasma deflection, which is why it was taken out. Nevertheless, comparison with this analytical model allowed us to ensure we were on the right track.
The analytical model we used to ensure code validity at an early stage of model development is given in Figure 3.6. The model assumes a plasma of finite conductivity and radius expanding in a uniform magnetic field.

For the test case, \( R_0 = 1.0 \) cm, \( L_0 = 10.0 \) cm, \( v_0 = 50.0 \) km/s, and \( B_0 = 400 \) T (see Figure 3.7). To simplify the model, it is instructive to assume the plasma is uniformly comprised of tungsten, at an initial density of 1000 kg/m\(^3\) (yields a total
plasma mass of 31.4 g). In this case, if one assumes the plasma motion to be entirely in the radial direction, then the maximum radius the plasma can expand is where the gasdynamic pressure equals the magnetic field pressure. This pressure-focused approach is stated in Eq. (3.67).

$$\frac{B_0^2}{2\mu_0} = \frac{1}{2}\rho v^2$$  \hspace{1cm} (3.67)

If one assumes the applied magnetic field strength does not change over the course of the plasma expansion, and the initial expansion velocity does not change over the course of the simulation, then one can solve for the plasma density at the point of maximal expansion using Eq. (3.67). Further, if one assumes the length of the plasma column does not change \( (L(t) = L_0) \), then one can determine the maximum expansion radius using Eq. (3.68).

$$\rho = \frac{m}{\pi r_{max}^2 \lambda_0}$$  \hspace{1cm} (3.68)

In Eq. (3.68), \( m \) is the mass of the plasma column (31.4 g), and \( \lambda_0 \) is the initial length of the plasma column (10 cm).

We ensure the assumptions are met for Eq. (3.67) and Eq. (3.68) as follows. To keep the magnetic field strength roughly constant over the course of the simulation, the SPFMax test cases use a solenoidal to generate the magnetic field, and ensures the plasma stay entirely in the solenoid; this keeps field strength at a constant 400 T. To ensure the length of the plasma column does not change, plasma motion is
Figure 3.8: Results from comparison of SPFMax with analytical model restricted to the xy plane (this is a switch turned on in SPFMax). In the results, we found that the expansion velocity does not appreciably change, until the plasma starts being deflected, thus verifying this assumption as well.

As the relevant assumptions are met, using Eq. (3.67) and Eq. (3.68) with the earlier mentioned plasma parameters results in an $R_{\text{max}}$ of 4.4 cm (note that this disagrees with the energy-focused approach in [88] which finds an $R_{\text{max}}$ of 5.1 m). This should be reflected in the SPFMax results.
In the SPFMax simulation, the plasma starts with the initial dimensions specified previously, and a 20 turn solenoid with a length of 50.0 cm, radius of 15 cm, and current of 8 MA is used to generate the 400 T field. The initial setup with the plasma and solenoid is given in Figure 3.8a. Results are given in Figure 3.8b-Figure 3.8d.

As shown in the results in Figure 3.8, the plasma starts at its initial position, and expands to the maximum extent in the nozzle, before collapsing in on itself, and then expanding again. These results are consistent with what is expected. Additionally, the plasma reaches its maximal radius of 5.0 cm at 1.5 µs, giving an average expansion velocity of 33 km/s (fairly close to the initial value of 50.0 km/s). More importantly, the maximal expansion radius from SPFMax (approximately 5.0 cm) is comparable to what was predicted using the analytical model, over-predicting by only 14%. Additionally, the maximal expansion radius is fairly stable over the course of the simulation, as shown in Figure 3.8d where the plasma yet again reaches its expansion radius of 5 cm.

Overall Figure 3.8 shows fairly close agreement between SPFMax and the analytical model. This gave us confidence that we were on the right track with developing SPFMax, and lead to comparison with an experiment.

3.2.2.4 Experiment

For our experiment, instead of using prior work, we used a study conducted at UAH by White et al. [7,8]. Instead of a laser-ablation plasma, like in H. Nakashima’s work, we used a plasma-jet as this has properties (namely, lower density [8]) that make it more applicable to a wider-variety of pulsed nuclear plasmas.
The experiment is similar to the one in Ref. [33], and involves a plasma interacting with a permanent magnet. The permanent magnet functions as a simpler version of a magnetic nozzle. A schematic of the experimental setup is given in Figure 3.9.

The tube at (0,150,0) mm is the plasma jet source. The magnet is placed in multiple configurations. Configuration 1 has the magnet in the same plane as the plasma jet (y=150 mm), but the magnet is 1 inch in the +x-direction and 1 inch in the +z-direction [7], and so the magnet center is at (25.4,150,25.4) mm. Because the magnet and jet are co-planar, in subsequent figures the entire coordinate system is moved -y 150 mm to put the origin at the plasma source. Configuration 1 also has the North pole of the magnet aligned in the +z-direction. The next configuration has the magnet in the same place as configuration 1, but the poles are reversed, so the North pole faces in the -z direction. The last configuration (Configuration 2 in Figure 3.9) has the magnet located 2 inch downstream from the plasma source and North pole of
the magnet facing in the -x direction. This last one has the magnetic field oriented parallel to the motion of the plasma, whereas the first two have the magnetic field oriented perpendicular to the motion of the plasma. In all cases the magnet is 2 inch long, 2 inch wide, and 0.5 inch deep.

The main difference between the experimental setup and SPFMax is that SPFMax approximates permanent magnets as solenoids. For now, we have elected to use a solenoid with the same dimensions as the magnet (2x2x1 inch), 50 turns, an a current of 250 A. This gives us a field strength of 500 G 1 inch off the surface of the magnet, which matches with experiment [7]. To match the different magnet configurations, the current in the solenoid is reversed, or the solenoid is rotated. For example, to match the second magnet configuration (North pole in -z direction), the current is reversed, but to match the last configuration (North pole in -x direction), the solenoid is rotated 90° counter-clockwise along the y-axis. Also, in SPFMax, the center of the magnet is moved 2 inch in the +y-direction for the third (last) magnet configuration to space out the plasma and magnet more.

A secondary difference between the experimental setup and the SPFMax simulation is the difference in jet expansion speeds. In SPFMax the initial velocity of the jet is 10 km/s, which is higher than the ion thermal velocity. This is to prevent appreciable thermal spreading of the jet, and does not have any effect on deflection.

For these comparisons, we used the version of SPFMax with the Ohmic heating term added (see the previous section) but this should not change the results significantly.
Figure 3.10: Reproduction of Figure 5 in Ref. [8]. a) shows plasma jet trajectory without permanent magnet, b) shows trajectory with magnet in first configuration (North pole +z), c) shows trajectory with magnet in second configuration (North pole -z).

The results of the setup in Figure 3.9 show significant deflection of the electrons of the plasma jet in the presence of the external magnetic field - the magnetic field is so strong that the electrons are magnetized and the plasma jet turns around completely [7] as shown in Figure 3.10. Here, the electrons collide with neutral particles ejecting photons, giving rise to the bright spots in Figure 3.10. The electrons are magnetized because the value of electron Hall parameter is $10^5$, which is much greater than 1. However, the value of the ion Hall parameter is approximately 10, which is not significantly more than 1 [8], indicating the ions are unmagnetized. This means they will not deflect in the presence of the field like the electrons will. This is shown in the plots of ion current (Figure 3.11 and Figure 3.12)

In all cases, the ions do not appreciably deflect. In Figure 3.11b-c, it appears the presence of a magnetic field leads to the plasma column being somewhat more compact. In Figure 3.12b it appears that the magnetic field causes some diffusion of the plasma in the -y direction of the graph. Again, this is consistent with the results expected based off the ion Hall parameter, which is not that much greater than 1.
Figure 3.11: Reproduction of Figure 6 in Ref. [8].  
a) shows ion current distribution in the case of no magnet,  
b) shows in the case of the magnet in the first configuration (North pole +z),  
c) shows in the case of the magnet in the second configuration (North pole -z)

Figure 3.12: Reproduction of Figure 7 in Ref. [8].  
a) shows ion current distribution in the case of no magnet,  
b) shows in the case of the magnet in the third configuration (North pole -x)
Figure 3.13: Results of SPFMax simulation for a) first magnet configuration (North pole +z), b) second magnet configuration (North pole -z) c) third magnet configuration (North pole -y). Plasma jet is black with yellow velocity vectors while magnet is yellow.

Since SPFMax only simulates the motion of the ions, we are only able to verify SPFMax based off ion motion. Therefore, SPFMax should show no deflection. This is the case for the 3 magnet configurations Figure 3.13.

As shown in Figure 3.13a-c, the black plasma jet fails to deflect in the presence of the magnetic field produced by the solenoid. In all cases, the plasma jet ends downstream of where it started, and its path has not deviated at all. While in some cases, the plasma is allowed to travel farther than in Figure 3.11 or Figure 3.12 (25.4 mm vs. 120 mm) the end result, of no deflection, is the same. In the experiment, because the ions are relatively low temperature, they have many collisions with the background neutral gas, and so they cannot deflect appreciably before hitting a neutral particle. While not requiring a background gas, SPFMax can reproduce these results just the same.

However, if the ions were significantly hotter, such that they had conductivity comparable to that of copper (based on Spitzer resistivity), they would have fewer
Figure 3.14: SPFMax results in the case of high-conductivity plasma \textit{Conductivity of} $1 \times 10^7$ Si/m. \textit{Magnet in gold, magnetic field lines in black, plasma particles in white}. 

Collisions with the neutral gas, giving a higher hall parameter, and resulting in deflection. In this case, SPFMax is able to produce deflection, as shown in Figure 3.14.

In this simulation, the coordinate system is slightly altered from Figure 3.9; such that the x and z axes are swapped. The magnet is also slightly altered from the usual third magnet configuration, and is moved +1 inch out of line with the plasma. Here, the magnetic is the gold coils, and several slices through the 3D geometry as shown that have the field strength and magnetic field lines plotted. However, these results show, that in the case of a high-conductivity plasma, SPFMax is able to reproduce expected behavior; the plasma deflected (appreciably), and the plasma particles even follow the magnetic field lines. This is consistent with the 'frozen-in-flow' assumption, whereby at high plasma conductivity, the plasma motion tracks with magnetic field lines [89]. The plasma is 'frozen-in' to the magnetic field lines. So, for deflection the plasma needs to have high conductivity.
However, higher plasma conductivities usually track with higher plasma temperatures. So, for the nozzle trade study, these results tell us plasma needs to be sufficiently hot.

### 3.2.3 Axial Nozzle study

After comparing SPFMax results with analytical models, computational models, and experiments, we wish to look at the plasma behavior in an axial nozzle for a baseline case. For the nozzle study, we considered plasmas composed of 0.5 kg Lithium and 0.1 kg Tungsten (Uranium stimulant), giving an effective molecular weight of 8 g/mol. This molecular weight is in between the low molecular weight of a fusion plasma and the high molecular weight of a fission plasma. Both materials start with an initial temperature of 69.3 eV. This is the resulting temperature from assuming the Tungsten starts at 1 keV and thermally equilibrates with the Lithium.

We use a constant conductivity of $10^5$ Si/m, an average value between the initial and final values, to reduce computational overhead. The plasma will be cylindrical in shape, with a radius of 3.6 cm and a length of 7.0 cm, and will start 1 m away from the vertex of the nozzle (see Figure 3.15). The plasma will be composed of 1,238 SPH particles (see Section 4.2.1.1).

The nozzle is composed of 32 struts, with an axial length (along the z-axis) of 3.5 m, and will have a major radius of 3.5 m for compatibility with current heavy-launch vehicle architectures [90]. For the baseline case, each nozzle strut has a current of 15 MA. We look at excursions from this case by raising and lowering this current, and looking at the resulting effects on performance. Note that this axial nozzle does
(a) Isometric view of nozzle and plasma (b) XY plane view of nozzle and plasma used in nozzle study. Nozzle in gold, used in nozzle study. Nozzle in gold, plasma in black

(c) XZ plane view of nozzle and plasma used in nozzle study. Nozzle in gold, plasma in black

Figure 3.15: Axial magnetic nozzle used in nozzle study. a) Isometric view, b) XY view, c) XZ view

not have a pusher plate; this is to reduce heat loads on the vehicle and therefore cooling system requirements. Additionally, radiative cooling is turned off from these runs.
Figure 3.16: Color plot of magnetic field inside and outside axial nozzle baseline case
CHAPTER 4

RESULTS

4.1 Power Generation System

For the power generation system, we will first present the trade study results, then comparison with prior work; the next section presents these in opposite order. Here, we present the comparison last because data from prior work is high-level.

4.1.1 Trade study results

To recap, for the power generation system, the design parameters we considered are: the number of turns on the stator coil $N_{\text{stat}}$, the initial current in the stator coil $I_{10}$, the length scale of the stator coil $r_{\text{stat}}$, the number of turns on the primary side of the transformer $N_{T1}$, the number of turns on the primary side of the transformer $N_{T2}$, and the energy storage system capacitance $C$. System performance is determined by the energy in the energy storage system capacitor when it is fully charged $E_{\text{charged}}$. We found that $N_{T2}$ and $r_{\text{stat}}$ scale system performance differently than the other design variables, and so consider them first.
4.1.1.1 Characterizing the number of turns on the secondary side of the transformer

Beginning with $N_{T_2}$, we found there does exist an optimal value of $N_{T_2}$, that maximizes $E_{\text{charged}}$, regardless of the values of the other 5 inputs ($N_{\text{stat}}$, $r_{\text{stat}}$, $I_{\text{o}}$, $N_{T_1}$, and $C$). As shown in Figure 4.1, the ideal $N_{T_2}$ is one (this corresponds to a single turn on a transformer secondary) for plasmas with sufficiently initial high expansion velocity. The expansion velocity in Figure 4.1 is changed by modifying the initial plasma temperature (which is related to the initial expansion speed via the expression in Table 3.1). As shown in Figure 4.1, for a speed of 177 km/s, the ideal $N_{T_2}$ is fairly flat from 1 turn to 1.75 turns, with a bump around 1.75 turns. The curve is flat in this region because the plasma is impinging on the stator. Because of how the simulation handles this impingement numerically, the curve is flat. The impingement stops around 1.75 turns, giving the bump. For $v_{p_0} = 79 \text{km/s}$ and $v_{p_0} = 56 \text{km/s}$ the optimal $N_{T_2}$ increases to 1.75 and then 2 turns. Therefore, the ideal number of...
secondary turns is fairly insensitive to expansion speed and choosing $N_{T_2} = 1$ for the study is justified. Note that for our PuFF armature the temperature $T_0$ will be 1 keV, which corresponds to a speed of 177km/s, the highest of the velocities considered in Figure 4.1.

A similar trend emerges when varying $N_{stat}$ (shown in Figure 4.2). The optimal $N_{T_2}$ decreases to about 3 turns for low values of $N_{stat}$, but swiftly approaches 1 for higher values. Additionally, increasing $N_{stat}$ decreased the maximum output energy; this implies that lower $N_{stat}$ is better, and ideal $N_{T_2}$ is between 1 and 3.
An interesting trend emerges when varying $N_{T_1}$ as shown in Figure 4.3. The optimal $N_{T_2}$, for maximal charged energy in the capacitor, stays constant at about 1 for all values of $N_{T_1}$, but the maximal energy trend is different. Changing $N_{T_1}$ changes the maximum amount of energy stored in the capacitors (the peak), but the peak increases for $N_{T_1}$ from 1 to 10, but then decreases for $N_{T_1}$ from 10 to 1000. This implies an ideal $N_{T_1}$ of about 10. The trend of $N_{T_2} = 1$ for highest energy stored in the electrical power storage system holds true for varying $C$, $I_{10}$, and $r_{stat}$ as well.

