2011

Neighborhood sum parameters on graphs

Frank Allen O'Neal

NEIGHBORHOOD SUM PARAMETERS ON GRAPHS

by

FRANK ALLEN O'NEAL

A DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy
in
The Department of Mathematical Sciences
to
The School of Graduate Studies
of
The University of Alabama in Huntsville

HUNTSVILLE, ALABAMA

2011

In presenting this dissertation in partial fulfillment of the requirements for a doctoral degree from The University of Alabama in Huntsville, I agree that the Library of this University shall make it freely available for inspection. I further agree that permission for extensive copying for scholarly purposes may be granted by my advisor or, in his/her absence, by the Chair of the Department or the Dean of the School of Graduate Studies. It is also understood that due recognition shall be given to me and to The University of Alabama in Huntsville in any scholarly use which may be made of any material in this dissertation.

Allen O'Neal 10/31/2011

Frank Allen O'Neal (date)

DISSERTATION APPROVAL FORM

Submitted by Frank Allen O'Neal in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Applied Mathematics and accepted on behalf of the Faculty of the School of Graduate Studies by the dissertation committee.

We, the undersigned members of the Graduate Faculty of The University of Alabama in Huntsville, certify that we have advised and/or supervised the candidate of the work described in this dissertation. We further certify that we have reviewed the dissertation manuscript and approve it in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Applied Mathematics.

Peter J Slater	11 Oct 11	Committee Chair
Dr. Peter J. Slater	(Date)	
Alan Sprague	Oct 11, 2011	
Dr. Alan Sprague	(Date)	
	Oct. 11, 2011	
Dr. Min Sun	(Date)	
	Oct 11, 2011	
Dr. Dongsheng Wu	(Date)	
	Oct 11, 2011	
Dr. Huaming Zhang	(Date)	
	Oct 11, 2011	
Dr. Grant Zhang	(Date)	
	10-12-2011	Department Chair
Dr. Jia Li	(Date)	
	10/18/11	College Dean
Dr. John D. Fix	(Date)	
Rhonda Kay Gaede	11/28/11	Graduate Dean
Dr. Rhonda Kay Gaede	(Date)	

ABSTRACT
School of Graduate Studies
The University of Alabama in Huntsville

Degree Doctor of Philosophy College/Dept. Science/Mathematical Sciences

Name of Candidate Frank Allen O'Neal

Title Neighborhood Sum Parameters on Graphs

This work considers the problem of distributing weights onto the vertices of a graph in an equitable fashion. If weights can be distributed in a manner such that each vertex has exactly the same total amount of weight in some specified neighborhood, then the graph is a vertex magic graph, and the equivalent amount of weight is a vertex magic constant. Vertex magic graphs have been extensively studied within the field of graph labeling, and, in general, graphs will not be vertex magic. We develop measures that describe how close an arbitrary graph is to vertex magic.

This work makes three major contributions to the body of knowledge within graph labeling. The first is that it connects the graph labeling problem of finding vertex magic graphs to the study of domination parameters. Bounds that are based on the fractional domination number and that state how close to vertex magic an arbitrary graph may be are developed. These bounds are used to establish that vertex magic constants, when they exist, are unique.

The second major contribution of this work is that it develops precise statements of how close to vertex magic some specific families of graphs are. For example, 2-regular graphs are not generally vertex magic. This work provides precise measures of how far

from vertex magic many of the 2-regular graphs are. Other families of graphs including the union of complete graphs and bipartite graphs are examined as well.

The third major contribution of this work is that it provides a framework for studying graph labeling problems that is richer than simply classifying graphs as exhibiting a certain property or not. This framework is developed in the context of vertex magic labelings. However, the idea of studying graph labeling problems in this manner is not limited to vertex magic labelings. An extension of this framework to the edge-magic labelings is suggested.

Abstract Approval: Committee Chair Peter J. Slater 21 Oct 11
Dr. Peter J. Slater

Department Chair 10-31-2011
Dr. Jia Li

Graduate Dean Rhonda Kay Gaede 11-28-2011
Dr. Rhonda Kay Gaede

ACKNOWLEDGMENTS

To the numerous individuals, whose various efforts over many years have made this undertaking possible, I am sincerely grateful. To Mr. Don Newman, thank you for sowing and nurturing the vine of mathematical thought many years ago, when there was no evidence it would ever bear fruit. To Dr. Slater, thank you for your tireless efforts to prune and feed the plant as it grew. To all the members of the committee, thank you for your assistance in harvesting the fruit of this effort.

I wish also to thank my wife, Dr. Pam O'Neal, for her continual support and encouragement during this journey. To my parents, who enabled the undertaking and completion of this journey, thank you. To all my family and friends, thank you for always cheering me on throughout the entire course.

Every good and perfect gift is from God, and it is Him I must thank foremost. Thank You for placing in my life each of the individuals above; each has proven to be a good and perfect gift.

TABLE OF CONTENTS

LIST OF FIGURES

FIGURE PAGE

LIST OF TABLES

TABLE PAGE

LIST OF SYMBOLS

SYMBOL	DEFINITION
A	Adjacency matrix of a graph
A_D	Distance D adjacency matrix of a graph
C_n	The cycle of order n
kC_t	The union of k cycles of order t
$d(u,v)$	Distance between vertices u and v
$deg(v)$	The number of vertices adjacent to vertex v
$deg_D(v)$	The size of the distance D neighborhood of vertex v
$\delta(G)$	Minimum degree of a vertex in graph G
$\delta_D(G)$	Minimum size of a distance D neighborhood
$\Delta(G)$	Maximum degree of a vertex in graph G
$\Delta_D(G)$	Maximum size of a distance D neighborhood
G^C	The complement of graph G
$\gamma(G)$	Domination number of the graph G
$\gamma(G;D)$	D neighborhood domination number of the graph G
$\gamma_f(G)$	Fractional (closed) domination number of the graph G
$\gamma_f^o(G)$	Fractional open domination number of the graph G

$\gamma_f(G;D)$	D neighborhood fractional domination number of the graph G
K_n	The complete graph on n vertices
pK_t	The union of p complete graphs on t vertices
$K_{p_1,p_2,\ldots,p_t}$	The complete multi-partite graph with partite sets having cardinality $p_1, p_2, \ldots, p_t$
m	Size of a graph; that is, the number of edges in the graph
n	Order of a graph; that is, the number of vertices in the graph
N	Closed neighborhood matrix of a graph
$N(v)$	Open neighborhood of vertex v
$N[v]$	Closed neighborhood of vertex v
$N_D(v)$	Set of vertices whose distance from v is in set D
$\mathbb{N}$	The set of all non-negative integers
$\mathbb{R}^+$	The set of all non-negative real numbers
$\rho(G)$	Packing number of the graph G
$\rho(G;D)$	D neighborhood packing number of the graph G
$\rho_f(G)$	Fractional (closed) packing number of the graph G
$\rho_f^o(G)$	Fractional open packing number of the graph G
$\rho_f(G;D)$	D neighborhood fractional packing number of the graph G
σ_W	Sum of the elements of a set W

W_{MAX} Maximum element of a set W

W_{MIN} Minimum element of a set W

To my parents

Faith is different from proof; the latter is human, the former is a Gift from God.