In summary, to design the power generation system to maximize output energy, we should minimize the number of turns on the secondary windings of the transformer. This makes sense because the power input into the transformer, from the stator, is often higher voltage than the initial seed current. Therefore, the number of turns on the secondary should be minimized to step down the voltage to the relatively lower voltage capacitor. For these reasons, in subsequent analyses, $N_{T_2}$ will be set to 1.
4.1.1.2 Characterizing device length scale

As will be shown, there is an optimal value of \( r_{\text{stat}} \) that depends on the other design parameters, but in most cases is higher than what could be launched by current heavy-life rocket architectures (SLS Block 2B). Additionally, the trends determined by studying \( r_{\text{stat}} \) can be applied to other variables as well.

Figure 4.4 illustrates that increasing the initial stator current up to 100 kA does not change the optimal stator radius, but it does increase the maximum output energy. At 1,000 kA, the optimum shifts leftward. Additionally, in Figure 4.4a, as

\[ E_{\text{charged}} = \begin{cases} \text{constant} & \text{for } I_0 = 10 \text{ kA} \\ \text{increasing} & \text{for } I_0 = 100 \text{ kA} \\ \text{decreasing} & \text{for } I_0 = 1,000 \text{ kA} \end{cases} \]

\[ r_{\text{stat}} = \begin{cases} \text{constant} & \text{for } C = 1 \text{ \( \mu \)F} \\ \text{decreasing} & \text{for } C = 10 \text{ \( \mu \)F} \\ \text{increasing} & \text{for } C = 100 \text{ \( \mu \)F} \\ \text{decreasing} & \text{for } C = 1,000 \text{ \( \mu \)F} \\ \text{decreasing} & \text{for } C = 10,000 \text{ \( \mu \)F} \end{cases} \]
well as Figure 4.4b-d, the graphs show a step drop or rise, or discontinuous peak in $E_{\text{charged}}$; this is due to a change in operating regimes. In one regime, plasma impinges on the device walls, and in the other regime, it does not.

In Figure 4.4b, there is a similar trend to Figure 4.3, where there is an optimal $N_{T_1}$; the maximum output energy is around 10 turns. Additionally, the optimal $r_{stat}$ does shift somewhat with changing values of $N_{T_1}$; the optimal $r_{stat}$ increases from $N_{T_1} = 1$ to $N_{T_1} = 10$, but then decreases from $N_{T_1} = 10$ to $N_{T_1} = 1000$. The $r_{stat}$ that results in the highest energy in the capacitor looks to be around 10, and is for $N_{T_1} = 10$.

Figure 4.4c is similar to Figure 4.4b; higher $C$ results in higher $E_{\text{charged}}$, until $C = 100 \mu F$ is reached. The $r_{stat}$ that results in the highest energy in the capacitor looks to be around 7, and is for $C = 100 \mu F$. The shifting optimum in Figure 4.4c is similar to Figure 4.4d.

In Figure 4.4d it is shown that the optimal $r_{stat}$ shifts much like for the variation with $N_{T_1}$ in Figure 4.4b; the optimum $r_{stat}$ increases for $N_{stat}$ from 1-10 but decreases for $N_{stat}$ from 10-1000. The $r_{stat}$ that results in the highest energy in the capacitor looks to be around 7, and is for $N_{T_1} = 10$. In this way the results of Figure 4.4c corroborate the results of Figure 4.4b.

The desired values for $r_{stat}$ shifts around from 7-10. However, the maximal usable payload fairing diameter for current heavy-lift rockets is around 7.5m [90]. To be conservative, we take this diameter to be 7m, giving the upper bound of $r_{stat}$ to be 3.5m. Therefore, in subsequent investigations, we use $r_{stat} = 3.5 m$. 

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4.1.1.3 Characterizing remaining design parameters

A 4-dimensional trade study was undertaken to characterize the relationships between the other four parameters. This study traded $N_{stat}$, $C$, $I_{1a}$, and $N_{T1}$ to determine how each would affect the charged energy in the capacitor. The other two parameters, $N_{T2}$ and $r_{stat}$, were set to 1 turn and 3.5 m respectively. Results from this study are presented in non-dimensional form using the non-dimensional parameters developed in Section 3.1.6. Results are presented in Figure 4.5.

As shown in Figure 4.5a-e the curves are completely coincident. This validates the non-dimensionalization. The non-dimensionalization also reduces the trade space from four parameters ($N_{stat}$, $I_{1a}$, $N_{T1}$, $C$) to three ($\pi_1$, $\pi_2$, $\pi_3$). $C$ is included in $\pi_3$. The trade space is reduced further to only two parameters ($\pi_1$, $\pi_3$) for $\pi_2 < 0.01$.

It seems that there is a $\pi_1$ that maximizes $\eta$ (and thereby output energy) for each value of $C$ - this value seems to be around $\pi_1 = 1$ for all values of $C$. In the context of the power generation system, $\pi_1 = 1$, corresponds to $L_{stat} = L_1$. This means that, to maximize $\eta$, the inductance of the stator and the primary side of the transformer must be essentially identical. The literature refers to an "impedance-matching transformer" [64], and these results indicate that for optimal system design, the impedance of the primary side of the transformer must match the impedance of the stator to the impedance of its own primary side; the two impedances go hand in hand. This result clarifies the statement in the literature.

For the relationship between $\pi_2$ and $\eta$ it seems that for $\pi_2 > 0.01$ there is a moderate decline in $\eta$ until $\pi_2 = 1$, after which there is a sharp decline in $\eta$. This
Figure 4.5: Ratio of output energy to stator energy ($\eta$) vs. ratio of primary inductance to stator inductance ($\pi_1$) vs. ratio $\pi_2$ for a) 1 $\mu$F capacitance b) 10 $\mu$F capacitance c) 100 $\mu$F capacitance d) 1,000 $\mu$F capacitance e) 10,000 $\mu$F capacitance. f) is a common legend for sub-figures a-e.
is because, for $\pi_2 > 1$ the stator has a higher initial energy than the plasma, so instead of energy naturally wanting to transfer from the stator to the plasma, energy wants to transfer the opposite way (from the plasma to the stator). Additionally, for $\pi_2 < 0.01$, the results suggest performance is linear with $\pi_2$ - meaning the higher the initial energy of the stator, the more performing ($E_{charged}$) the system is. This gives scaling for the system - the initial energy in the stator coils linearly increases the output energy, until the initial energy in the stator coils goes past 1% the initial plasma energy. Then, increasing the initial energy in the stator coils non-linearly (more gradually) increases the output energy, until the initial energy in the stator coils exceeds the plasma energy; then increasing energy in the stator coils decreases the output energy. Therefore, for ideal system design the initial stator energy should be 1% the initial plasma energy.

4.1.2 Comparison with prior work

Now that we have determined system operation and scaling we would like to determine the mass of our system to compare with prior work. For this, we use a simple bottoms-up mass estimation approach given in Appendix A.

The approach gives two constraints, the magnetic pressure constraints and the geometric constraints, that must be satisfied for the design to be feasible. For the selected wire material (pure copper), we need to modify the system design found in the previous section. In the previous section, careful inspection of Figure 4.5 reveals that performance is maximized for a system design that uses (in addition to the parameters in Table 3.2), $N_{T_2} = 1$, $r_{stat} = 3.5$ m, $I_{1a} = 100$ kA, $N_{stat} = 100$,
$N_{T_1} = 24$, and $C = 100 \ \mu F$. However, these have be modified to meet the two constraints. The system design that meets these constraints is given in Table 4.1.

### Table 4.1: Model parameters for mass estimation

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m$</td>
<td>600 grams</td>
<td>$N_{stat}$</td>
<td>100</td>
</tr>
<tr>
<td>$MW$</td>
<td>8 grams/mol</td>
<td>$I_{10}$</td>
<td>400 kA</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>1.3</td>
<td>$r_{stat}$</td>
<td>3.75 m</td>
</tr>
<tr>
<td>$r_{p_0}$</td>
<td>1 cm</td>
<td>$\mu_r$</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$k$</td>
<td>0.9</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\lambda_{T_1}$</td>
<td>0.5 m</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$r_T$</td>
<td>0.1 m</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\lambda_{T_2}$</td>
<td>0.1 m</td>
</tr>
<tr>
<td>$N_{T_1}$</td>
<td>15</td>
<td>$l_1$</td>
<td>10 nH</td>
</tr>
<tr>
<td>$N_{T_2}$</td>
<td>1</td>
<td>$l_2$</td>
<td>100 nH</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$C$</td>
<td>100 $\mu F$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$R_1$</td>
<td>0 $\Omega$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$R_2$</td>
<td>0 $\Omega$</td>
</tr>
</tbody>
</table>

Primarily, $I_{10}$ was increased to 400 kA, $r_{stat}$ was increased to the maximal SLS payload envelope, $N_{T_1}$ was lowered to 15 turns, and $\lambda_{T_1}$ was increased to 0.5m. This new design results in a sharp decline of $E_{charged}$ to 1.2 MJ (compare with 40 MJ found with the previous design). However, this realistic design allows us to vary the wire diameter, and estimate the mass of the power generation system.

The results are plotted in Figure 4.6. Component resistances calculated as part of this trade study were relatively low on the secondary side (around 100 $\mu \Omega$), while fairly high on the primary side (around 50 m$\Omega$). However, these line resistances were not found to affect the final energy on the capacitors ($E_{charged}$) that much. It seems that resistances on the primary side of the circuit do not affect the final capacitor energy appreciably (more than 1 %), while high resistances on the secondary side do
Figure 4.6: Wire width ($D_{wi}$) vs. Component total mass ($m_{wi} + m_{radiator}$) for power generation system with parameters in Table 4.1.

Figure 4.6 illustrates that the lowest-mass design uses 3 cm diameter wires for the stator, primary transmission lines, and primary transformer windings, and 4 cm diameter wires for the secondary transformer windings and transmission lines. This design results in a system mass of 35 t (where 1 t = 1000 kg), which results in a specific energy of about 35 J/kg, which is 33% of the 2003 HOPE vehicle, and 100x smaller than the 2010 HOPE vehicle. Looking at specific power, given the system pulse time of 1 s, the system is able to produce 1.2 MW, and for 45 mT (10 mT for capacitors [14]), this results in a total system specific mass of about 37.5 kg/kW, which is two orders of magnitude worse than the VISTA vehicle [14] and one order of

affect $E_{charged}$, but in this case the secondary resistances are low enough to not affect $E_{charged}$ greatly.

affect $E_{charged}$, but in this case the secondary resistances are low enough to not affect $E_{charged}$ greatly.
magnitude worse than the 2003 HOPE vehicle (it is four orders of magnitude worse than the 2010 HOPE vehicle).

While the design here has worse performance than previous work, the model here is significantly more detailed than the VISTA and HOPE work, and as endemic in such things, a more detailed design increases system mass. Also, meeting the minimum wire diameter constraint drives the system mass up; using a different material with a higher yield strength but still fairly high conductivity would reduce system mass. Using hollow transmission lines might also reduce system mass. Lastly, these results were generated with a 1-D plasma model. A more detailed (3-D) model of the plasma, that properly include transport processes, and considers the roles of drifts, turbulence, and kinetic effects, on transport is necessary to capture the plasma-magnetic field interaction and potentially reduce system mass. This model would illustrate novel ways to increase the time the plasma and stator interact; increasing the plasma-stator interaction time would reduce peak currents, thereby reducing system mass. However, based on the results of our simplified 1-D modeling, the complexity and cost of using a full 3-D model that includes kinetic and fluid effects may be justified to further explore plasma FCGs. This model would be synergistic with broader magnetic nozzle modeling. These issues can be explored in future works.

4.2 Magnetic Nozzle Subsystem

To start off with, we must ensure SPFMax solves the differential equations correctly; to this end we performed a convergence study. Those results will be presented first, followed by a performance comparison with the solenoidal nozzle to determine
the ideal particle resolution. Next, we will compare the results of SPFMax to prior work, focusing on results from the 3D hybrid PIC code. Lastly, using the particle resolution from the convergence study, we will present the results of the nozzle trade study.

4.2.1 Comparison with prior work

4.2.1.1 Convergence

To start with it is necessary to determine if the differential equations are implemented properly and the method converges. To assist with this, we define the error as follows

\[ \epsilon_N = \frac{1}{N} \left( \frac{K E_{zN} - K E_{zref}}{K E_{zref}} \right) \quad (4.1) \]

Here, \( N \) is the total number of particles in the simulation, \( K E_{zN} \) is the final kinetic energy in the z-direction of the plasma at resolution \( N \), and \( K E_{zref} \) is the final kinetic energy in the z-direction of the plasma at the reference value. We elected to use \( K E_z \) as our figure of merit for the convergence study because it is intrinsically linked to nozzle performance, in that higher \( K E_z \) results in higher nozzle performance. Additionally, for our convergence study, we decided to use the solenoidal nozzle test case (see Section 4.2.1.2), and while Ref. [9] does give a nozzle efficiency, because of our SPH method, we cannot calculate nozzle efficiency in the same way Nagamine and Nakashima do. However, Nagamine and Nakashima do list a final \( K E_z \), hence we use that. As shown in Fig. 8 of Ref. [9], the final kinetic energy in the z-direction is 2.5 Mega-J, therefore \( K E_{zref} = 2.5 \text{ MJ} \).
Results of the study are shown in Figure 4.7. They clearly show a trend of reduced error for increasing SPH particle resolution; the method seems to be working and the results are converging. However, if we define the relative error as follows

$$
\epsilon_{rel} = \frac{KE_{z_N} - KE_{z_{ref}}}{KE_{z_{ref}}} 
$$

where quantities are the same as before, we get the results in Figure 4.8. The figure shows that a particle resolution of about 1,000 SPH particles results in the smallest relative error. This is because 1,000 particles is not too small that important plasma features are unresolved, but not too great that numerical error takes over the solution. Therefore, in simulations, the plasma should have around 1,000 particles; this why we use 1,238 SPH particles for the axial nozzle study. Figure 4.8 also illustrates that the relative error does not vary greatly as the number of particles is increased (varying only by 0.02), meaning the method is stable.
Lastly, we look at the energies over time (Figure 4.9) for the medium resolution (1,151 particle) solenoidal case to illustrate plasma processes. The figure shows that the plasma starts out with 4.3 MJ of energy, almost all of it thermal energy; kinetic energy is small compared to thermal. As the plasma expands, its kinetic energy increases, its thermal energy decreases, and $E_{\text{current}}$ increases. At 3 $\mu$s, the kinetic energy starts to decrease slightly as the plasma enters the high field region of the
nozzle and begins to slow down. A diamagnetic current develops, increasing $E_{current}$ to its maximum value at 4.5 $\mu$s. The diamagnetic current creates a diamagnetic cavity that reduces the field inside the plasma by 10% - less than prior work [41]. After $E_{current}$ reaches its maximum, the plasma begins to detach from the nozzle, leading kinetic energy to an increase again and resulting in performance. Thermal energy increases slightly as the diamagnetic current has caused Ohmic heating in the plasma.

This cycle (plasma starts off high in thermal energy, expands increasing kinetic and $E_{current}$ while reducing thermal energy, enters the high-field region of the nozzle increasing $E_{current}$ and decreasing kinetic energy, and lastly, detaching from the nozzle and decreasing $E_{current}$ while increasing kinetic energy) is be present, in some form, in all cases. However, variations in the nozzle will cause variations in this cycle.

4.2.1.2 Solenoidal nozzle test case

For this test case, we wish to compare our results with the results from Nagamine and Nakashima. It is instructive to start with comparing the kinetic energies. Starting with Figure 4.10a-b, as shown in the figure, changing particle resolution seems to increase consistency at the expense of potential accuracy. While the high resolution case (8,186 particles) has the highest $KE_x$ and $KE_y$ until 2 $\mu$s (and therefore it is closest to Ref. [9]), it is overtaken by the medium-resolution case after this point. Additionally, it seems that the medium resolution case has the highest energy in all components at the conclusion of the simulation, hence its reduced relative error in the convergence study. This reduced relative error corresponds to somewhat higher
accuracy. However, this accuracy sacrificed for consistency; the medium resolution case (and the low resolution 179 particle case) both feature small oscillations in the energy (decreasing around 4 $\mu$s and then increasing again around 6 $\mu$s), but this is damped out in the high-resolution case. Therefore, while the medium resolution case is more accurate, the high-resolution case is more consistent.