—Pascal

CHAPTER 1

INTRODUCTION

1.1 Purpose

In a graph labeling problem, the vertices and/or edges are assigned values from a fixed set. The assignment of labels to a graph induces values that are a result of the labels used and the structure of the graph itself. For example, on a graph with n vertices and m edges, where $n \leq m+1$, one can assign to each vertex a unique label from the set of integers $\{0,1,\ldots,m\}$. Once these assignments are made, one can associate with each edge the absolute difference of the labels assigned to endpoints of the edge. An assignment of labels in this fashion that produces distinct induced values for the edges is called a *graceful labeling* of the graph; a graph for which such a labeling is possible is called a *graceful graph*. It is this type of labeling that has led to the conjecture that all trees are graceful, a conjecture which remains one of the most significant open problems in all of discrete mathematics.

In Gallian [5] there are nearly 1200 references cited for graph labeling publications. As stated by Gallian in the introduction of his survey, despite the large number of publications, few results of a general nature exist. Many of the published results demonstrate that a specific family of graphs possesses a specific property or not. For example, in Huang [10]

it is shown that all trees with at most four endpoints are graceful, and hence, all paths are graceful. This type of YES/NO classification is common to many of the works cited by Gallian [5].

The primary purpose of this research is to develop a more general framework for studying graph labeling problems than is provided by a simple YES/NO classification. For example, one might be interested in answering the question of how close to graceful is a non-graceful graph. In order to develop our framework, we will consider the graph labeling problem where the vertices of the graph are labeled, and then for each vertex, the labels a specified distance from the vertex are summed. Like graceful labelings, this latter type of labeling has been studied as a YES/NO type of problem, where the question asked is whether labels can be assigned in a fashion such that all the induced vertex sums are equal. By studying this problem in a more general context, we have been able to make some strong connections to other well known graph theoretic results, and thus provide results of a general nature. Moreover, the idea behind our approach can be extended to other graph labeling problems, and it is hoped, that by doing so, more results of a general nature can be achieved for a broader set of graph labeling problems.

In the remainder of Chapter 1, we will first describe some specific applications of the graph labeling problem with which we are concerned. Secondly, we will provide the requisite graph theory background on which the remainder of this dissertation is based. In Chapter 2, we will formally define the neighborhood sums labeling problem and our framework for analyzing such problems. In Chapter 3, we present the general results that have been achieved using our framework. It is here that the most significant contribution of this dissertation is made. We establish a connection between the neighborhood sums labeling

problem and the fractional domination and packing parameters of the graph. Domination and packing parameters have been studied extensively on their own, apart from any graph labeling context. By creating this connection, we are able to answer one of the more significant open graph labeling problems that was presented by Arumugam [2] at the 2010 International Workshop on Graph Labeling (IWOGL). In Chapter 4, we apply our framework to several different families of graphs. In particular, we present a nearly complete set of neighborhood sum parameters for cycles. In Chapter 5, we extend our framework for the notion of competitive games. This research exposes as many unanswered questions as it solves; several of the most notable topics for future research will be presented as part of our conclusion in Chapter 6.

1.2 Motivation and Applications

Many problems in operations research involve optimizing the benefit that is received from a constrained set of resources. Suppose we have a geographical area that can be modeled using a graph. The vertices in the graph would represent specific locations, and the edges in the graph would represent routes that can be used to travel or communicate in some fashion between locations. For example, the graph could be used to represent townships in a county, and the edges the street networks connecting the townships. As as another example, the vertices could represent potential sites for missile detection devices in a large geographical areas such as the Pacific Ocean or Europe, and the edges could represent some standardized distance between locations. Suppose we also have resources with different capabilities that can be placed at the locations. For example, in the first instance

the resource may be firetrucks with different water capacities; in the latter instance, the resource may be radars with different detection capabilities. In both cases, it is reasonable to assume that the resources will be effective at the location where they are positioned, as well as at locations that are close in proximity, for example, at all locations that can be reached by traversing a single edge. A reasonable objective in either case is to position the resources in a such a fashion that each location receives equitable benefit from the total set of resources.

In the strictest sense, equitable would mean each location receives exactly the same benefit from the resources. However, in many cases it will not be possible to provide equal coverage, hence, we need some way to measure the equitability of a particular arrangement. One measure of equitability would be to ensure that no one location had too much coverage; in this case we may seek to arrange resources so as to minimize the maximum amount of coverage provided to any one location. Another measure of equitability would be to ensure that no single location receives too little coverage; to measure equitability in this fashion we could arrange resources in such a fashion so as to maximize the minimum amount of coverage present at any one location. A third measure of equitability would be to ensure that the maximum disparity between coverage at any two locations was as small as possible, that is, we would minimize the difference between the maximum and minimum amounts of coverage provided to different locations. As we will see, these measures of equitability are not, in general, the same. That is, an arrangement of resources that minimizes the maximum amount of coverage will not, in general, be the same arrangement that maximizes the minimum amount of coverage.

Consider the specific example where we need to protect a sensitive site such as a military installation. Suppose the site is surrounded by two rings of boundary defenses such as fencing. Along each ring are placed 8 guards (16 in total) of various skill levels and experience. For the sake of argument suppose we rank order the capabilities of the guards from 1 to 16 and use that ranking as a measure of their ability to respond. In this case the most skilled guard would get the ranking of 16 and have 16 times the response capability as the guard who received the ranking of 1. Further suppose that once we position a guard, they can detect and respond to incidents at their location and at adjacent locations. The total response capability at any location could then be measured as the sum of the skill levels of the guards who can respond to an incident at that location. The layout of our boundary detection layers is shown in Figure 1.1 where the two layers of fencing are represented by the concentric circles. The smaller circles along each ring represent the locations of the guard posts. If two guard posts are connected by a line, then the guards positioned at those posts can mutually reinforce one another.

In the interest of providing equitable protection at all points along the perimeter of the installation, we need to determine how one should position the guards at the various guard posts. As it turns out in this case, there is a way to arrange the guards at the various posts so as to optimize all three measures of equitability simultaneously. That is, we can arrange the guards such that we

(i) minimize the maximum amount of coverage at any one location,

(ii) maximize the minimum amount of coverage at any one location, and

(iii) minimize the difference between the maximum and minimum amounts of coverage provided.

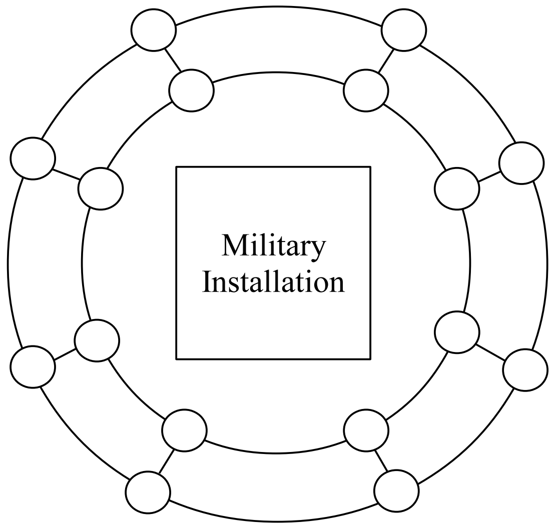

Figure 1.1: Application of neighborhood sums to facility protection

An arrangement of guards that achieves the objectives is shown in Figure 1.2. The labels at each guard post represent the capability of the guard stationed at that post. With the arrangement shown, the maximum amount of coverage provided to any one location is 35. Consider for example the location labeled 12; the total amount of protection provided at that location is $12+1+16+6=35$. The minimum amount of protection provided at any one location is 33. Consider for example the location labeled 1; the total amount of protection provided at this location is $1+15+5+12=33$. Hence, the difference between the minimum and maximum amounts of coverage provided is 2. We will establish in Theorem 4.70, that for this particular scenario, the guards cannot be arranged in a fashion that would be more equitable. That is, the guards cannot be arranged in a fashion such that the maximum amount of coverage is less than 35, or such that the minimum amount of coverage is more than 33, or in such a fashion that the difference between the maximum and minimum amounts of coverage is less than 2.

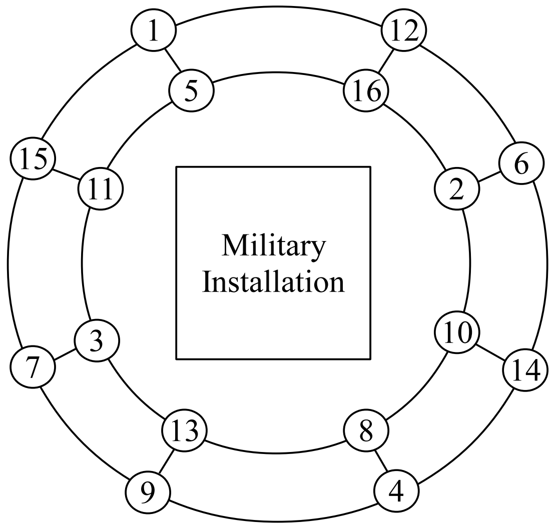

Figure 1.2: Optimal facility protection solution

At this point it bears mentioning that the layout of the guard posts above can easily be represented by a graph. One may have also wanted to use labels other than the integers $\{1,2,\ldots,16\}$ for representing the capabilities of the guards. Another generalization that could have been considered is the scenario where the guards can respond to incidents within some arbitrary distance from their station. For example, maybe the guards can mutually reinforce not only their neighbors, but also their neighbors' neighbors. We will consider each of these types of generalizations in our development.

In light of our purpose of developing a generalized framework for considering these types of problems, we should mention that seeking strict equality leads to a YES/NO type of classification. That is, for a particular scenario, strict equality can either be achieved, or it cannot. However, for all scenarios, regardless of the underlying graphical model, the labels used, or the distances at which locations can mutually reinforce one another, the measures of equitability introduced exist. Moreover, as we will show, in most scenarios

strict equality cannot be achieved, and among these scenarios, there is wide variation in how close to strict equality the most equitable arrangement can be.

1.3 Graph Theory Essentials

Before formally defining the neighborhood sum problem and our approach to analyzing it, we present the essential definitions and results from graph theory on which our results are based. Existing notation, terminology, and definitions will be consistent with Haynes et al. [9].

A *graph* $G = (V, E)$ is defined as an ordered pair of a set $V(G)$, whose elements are called *vertices*, and a set $E(G)$ whose elements, which are unordered pairs of vertices, are called *edges*. The cardinality of $V(G)$ is called the *order* of the graph G and will be denoted by n. The cardinality of $E(G)$ is called the *size* of the graph and will be denoted by m. For example, the graph H shown in Figure 1.3 has vertex set $V(H) = \{v_1, v_2, v_3, v_4, v_5\}$ and edge set $E(H) = \{v_1v_2, v_1v_3, v_2v_3, v_2v_4, v_3v_5, v_4v_5\}$. Graph H has order five and size six.

If G is a graph and $e = uv \in E(G)$, then vertices u and v are said to be *adjacent* to each other, and *incident* with edge e. For a vertex $v \in V(G)$, the set of all vertices adjacent to v is called the *open neighborhood* of v and is denoted by $N(v)$. The cardinality of $N(v)$ is called the *degree* of v and is denoted by $deg(v)$. If $deg(v) = 0$, then v is said to be *isolated*. The values of the minimum and maximum degrees of a vertex in G are denoted by $\delta(G) = min\{deg(v) : v \in V(G)\}$ and $\Delta(G) = max\{deg(u) : u \in V(G)\}$ respectively. If there exists a constant r such that for all $v \in V(G)$, $deg(v) = r$ (or equivalently, $\delta(G) = \Delta(G) = r$),

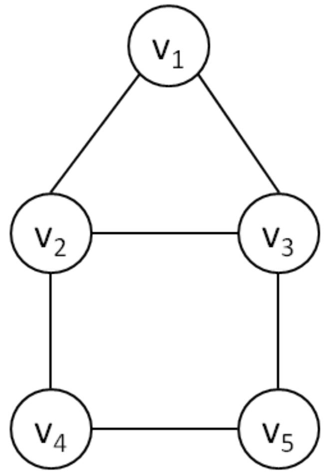

Figure 1.3: House Graph

graph G is said to be *r-regular*. If $V(G) = \{v_1, v_2, \ldots, v_n\}$, then the *degree sequence* of the graph G is $(deg(v_1), deg(v_2), \ldots, deg(v_n))$ and will usually be written in non-increasing order. The *closed neighborhood* of a vertex v is $N[v] = N(v) \bigcup \{v\}$. If $f : V(G) \to \mathbb{R}$ is any function, then we define $f(N(v)) = \sum_{u \in N(v)} f(u)$ and $f(N[v]) = \sum_{u \in N[v]} f(u)$.

Let the set of vertices of graph G be $V(G) = \{v_1, v_2, \ldots, v_n\}$. The *adjacency matrix* of a graph G is the $n \times n$ binary matrix $A = [a_{i,j}]$, where $a_{i,j} = 1$ if and only if $v_i v_j \in E(G)$. The *closed neighborhood matrix* of graph G is the matrix $N = I_n + A$, where I_n is the $n \times n$ identity matrix. Notice that the adjacency or closed neighborhood matrix for any graph is symmetric. For graph H from Figure 1.3 we have adjacency and closed neighborhood matrices as shown in Figure 1.4.

A *path* is a sequence of distinct adjacent vertices, and the *length* of the path is one less than the number of vertices in the path. Assume $u, v \in V(G)$ are distinct vertices of G, and that there exists a path beginning at u and terminating at v. The minimum length of such a path is called the *distance* from u to v and is denoted by $d(u, v)$. If for all $x, y \in V(G)$

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \qquad N = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Figure 1.4: Adjacency and closed neighborhood matrix of a graph

there exists a path from x to y, then graph G is said to be *connected*. If graph G is connected, then for each vertex $v \in V(G)$, the *eccentricity* of v is $ecc(v) = max\{d(v,u) : u \in V(G)\}$. For a connected graph G, the radius of G is $rad(G) = min\{ecc(v) : v \in V(G)\}$, and the diameter of G is $diam(G) = max\{ecc(u) : u \in V(G)\}$.