Additionally, as shown in Figure 4.10a-b overall energy magnitudes compare well, but trends do not. Both $KE_x$ and $KE_y$ have a final value of 0.67 MJ in Ref. [9], but this is increased to 1 MJ in the SPFMax results. This is within 50%. However, with trends, Nagamine and Nakashima get higher $KE_x$ and $KE_y$ early on and show them decreasing, whereas the SPFMax results show a fairly consistent increase in both. This is most likely due to differences in the methodologies of simulations; initially the PIC code puts all energy in kinetic energy of the ions and calculates plasma motion from there, whereas SPFMax puts all energy in the thermal components, before transferring that to kinetic (as shown in Figure 4.9). What this means is, early on in the simulation, SPFMax has reduced kinetic energy (compared with Nagamine and Nakashima). This is rectified as the plasma expands and thermal energy decreases, but a discrepancy still remains. This is because while thermal energy is decreasing in the SPFMax simulation, kinetic energy is not. Therefore in the SPFMax energy curves, we do not see a decrease in $KE_x$ and $KE_y$ like in Ref. [9].

As shown in Figure 4.10c, for $KE_z$ trends are different. Nagamine and Nakashima show $KE_z$ decreasing from 0-2.5 $\mu$s, after which the plasma impacts the nozzle and the energy starts to increase. In contrast, SPFMax shows an increase in $KE_z$ as thermal energy is turned into kinetic energy, until 2.5 $\mu$s wherein the plasma impacts
Figure 4.10: Comparison of Ref. [9] energy results to SPFMax energy results for the solenoidal nozzle test case at various SPH particle resolutions.
the nozzle and $KE_z$ starts to decrease slightly. At around $5\mu s$ $KE_z$ starts to increase again, with all resolutions increasing similarly. This delay is most interesting, and might be due to how SPFMax keeps diamagnetic current energy separate (with $E_{\text{current}}$) as the decrease in kinetic energy corresponds to a peaking in $E_{\text{current}}$ (see Figure 4.9). Some kinetic energy is converted into diamagnetic current energy, which then has to be converted back into kinetic energy, causing the delay. The PIC code does not have a separate term for diamagnetic current energy; this energy is captured in the motion of the plasma particles themselves.

For quantitative comparison, SPFMax results are below the Nagamine and Nakashima $KE_z$ of 2.5 MJ. The highest is the medium-resolution case, which manages 1.2 MJ - around half. This is similar to $KE_x$ and $KE_y$, where SPFMax results are also within 50%. Putting the $KE_x$, $KE_y$, and $KE_z$ results together suggests quantitative SPFMax results are within 50% for the solenoidal nozzle case.

Next, we compare qualitative results (plasma behavior). Beginning with the 3D density plots, the Nagamine and Nakashima results are shown in Figure 4.11a. These results show a plasma that looks vaguely mushroom shaped with a large round head in front of $z > 0$ and a slim stem for $z < 0$. This is not reproduced in the SPFMax results, which feature a roughly spherical plasma, with maybe a bit of a tail for 1,151 and 8,186 particles (see Figure 4.11b-c). Nagamine and Nakashima give a density plot at $3\mu s$, right when the Nagamine and Nakashima plasma is getting deflected, but while $E_{\text{current}}$ is still increasing for SPFMax; therefore the discrepancy might be due to how the two codes consider energy.
Figure 4.11: Comparison of Ref. [9] and SPFMax solenoidal nozzle test case 3D density plots at different SPH particle resolutions.
Next, Nagamine and Nakashima provide 2D contour density plots of their plasma at different times and in different planes. They provide plots for $t = 0, 2, 4, 8\mu s$ in both the $XY$ and $XZ$ planes. Contour levels are not provided for any of these plots; therefore we generated our own plots at several contour levels to compare. The plots for $t = 0, 2$ are roughly circular and there is rough agreement between Ref. [9] and SPFMax.

Beginning with the $t = 4\mu s$ case, in the $XY$ plane, results from Nagamine and Nakashima (Figure 4.12a) show the contours that are mostly circular with $r = 1$ m. Contour plots from SPFMax at all particle resolutions reproduce these results exactly at the $10^{-5} \text{kg/m}^3$ contour level. The high-resolution particle case matches better at the $8 \times 10^{-6} \text{kg/m}^3$ contour level, however this is not a large discrepancy.

Changing to the $XZ$ plane, agreement is not as strong. Nagamine and Nakashima results clearly show a mushroom-shaped plasma starting to form, with a stem from $z < -0.5$, and a head with $r = 1.25$ m from $z > -0.5$ (see Figure 4.13a). Here and subsequently in this section, $r$ is defined in the cylindrical sense, i.e., $r = \sqrt{x^2 + y^2}$. The low and medium particle resolution cases, Figure 4.13b-c, do not show any mushroom formation; the plasma appears bullet shaped and does not have a long stem. The mushroom head also only extends to $r = 1$ m. The higher particle resolution case, Figure 4.13c, has a bit of a mushroom head and tail forming, with elliptical contours at the $8 \times 10^{-6} \text{kg/m}^3$ level extending back to $z = -1.5$ m. However, the stem is not as thin, being about 2 m wide (in Figure 4.13a the stem is 0.5 m wide). In Figure 4.13b-d, results match most well with Figure 4.13a at the $\rho = 10^{-5} \text{kg/m}^3$ contour level, like for the $XY$ plane.
Figure 4.12: Comparison of Ref. [9] and SPFMax solenoidal nozzle test case 2D contour density plots at $t = 4\mu s$ in the XY plane at different SPH particle resolutions.
At $t = 8\mu s$ in the XY plane, results from Nagamine and Nakashima show a plasma that has mostly exited this plane. As shown in Figure 4.14a, at $t = 8\mu s$, the mushroom head has gone forward and the tail pierces the XY plane here, leading to a reduction in the circular radius to $r = 0.5$ m. Results here for SPFMax match well if the contour levels in Figure 4.14a are taken to be at $\rho = 10^{-5} \text{ kg/m}^3$. There are some pretty large circles of $r = 2$ m at the $\rho = 10^{-6} \text{ kg/m}^3$ level, but none at $\rho = 10^{-5}$.
Figure 4.14: Comparison of Ref. [9] and SPFMax solenoidal nozzle test case 2D contour density plots at $t = 8 \mu s$ in the XY plane at different SPH particle resolutions.
Figure 4.15: Comparison of Ref. [9] and SPFMax solenoidal nozzle test case 2D contour density plots at \( t = 8 \mu s \) in the XZ plane at different SPH particle resolutions.

\( \text{kg/m}^3 \). Overall, SPFMax results show plasma that is significantly less dense here, consistent with Ref. [9].

In the XZ plane, we wish to reproduce the full stem-and-head mushroom shape in Figure 4.15a, with a head roughly \( r = 2 \text{ m} \), and a tail that stretches from \( z = 0 \) to \( z = -2.5 \text{ m} \). SPFMax is better able to reproduce this shape with increasing
particle resolution. The low resolution case in Figure 4.15b shows a vaguely spherical plasma, with high density in the region $z < -1.75 \, \text{m}$. The medium resolution case in Figure 4.15c shows the beginnings of a head at the $\rho = 10^{-6} \, \text{kg/m}^3$ contour level, with $r = 2 \, \text{m}$ (matching with Ref. [9]). The beginnings of a tail are also present in Figure 4.15c: a rather wide tail, 2 m in width, extending from $z = -3$ to $z = -2 \, \text{m}$. However, the high-resolution case in Figure 4.15d shows the best agreement, with a clear tail from $z = -0.5 \, \text{m}$ to $z = -2.5 \, \text{m}$, and a mushroom head just over $r = 2 \, \text{m}$ at the $\rho = 10^{-6} \, \text{kg/m}^3$ contour level. Note that taking these results at the $\rho = 10^{-6} \, \text{kg/m}^3$ contour level negates the results from the XY plane - curiously it seems that because SPFMax has the tail starting at a smaller $z$ value ($z = -0.5 \, \text{m}$) slices through the XY plane show a wider circle. Maybe running SPFMax on a higher-performing machine with a higher particle count will move the tail to $z = 0 \, \text{m}$ and show agreement with Figure 4.14a. Lastly, in Figure 4.15d there are regions of high density due to numerical discontinuities.

Taken together, these results suggest that SPFMax is able to reproduce qualitative results (plasma behavior) accurately. For higher particle resolutions, results match closer, but medium-particle resolutions match acceptably. Overall, quantitative performance is around half of what is found in the reference.

4.2.1.3 Axial nozzle test case

For the axial nozzle test case, we wish to compare SPFMax results with results from Cassibry et al. [46] as well as from T. Morita et al. [91], with the former generating their results using the latest version of the 3D hybrid PIC code.

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To begin with, we must analyze the energy graph to determine the plasma behavior in the nozzle. Behavior here is similar to before; the plasma starts with primarily thermal energy, which is converted to kinetic energy fairly quickly (0-0.1 µs as shown in Figure 4.16). \( E_{\text{current}} \) rises as the diamagnetic current increases, reducing kinetic energy (0.1-0.2 µs in Figure 4.16), but then falls as the plasma reaches its radius of maximum extent. At this point, \( E_{\text{current}} \) is converted to thermal energy (0.2-0.4 µs in Figure 4.16), but due to the axial nozzle geometry, there is still some plasma left in the nozzle after 0.5 µs; this continues to interact with the nozzle, leading to a slow increase in diamagnetic current and kinetic energy (0.5-2.0 µs in Figure 4.16). The rise in \( KE_z \) during this time creates a performance increase here.

Having analyzed the energy graph, we now wish to compare performance (specific impulse) with Cassibry et al. (Ref. [46]) and Morita et al. (Ref. [91]). As shown in Figure 4.17a, Cassibry et al. show a peak in specific impulse of 8,000 sec at 0.1 µs,

**Figure 4.16**: Energies over time for the axial nozzle test case. \( Total, E_{\text{current}}, Thermal, KE_x, KE_y, KE_z \)
before a slight drop and another peak of 9,000 sec. At this point, the older version of SPFMax crashes, leading to inaccurate results. Cassibry et al. also show an increase in specific impulse of 3,000-4,000 sec with the nozzle turned on (blue line minus orange line). In contrast, Morita et al. show a peak in specific impulse of 16,000 sec at 0.5 µs, before decreasing to a steady-state value of 13,500 sec, both shown by the dashed line in Figure 4.17b (the nozzle has a current of 10MAcoil - see Section 3.2.2.2). Finally,
there are the results from the current version of SPFMax in Figure 4.17c. These show an initial increase to a peak of around 15,000 sec at 0.1 \( \mu s \) (which is not better than the performance from the plate), before performance falls to worse than the plate at 0.4 \( \mu s \). However, after this point, performance improves to 16,000 sec, with some oscillation in the performance curves as particles enter and exit the region of the nozzle apex. 16,000 sec Isp is in line with the peak value from Morita et al., but more than their steady-state value of 13,500 sec. This is much more than the value from Cassibry et al., but this makes sense as the Cassibry et al. case was not able to run to full completion. Additionally, at 2.0 \( \mu s \), performance of SPFMax improves over the base pusher plate by over 9,000 sec, which is higher than Cassibry et al. and Morita et al.

It is quite curious that the current version SPFMax, bare plate results are so high; 15,000 sec is more than both Cassibry et al. and Morita et al. Additionally, the performance decays from 15,000 sec to 7,000 sec, which also does not match. However, the performance decay behavior in Figure 4.17c makes sense when one considers that the performance decays as the plasma expands beyond the bounds of the plate and begins to slip behind the plate, leading to reduced performance. This occurs after SPFMax in Cassibry et al. crashes, and is not in the Morita et al. results for some reason.

It is interesting that the quantitative results from the axial nozzle are more in line with the PIC code results (and much higher than the Cassibry et al. results) than the results from the solenoidal nozzle. We suppose it helps that the model used in Ref. [91] is newer, and might be more updated than the model in Ref. [9]. Regardless,
these results suggest that final SPFMax performance values are within 20%, but there is a discrepancy in behavior over time (like with the solenoidal case energy graphs).

The last area of comparison is in plasma motion. Cassibry et al. do not show plasma motion in the nozzle, but Morita et al. do. As shown in Figure 4.18a-c, first the plasma (black) starts in its initial position in the nozzle (red), right next to the pusher plate (green). It then begins to expand at $t = 0.12 \mu s$ (see Figure 4.18b), and thereafter, begins to interact with the nozzle. Some of the plasma even begins to slip behind the pusher plate (shown in Figure 4.18b). However, it seems to form two spiral vortexes near where it starts to interact with the nozzle (shown in the circled blue part of Figure 4.18c). These keep the plasma column tight and away from the nozzle. They might develop due to a high-frequency oscillation in the magnetic field Morita et al. report during their simulation.

As shown in Figure 4.19, the SPFMax results are initially similar and progress somewhat differently, but end in a similar place. Figure 4.19a-b are essentially the same as Figure 4.18a-b, this is partially because these images are early on in the simulation and at similar times. However, past this point plasma motion starts to change. More plasma particles get behind the nozzle in Figure 4.19c, leading to the drop in performance in Figure 4.17c and the drop in kinetic energy in Figure 4.16. These particles then get trapped behind the pusher plate, leading to recirculation in Figure 4.19d; it also appears some of the plasma is moving perpendicular to the plate in this figure. Eventually, enough collisions with the plate and other plasma particles, as well as natural thermal expansion, kick enough of the plasma particles away from the plate for performance to start of increase again (shown in Figure 4.19e).
Figure 4.18: Plasma motion in axial nozzle test case, T. Morita et al. results. Nozzle in red, plasma in black, pusher plate in green.
Also shown in Figure 4.19e is two plasma vortexes forming, similar to Figure 4.18c - however these vortexes are in the XZ plane as opposed to the XY plane. This is evidenced by the velocity vectors of particles near \( x = -0.05 \) m and \( x = 0.05 \) m being perpendicular to the plasma column. These vortexes continue in Figure 4.19f, curiously with the bulk of the plasma velocity vectors pointing in the +x direction, not in the +z direction (out the back of the nozzle) as expected. Therefore, the plasma might be forming a single vortex. Regardless, the formation of vortexes, even if they are spinning the wrong way, matches with the Morita et al. results, illustrating qualitative similarity between SPFMax and the literature.

In summary, current-version SPFMax results show good qualitative comparison but not as strong quantitative comparison with the 3D hybrid PIC code for the axial nozzle test case (much as with the solenoidal nozzle test case). The former is evidenced by the formation of vortexes in the plasma motion results, despite increased particles behind the nozzle. The latter is evidenced by the discrepancy in how performance evolves over time between the two simulations, but this is ameliorated by the similar peak values (16,000 sec Isp) between the two simulations. Having compared SPFMax results to prior work, and found acceptable comparison, we now turn our attention to the study results.

### 4.2.2 Axial nozzle study results

For the axial nozzle study, it is instructive to start with the base case, which has a per-strut current of 15 MA. We will then look at deviations from this case.
(a) Current version SPFMax axial nozzle test case plasma initial position.

(b) Current version SPFMax axial nozzle test case plasma at 0.06 $\mu s$.

(c) Current version SPFMax axial nozzle test case plasma at 0.14 $\mu s$.

(d) Current version SPFMax axial nozzle test case plasma at 0.25 $\mu s$.

(e) Current version SPFMax axial nozzle test case plasma at 0.5 $\mu s$.

(f) Current version SPFMax axial nozzle test case plasma at 1.25 $\mu s$.