The *complement* of a graph G is the graph G^C, where $V(G^C) = V(G)$ and $uv \in E(G^C)$ if and only if $uv \notin E(G)$. Some common families of graphs that we will encounter in this dissertation are complete graphs, paths, cycles, and bipartite graphs. The *complete graph* of order n is the graph K_n, where, for all $u, v \in V(K_n)$, we have that $uv \in E(K_n)$; that is, all $\binom{n}{2}$ pairs of vertices are connected by an edge. K_n^C is the graph of order n that has no edges; that is, all vertices in the graph are isolated. The *path* of order n is the graph P_n, where $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(P_n) = \{v_i v_{i+1} : 1 \leq i \leq n-1\}$. The *cycle* of order n is the graph C_n, where $V(C_n) = V(P_n)$ and $E(C_n) = E(P_n) \bigcup \{v_n v_1\}$. A graph G is said to be *bipartite* if $V(G)$ can be partitioned into two sets S_1 and S_2, called the partite sets of G, such that for all $uv \in E(G)$, u and v are in different partite sets. The *complete bipartite graph* $K_{a,b}$ is a bipartite graph, where $V(K_{a,b}) = \{v_{1,1}, v_{1,2}, \ldots, v_{1,a}, v_{2,1}, v_{2,2}, \ldots, v_{2,b}\}$ and $E(K_{a,b}) = \{v_{1,i} v_{2,j} : 1 \leq i \leq a, 1 \leq j \leq b\}$. More generally, the *complete mutipartite graph* $K_{a_1,a_2,\ldots,a_k}$ has vertex set $V(K_{a_1,a_2,\ldots,a_k}) = \{v_{1,1}, \ldots, v_{1,a_1}, v_{2,1}, \ldots, v_{2,a_2}, \ldots, v_{k,1}, \ldots, v_{k,a_k}\}$ and edge set $E(K_{a_1,a_2,\ldots,a_k}) = \{v_{i,s} v_{j,t} : 1 \leq i < j \leq k, 1 \leq s \leq a_i, 1 \leq t \leq a_j\}$.

$$\begin{array}{rr}\gamma(G) = \text{MIN} & \sum_{i=1}^{n} X_i \\ \text{subject to} & NX \geq 1 \\ & x_i \in \{0,1\}\end{array} \qquad \begin{array}{rr}\rho(G) = \text{MAX} & \sum_{i=1}^{n} X_i \\ \text{subject to} & NX \leq 1 \\ & x_i \in \{0,1\}\end{array}$$

Figure 1.5: 0-1 Integer Programs for $\gamma(G)$ and $\rho(G)$

Given a graph G, a set of vertices $S \subset V(G)$ is said to be a *dominating set* on G if for all $v \in V(G)$ we have that $N[v] \bigcap S \neq \varnothing$. The minimum cardinality of a dominating set on G is called the *domination number* of the graph and is denoted by $\gamma(G)$; if a set S is a dominating set for G and has cardinality $\gamma(G)$, we say that S is a $\gamma(G)$-set. A set of vertices $T \subset V(G)$ is said to be a *packing* on G if for all $v \in V(G)$ we have that $|N[v] \bigcap T| \leq 1$. The maximum cardinality of a packing on G is called the *packing number* of the graph and is denoted by $\rho(G)$; if a set T is a packing on G and has cardinality $\rho(T)$, we say that T is a $\rho(G)$-set. If a set $R \subset V(G)$ is both a dominating set and a packing, we say that R is an *efficient dominating set* for G.

Given a graph G, the problem of determining the parameters $\gamma(G)$ and $\rho(G)$ can be represented as 0-1 integer programs. If $V(G) = \{v_1, v_2, \ldots, v_n\}$, let $X = [x_1, x_2, \ldots, x_n]^T$ be a vector of variables, where $x_i \in \{0,1\}$ and $x_i = 1$ if and only if the vertex v_i is an element of the dominating set, or packing set. The parameters $\gamma(G)$ and $\rho(G)$ are the solutions to the 0-1 integer programs in Figure 1.5.

We can replace the constraint that $x_i \in \{0,1\}$ by the constraint $x_i \in \mathbb{N} = \{0,1,2,\ldots\}$. In the domination problem, there is never any benefit to setting a value $x_i > 1$; if a constraint is satisfied by setting $x_i > 1$, then it will also be satisfied by setting $x_i = 1$, and the objective function will always be less by setting $x_i = 1$. The constraints of the packing problem will

$$\begin{array}{rlcrl} \gamma(G) = \text{MIN} & \sum_{i=1}^{n} X_i & \quad & \rho(G) = \text{MAX} & \sum_{i=1}^{n} X_i \\ \text{subject to} & NX \geq 1 & & \text{subject to} & NX \leq 1 \\ & x_i \in \mathbb{N} & & & x_i \in \mathbb{N} \end{array}$$

Figure 1.6: Integer programs for $\gamma(G)$ and $\rho(G)$

$$\begin{array}{rlcrl} \gamma_f(G) = \text{MIN} & \sum_{i=1}^{n} X_i & \quad & \rho_f(G) = \text{MAX} & \sum_{i=1}^{n} X_i \\ \text{subject to} & NX \geq 1 & & \text{subject to} & NX \leq 1 \\ & x_i \geq 0 & & & x_i \geq 0 \end{array}$$

Figure 1.7: Linear programs for $\gamma_f(G)$ and $\rho_f(G)$

ensure that $x_i \leq 1$. Hence equivalent formulations for determining $\gamma(G)$ and $\rho(G)$ are the integer programs shown in Figure 1.6.

If we relax the constraint that $x_i \in \mathbb{N}$ and simply require that $x_i \geq 0$, then we can define the parameters $\gamma_f(G)$ and $\rho_f(G)$ using the linear programs shown in Figure 1.7. The parameter $\gamma_f(G)$ is called the *fractional domination number* for the graph G. Similarly, the parameter $\rho_f(G)$ is called the *fractional packing number* for G. Since any dominating set is also a fractional dominating set, we have that $\gamma_f(G) \leq \gamma(G)$. Likewise, since any packing on G is also a fractional packing on G, we have that $\rho(G) \leq \rho_f(G)$. Moreover, the linear programs in Figure 1.7 are duals of one another. Since all graphs will have a set of vertices that is a dominating set, and since all graphs will have a set of vertices that is a packing, these linear programs always have a solution. Therefore, it follows that $\rho(G) \leq \rho_f(G) = \gamma_f(G) \leq \gamma(G)$. A detailed discussion of linear programming duality applied to domination and packing problems can be found in Haynes et al. [8].

For a graph $G=(V,E)$ and a function $f:V(G)\to\mathbb{N}$, f is said to be a *dominating function* if for every vertex $v\in V(G)$, $f(N[v])\geq 1$. A function $g:V(G)\to\mathbb{R}^{+}=[0,\infty)$ is said to be a *fractional dominating function* if for every $v\in V(G)$, $g(N[v])\geq 1$. It follows that $\gamma(G)=min\{\sum_{v\in V(G)} f(v): f$ is a dominating function$\}$ and $\gamma_f(G)=min\{\sum_{v\in V(G)} g(v):$ g is a fractional dominating function$\}$.

Similarly, a function $f:V(G)\to\mathbb{N}$ is said to be a *packing function* if for every $v\in V(G)$, $f(N[v])\leq 1$. A function $g:V(G)\to\mathbb{R}^{+}$ is said to be a *fractional packing function* if for every $v\in V(G)$, $g(N[v])\leq 1$. It follows that $\rho(G)=max\{\sum_{v\in V(G)} f(v):$ f is a packing function$\}$ and $\rho_f(G)=max\{\sum_{v\in V(G)} g(v): g$ is a fractional packing function$\}$.

Let G and H be two graphs. If $V(H)\subset V(G)$ and $E(H)\subset E(G)$, then H is said to be a *subgraph* of G, which we denote by writing $H<G$. If $H<G$ and $V(H)=V(G)$ we say that H is a *spanning subgraph* of G. There are several ways to combine graphs G and H to create a new graph K. The *union* of disjoint graphs G and H is defined to be the graph $K=G\bigcup H$, where $V(K)=V(G)\bigcup V(H)$ and $E(H)=E(G)\bigcup E(H)$. The *join* of disjoint graphs G and H is defined to be the graph $K=G+H$, where $V(K)=V(G)\bigcup V(H)$ and $E(K)=E(G)\bigcup E(H)\bigcup\{uv:u\in V(G),v\in V(H)\}$. The *(cartesian) product* of graphs G and H is defined as $K=G\times H$, where $V(K)=V(G)\times V(H)$ and where vertices (u_1,v_1) and (u_2,v_2) are adjacent if and only if $u_1=u_2$ and $v_1v_2\in E(H)$, or $v_1=v_2$ and $u_1u_2\in E(G)$. The *strong product* of graphs G and H is the graph $K=G*H$, where $V(K)=V(G)\times V(H)$ and where the vertices (u_1,v_1) and (u_2,v_2) are adjacent if and only if $u_1=u_2$ and $v_1v_2\in E(H)$, or $v_1=v_2$ and $u_1u_2\in E(G)$, or $u_1u_2\in E(G)$ and $v_1v_2\in E(H)$. Notice that $G\times H$ is a spanning subgraph of $G*H$.

In Chapter 3 we will generalize several of the definitions we have presented. In particular, we will extend the definition of the neighborhood of a vertex. The open (closed) neighborhood of a vertex is the set of vertices at distance one (no more than one). Our extension will define the neighborhoods for an arbitrary set of distances. Additionally, we will extend the notion of dominating and packing sets, and the associated linear programs, for these extended neighborhoods. By so doing, we will be able to relate the measures of equitable distribution of weights that have been introduced to the fractional domination and fractional packing parameters. The relationships that we establish, and the results that will follow from them, will be one of the most significant contributions of this dissertation.

CHAPTER 2

DEFINITIONS AND EXAMPLES

The purpose of this chapter is to extend, as necessary, the definitions from graph theory that were presented in Chapter 1, and to formalize the notions of equitable distributions of weights. In particular, we will formalize the three measures of equitable distribution that were discussed in Section 1.2. In this chapter we will also mention many of the notational conventions we will follow.

2.1 Notational Conventions

The order of a graph G will be denoted by the letter n. The size of a graph G will be denoted by m. Unless specifically restricted otherwise, the letter D will be used to denote an arbitrary subset of $\mathbb{N}$. The elements in the set D will represent a set of distances we are considering. For example, if we are concerned with closed neighborhoods, we will restrict D to be the set $\{0,1\}$. We will use the letter W to denote a multiset $W = \{w_1, w_2, \ldots, w_n\}$, where each $w_i \in \mathbb{R} = (-\infty, \infty)$ is referred to as a weight (or label). Occasionally we will order the weights in a specific manner, restrict W to being a set, or place some other restriction on W. When any such restriction is present, it will be specifically stated. Frequently the set W of weights we will use will be $\{1, 2, \ldots, n\}$; by convention, we will represent

this set as $[n] = \{1, 2, \ldots, n\}$. The sum of the elements from W will be denoted by σ_W, the minimum element of W will be denoted by W_{MIN}, and the maximum element of W will be denoted by W_{MAX}.

2.2 Extension of Graph Theory Definitions

Next we extend the basic graph theory definitions to encompass neighborhoods of arbitrary distance.

Definition 2.1 ([13])**.** For a graph G and a vertex $v \in V(G)$, the *D-neighborhood* of v, denoted by $N_D(v)$, is defined as $N_D(v) = \{u \in V(G) : d(v,u) \in D\}$. The *D-degree* of v is $deg_D(v) = |N_D(v)|$.

If $D = \{1\}$, then $N_D(v)$ is the open neighborhood of the vertex v. We will adopt the notation that $N(v) = N_{\{1\}}(v)$. Since $d(v,v) = 0$, if $D = \{0,1\}$, then $N_D(v)$ is the closed neighborhood of the vertex v. We will adopt the notation that $N[v] = N_{\{0,1\}}(v)$. We will also adopt the convention that $N_\varnothing(v) = \varnothing$. When $D = \{1\}$ we will write $deg(v) = deg_{\{1\}}(v)$, where $deg(v)$ is the usual distance 1 degree of the vertex.

Notice that if G has diameter d and $D \subset \{d+1, d+2, \ldots\}$, then for all $v \in V(G)$ we have that $N_D(v) = \varnothing$.

Definition 2.2 ([13])**.** Let graph G have vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. The *distance D adjacency matrix* of G, denoted by $A_D = [a_{i,j}]$, is defined to be the $n \times n$ binary matrix with $a_{i,j} = 1$ if and only if $d(v_i, v_j) \in D$.

If $D = \{1\}$, then $A_D = A$ is the ordinary adjacency matrix of G. If $D = \{0,1\}$, then $A_D = N$, where N is the closed neighborhood matrix of G. By convention we will take $A_\emptyset$ to be the $n \times n$ zero matrix. Also note that for any set $D \subset \mathbb{N}$ we have that A_D is symmetric.

Definition 2.3 ([13])**.** Graph G is defined to be (D,r)-*regular* if for all $v \in V(G)$ we have that $deg_D(v) = r$.

Definition 2.4. The *distance D degree sequence* of graph G is defined to be the sequence $(deg_D(v_1), deg_D(v_2), \ldots, deg_D(v_n))$.

We will usually write the distance D degree sequence of the graph in non-increasing order. When $D = \{1\}$ the distance D degree sequence is the ordinary degree sequence of the graph.

Definition 2.5. The *minimum distance D degree* of the graph G, denoted by $\delta_D(G)$, is defined as $\delta_D(G) = min\{deg_D(v) : v \in V(G)\}$.

When $D = \{1\}$ we will write $\delta(G) = \delta_{\{1\}}(G)$, where $\delta(G)$ is the ordinary minimum degree of the graph. Also notice that $\delta_{\{0,1\}}(G) = \delta(G) + 1$.

Definition 2.6. The *maximum distance D degree* of the graph G, denoted by $\Delta_D(G)$, is defined as $\Delta_D(G) = max\{deg_D(v) : v \in V(G)\}$.

For $D = \{1\}$ we write $\Delta(G) = \Delta_{\{1\}}(G)$, where $\Delta(G)$ is the ordinary maximum degree of the graph. We also have that $\Delta_{\{0,1\}}(G) = \Delta(G) + 1$.

2.3 Formalization of Equitable Distributions of Weight

We are now in a position where we can formalize the notions of equitable distributions of weight. The ideas behind these definitions were first introduced in the articles

by Schneider and Slater [16, 17]; these papers defined the minimax open and closed neighborhood sums. The notion of the maximin neighborhood sums and the spread parameters of a graph were introduced by O'Neal and Slater in [12]. In each of these references, the weight set W is allowed to be any multiset that has the same cardinality as the vertex set of the graph. These definitions were extended for arbitrary distances in O'Neal and Slater [13].

The first notion of equitable distribution of weights that we formalize is the idea that the maximum amount of weight that a location (vertex) receives should be made as small as possible.

Definition 2.7 ([16]). For graph G , a bijection $f : V(G) \to W$, and a subset $S \subset V(G)$, the *weight of S under f*, denoted by $f(S)$, is defined as $f(S) = \sum_{v \in S} f(v)$. If $S = \emptyset$, then we define $f(S) = f(\emptyset) = 0$.

Notice that if graph G has diameter d, then for all $v \in V(G)$, we have that $\sum_{i=0}^{d} f(N_{\{i\}}(v)) = \sum_{j=1}^{n} w_j = \sigma_W$. Hence, $\sum_{v \in V(G)} \sum_{i=0}^{d} f(N_{\{i\}}(v)) = n \sum_{j=1}^{n} w_j = n\sigma_W$.