Figure 4.19: Plasma motion in axial nozzle test case, SPFMax results. Nozzle in gold, plasma in black, plasma velocity in yellow arrows, pusher plate in black.
4.2.2.1 Axial nozzle base case

Firstly, as before, we need to start with the energy graph to establish what is happening in the nozzle. Shown in Figure 4.20 is a similar cycle of plasma behavior to the test cases. Initially the plasma starts of entirely with thermal energy. This decreases as kinetic energy increases \((t = 0 – 12\, \mu s)\), along with \(E_{\text{current}}\). Here as before kinetic energy increases faster than \(E_{\text{current}}\), before decreasing as \(E_{\text{current}}\) continues to increase. The diamagnetic current energy exceeds both thermal and kinetic energy at \(t = 13\, \mu s\), when the plasma reaches is maximal point of expansion and starts to be deflected by the nozzle, which does not happen in either of the test cases. The rise in \(E_{\text{current}}\) here corresponds to an increase in Ohmic heating in parts of the plasma, causing the thermal energy to rise \((t = 13\, \mu s – 27\, \mu s)\). The thermal energy and \(E_{\text{current}}\) stay roughly the same throughout the course of the simulation, unlike in the test cases where they decrease. This occurs because some plasma has been trapped in
(a) Axial nozzle plasma $t = 0\mu s$.

(b) Axial nozzle plasma $t = 11.55\mu s$.

(c) Axial nozzle plasma $t = 24.07\mu s$.

(d) Axial nozzle plasma $t = 49.01\mu s$.

(e) Axial nozzle plasma $t = 74.08\mu s$.

**Figure 4.21:** Axial nozzle base case plasma motion. *Nozzle in gold, plasma in black.*

the nozzle apex and stays there, continuing to interact with the nozzle and produce
diamagnetic current. However, $KE_z$ continues to increase (at the expense of $KE_y$
and $KE_x$) as some of the plasma expands out the top of the nozzle. $KE_z$ is where
we get our performance.
Next, we look at plasma motion to get a better idea of what is going on inside the nozzle. In Figure 4.21, we see a visualized sequence of events that supports Figure 4.20. The plasma starts off in the nozzle in Figure 4.21a high in thermal energy. It reaches its maximum point of expansion in Figure 4.21b increasing the diamagnetic current, and interacting with the nozzle (as evidenced by the particles that have their velocity reduced at the bottom of Figure 4.21b). Interestingly, the point of maximum expansion for an axial nozzle has some of the plasma outside the nozzle. This is because the magnetic field for an axial nozzle is higher outside the nozzle struts than inside. The plasma maximizes its diamagnetic current in Figure 4.21c and plasma in the nozzle apex (the bottom) begins to be deflected. However, some of this plasma gets trapped in the nozzle apex, as illustrated by the tight bunch of black particles in Figure 4.21d. As shown in Figure 4.21e this confined plasma either escapes out the top of the nozzle, or goes through the nozzle sides/bottom. The latter results in reduced performance.

To support our explanation of plasma motion, it is instructive to look that the pressure of the plasma over time. As before, initially in Figure 4.22a, the plasma is at high pressure. It expands, and its pressure largely decreases, as shown in Figure 4.22b. However, particles that have run into the high-field region at the bottom of the nozzle apex, the particles at pressure $10^7$ Pa, are beginning to be deflected by the nozzle. The high-pressure region at the bottom of the nozzle grows in Figure 4.22c, with particles in the region increasing their pressure to $10^8$ Pa. Effectively, the particles at the bottom create a pressure wave that travels up the length of the nozzle and causes the plasma to expand out the nozzle top. This wave is evidenced by the increased
(a) Axial nozzle plasma pressure and velocity $t = 0 \, \mu s$.

(b) Axial nozzle plasma pressure and velocity $t = 11.55 \, \mu s$.

(c) Axial nozzle plasma pressure and velocity $t = 24.07 \, \mu s$.

(d) Axial nozzle plasma pressure and velocity $t = 49.01 \, \mu s$.

(e) Axial nozzle plasma pressure and velocity $t = 74.08 \, \mu s$.

**Figure 4.22:** Axial nozzle base case plasma pressure and velocity throughout time. *Plasma Temperature color plot, velocity orange vector plot, nozzle in gold.*
pressure near the nozzle apex (10^7 Pa) and the expanding low pressure particles near $z = 3.5$ m in Figure 4.22d-e.

We end with an explanation for the physical process occurring inside a purely axial nozzle. First, the expanding plasma encounters the high-field region in the nozzle apex, where the diamagnetic current is maximized in this region. Second, the increase in diamagnetic current causes Ohmic heating and an increase in the temperature and pressure of the plasma there at the apex. This pressure wave traverses the length of the nozzle and causes augmented expansion of the plasma, but crucially, some particles are still left in the bottom of the nozzle. These high pressure particles are also high temperature, due to the Ohmic heating, meaning they have high conductivity and high diamagnetic current. A feedback loop develops where the particles at the apex have high $E_{current}$ due to the high field, which turns into Ohmic heating, which feeds diamagnetic current, which increasing Ohmic heating etc. The particles get their high $E_{current}$ from their own kinetic energy, meaning their kinetic energy is decreasing throughout this process. Effectively, the plasma particles at the nozzle apex are confined there, and will not leave the nozzle. This incurs high heat loads on the nozzle and vehicle, but these can be ameliorated by putting coils near the apex to repulse the plasma particles and prevent them from being confined (maybe solenoidal coils).

We end by presenting the performance from this run. Here, performance is quantified by impulse bit (which for a system operating at 1 Hz is the same as thrust), specific impulse, and nozzle efficiency. Eq. (2.10) is the definition of nozzle efficiency. Graphs of each quantity are presented over time in Figure 4.23.

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Interestingly, performance increases rapidly, leveling off somewhat around $t = 25 \mu s$, before increasing again. Performance decreases slightly from $t = 40 - 65 \mu s$, before increases again as the pressure wave has fully traversed the nozzle. Performance continues to increase until the simulation cut-off time of 100 $\mu s$. Performance here is pretty good; 16 kNs impulse bit, 2,400 sec specific impulse, and a nozzle efficiency of 0.34. However, it looks like the plasma has not fully expanded after the simulation cut
4.2.2.2 Axial nozzle double current

For this simulation, instead of using 15 MA/strut, we doubled the current in each strut to yield 30 MA/strut. As we will see, doubling the current greatly increased the magnetic field, leading to increased magnetic drag.

Firstly, starting with the graph of energy over time, Figure 4.24, we see similar trends to previous cases. Initially, thermal energy is high and all others are low. Kinetic and diamagnetic energies increase as thermal energy decreases, from 0-10 $\mu$s. Then, the plasma starts to reach its point of maximum expansion, slow down, and interact with the nozzle. This decreases kinetic energy while increasing diamagnetic energy (10-20 $\mu$s). Simultaneously, the diamagnetic current produces ohmic heating in the plasma that increases thermal energy. In this case, due to the high field,

\[ E_{\text{current}}, \text{ Thermal, } KE_x, KE_y, KE_z \]
the plasma continues to interact/stay within the nozzle, slowly leaving it and slowly increasing $KE_z$ while thermal energy decreases (20-100 $\mu$s). Bumps in kinetic and diamagnetic energy are due to individual SPH particle interactions.

Figure 4.25 expands upon the story told in Figure 4.24; initially, motion is the same as in the base case where the plasma expands and begins to interact with the nozzle (Figure 4.25a-b). However, when the plasma reaches its point of maximum expansion and begins to collapse in (Figure 4.25c) the magnetic field in the nozzle is higher. The bulk of the plasma in the nozzle apex ($z < 1$ m) is magnetized, and some of it is actually drawn back into the nozzle (Figure 4.25d). This effect, where plasma is drawn back into the nozzle, functions similarly to magnetic drag for steady-state nozzles [3, 4] and reduces performance. The effect of magnetic drag is illustrated in Figure 4.25e, where there are more particles than in Figure 4.21e.

These plasma behaviors result in the performance curves in Figure 4.26. Performance increases to a local maximum around 26 $\mu$s as the plasma reaches its point of maximum expansion, but after this the plasma is drawn back into the nozzle and performance decreases. Performance increases again as some plasma is able to exit the nozzle, but after 75 $\mu$s performance has flat-lined; final performance is 13 kNs impulse bit, 1,900 sec specific impulse, and a nozzle efficiency of 0.27, which is a decrease from before. Doubling the current did not result in a more efficient nozzle design; so next we tried halving it.
Figure 4.25: Axial nozzle double current case (30 MA/strut) plasma motion. Nozzle in gold, plasma in black.
(a) Axial nozzle double current case impulse bit (Ns).

(b) Axial nozzle double current case specific impulse (sec).

(c) Axial nozzle double current case nozzle efficiency.

**Figure 4.26**: Axial nozzle double current case (30 MA/strut) performance through time.

### 4.2.2.3 Axial nozzle half current

For this case we halve the current to 7.5 MA/strut. As we will see, halving the current reduced magnetic drag, but created insufficient plasma deflection.

Beginning with the energy graph (Figure 4.27) we see similar behavior to before. Plasma initially starts off with high thermal energy, and kinetic and diamagnetic
energy increase rapidly from 0-10 $\mu$s. Then, the plasma starts to reach its point of maximum expansion, slow down, and interact with the nozzle. This decreases kinetic energy while increasing diamagnetic energy (10-25 $\mu$s). Note that this process takes longer than in previous cases. Simultaneously, the diamagnetic current produces Ohmic heating in the plasma that increases thermal energy. After this, the plasma leaves the nozzle, kinetic energy slowly increases, diamagnetic energy slowly decreases, and thermal energy stays roughly constant due to Ohmic heating (25-100 $\mu$s). The slow increase in kinetic energy gives performance.

Next we consider plasma motion. Initially (like in all cases) the plasma starts off compact with high thermal energy as shown in Figure 4.28a. The plasma expands, however it reaches a farther radius of maximum extent than previous cases with more plasma outside the nozzle than in previous cases (Figure 4.28b-c). This is because the field from this nozzle is much lower, so low in fact that most of the plasma is directed outside the nozzle. The lower field induces a weaker diamagnetic current that does

**Figure 4.27:** Energy over time in J in the plasma for the axial nozzle half current case (7.5 MA/strut). Total, $E_{\text{current}}$, Thermal, $KE_x$, $KE_y$, $KE_z$
(a) Axial nozzle half current plasma \( t = 0 \mu s \).

(b) Axial nozzle half current plasma \( t = 11.55 \mu s \).

(c) Axial nozzle half current plasma \( t = 24.07 \mu s \).

(d) Axial nozzle half current plasma \( t = 49.01 \mu s \).

(e) Axial nozzle half current plasma \( t = 74.08 \mu s \).

Figure 4.28: Axial nozzle half current case (7.5 MA/strut) plasma motion. Nozzle in gold, plasma in black.
not arrest plasma motion as much. Since the field is stronger outside the nozzle than inside it [68], the plasma is deflected by this high-field region. Here, the plasma is deflected back into the nozzle and out the top (Figure 4.28d-e). This continues until most of the plasma has left the nozzle.

With regard to performance, this case shows higher performance than the double current case, but not as much as the base case. This is because the plasma

**Figure 4.29**: Axial nozzle half current case (7.5 MA/strut) performance through time.
is deflected too late (outside the nozzle). As shown in Figure 4.29, maximal impulse bit is 14 kNs, specific impulse is 2,100 sec, and nozzle efficiency is 0.30. Performance decreases slightly around 75 µs as plasma escapes out the bottom of the nozzle. Here, magnetic drag is not too high, but the current is too low to effectively deflect all of the plasma. Regardless, there is still some performance left as evidenced by the relatively low nozzle efficiency, but this might be realized with further optimization.

4.2.3 Summary

<table>
<thead>
<tr>
<th>Per-strut Current (MA)</th>
<th>Impulse Bit (kNs)</th>
<th>Specific Impulse (sec)</th>
<th>ηₚₜᵣₑ</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.5</td>
<td>14</td>
<td>2,100</td>
<td>0.30</td>
</tr>
<tr>
<td>15</td>
<td>16</td>
<td>2,400</td>
<td>0.34</td>
</tr>
<tr>
<td>30</td>
<td>13</td>
<td>1,900</td>
<td>0.27</td>
</tr>
</tbody>
</table>

In summary, SPFMax is able to reproduce features important for accurately modeling plasmas in magnetic nozzles. It shows pretty good agreement with prior work ([9,46]), especially with qualitative effects. Quantitative agreement is within 50%; part of this might be due to modeling differences inherent in the two methods, but this difference is acceptable given that the PIC code does not perfectly align with experimental results. Lastly, using SPFMax for a pulsed nuclear nozzle design results in an axial nozzle with 32 struts, 15 MA/strut, with the fission-fusion plasma that starts 1 m away from the nozzle apex. This design effectively reduces magnetic drag while still effectively deflecting the plasma, resulting in 16 kNs impulse bit,
2,400 sec specific impulse, and a nozzle efficiency of 0.34. The design study suggests that minimizing magnetic drag, while still ensuring plasma deflection, is crucial in magnetic nozzle design for fission, fusion or hybrid plasmas.
5.1 Overall conclusions

We investigated theoretical feasibility of a power and propulsion producing magnetic nozzle for pulsed nuclear propulsion systems. Overall, this work suggests such a nozzle is feasible. For the power generation side, using a plasma FCG system potentially offers increased specific energy and specific power, as compared to thermionic, thermodynamic, and photovoltaic power systems. A review of relevant literature lead to a first research question: how does system design of a plasma FCG system affect performance scaling? To answer this question, using magnetic flux compression theory, we developed a multi-input mathematical model of a plasma FCG circuit connected to a prototypical fission-fusion power storage system (capactive load). We used this model to discover two important non-dimensional parameters that must be considered during the design of any power generation system; the ratio of the inductance of the primary side of the transformer to the initial inductance of the stator ($\pi_1 = L_1/L_{stat_0}$), and the ratio of the initial energy in the stator to the initial plasma energy ($\pi_2 = E_{stat_0}/E_{plasma_0}$). Results suggest that $\pi_2$ should be around 0.01 to maximize output energy. It also seems that for lower values of $\pi_2$ and $C$, $\pi_1$
should be about 1, but this relationship changes for higher values of \( C \), indicating \( C \) should be incorporated into \( \pi_1 \) somehow. Taken together, these two points serve to answer our research question. Additionally, through doing mass estimation of a point design, we demonstrated a specific energy of 35 J/kg and a specific power of 37.5 kg/kW\(_e\). This design is more detailed but performs worse than prior work [2, 14, 17].

For the magnetic nozzle system itself, we were aware that using a magnetic nozzle potentially offers higher performance than conventional propulsion schemes (chemical, electric propulsion) for reduced heat loads on the vehicle. After a review of the literature we developed our next research question: what are some principles of purely axial nozzle design for a fission-fusion plasma? To answer this question we built our own in-house magnetic nozzle computer model and compared it with the state of the art. To answer these questions, we developed a fully 3D plasma modeling code called SPFMax, and we compare the results from SPFMax with two cases from prior work: a case in Ref. [9] and a case in Ref. [46]. Qualitative results compare favorably for both cases, with quantitative results being within 50% of Ref. [9] and subsequent derived results. Having done the comparison, we investigate the effect of magnetic nozzle topology on a fission-fusion plasma, and find that two effects must be balanced for an effective design; magnetic drag and deflection. The field strength in the nozzle must not be too high, or else magnetic drag dominates, but not too low, or else the plasma remains relatively undeflected. For a nozzle with 32 struts and a 0.1 kg Tungsten, 0.5 kg Lithium 69.3 eV plasma that starts 1 m away from the apex of the nozzle, we find that to balance these effects, each strut of the nozzle should have 15 MA. Performance here is a specific impulse of 2,400 sec, an impulse of 16 kNs
(which for a system with a mass of 200 mT and a pulse rate of 1 Hz [15], corresponds
to a thrust-to-weight of 0.01), and a nozzle efficiency of 0.34. While somewhat low,
this relatively low efficiency shows there is great room for improvement.