Definition 2.8 ([13]). For a graph G and a bijection $f : V(G) \to W$, we define the *D-neighborhood sum* of f, denoted by $NS(f;D)$, as $NS(f;D) = max\{f(N_D(v))|v \in V(G)\}$.

When $D = \{1\}$ we will by convention shorten the notation to $NS(f) = NS(f;D)$. Similarly, when $D = \{0,1\}$ we will by convention shorten the notation to $NS[f] = NS(f;D)$.

Definition 2.9 ([13]). The *W-valued D-neighborhood sum* of graph G, denoted by $NS_W(G;D)$, is defined as $NS_W(G;D) = min\{NS(f;D)|f : V(G) \to W \text{ is a bijection }\}$.

When $D = \{1\}$ we will adopt the previous convention and shorten the notation to $NS_W(G) = NS_W(G;D)$. Likewise, when $D = \{0,1\}$ we shorten the notation to $NS_W[G] = NS_W(G;D)$.

For a graph G of order $|V(G)| = n$, we are generally interested in the set of weights $W = [n] = \{1,2,\ldots,n\}$. When this is the case, we will by convention shorten the notation to $NS(G;D) = NS_{[n]}(G;D)$. So for the open neighborhood sum case (that is, when $D = \{1\}$) with weight set $W = [n]$ our notation will be $NS(G) = NS_{[n]}(G;\{1\})$. For the closed neighborhood sum case (that is, when $D = \{0,1\}$) with weight set $W = [n]$ our notation will be $NS[G] = NS_{[n]}(G;\{0,1\})$. By following these conventions, our notation is consistent with what has been introduced by Schneider and Slater [16, 17].

Example 2.10. Consider graph H in Figure 1.3 which has diameter 2. We claim that $NS(H) = 8$, $NS[H] = 11$, and $NS(H;\{2\}) = 6$.

To see that $NS(H) = 8$, consider that $N(v_2) \bigcup N(v_4) = V(H)$ and $N(v_2) \bigcap N(v_4) = \emptyset$. Hence for any bijection $f : V(H) \to [5]$ we must have that $f(N(v_2)) + f(N(v_4)) = 15$. It follows that one of $\{f(N(v_2)), f(N(v_4))\}$ is greater than or equal 8. From this we have that $NS(H) \geq 8$. The bijection shown in Figure 2.1 demonstrates that $NS(H) \leq 8$. Therefore we conclude that $NS(H) = 8$.

To see that $NS[H] = 11$, let $f : V(H) \to [5]$ be an arbitrary bijection. Notice that one of $\{f(v_4), f(v_5)\}$ is no more than 4. If $f(v_4) \leq 4$, then $f(N[v_3]) = 15 - f(v_4) \geq 11$. Similarly, if $f(v_5) \leq 4$, then $f(N[v_2]) = 15 - f(v_5) \geq 11$. Thus $NS[H] \geq 11$. The bijection shown in Figure 2.2 demonstrates that $NS[H] \leq 11$. Therefore $NS[H] = 11$.

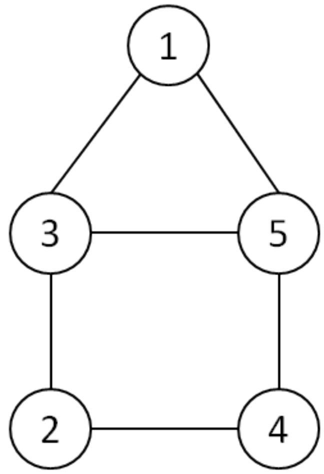

Figure 2.1: $NS(H) = 8$, $NS(H;\{2\}) = 6$, $NS^{-}(H) = 7$, $NS^{sp}(H) = 1$, $NS^{sp}[H] = 4$, and $NS^{sp}(H;\{2\}) = 4$

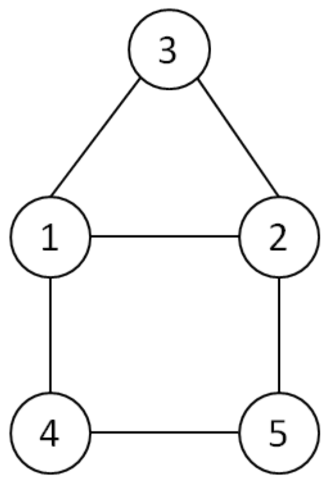

Figure 2.2: $NS[H] = 11$ and $NS^{-}(H;\{2\}) = 4$

Finally, to see that $NS(H;\{2\}) = 4$, let $f : V(H) \to [5]$ be an arbitrary bijection. Notice that if $f(v_1) = 5$ or $f(v_2) = 5$, then $f(N_{\{2\}}(v_5)) = f(v_1) + f(v_2) \geq 6$. If $f(v_3) = 5$, then $f(N_{\{2\}}(v_4)) = f(v_1) + f(v_3) \geq 6$. If $f(v_4) = 5$ or $f(v_5) = 5$, then $f(N_{\{2\}}(v_1)) = f(v_4) + f(v_5) \geq 6$. Thus $NS(H;\{2\}) \geq 6$. The bijection shown in Figure 2.1 demonstrates that $NS(H;\{2\}) \leq 6$. Therefore $NS(H;\{2\}) = 6$.

Next we formalize the notion of equitably distributing weight by ensuring that the minimum amount of weight at a location (vertex) is made as large as possible.

Definition 2.11 ([13])**.** For graph G and a bijection $f : V(G) \to W$, the *lower D-neighborhood sum* of f, denoted by $NS^{-}(f;D)$, is defined as $NS^{-}(f;D) = min\{f(N_D(v))|v \in V(G)\}$.

Following the previous pattern for notation, we will let $NS^{-}(f) = NS^{-}(f;\{1\})$ and $NS^{-}[f] = NS^{-}(f;\{0,1\})$.

Definition 2.12 ([13])**.** The *W-valued lower D-neighborhood sum* of graph G, denoted by $NS_W^{-}(G;D)$, is defined as $NS_W^{-}(G;D) = max\{NS^{-}(f;D)|f : V(G) \to W$ is a bijection $\}$.

For the open and closed neighborhood cases, we let $NS_W^{-}(G) = NS_W^{-}(G;\{1\})$ and $NS_W^{-}[G] = NS_W^{-}(G;\{0,1\})$ respectively. When $W = [n]$, we drop the W from the notation altogether.

Example 2.13. Consider again the graph H from Figure 1.3. We claim that $NS^{-}(H) = 7$, $NS^{-}[H] = 9$, and $NS^{-}(H;\{2\}) = 4$.

To see that $NS^{-}(H) = 7$, again notice that $N(v_2) \bigcup N(v_4) = V(H)$ and $N(v_2) \bigcap N(v_4) = \emptyset$. For any bijection $f : V(G) \to [5]$ we must have that $f(N(v_2)) + f(N(v_4)) = 15$. It follows that one of $\{f(N(v_2)), f(N(v_4))\}$ is less than or equal 7; hence, we have that $NS^{-}(H) \leq 7$. The bijection shown in Figure 2.1 demonstrates that $NS^{-}(H) \geq 7$. Therefore we conclude that $NS^{-}(H) = 7$.