5.2 Future work

While we have answered our research questions, our work has left additional
questions unanswered. For example, for the power generation system, we have not
determined mass scaling. One question might be, how does the mass of the power
generation system scale with system design? Additionally, we have left open further
system mass reduction and optimization. This can be accomplished through using
better materials, hollow transmission lines, or a 3-D plasma model that accurately
captures transport processes. This model might validate (or refute) the results ob-
tained with the 1-D plasma model.

For the magnetic nozzle, undertaking an experimental test program to verify
the theory and computational models is of utmost importance. This is because both
SPFMax, and codes in the state of the art do not have a lot of data to bench off
of; in thus we are unsure if our models are incorporating the right theory and our
results are valid. And despite the fecundity of H. Nakashima and co-authors, more
experiments must be done. Of secondary concern are modeling additional effects,
such as incorporating the difference in temperature of the two plasma species, con-
sidering radiative cooling along with radiative re-absorption, and considering targets
of varying composition. The later is important to model more modern systems that
vary the contents of their exhaust to change performance, such as PuFF [15]. Also of
importance is extending this work to rigorously compare axial and solenoidal nozzle designs, and determine non-dimensional parameters that characterize design, like $\kappa$ and $\epsilon_b$ from Ref. [78]. Lastly, combining the plasma FCG and magnetic nozzle in one 3D simulation would greatly mature design; however this should be combined with experimental verification of some sort.

But, in totality, these data in this dissertation demonstrate the promise of power generating magnetic nozzles. Through undertaking future work, researchers can increase their performance until the technology is fully realized. With such a nozzle, pulsed nuclear propulsion systems can finally realize their full potential, and revolutionize the way humanity gets around the solar system.
REFERENCES


APPENDICES
APPENDIX A

MASS ESTIMATION METHODOLOGY

For the bottoms-up system mass analysis, the system is divided into several components; 1) Stator 2) Primary side of the transformer transmission lines 3) Primary side of the transformer windings 4) Secondary side of the transformer windings 5) Primary side of the transformer transmission lines and 6) Energy storage system. Prior work discusses the overall system and the energy storage system separately, so the energy storage system mass is not analyzed here. The first components 5 be broke into two sub-components: 1) a mass of wire that carries the currents/voltages and 2) a cooling system to cool down the wire after each pulse.

A.1 Wiring mass estimation

The wire mass is assumed to be pure copper (no insulation), with a circular cross-section. The cross section for the wiring in each component is assumed to vary independently. Since the cross section for all wires is circular, the wire cross-section can be characterized by the diameter of the wire. The wiring mass for the $i$th
component can be characterized by

\[ m_{\text{wiring}_i} = \rho_{\text{cu}} \lambda_{w_i} \frac{1}{4} \pi D_{w_i}^2 \]  \hspace{1cm} (A.1)

where \( m_{\text{wiring}_i} \) is the mass of wire associated with the component, \( \rho_{\text{cu}} \) is the density of copper (taken to be 8960 kg/m\(^3\)), \( \lambda_{w_i} \) is the length of wiring in the component (not the same as the length of the component), and \( D_{w_i} \) is the diameter of the wire in the component.

The length of wiring in each component, \( \lambda_{w_i} \), varies based off the component; for the stator, primary transformer windings, and secondary transformer windings, the helical length of wire is approximated as a series of rings, given by

\[ \lambda_{w_i} = 2\pi r_{w_i} N_i \]  \hspace{1cm} (A.2)

Here, \( r_{w_i} \) is the radius of the component, either \( R_{\text{stat}} \) for the stator or \( r_T \) for the transformer, and \( N_i \) is the number of turns in the component. \( N_i \) is \( N_{\text{stat}} \) for the stator, \( N_{T_1} \) for the primary transformer windings, or \( N_{T_2} \) for the secondary transformer windings. For the primary/secondary transmission lines, length is taken to in the worst-case scenario where the wires must run the entire length of the 8.4m diameter SLS Payload fairing, minus the 3.5 length of the stator. This results in a length of 21.33 m, which is fairly substantial. It was found that to minimize system mass, the primary transmission lines should run 9/10ths or 19.20m of this length, and the secondary transmission lines should run the other 2.13 m of this length. One final
note is that the overall transmission line lengths should be doubled; one line carries the load and one is the current return line. This results in values for \( \lambda_{wi} \) of 38.39 m for the primary transmission lines and 4.27 m for the secondary transmission lines. This characterizes \( \lambda_{wi} \) for all components. All that is needed in order to characterize the wiring mass of a component is the cross-sectional area, which for circular wires is given by

\[
XC_{wi} = \frac{1}{4} \pi D_{wi}^2
\]  

(A.3)

A.2 Cooling system mass estimation

The cooling system mass is estimated by assuming the cooling system mass is dominated by the radiator mass. The radiator mass is calculated in the following manner. First, a component resistance is calculated using its definition, given by

\[
R_i = \frac{\lambda_{wi}}{\sigma_{cu} XC_{wi}} = \frac{\lambda_{wi}}{\sigma_{cu} \frac{1}{4} \pi D_{wi}^2}
\]  

(A.4)

Here, \( R_i \) is the resistance of the component, \( \sigma_{cu} \) is the conductivity of copper (we use \( 5.95 \times 10^7 \) S/m [92]), \( \lambda_{wi} \) is again the length of wire in the component (not the length of the component), \( XC_{wi} \) is again the cross-sectional area of the wiring in the component, and \( D_{wi} \) is again the diameter of the wire in the component. The value of \( \lambda_{wi} \) can be found using the previous section. After the component resistance is calculated, the heating power of the component is calculated assuming heating only
occurs through the primary $I^2R$ joule heating. This is summarized as

$$Q_i = R_i \int_0^t I_i(t')dt'$$  \hspace{1cm} (A.5)

where $Q_i$ is the heat load from joule heating in the $i$th component, $R_i$ is the resistance in the $i$th component, $I_i$ is the current in the $i$th component; $I_i$ is $I_1$ for the stator, primary transmission lines, and primary transformer windings. $I_i$ is $I_2$ for the secondary transmission lines and secondary transformer windings. To calculate the radiator mass, the heating energy from joule heating is smeared over the maximum time interval of system operation, and multiplied by the radiator specific mass. The radiator specific mass or alpha is taken to be 0.034 kg/W, from using 350W/m$^2$ and $12\text{kg/m}^2$ in Ref. [93]. We codify this relationship in

$$m_{\text{radiator},i} = 0.034\dot{Q}_i = 0.034\frac{Q}{\frac{1}{J} - \tau}$$  \hspace{1cm} (A.6)

where $m_{\text{radiator},i}$ is the radiator mass, $\dot{Q}_i$ is the heating power of the joule heating in the wires, $J$ is the pulse frequency, and $\tau$ is the burnout time. Taking $J = 1$ Hz and given that $\tau$ for our device is so small, the denominator of the fraction reduces to 1 ($\frac{1}{J} - \tau \rightarrow 1$).

### A.3 System mass estimation and trade study

Having characterized all other aspects of the system, the total mass of the system is found by adding up the wiring masses ($m_{\text{wiring},i}$) and radiator masses ($m_{\text{radiator},i}$).
for each component in the system. Both wiring mass and radiator mass are depen-
dent on the diameter of the wires used in each component. Since the wiring mass and
radiator mass follow opposite trends with regard to wiring diameter (wiring mass in-
creases with increasing wire diameter whereas radiator mass decreases with increasing
wire diameter, and these trends switch for decreasing wire diameter), one can imagine
an optimal wire diameter that minimizes total mass. These trends necessitate a trade
study with regard to wire diameter. However, there are two constraints with regard
to the wire diameter; 1) the wire must be thick enough to withstand the magnetic
pressure of the current running through it, and 2) the wire must be able to fit inside
the geometry proscribed by the inductive power conversion system model.

A.4 Minimum wire diameter - magnetic pressure constraint

To satisfy the magnetic pressure constraint, the wire must have a minimum
width or else it will break under the magnetic pressure. The yield strength of high-
conductivity copper was determined to be 50 MPa [94], so the maximal magnetic
pressure in the wire needs to be smaller than this. The magnetic pressure is given by

\[ P_B = \frac{B^2}{2\mu_0} \]  

(A.7)

where \( P_B \) is the magnetic pressure, \( B \) is the magnetic field strength, and \( \mu_0 \) is the
permeability of free space.
The magnetic field from the current in the wire is approximated as the magnetic field from an infinitely long, straight, current-carrying wire, which is expressed mathematically in Eq. (A.8). This is somewhat of an over-estimation.

\[ B = \frac{\mu_0 I}{2\pi r} \]  

(A.8)

Here, \( B \) is the magnetic field strength at a location away from the center-line of the wire, \( \mu_0 \) is the permeability of free space, \( I \) is the current the wire carries, and \( r \) is the distance away from the wire center-line. Rearranging Eq. (A.7), (A.8) and plugging in the yield strength of copper, 50MPA, with a factor of safety of 2 for \( P_B \) yields the following

\[ D_{w_{\text{min}}} = \sqrt{\frac{\mu_0 I_{\text{imax}}}{50 \pi}} \]  

(A.9)

where \( D_{w_{\text{min}}} \) is the smallest possible diameter the wire in the ith component can be; any smaller and the wire would fail (within the required safety factor) and be crushed by the magnetic pressure force. \( D_{w_{\text{min}}} \) is a function of \( I_{\text{imax}} \), the maximal current in the ith component. For the stator, primary transmission lines, and primary transformer windings, \( I_{\text{imax}} \) is \( I_{1\text{max}} \), but for the secondary transmission lines and secondary transformer windings \( I_{\text{imax}} \) is \( I_{2\text{max}} \).

### A.5 Maximum wire diameter - geometric constraints

For the largest possible wire diameter, geometric constraints come into play. These constraints are not applicable to the transmission lines, practicably, because the
fairing could fit wires over a meter in diameter, but these constraints are applicable to
the stator, the primary transformer windings and the secondary transformer windings.
All three have a length and number of turns associated with them, with the maximal
possible wire width given by the total component length divided by the number of
turns. Mathematically, this statement is expressed as follows

\[ D_{w_{\text{max}}} = \frac{\lambda_i}{N_i} \]  \hspace{1cm} (A.10)

Here, \( D_{w_{\text{max}}} \) is the maximum wire width for the specified component, \( \lambda_i \) is the length
of the component (\( \lambda_i \) is \( r_{stat} \) for the stator, \( \lambda_{T_1} \) for the primary transformer windings,
and \( \lambda_{T_2} \) for the secondary transformer windings), and \( N_i \) is the number of turns in
the component (\( N_i \) is \( N_{stat} \) for the stator, \( N_{T_1} \) for the primary transformer windings,
and \( N_{T_2} \) for the secondary transformer windings). Note that if \( D_{w_{\text{min}}} > D_{w_{\text{max}}} \) then
the design is infeasible.
APPENDIX B

RESULTS FROM OLDER VERSION OF SPFMAX

Results presented here are from an older version of SPFMax that did not conserve energy as well (as will be shown). This version did not include the $\frac{1}{\rho_0} j_2^2$ term in the energy equation, so energies were conserved using $E_{\text{current}}$ in the following manner.

When $E_{\text{current}}$ is calculated, it is compared with all other types of energy, and current is regulated as needed to ensure energy is conserved. What this means is, at a specific time, if the sum of $E_{\text{current}}$ and the right-hand side of Eq. (3.58) is greater than 0 (meaning $E_{\text{current}}$ is too big), $\vec{j}$ is systematically reduced until the quantity does equal 0. On the other hand, if the sum of $E_{\text{current}}$ and the right-hand side of Eq. (3.58) is less than 0 (meaning $E_{\text{current}}$ is too small), $\vec{j}$ is systematically increased until the quantity does equal 0. This ensures the electromagnetic field energy is conserved along with the fluid energies.

The following is are results using this older version of SPFMax. The older version over-predicts magnetic drag relative to the new version, but generates a stronger diamagnetic cavity.
B.1 Magnetic Nozzle Subsystems

To recap magnetic nozzle operation, as explained in the literature review section, firstly the hot fission-fusion plasma begins isentropic expansion in the presence of the high-strength applied magnetic field from the nozzle. The isentropic expansion coupled with the high strength field begins larmor motion in the plasma, creating a diamagnetic current that excludes a significant portion of the applied field from the nozzle. The portion of the plasma where the applied field is excluded is called the "diamagnetic cavity", and it effectively turns the plasma into a giant magnet. However, if the plasma can be thought of as a giant magnet, so too can the nozzle. Because of the diamagnetic current and cavity, the polarity of the plasma-magnet will be the exact opposite polarity of the nozzle-magnet. Because opposites repel, the nozzle pushes the plasma away from it, and by Newton’s third law of motion, the nozzle generated thrust and specific impulse.

The explanation gives us some benchmarks to ensure SPFMax is working correctly. Firstly, the code should reproduce the formation of a diamagnetic current and diamagnetic cavity. This diamagnetic current increases the field strength outside the plasma, but decreases the field strength inside the plasma (up to 40%). This effect must be present in any accurate magnetic nozzle simulation.

Additionally, any accurate magnetic nozzle simulation must properly vary the plasma energies over time. In terms of energy, the plasma will start out with high thermal energy and not much else. As the plasma expands its thermal energy will decrease, and its current energy (the energy in the diamagnetic current of the plasma,
or $E_{\text{current}}$ will increase. When the plasma reaches its maximal point of expansion, the current energy will start decreasing, and the kinetic energy of the plasma will increase as it is repelled away from the nozzle and it undergoes free expansion.

**B.2 Comparison with prior work**

**B.2.1 High-level physics**

For this work, we compare results from SPFMax with results from the solenoidal test case and axial nozzle test case (as detailed in the methodology section). Beginning with the solenoidal nozzle test case, both benchmarks are illustrated. For the first effect we wished to verify, the diamagnetic cavity is shown in Figure B.1 and Figure B.2.

Initially, there is no cavity present as the simulation is just getting started. However, as the plasma expands, the diamagnetic current increases and the diamagnetic cavity slowly expands. This is evidenced by the region of lower-field near the origin in Figure B.1b-c and Figure B.2b-c. In Figure B.1b-c this low field region is initially the green color (0.5 T - Figure B.1b) but the center region decreases in field strength further to a blue color (0.07 T - Figure B.1c). In Figure B.2b-c the low-field region the colors are slightly different (yellow and blue respectively), but the magnitudes are the same. The cavity deflates somewhat at $t = 3.0\mu s$ with field intensity increasing at the origin in Figure B.1d and Figure B.2d to about 0.1 T, though a boundary of the cavity forms at $r = 0.5$ m.
On the other side of the boundary, the field is amplified, as shown via the red region in Figure B.1d and Figure B.2d. Here the field increases from 1 T to 10 T. This is an increase by an order of magnitude.

The applied field in the cavity is around 1 T, but with the advent of the diamagnetic current, this is reduced to around 0.1 T. This reduction is illustrated most vividly in Figure B.2c and Figure B.1c, where the field is reduced by up to
Figure B.2: Diamagnetic cavity formation over time in the xz plane with 8,186 SPH particles, solenoidal nozzle test case. *B* field as color plot, nozzle in black.

0.001 T. However, across all figures (Figure B.1b-d and Figure B.2b-d) the field is reduced to an average of 0.1 T. This reduction is consistent with prior work (see Fig. 3 of Ref. [9]). Additionally, Figure B.2 shows an increase in the field for particles not in the cavity but $z < 0$, where the field in this region seems slightly increased. This is especially obvious in the red regions of Figure B.2c-d, where the field strength is increased to 10 T. However, some of this increase might be explained by the plasma...
getting closer to the nozzle. Regardless, this is another feature of the diamagnetic cavity that our code reproduces.

The applied field for this nozzle is uniformly in the +z direction, so by the Right-Hand Rule for currents [68], the currents should move clockwise when viewed from above to cancel out the applied field, which indeed they do, as shown in Figure B.3 (black arrows move clockwise when viewed from above).