Figure 2.3 demonstrates that $NS^{-}[H] \geq 9$. If there exists a bijection $f : V(H) \to [5]$ such that $NS^{-}[f] \geq 10$, then $f(N[v_1]) \geq 10$, and hence $f(v_4) + f(v_5) \leq 5$. However, this

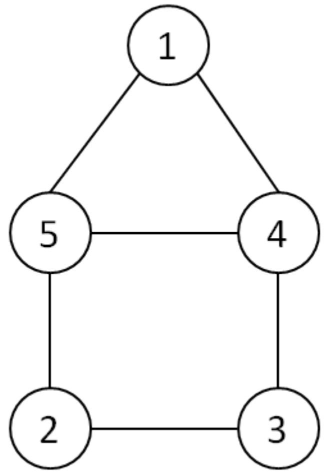

Figure 2.3: $NS^{-}[H] = 9$

would imply that one of $\{f(N[v_4]), f(N[v_5])\}$ is no more than 9. Therefore, such a bijection does not exist, and we conclude that $NS^{-}[H] = 9$.

Finally, we show that $NS^{-}(H;\{2\}) = 4$. Consider that one of $\{f(v_4), f(v_5)\}$ is less than or equal 4. If $f(v_4) \leq 4$, then $f(N_{\{2\}}(v_3)) = f(v_4) \leq 4$. If $f(v_5) \leq 4$, then $f(N_{\{2\}}(v_2)) = f(v_5) \leq 4$. Hence $NS^{-}(H;\{2\}) \leq 4$. The bijection shown in Figure 2.2 demonstrates that $NS^{-}(H;\{2\}) \geq 4$. Therefore $NS^{-}(H;\{2\}) = 4$.

The third notion of an equitable distribution of weights is to make as small as possible the difference between the maximum amount of weight at a location (vertex) and the minimum amount of weight at a location (vertex).

Definition 2.14 ([13]). For graph G and a bijection $f : V(G) \to W$, we define the *D-neighborhood spread* of f, denoted by $NS^{sp}(f;D)$, as $NS^{sp}(f;D) = NS(f;D) - NS^{-}(f;D)$.

We let $NS^{sp}(f) = NS^{sp}(f;\{1\})$ and $NS^{sp}[f] = NS^{sp}(f;\{0,1\})$.

Definition 2.15 ([13]). The *W-valued D-neighborhood spread* of graph G, denoted by $NS^{sp}_W(G;D)$, is defined as $NS^{sp}_W(G;D) = min\{NS(f;D) - NS^{-}(f;D) | f : V(G) \to W$ is a bijection $\}$.

Consistent with our previous notation, we let $NS^{sp}_W(G) = NS^{sp}_W(G;\{1\})$, $NS^{sp}_W[G] = NS^{sp}_W(G;\{0,1\})$, and drop the W when $W = [n]$.

Example 2.16. For the graph H from Figure 1.3, we claim that $NS^{sp}(H) = 1$, $NS^{sp}[H] = 4$, and $NS^{sp}(H;\{2\}) = 4$.

Since $NS(H) = 8$ and $NS^{-}(H) = 7$, then we must have $NS^{sp}(H) \geq 1$. The bijection shown in Figure 2.1 demonstrates that $NS^{sp}(H) \leq 1$. Therefore $NS^{sp}(H) = 1$.

The bijection g from Figure 2.1 also demonstrates that $NS^{sp}[H] = g(N[v_3]) - g(N[v_2]) \leq 4$. Assume there exists a bijection $f : V(H) \to [5]$ such that $NS^{sp}[f] \leq 3$. Since $f(N[v_2]) - f(N[v_1]) = f(v_4)$, we must have that $f(v_4) \leq 3$. Similarly, we must have that $f(v_5) \leq 3$. Hence, one of $\{f(v_4), f(v_5)\}$ is less than or equal 2. If $f(v_4) \leq 2$, then $f(N[v_3]) \geq 13$. If $f(v_5) \leq 2$, then $f(N[v_2]) \geq 13$. But since $NS^{-}[H] \leq 9$, we have that $NS^{sp}[f] \geq 4$, which is a contradiction. Therefore $NS^{sp}[H] = 4$.

Finally, notice that the bijection h from Figure 2.1 also demonstrates that $NS^{sp}(H;\{2\}) = h(N_{\{2\}}[v_1]) - h(N_{\{2\}}[v_3]) \leq 4$. Assume there exists a bijection $f : V(H) \to [5]$ such that $NS^{sp}(H;\{2\}) \leq 3$. In this case $f(N_{\{2\}}(v_1)) - f(N_{\{2\}}(v_2)) = f(v_4) \leq 3$. Similarly we have that $f(N_{\{2\}}(v_1)) - f(N_{\{2\}}(v_3)) = f(v_5) \leq 3$. Thus, one of $\{f(v_4), f(v_5)\}$ is less than or equal 2. If $f(v_4) \leq 2$, then we have $f(N_{\{2\}}(v_3)) = f(v_4) \leq 2$. If $f(v_5) \leq 2$, then we have $f(N_{\{2\}}(v_2)) = f(v_5) \leq 2$. In either case, since $NS(H;\{2\}) \geq 6$, we have that $NS^{sp}(f;\{2\}) \geq 4$, which is a contradiction. Therefore, $NS^{sp}(H;\{2\}) = 4$.

Example 2.17. Since the diameter of the graph H in Examples 2.10, 2.13 and 2.16 is two, when $D = \{0,1,2\}$, we have that $NS(H;D) = NS^{-}(H;D) = 15$ and that $NS^{sp}(H;D) = 0$. Indeed, for any graph G of order n and diameter d, when $D = \{0,1,2,\ldots,d\}$, we have $NS^{-}(G;D) = NS(G;D) = \frac{n(n+1)}{2}$ and $NS^{sp}(G;D) = 0$. For such a graph and for any weight set W, we have $NS_W^{-}(G;D) = NS_W(G;D) = \sigma_W$ and $NS_W^{sp}(G;D) = 0$.

Examples 2.10, 2.13 and 2.16 demonstrate that there are graphs where $NS^{sp}[G] > NS[G] - NS^{sp}[G]$. When this is the case, there cannot be a single bijection that achieves both $NS[G]$ and $NS^{-}[G]$. A similar result holds for the open neighborhood case; that is, there are graphs where $NS^{sp}(G) > NS(G) - NS^{sp}(G)$. In these examples we also had that $NS[H] + NS^{-}(H;\{2\}) = \frac{5(5+1)}{2} = 15 = NS^{-}[H] + NS(H;\{2\})$ and that $NS^{sp}[H] = NS^{sp}(H;\{2\})$. These latter results were not coincidental, and we will prove a more general result in Chapter 3.

It should be noted that our definitions of an equitable distribution of weight do not preclude the typical YES/NO classification of graphs that has been traditionally studied. To connect our work to the research that others have done, we make the following definition.

Definition 2.18 ([13])**.** Graph G is said to be *D-vertex magic*, or equivalently *D-distance magic*, if there exists a bijection $f : V(G) \to [n]$ and a constant c such that for all $v \in V(G)$, $\sum_{u \in N_D(v)} f(u) = c$. When such a constant exists, it is called a *D-vertex* (or equivalently, *D-distance*) *magic constant* of G.