Initially, as shown in Figure B.3a, there are no currents present as the simulation is still being set up. However, as the plasma expands, the diamagnetic current sphere expands as well, from \( r = 0.5 \text{ m} \) (Figure B.3b) to \( r = 0.6 \text{ m} \) (Figure B.3c) to the width of the nozzle \( (r = 1 \text{ m} \) - Figure B.3d).

With regard to the energies, the energy over time in the plasma is shown in Figure B.4. The plasma starts out with 4.3 MJ of energy, almost all of it thermal energy (kinetic energy is small compared to this). As the plasma expands, its kinetic energy increases to a point, its thermal energy decreases, and \( E_{current} \) increases more rapidly, from near 0 to basically even with the kinetic energy after about 3 \( \mu \text{s} \). At this point, the plasma has reached its maximum expansion, and the diamagnetic current decreases as the kinetic energy increases (here the thermal energy increases as well as the diamagnetic current produced Ohmic heating in the plasma) until around 5 \( \mu \text{s} \). At this point, the solver starts diverging and both the total and thermal energy start erroneously increasing, making results past this point of dubious quality.

However, the diamagnetic current, development of a diamagnetic cavity, and overall energy trends are present in SPFMax results from the solenoidal nozzle test case. They should be true in all other cases as well.
Figure B.3: Diamagnetic current vector plot over time with 8,186 SPH particles. 
*Currents in black with nozzle in gold*
B.2.2 Solenoidal nozzle test case

For this test case, we wish to compare our results with the results from Nagamine and Nakashima. Their results will be in the top right of the following plots.

It is instructive to start with comparing energies. This comparison is displayed in Figure 4.10.

For the solenoidal nozzle test case, Nagamine and Nakashima show a plasma that has initially decreasing kinetic energy (from $t = 1 - 3\mu s$) as the gas cools and expands. While the directed kinetic energy could be increasing, in this case the total kinetic energy is decreasing because the thermal energy decreases as the gas expands. In Particle-in-Cell codes, due to the use of ion macroparticles, this thermal energy is
(a) Nagamine and Nakashima base case component-wise split kinetic energy. Reproduction of Fig. 8 in Ref. [9]

(b) SPFMax plasma energies over time 1,151 SPH particles

(c) SPFMax plasma energies over time 8,186 SPH particles

(d) SPFMax plasma energies over time 41,636 SPH particles

Figure B.5: Comparison of Ref. [9] energy results to SPFMax energy results for the solenoidal nozzle test case at various SPH particle resolutions.
reflected in the kinetic energy of individual ions, meaning that if the thermal energy
decreases, the kinetic energy should decrease uniformly. Interestingly, kinetic energy
does decrease uniformly for $KE_x$ and $KE_y$ (compare the line with squares to the solid
line in Figure B.5a), but $KE_z$ seems to decrease faster before increasing again. This
might be because the magnetic field from the nozzle is uniformly in the z-direction,
so the diamagnetic current forms with ion motion in the x & y directions, increasing
$KE_x$ and $KE_y$ early on at the expense of $KE_z$.

Then, at $t = 3\mu s$, after the nozzle begins to interact with the field, energy is
re-directed into kinetic energy in the z-direction, leading to thrust. Kinetic energy
continues to decrease along the x and y directions as the plasma is discouraged to
cell in those directions. Other components of energy are present (current, field),
but as Nagamine and Nakashima generate their results using a Particle-in-Cell code,
those energies must be derived from particle position/motion and so are not reported.
However, based on our understanding of the physical mechanisms, when the total
kinetic energy is decreasing from $t = 1 - 3\mu s$, this is matched by an increase in the
diamagnetic current and therefore the current energy. When $KE_z$ begins to increase,
the current energy should decrease.

In comparing the Nagamine and Nakashima results with the SPFMax results,
we must first note a discrepancy. The total energy in the simulation (magenta line
in Figure B.5b-d) does not stay constant throughout line (energy is not conserved
throughout total time). The authors went through 10 different versions of the SPF-
Max code to produce one that would keep energy constant while not underestimating
performance and meeting other sanity checks, and we found the current version per-
formed the best. Nevertheless, valid results can only be taken from the start of the simulation to the point at which the energy starts to diverge. This is from $t = 0–5.5\mu s$ for the 1,151 particle case, $t = 0–5\mu s$ for the 8,186 particle cases, and $t = 0–2.3\mu s$ for the 41,636 particle case. This indicates that increasing particle resolution decreases the amount of time the simulation is valid. Interestingly, sometimes the energy will come back to its initial value for a time period, then diverge again (see the magenta line from $t = 2–4.8\mu s$ and from $t = 4.8–6.2\mu s$ in Figure B.5d). Nevertheless, results should not be considered outside this range.

Additionally, for the 1,151 particle and 8,186 particle cases, it seems energy is fairly consistent until current energy starts dropping and/or thermal energy starts to increase. In these cases, the increase in energy seems to be due to the thermal energy increasing. This is illustrated at $t = 5.5\mu s$ in Figure B.5b and at $t = 5\mu s$ in Figure B.5c, when increases in the magenta line correspond to increases in the black (thermal energy) line. These spikes in thermal energy could be because of equation of state issues (such as the equation of state subroutine reporting a negative thermal energy due to a temperature being outside of a tabular data set). This is outside the electromagnetic field solver.

Sometimes current energy drops because thermal energy spikes irregularly - when thermal energy changes the solver is designed to compensate by reducing current energy and therefore current accordingly. This is illustrated in the spikes in thermal energy from $t = 2.3–3.1\mu s$ in Figure B.5d. The black line increases and the yellow $E_{current}$ line decreases the exact amount at the exact time, increasing when the back
line decreases again. The peak in the black line can exactly fit inside the valley in the yellow line.

However, comparison of Figure B.5a with Figure B.5b-d reveals that SPFMax is not getting the same results as Nagamine and Nakashima with regard to energy. Nagamine and Nakashima get decreasing $KE_x$ and $KE_y$, and increasing $KE_z$, with the former two having final values around 0.58 MJ, and the latter having a value of 2.5 MJ. In contrast, for valid times, SPFMax has $KE_x$, $KE_y$, and $KE_z$ all uniformly increasing, and at roughly the same magnitude of 1 MJ. This rises to 2MJ when thermal energy is included, but thermal energy would roughly increase all components evenly, so there is no favoritism of $KE_z$. However, as we will see, more plasma is directed along the z-axis than the other two axis, indicating SPFMax is modeling the plasma as intended. But this is not shown in the energy, mostly likely due to differences in modeling approaches (PIC vs. SPH). Nevertheless, discussing energy first is important to establish times for when the SPFMax results can be considered valid.

Comparing the 3D density plots, the Nagamine and Nakashima results are shown in Figure B.6a. These results show a plasma that looks vaguely 'mushroom' shaped with a large round head in front of $z>0$ and a slim stem for $z<0$. This is not reproduced in the SPFMax results, which feature a roughly cylindrical plasma, with maybe a bit of a tail for 1,156 and 8,186 particles (see Figure B.6b-c). The 41,636 particle case (Figure B.6d) more strongly resembles Nagamine and Nakashima (Figure B.6a), with the formation of a front part and stem, despite being outside the temporal range of validity. These results suggest that, despite energy conservation
Figure B.6: Comparison of Ref. [9] and SPFMax solenoidal nozzle test case 3D density plots at different SPH particle resolutions.
issues, for increasing particle resolution, SPFMax produces results closer to the PIC code.

Next, Nagamine and Nakashima provide 2D contour density plots of their plasma at different times and in different planes. They provide plot for $t = 0, 2, 4, 8 \mu s$ in the $xy$ and $xz$ planes. Contour levels are not provided for any of these plots. The plots for $t = 0, 2$ are roughly circular and nothing interesting is going in between Ref. [9] and SPFMax.

Beginning with the $t = 4 \mu s$ case, in the $xy$ plane, results seem consistent with $t = 0, 2$; nothing much interesting going on. Results from Nagamine and Nakashima (Figure B.7a) show the plasma is mostly in a ring 1 m in radius. Contour plots from SPFMax at all particle resolutions reproduce these results; although the 41,636 particle case - Figure B.7 shows small localized areas of high density. These regions are because the data for the 41,636 particle case is not valid from an energy conservation perspective. And the contour level does change slightly from resolution to resolution - it decreases a bit from $10^{-5} \text{ kg/m}^3$ to $5 \times 10^{-6} \text{ kg/m}^3$ as particle resolution increases, but it stays remarkably consistent. But overall, these results show good agreement with Nagamine and Nakashima

Flipping over to the $xz$ plane though, the agreement is not as strong for the lower particle resolution case. The Nagamine and Nakashima results show a mushroom-shaped plasma starting to form, with a clear stem for $z < -0.5$, and a head with $r = 1.25 \text{ m}$ otherwise (see Figure B.8a). Here and subsequently for this subsection, $r$ is defined in the cylindrical sense, $r = \sqrt{x^2 + y^2}$. The low particle resolution case (1,151 particles - Figure B.8b) does not show any mushroom formation;
Figure B.7: Comparison of Ref. [9] and SPFMax solenoidal nozzle test case 2D contour density plots at $t = 4\mu s$ in the xy plane at different SPH particle resolutions.
Figure B.8: Comparison of Ref. [9] and SPFMax solenoidal nozzle test case 2D contour density plots at $t = 4\mu s$ in the xz plane at different SPH particle resolutions.
it appears bullet shaped and does not have a long stem. Its mushroom head also only extends to $r = 1$ m. For the higher particle resolutions (Figure B.8c-d) they have mushroom heads that do extend to $r = 1.25$ m, and they show stem formation, albeit the stem is not as thin. It is more gently tapering in both cases. Here again, in Figure B.8b-d, results match most well with Figure B.8a at the $\rho = 10^{-5} \text{ kg/m}^3$ contour line.

Figure B.8d again has discontinuous regions of high density. This is again because energy is not being conserved at this time.

Moving ahead to $t = 8\mu s$ and flipping back to the xy plane, results from Nagamine and Nakashima show a plasma that has mostly exited this plane. In Ref. [9], at $t = 8\mu s$, the mushroom head has gone forward and the tail pierces the xy plane here, leading to a reduction in the circular radius to $r = 0.5$ m (see Figure B.9a). Results here for SPFMax are mixed. All simulations are outside their temporal range where energy is conserved, leading to discontinuous regions of high density (internal spots with lots of contour lines in Figure B.9b-d), but as we saw with Figure B.7d, this just seems to cloud results. Indeed, looking at the $\rho = 10^{-5} \text{ kg/m}^3$ contour line, for all cases results in a circle of radius of $r = 0.75$ m, which is not that far off from $r = 0.5$ m. However, the points of high density make the shape uneven; especially in Figure B.9d the circle looks lumpy an uneven, more blob-like, than a true circle.

Flipping to the xz plane, we wish to reproduce the full stem-and-head mushroom shape Nagamine and Nakashima present in Figure B.10a, with a head roughly $r= 2$ m, and a tail that stretches from $z = 0$ to $z=-2.5$ m. Not all the SPFMax plots reproduce this shape, and none do it as cleanly, but all have some of its important
(a) Nagamine and Nakashima results - 2D density contour plot at $t = 8\mu s$ in the $xy$ plane. *Reproduction of Fig. 2a in Ref. [9]*

(b) SPFMax solenoidal case results - 2D contour density plot at $t = 8\mu s$ in the $xy$ plane. 1,151 SPH particles

(c) SPFMax solenoidal case results - 2D contour density plot at $t = 8\mu s$ in the $xy$ plane. 8,186 SPH particles

(d) SPFMax solenoidal case results - 2D contour density plot at $t = 8\mu s$ in the $xy$ plane. 41,636 SPH particles

**Figure B.9**: Comparison of Ref. [9] and SPFMax solenoidal nozzle test case 2D contour density plots at $t = 8\mu s$ in the $xy$ plane at different SPH particle resolutions.
Figure B.10: Comparison of Ref. [9] and SPFMax solenoidal nozzle test case 2D contour density plots at $t = 8\mu s$ in the xz plane at different SPH particle resolutions.

features. Figure B.10b-c (both 1,151 particles and 8,186 particles) have a head-and-tail, with the 1,151 particles case having a $r = 2$ m head and stem that extends from $z = -0.5$ m to $z = -2$ m. These features are apparent for the contour level $\rho = 5 \times 10^{-6} \ kg/m^3$. For the 8,186 particle case, the head looks bigger, extending out to $r = 2.5$ m, but the stem matches better, extending from $z = 0$ to $z = -3$ m. These features are at
a lower contour level than in previous comparison figures, at $\rho = 10^{-6} \text{kg/m}^3$. Stems for both are much thicker at 1 m across, as opposed to the 0.5 m in Figure B.10a.

In contrast to Figure B.10b-c, Figure 4.15d does not seem to match with Figure B.10a much. The mushroom head is too large at $r = 3$ m, the stem is too short, extending from $z = -1$ m to $z = -2$ m and all the spots of high density make it difficult to determine overall shape and relevant contour levels (maybe $\rho = 10^{-6}$ kg/m$^3$ can work). The signal to noise ratio, due to these results being temporally far removed from when energy is being conserved, is too low.

Taken together, the 2D and 3D results suggest that while higher particle resolutions are better for resolving features of the plasma in early simulation time, they are less accurate later. This is because high particle resolutions cannot conserve energy for as long as low particle resolutions. Meaning, high SPH particle resolutions are better for earlier in simulation time, but low resolutions are better for later times. Additionally a particle resolution between 8,186 and 41,636 offers a reasonable trade between relevant feature capture, computational times, energy conservation and accumulated precision error.

Lastly, we compare the line-integrated plasma densities. Nagamine and Nakashima integrate the plasma density in the xy plane at some point $z$ to get $n$ in $1/m$. We do not do that, but the trends will still be instructive.

These results are largely in line with previous results; Nagamine and Nakashima show most of the plasma is ahead of the coil in Figure B.11, with a number density spike at $n = 10^{20}$ at $z = -0.4$ m. With the SPFMax results, spikes are between $n = 0.7 - 0.9 \times 10^{20}$, and with higher particle resolution, the spike seems to resolve
Figure B.11: Comparison Ref. [9] and SPFMax line density plots at $t = 4\mu s$.

closer and closer to $z = -0.4$ m. Note though that for the lowest particle resolutions, the plasma does not seem to be affected by the nozzle much. The density looks roughly Gaussian (see Figure B.11b) for the 1,156 particle case, and for the 8,186 particle case the curve has two bumps (one at $z = 0.5$ m and one at $z = -0.5$ m) and is a double-Gaussian. The high resolution case, Figure B.11d shows the least
(a) Ref. [9] base case plasma number density line plot along the z-axis at $t = 6\mu s$.

(b) SPFMax plasma number density line plot along the z-axis at $t = 6\mu s$. 1,151 SPH particles

(c) SPFMax plasma number density line plot along the z-axis at $t = 6\mu s$. 8,186 SPH particles

(d) SPFMax plasma number density line plot along the z-axis at $t = 6\mu s$. 41,636 SPH particles

Figure B.12: Comparison Ref. [9] and SPFMax line density plots at $t = 6\mu s$.

Symmetrical behavior, but the sharp spikes are due to the data being outside the temporal range of energy conservation. So, the higher-resolution case matches better here.

These results are mostly in line with previous integrated density line plots, but differ in other ways. Nagamine and Nakashima show a clear spike of height
\( n = 2.3 \times 10^{20} \) at \( z = 0 \) m (see Figure B.12a), which is absent from the 1,156 and 8,186 particle cases (see Figure B.12b-c), neither of which have clear spikes in the correct \( z \) location. However, the 41,636 particle case has a spike near \( z = 0 \) (see Figure B.12d) but it is only \( n = 0.8 \times 10^{20}, 65\% \) lower. In all cases, energy is not conserved and the temporal range is exceeded.

So for the 1D plots, it appears that, despite exceeding the temporal range, the high resolution plot is best in all cases, as opposed to only being best at earlier times, with the lower particle densities matching better at later times. This might be because we did not integrate the line densities, and if we did, we would get results similar to the ones from the 3D and 2D plots.

B.2.3 Axial nozzle test case

For the axial nozzle test case, we wish to compare SPFMax results with results from Cassibry et al. [46] as well as from T. Morita et al. [91], with the former generating their results using the latest version of the 3D hybrid PIC code based on our guidance.

To begin with, we must start with looking at the energy graph and determine the time period over which the simulation is valid (conserves energy).

From Figure B.13, we can see that energy is conserved from \( t = 0 - 0.150 \mu s \). During this time period, we see similar trends in energy compared to the solenoidal case, where initially thermal energy is high and directed kinetic/current energies are low. Then, thermal energy decreases as the plasma expands, with that energy primarily going into diamagnetic current energy, so \( E_{\text{current}} \) increases sharply.
Figure B.13: Energies over time for the axial nozzle test case. *Total, $E_{\text{current}}$, Thermal, $KE_x, KE_y, KE_z$* 

$(t = 0 - 0.05\mu s)$. Kinetic energy increases as well. Then the current energy starts to decrease and directed kinetic energy ($KE_z$) increases $(t = 0.05 - 0.1\mu s)$. However, at this point, thermal energy starts to drastically decrease (in a divergence from the solenoidal case), leading the simulation to put all energy in current $(t = 0.1 - 0.15\mu s)$; then a feedback loop in the thermal conduction subroutine causes unstable, unwanted oscillations in thermal energy. These oscillations mess up energy conservation, leading total energy in the simulation to diverge at $t = 0.15\mu s$.

With temporal validity established, it is instructive to look at the performance output by the three different simulations. All references feature specific impulse over time, so it is prudent to use that measure.

As shown in Figure B.14a, Cassibry *et al.* has a peak around 8,000 sec at $t = 0.100\mu s$, before a slight drop and another peak at 9,000 sec. At this point, the older version of SPFMax crashes, leading to inaccurate results. Cassibry *et al.* also shows an increase of 3,000-4,000 sec with the nozzle turned on. In contrast, T. Morita
et al. shows a peak performance of 16,000 sec around $t = 0.500 \mu s$, before performance decreases to a steady-state value of 13,500 sec, both shown by the dashed line in Figure B.14b (the nozzle has a current of 10MA/coil - see Section 3.2.2.2). Finally, there are the results from the current version of SPFMax in Figure B.14c. These show an increase to a peak of around 7,000 sec at $t = 0.15 \mu s$. This is a 3,000 sec
increase over the bare pusher plate. This is more in-line with Cassibry et al. than T. Morita et al.

It is also consistent with the energy results from the solenoidal nozzle test case - there the Kinetic energy output by SPFMax as around half the energy output by the 3D hybrid PIC code. Here, the performance output by SPFMax is around half the performance of the 3D hybrid PIC code. The performance is around half probably because the PIC code considers more macroparticles, and is therefore able to simulate the plasma with higher granularity. This allows the PIC code to take into account more effects. We have found using past versions of SPFMax that performance tends to rise as particle resolution increases (but this is necessarily more computationally expensive). However, the fact that (as explained in the literature review), results from the PIC code come in 3-4x higher than results from experiments suggests that the PIC code might be overestimating the strength of forces in the plasma somewhat. The reality might be closer to SPFMax than the PIC code.

Regardless, the last point of comparison is plasma motion. Cassibry et al. do not show plasma motion in the nozzle, but T. Morita do. Their results are given in Figure B.15.

As shown in Figure B.15a-c, first the plasma (black) starts in its initial position in the nozzle (red), right next to the pusher plate (green). It then begins to expand around $t = 0.12\mu$s (see Figure B.15b), and thereafter, begins to interact with the nozzle (not shown). However, it seems to form two spiral vortexes near where it starts to interact with the nozzle (shown in the circled blue part of Figure B.15c). These keep the plasma column tight and away from the nozzle. They might develop
(a) Ref. [91] axial nozzle test case plasma initial position.

(b) Ref. [91] axial nozzle test case plasma $t = 70$ns.

(c) Ref. [91] axial nozzle test case plasma $t = 2000$ns (steady-state). *Particle spiral vortexes emphasized with blue annotations*

**Figure B.15:** Plasma motion in axial nozzle test case, T. Morita *et al.* results. *Nozzle in red, plasma in black, pusher plate in green.*
due to a high-frequency oscillation in the magnetic field T. Morita et al. report occurred during their simulation.

As shown in Figure B.16, the results SPFMax generates are initially the same, but progress fairly differently, especially when we look at images past the time when energy is conserved. The plasma/plate (black) and nozzle (gold) start at the same initial position (see Figure B.16a), and the plasma expands off the plate roughly the same amount as in the results from T. Morita et al. As shown in Figure B.16b, the plasma roughly expands to $z = 0$ m, which is roughly in line with where it is in Figure B.15b. However, also shown in Figure B.16b are several particles that escape through the pusher plate toward the nozzle coils (particles with $z < -0.05$ m).
(a) Current version SPFMax axial nozzle test case plasma initial position.

(b) Current version SPFMax axial nozzle test case plasma $t = 70\text{ns}$.

(c) Current version SPFMax axial nozzle test case plasma $t = 1230\text{ns}$ (steady-state).

Figure B.16: Plasma motion in axial nozzle test case, SPFMax results. *Nozzle in gold, plasma in black, pusher plate in black.*
Unfortunately, this is non-physical but an unavoidable aspect of the simulation. Some particles traveled through the pusher plate in all runs we did, but at least fortunately in this run they were negligible in number compared to the bulk of the plasma.

Past the time when energy is conserved, plasma motion stops varying and reaches a steady constant value shown in Figure B.16c, with the plasma off the nozzle and pusher plate, and its bulk centered at (0,0). This is somewhat similar to Figure B.15c, but plasma motion is not as directed. It mostly just stays in one spot. Though it does stay away from the nozzle like in Figure B.15c.

In summary, for the axial nozzle test case, with a temporal range of $t = 0 - 0.15\mu s$, SPFMax show performance results about half of what is found in Ref. [91] by T. Morita et al. This is in line with other, peer-reviewed results from SPFMax [46], and in line with previous comparisons between results from the 3D hybrid PIC code and SPFMax. However, SPFMax might be closer to reality as results from the 3D hybrid PIC code tend to exceed experiments by a factor of 3-4x. With regard to plasma behavior, it is difficult to tell as the temporal range is quite short, but even outside the temporal range, plasma position at least seems similar.

### B.3 Axial nozzle study results

For the axial nozzle study, we start with the base case. It has a per-strut current of 15 MA.
B.3.1 Axial nozzle base case

Firstly, as before, we need to start with the energy graph to establish the temporal range of validity. As shown in Figure B.17, energy mostly is conserved until $t = 28.20\mu s$ (flat purple line), then it increases. There is a slight bump in the energy curve at $t = 12.52\mu s$ but this represents a change in total energy of 7%, and is therefore negligible.

Additionally, the behavior of the energy curves follows the same pattern as previous cases; initially thermal energy is high, but decreases as the plasma cools and energy goes into the diamagnetic current ($E_{\text{current}}$ increases from $t = 0 - 12.52\mu s$). Eventually $E_{\text{current}}$ reaches a maximum as the plasma reaches its farthest point of expansion (see Figure B.16c below). Then, directed kinetic energy ($KE_z$) increases as bit as thermal energy and current energy fall ($t = 12.52 - 13.1\mu s$). Interestingly, as this point thermal energy discontinuously increases and steals energy from $KE_z$,

Figure B.17: Energies over time for the axial nozzle base case. Total, $E_{\text{current}}$, Thermal, $KE_x$, $KE_y$, $KE_z$
resulting in a decrease in $KE_z$ and eventually an increase in thermal energy again (from $t = 15 - 28.20\mu s$). At this point, thermal energy discontinuously decreases along with current energy and directed kinetic energy dominates - this is non-physical and probably due to an error in the equation of state somewhere. The total energy curve falls back down to its initial value, and goes up and down a couple of times before thermal energy discontinuously decreases toward the end of the simulation, breaking energy conservation for the rest of the simulation. Nevertheless, Figure B.17 establishes the temporal range of the simulation, $t = 0 - 28.20\mu s$.

Having established temporal range, let us now look at plasma motion. In Figure B.18, we see a visualized sequence of events that tells the same story as Figure B.17. The plasma starts off in the nozzle in Figure 4.21a high in thermal energy. It expands in Figure B.18b increasing the diamagnetic current, reaching its point of maximum expansion in Figure B.18c. Interestingly, the point of maximum expansion for an axial nozzle has some of the plasma outside the nozzle. This is because the magnetic field for an axial nozzle is higher outside the nozzle struts than inside. But after expanding to its maximum point outside the nozzle, the plasma starts to collapse in on itself as shown in Figure B.18d. The plasma expansion is guided, and particles begin to leave the nozzle through its top ($z > 3$) as shown in Figure B.18e, but also it looks like the plasma begins to expand again, leaving the nozzle through the sides and bottom. This is most curious, especially because energy conservation is broken after this event.

To further illustrate what is going on inside the nozzle, next we look at the temperature and pressure of the plasma as it changes throughout time. Beginning
Figure B.18: Axial nozzle base case plasma motion. *Nozzle in gold, plasma in black.*
(a) Axial nozzle plasma temperature and velocity $t = 0 \, \mu s$.

(b) Axial nozzle plasma temperature and velocity $t = 8.43 \, \mu s$.

(c) Axial nozzle plasma temperature and velocity $t = 10.46 \, \mu s$.

(d) Axial nozzle plasma temperature and velocity $t = 12.52 \, \mu s$.

(e) Axial nozzle plasma temperature and velocity $t = 21.30 \, \mu s$.

(f) Axial nozzle plasma temperature and velocity $t = 28.20 \, \mu s$.

**Figure B.19**: Axial nozzle base case plasma temperature and velocity throughout time. *Plasma Temperature color plot, velocity orange vector plot, nozzle in gold.*
with the temperature plots, as shown in Figure B.19a, the plasma starts not expanded and at a constant temperature (around 100 eV). Then, the plasma begins to expand and cool with particles reflecting off the base of the nozzle; see the grouping of 5 SPH particles at the bottom of the nozzle in a v-shape in Figure B.19b that have their expansion halted in Figure B.19c and are reflected back into the nozzle in Figure B.19d. This compression creates localized heating in the bottom of the nozzle, as illustrated by the ring of particles at 0.1 eV at the bottom of Figure B.19d, whereas the rest of the plasma in this figure is colder at 0.01 eV. This heating is due to the particle reflection, as a pressure wave travels from the bottom through the top of the nozzle.

The pressure wave is more easily seen in the pressure plots, but in these plots it is manifest as a heating wave (region of $T = 10$ eV) that travels up the nozzle, reaching $z = 1$ at $t = 21.30 \, \mu s$ as shown in Figure B.19e, and continuing up through the rest of the nozzle in Figure B.19f.

The pressure wave is more obvious in Figure B.20. In Figure B.20a, the outside of the plasma starts at constant pressure, over $10^9$ Pa. As the plasma expands, its pressure drops to $10^7$ Pa (Figure B.20b-c), with the outer edges at a slightly lower pressure of $10^6$ Pa. However, after the plasma impacts the bottom of the nozzle and reaches its point of maximal expansion, a pressure wave begins at the bottom of the plasma (region of $10^5$ Pa whereas bulk plasma is $10^4$ Pa in Figure B.20d) that traverses up the plasma column. This motion of the pressure wave is shown in Figure B.20e-f, with the bulk of the wave constituting the region that is $10^8$. 
(a) Axial nozzle plasma pressure and velocity $t = 0 \, \mu s$.

(b) Axial nozzle plasma pressure and velocity $t = 8.43 \, \mu s$.

(c) Axial nozzle plasma pressure and velocity $t = 10.46 \, \mu s$.

(d) Axial nozzle plasma pressure and velocity $t = 12.52 \, \mu s$.

(e) Axial nozzle plasma pressure and velocity $t = 21.30 \, \mu s$.

(f) Axial nozzle plasma pressure and velocity $t = 28.20 \, \mu s$.

Figure B.20: Axial nozzle base case plasma pressure and velocity throughout time. *Plasma Temperature color plot, velocity orange vector plot, nozzle in gold.*
We end our coverage of the base case of the axial nozzle with presenting the performance from this run. Here, performance is quantified by impulse bit (which for a system operating at 1 Hz is the same as thrust), specific impulse, and nozzle efficiency (see Eq. (2.10) for that definition). Graphs of each quantity are presented over time in Figure B.21.

Interestingly, the final values of quantities are less than the initial values, similar to Figure B.14b. This is most likely because, after reaching its point of maximum
expansion, some of the plasma gets drawn back into the nozzle (see Figure B.18c-d) and its thermal energy increases (see Figure B.19e-f). This ‘drawing in’ is because some of the plasma has become magnetized with insufficient energy to escape the nozzle. This is representative of the plasma detachment problem that is anticipated for both steady-state magnetic nozzles [4,75] and is present here. Because of this effect, the impulse produced in the z-direction decreases. This reduces specific impulse and nozzle efficiency accordingly. However, both remain fairly high; final impulse bit tops is 8 kNs (down from a high of 12 kNs), final specific impulse is around 1,000 sec (down from a high of about 2,000 sec), and nozzle efficiency ends at 0.16 (down from a high of 0.26). While already performing slightly better than an NTP engine, the efficiency numbers tell us that this design is not optimized. Therefore, in the next section we will look at departures from this design and see if performance improves.

B.3.2 Axial nozzle double current

For this simulation, instead of using 15 MA/strut, we doubled the current in each strut to yield 30 MA/strut. As we will see, doubling the current greatly increased the magnetic field, and lead to magnetic drag.

Firstly, starting with the graph of energy over time, Figure B.22, we see that the period for temporal validity is $t = 0 - 38 \, \mu s$. Within this period, the usual sequence of events happens; plasma thermal energy starts high, it decreases as the plasma expands, current energy increases along with the kinetic energy as the gas expands. Interestingly, the kinetic energy increases faster than the current energy, so the kinetic energy actually exceeds the thermal and current energy from $t = 1 - 12 \, \mu s$.
Figure B.22: Energies over time for the axial nozzle case, 30 MA/strut. Total, \( E_{current} \), Thermal, \( KE_x \), \( KE_y \), \( KE_z \).

in Figure B.22. Then, the current energy exceeds both and the kinetic energy starts falling; it is almost as if the diamagnetic current takes energy from the directed kinetic energy instead of the usual thermal energy during \( t = 12 - 18 \) µs. After the current energy starts to fall, the thermal and kinetic energy start to increase again, but the thermal energy increases faster until energy conservation ends at \( t = 38 \) µs. Thermal energy might start increasing because of Ohmic heating in the plasma from the high diamagnetic current; we use a resistive plasma model in contrast to previous authors. After \( t = 38 \) µs there are drops in the thermal energy, and the total energy returns to its initial value, but these regions are sufficiently outside the range of temporal validity that reporting results from them is not helpful.

Figure B.23 expands upon the story told in Figure B.22; initially, the motion is the same as before where the plasma expands (Figure B.23a-b). However, when the plasma reaches its point of maximum expansion and begins to collapse in (Figure B.23c) the magnetic field in the center is higher. The diamagnetic cur-
Figure B.23: Axial nozzle double current case (30 MA/strut) plasma motion over time. Plasma in position in black, velocity vectors in yellow, nozzle in gold.
rent increases to compensate for this increased applied field, but resistance is futile. The bulk of the plasma is magnetize, and so cannot escape (Figure B.23d). This is evidenced by the fact that most of those particles have not moved in Figure B.23e.

These plasma behaviors result in the performance curves in Figure B.24. Performance increases to a local maximum around $t = 10 \, \mu s$ as the plasma reaches its point of maximum expansion, and performance continues to increase haltingly ($t = 15$
Figure B.25: Energy over time in J in the plasma for the axial nozzle half current case (7.5 MA/strut). Total, $E_{\text{current}}$, Thermal, $KE_x$, $KE_y$, $KE_z$

$\mu s$) until the plasma is drawn back into the nozzle and performance decreases to 0 (the region without data in Figure B.24a-b). Performance increases again as some plasma is able to make it outside and leave the nozzle; final performance is 5 kNs impulse bit, 400 sec specific impulse, and a nozzle efficiency of 0.08, which is a significant decrease from before. Doubling the current did not result in a more efficient nozzle design; so next we tried halving it.

### B.3.3 Axial nozzle half current

Instead of 15 MA/strut, for this case we keep the nozzle and plasma geometry the same, but use 7.5 MA/strut instead. As we see, halving the current reduced magnetic drag, and lead to a modest increase in performance.

Beginning with the energy graph (Figure B.25) we see that energy is conserved from $t = 0 - 18.85 \mu s$. Within this range, a similar story to previous cases emerges. Thermal energy starts high, and directed kinetic ($KE_z$) and current energy ($E_{\text{current}}$)
start low. As the plasma expands, kinetic and current energy increase, but in this case kinetic energy increases faster than current energy, due to the weaker magnetic field present in the nozzle. The weaker magnetic field will induce a weaker diamagnetic current in the plasma (as the diamagnetic current developed to resist the applied field); this weaker current is present for the first 10 $\mu s$ of the simulation, until the current energy finally exceeds the thermal energy. Eventually, the current energy exceeds the kinetic energy (around 15 $\mu s$ or so) before the kinetic energy exceeds to match it ($t = 17.35 - 18.85 \mu s$) due to a spurious decrease in the thermal energy. This spurious decrease causes further increases in kinetic and current energy that break energy conservation.

But, in this case, it is interesting how low current energy stays relative to all other quantities; this suggests the plasma is 'overexpanded' relative to cases previous; the nozzle is not pushing on it hard enough to confine the plasma early. This is actually beneficial, as we shall see in the performance graphs.

To see the energy graph visualized in the plasma motion, initially (like in all cases) that plasma starts off compact and with high thermal energy as shown in Figure B.26a. The plasma expands, increasing its directed kinetic energy (Figure B.26b-c), and its expansion is not as arrested as earlier cases. It is allowed farther outside the nozzle in previous cases due to the reduced diamagnetic current, from the reduced applied field. However, it eventually reaches a point where the diamagnetic current is strong enough to deflect the plasma back into the nozzle (Figure B.26d-e) but at this point the thermal energy drops precipitously leading the plasma to expand in a non-physical manner - see Figure B.26f.
Figure B.26: Axial nozzle half current case (7.5 MA/strut) plasma motion over time. Plasma in position in black, velocity vectors in yellow, nozzle in gold.
(a) Axial nozzle half current case impulse bit (Ns).

(b) Axial nozzle half current case specific impulse (sec).

(c) Axial nozzle half current case nozzle efficiency.

**Figure B.27**: Axial nozzle half current case (7.5 MA/strut) performance through time.

With regard to performance, this case shows the highest performance of the cases, as shown in Figure B.27. Final and maximal impulse bit is 20 kNs, specific impulse is 2,000 sec, and nozzle efficiency is 0.3. These are close to the peak values of the base case, but here the magnetic field is not high enough to drag the plasma back into the nozzle. However, there is still some performance left as evidenced by the
relatively low nozzle efficiency, but this might be realized with an even lower current; it seems earlier cases suffered from magnetic drag.

B.4 Summary

In summary, SPFMax is able to reproduce features important for accurately modeling plasmas in magnetic nozzles. It is able to reproduce the diamagnetic cavity and diamagnetic current within temporal restraints imposed by energy conservation. It shows pretty good agreement with prior work ([9,46]), especially when low-particle resolution results are compared at later times and high-particle resolution results are compared at earlier times. It seems that particle resolution inversely affects the range of temporal validity; higher particle resolutions are able to conserve energy for smaller amounts of computational time (before small errors build up). Also, quantitative performance for these is around half of that from the PIC code. This might be due to modeling differences inherent in the two methods, but this difference is acceptable given that the PIC code over-predicts experimental results by 3-4x. Lastly, using SPFMax for a pulsed nuclear nozzle design results in an axial nozzle with 32 struts, 7.5 MA/strut, with the fission-fusion plasma that starts 1 m away from the nozzle apex. This design maximizes performance by minimizing magnetic drag, through delaying the formation of a strong diamagnetic field to later in the simulation. The design study suggests that minimizing magnetic drag, while still ensuring plasma deflection, is crucial in magnetic nozzle design for fission, fusion or hybrid plasmas.
APPENDIX C

DERIVATION OF EQUATION FOR THE CHANGE IN CURRENT
WITH RESPECT TO TIME

To review, SPFMax uses the following equation to calculate the current at each time step of the electromagnetic field solver

\[
\frac{d\vec{j}}{dt} = \frac{\lambda_p}{L_p A_p} \left( \vec{v} \times \vec{B} - \frac{1}{\sigma} \vec{j} \right) \quad (C.1)
\]

In SPFMax, this is implemented as

\[
\begin{align*}
dj_x &= H*(vy*Bz-vz*By) - jx*(1-exp(-(H/conductivity)*dt))/dt; \\
dj_y &= H*(vz*Bx-vx*Bz) - jy*(1-exp(-(H/conductivity)*dt))/dt; \\
dj_z &= H*(vx*By-vy*Bx) - jz*(1-exp(-(H/conductivity)*dt))/dt;
\end{align*}
\]

To derive Eq. (C.1) we take heritage from transmission-line matrix (TLM) [95] and transmission line (TL) modeling [96]. We treat the collection of SPH particles as an unstructured network of 3D transmission lines, with current flowing through each particle based on the particle’s inductance and resistance. Note that currents cannot flow across particles and capacitive effects are ignored here because plasma particles are assumed to be sufficiently conductive such that no charge separation occurs across
particle surface. For a 1D transmission line, the relevant equation is [68]

\[ RI + L \frac{dI}{dt} = \varepsilon_{\text{applied}} \]  \hspace{1cm} (C.2)

where \( R \) is the resistance of the line, \( I \) is the current, \( L \) is the inductance, and \( \varepsilon_{\text{net}} \) is the net voltage across the line [68]. Re-arranging this equation to solve for \( \frac{dI}{dt} \) yields

\[ \frac{dI}{dt} = \frac{\varepsilon_{\text{applied}} - RI}{L} \]  \hspace{1cm} (C.3)

To extend this to 3D, let us suppose that both the resistance and the inductance do not change based on direction (the resistance in the x-direction is the same as the resistance in the y-direction, which is the same as the resistance in the z-direction, etc.). The former is reasonable for our plasmas because they are high temperature and therefore high-conductivity (and therefore low resistivity). The latter is made to simplify the equation set. In 3D, the change in current through the line (SPH particle) will be

\[ \frac{d\vec{I}}{dt} = \frac{\varepsilon_{\text{applied}} - R\vec{I}}{L} \]  \hspace{1cm} (C.4)

Now, we expand the following terms. Firstly, for an SPH particle, the current is equal to the current density times the cross-sectional area of the plasma particle or

\[ \vec{I} = jA_p \]  \hspace{1cm} (C.5)
Second, the resistance $R$ is equal to the length scale of the plasma particle divided by the cross-sectional area times the conductivity or

$$R = \frac{\lambda_p}{\sigma A_p} \quad (C.6)$$

And third, we assume the inductance of a plasma particle is equal to that of a single-turn ideal solenoidal, with $\mu_r = 1$

$$L = L_p = \frac{\mu_0 A_p}{\lambda_p} \quad (C.7)$$

Note that we assume the cross-sectional area in all directions of an SPH particle is the same because we assume the particles are roughly spherical. Substituting Eq. (C.5) - (C.6) into Eq. (C.4) results in

$$A_p \frac{d \vec{j}}{dt} = \frac{\varepsilon_{\text{applied}} - \frac{\lambda_p}{\sigma} \vec{j}}{L_p} \quad (C.8)$$

We shall substitute Eq. (C.7) later. As a next step, we solve for $\frac{d \vec{j}}{dt}$

$$\frac{d \vec{j}}{dt} = \frac{\varepsilon_{\text{applied}} - \frac{\lambda_p}{\sigma} \vec{j}}{L_p A_p} \quad (C.9)$$

Now, the applied voltage across an SPH particle is equal to the applied electric field across the SPH particle times the length scale of the particle

$$\varepsilon_{\text{applied}} = \vec{E}_{\text{applied}} \lambda_p \quad (C.10)$$
Substituting this into Eq. (C.8) yields

\[ \frac{d\mathbf{j}}{dt} = \frac{E_{\text{applied}}\mathbf{j} - \lambda_p j}{L_p A_p} \]  

(C.11)

Next, it is necessary to determine electric field applied to the particle, which is the external field plus the Lorentz contribution

\[ E_{\text{applied}} = E_{\text{ext}} + \mathbf{v} \times \mathbf{B} \]  

(C.12)

This is the same as the modified Ohm’s law, as in other magnetohydrodynamic codes such as MACH2, but with the resistive term dropped (as we have already included this term in the transmission line formulation) [97]. SPFMax has the capability to model \( E_{\text{ext}} \neq 0 \), but for a magnetic nozzle \( E_{\text{ext}} = 0 \) as no electrodes are connected to the plasma. Accordingly, we drop \( E_{\text{ext}} \) in this derivation. After doing as such, substituting the result into Eq. (C.11), and factoring out \( \lambda_p \) the result is

\[ \frac{d\mathbf{j}}{dt} = \frac{\lambda_p}{L_p A_p} \left( \mathbf{v} \times \mathbf{B} - \frac{1}{\sigma} \right) \]  

(C.13)

This matches with Eq. (3.61) and Eq. (C.1).

Note that in the above equation the term \( \frac{\lambda_p}{L_p A_p} \) appears frequently. In the code we capture this term with \( H \), where

\[ H = \frac{\lambda_p}{L_p A_p} \]  

(C.14)
Note that, due to the presence of $\vec{J}$ in Eq. (C.13), if the time step in the electromagnetic field solver, $\Delta t$, exceeds the LR time constant, $\vec{J}$ will oscillate in sign rapidly, leading to divergence. Isolating just the resistive term yields

$$\frac{d\vec{J}}{dt_{res}} = -\frac{H}{\sigma} \vec{J}$$  \hspace{1cm} (C.15)$$

Therefore the time constant for this equation is

$$\tau_{LR} = \frac{\sigma}{H}$$ \hspace{1cm} (C.16)$$

For reference, for the highly conductive plasmas we use in the nozzle, $\sigma = 10^4 - 10^7$ Si/m, and for the runs we did $h_{ab} \approx 1$ mm in the earliest part of the simulation. Using $\lambda_p = h_{ab}$ and $A_p = \pi h_{ab}^2$ results in

$$H = \frac{\lambda_p}{L_p A_p} = \frac{h_{ab}}{\left(\frac{\mu_0 \pi h_{ab}^2}{h_{ab}}\right) \pi h_{ab}^2} = \frac{1}{\mu_0 \pi^2 h_{ab}^2} = \frac{1}{4\pi 10^{-7} \pi^2 0.001^2}$$ \hspace{1cm} (C.17)$$

which simplifies to $8.1 \times 10^{10}$ Si/(m-s). Note that at later times, $h_{ab}$ increases to $\approx 1$ m but because its in the denominator of $H$, this means $H$ decreases. At any rate, if $H = 8.1 \times 10^{10}$ Si/(m-s) at worst, and $\sigma = 10^4$ Si/m at worst, then

$$\tau_{LR} = \frac{10^4}{8.1 \times 10^{10}} = 120 \text{ ns}$$ \hspace{1cm} (C.18)$$

which is many times above typical time step length of the electromagnetic solver (about 0.5 ns). However, for smaller, low conductivity plasmas, or for when the
plasma has exited the nozzle and cools sufficiently to exit the plasma state, conductivity can be as low as $10^{-3}$ Si/m. Assuming the plasma has the same $h_{ab}$ as before results in

$$\tau_{LR} = \frac{10^{-3}}{8.1 \times 10^{10}} = 0.000012 \text{ ns}$$

(C.19)

This is much below the typical time length step of the electromagnetic solver. Additionally, because the conductivity is so low, the coefficient in front of the resistive term $\frac{H}{\sigma}$ is very high

$$\frac{H}{\sigma} = \tau_{LR}^{-1} = (0.000012 \times 10^{-9})^{-1} = 8.1 \times 10^{13} \text{ Hz}$$

(C.20)

especially when compared to the magnitude of the inductive term when the plasma has exited the nozzle (and has cooled)

$$H \left( \vec{v} \times \vec{B} \right) = 8.1 \times 10^{10} \times (10^{4} \times 10^{-3}) = 8.1 \times 10^{11} \text{ Hz}$$

(C.21)

The coefficient in front of the resistive term dominates the inductive term, therefore the inductive term can be dropped. Returning to Eq. (C.13)

$$\frac{d\vec{j}}{dt} = \frac{d\vec{j}}{dt_{res}} = -\frac{H}{\sigma} \vec{j}$$

(C.22)

Assuming $H$ and $\sigma$ do not vary significantly with time (which is reasonable given that the plasma has cooled and is going relatively slow, therefore its volume does not
change much) we can get an approximate solution for the differential equation as

\[ \vec{j}(t) = \vec{j}_0 e^{-\frac{H}{\sigma} t} \]  

(C.23)

Over a small \( \Delta t = t_2 - t_1 \), this can be approximated as

\[ \Delta \vec{j} = \vec{j}_0 e^{-\frac{H}{\sigma} t_2} - \vec{j}_0 e^{-\frac{H}{\sigma} t_1} \]  

(C.24)

Which can be simplified as follows

\[ \vec{j}_0 e^{-\frac{H}{\sigma} t_2} - \vec{j}_0 e^{-\frac{H}{\sigma} t_1} = \vec{j}_0 \left( e^{-\frac{H}{\sigma} t_2} - e^{-\frac{H}{\sigma} t_1} \right) = \vec{j}_0 e^{-\frac{H}{\sigma} t_1} \left( e^{-\frac{H}{\sigma} t_2} - e^{-\frac{H}{\sigma} t_1} \right) = \vec{j}_0 e^{-\frac{H}{\sigma} t_1} \left( e^{-\frac{H}{\sigma} \Delta t} - 1 \right) \]  

(C.25)

Putting this back into Eq. (C.13) means replacing the resistive term \(-\frac{H}{\sigma} \vec{j}\)

\[ \frac{\Delta \vec{j}}{\Delta t} = H \left( \vec{v} \times \vec{B} \right) - \vec{j} e^{-\frac{H}{\sigma} \Delta t} \]  

(C.26)

Expanding the cross product and implementing yields the code lines given at the beginning of this chapter

\[ \text{d}j_x/\text{d}t = H*(\text{vy}*\text{Bz}-\text{vz}*\text{By}) - jx*(1-exp(-\text{(H/conductivity)*dt}))/\text{dt}; \]
\[ \text{d}j_y/\text{d}t = H*(\text{vz}*\text{Bx}-\text{vx}*\text{Bz}) - jy*(1-exp(-\text{(H/conductivity)*dt}))/\text{dt}; \]
\[ \text{d}j_z/\text{d}t = H*(\text{vx}*\text{By}-\text{vy}*\text{Bx}) - jz*(1-exp(-\text{(H/conductivity)*dt}))/\text{dt}; \]
Note that $\Delta t$ is replaced by $dt$, and the $dt$ in the denominator at the end of each line is to compensate for when the quantity will be multiplied by $dt$ in the Runge-Kutta algorithm.