Consider the case where we restrict the distance set to be $D = \{1\}$ (open neighborhood case). Saying that a graph G is $\{1\}$-vertex magic is equivalent to saying that $NS^{sp}(G) = 0$. This notion of labeling the n vertices of a graph G in a fashion such that all

open neighborhood sums are equal was first introduced by Vilfred [21]. He referred to such a graph as a Σ-*labeled graph*, and the labeling (bijection) that produced equal open neighborhood sums as a Σ *labeling*. Miller et al. [11] referred to such a labeling as a *1-vertex magic labeling*. More recently Sugeng et al. [20] have referred to such a labeling as a *distance magic labeling*. A labeling of the n vertices of the graph G in such a fashion that all the closed neighborhood sums are equal was referred to by Beena [4] as a Σ' *labeling*, and a graph that permitted such a labeling as a Σ'-*labeled graph*. In her presentation at the 2010 IWOGL Conference [19], Rinovia Simanjuntak introduced the notion of distance magic labelings for a fixed distance other than one. Her presentation suggests a generalization of the notion of distance magic to an arbitrary set of distances.

What each of these works has in common is a YES/NO type of classification. This type of classification is unsatisfying in the sense that there seem to be very few Σ-labeled graphs. In fact, we list all Σ'-labeled graphs of order ten or less in Appendix A, and, as we will show, the only Σ-labeled graphs of order ten or less are the complements of these graphs. Hence, the YES/NO classification says nothing about the majority of graphs, except that they are not Σ or Σ'-labeled. For example, the only cycle that is Σ-labeled is C_3. For an arbitrary cycle C_n, it is reasonable to ask if $NS^{sp}(C_n)$ is small, or large. In particular, we would like to know how $NS^{sp}(C_n)$ can be bounded as n grows large. To a small degree in Chapter 3, and to a significant degree in Chapter 4, we will provide exact solutions and bounds for our measures of equitable distributions of weights for several families of graphs, including the cycles.

As we have argued, given a graph G, a weight set W, and a distance set D, each of $NS_W(G;D)$, $NS_W^-(G;D)$, and $NS_W^{sp}(G;D)$ is a measure of how close one can come to

achieving a perfectly equitable distribution of the weights. It is clear that $NS_W^{sp}(G;D) \geq 0$, and hence, there either exists a bijection $f : V(G) \to W$ such that $NS^{sp}(f;D) = 0$, or no such bijection is possible. So when considering neighborhood spreads, it is clear that there is a fixed value, namely zero, that establishes what a perfect distribution of weight would be. When we consider the parameters $NS_W(G;D)$ and $NS_W^-(G;D)$, it is not at all clear what value one should use to represent a perfect distribution of weights. Without a well established basis for comparison, we will be unable to say whether G, W, and D are such that a nearly equitable distribution can be achieved, or whether they are such that all arrangements of weight will be far from equitable. For example, in Example 2.10, we need a value that would have represented a perfectly equitable distribution of weights; otherwise, we do not know whether $NS[H] = 11$ is close to, or far from, what we could have hoped for.

Moreover, there are some other fundamental questions that can be asked regarding our measures. First of all, what (if any) is the relationship between $NS_W(G;D)$ and $NS_W^-(G;D)$; in particular, is it possible that we can have $NS_W(G;D) < NS_W^-(G;D)$? Secondly, is it possible that there could exists bijections $f, g : V(G) \to W$ such that $NS^{sp}(f;D) = NS^{sp}(g;D) = 0$, yet $\sum_{v \in V(G)} f(N_D(v)) \neq \sum_{v \in V(G)} g(N_D(v))$; or if $W = [n]$, can G be D-vertex magic and exhibit two distinct D-vertex magic constants? This latter question was raised by Arumugam [2] in his presentation at the 2010 IWOGL Conference. As we demonstrate next in Example 2.19, for a slightly modified graph labeling problem, similar questions would both have affirmative answers.

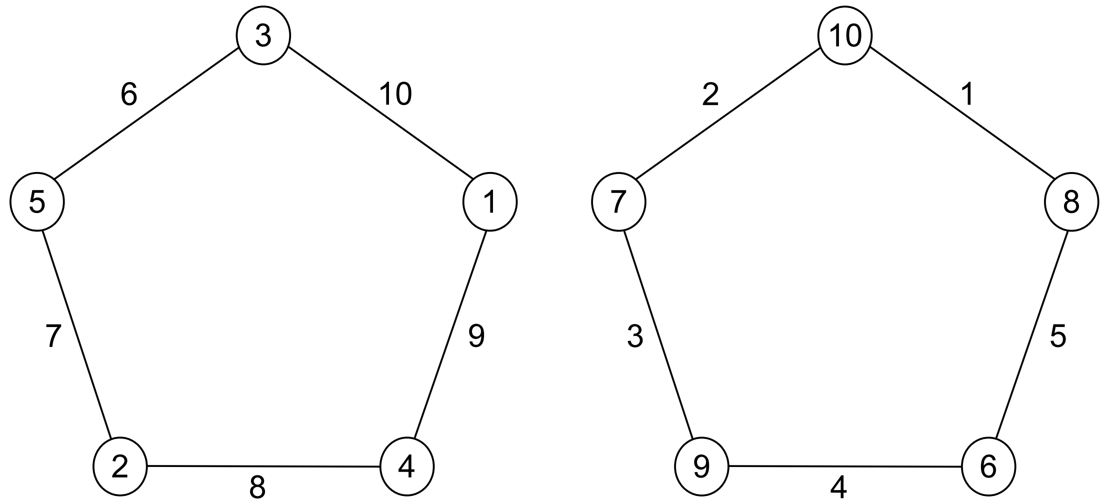

Figure 2.4: Distinct edge-magic labelings of C_5

Example 2.19. Let G be a graph. Consider the graph labeling problem where we bijectively label both the vertices and edges with the integers $[m+n]$, and then for each edge, we sum the labels of the edge and its two incident vertices. If there exists a bijection $f : V(G) \bigcup E(G) \to [m+n]$ and a constant c such that for all $uv \in E(G)$, $c = f(uv) + f(u) + f(v)$, then G is sometimes said to be *edge-magic*. Gallian [5] calls such a graph an *edge-magic total* graph, and such a labeling as an *edge-magic total labeling*. Godbold and Slater [6] established that all cycles have edge-magic labelings that achieve the minimum (respectively, maximum) feasible edge-magic constant. In Figure 2.4 we show two different labelings of the cycle C_5; the first has an edge-magic constant of 14, and the second has an edge-magic constant of 19. For all cycles, the maximin value of such a labeling exceeds the minimax value of such a labeling.

As we will establish in Chapter 3, for any graph G, weight set W, and distance set D, it is the case that $NS^-_W(G;D) \leq NS_W(G;D)$. This will imply that if G is D-vertex magic,

the D-vertex magic constant is unique. Moreover, there is a value that can be used as a firm basis of comparison for $NS_W(G;D)$ and $NS_W^-(G;D)$; this value will be determined using a generalization of the fractional domination number for the graph.

We close this chapter by stating two straightforward facts that follow directly from definitions we have introduced. For any graph G, weight set W, and neighborhood set D, it is clear that $NS_W^{sp}(G;D) \geq NS_W(G;D) - NS_W^-(G;D)$, though at this point, we have not established that the right hand side of the inequality is always positive. In particular, $NS^{sp}[G] \geq NS[G] - NS^-[G]$ and $NS^{sp}(G) \geq NS(G) - NS^-(G)$. Moreover if $f : V(G) \to W$ is such that $NS_W^-(G;D) = NS^-(f;D)$ and $NS_W(G;D) = NS(f;D)$, then we must have that $NS_W^{sp}(G;D) = NS^{sp}(f;D)$. For reference we state these two results in the following propositions.

Proposition 2.20. *For any graph G, $NS_W^{sp}(G;D) \geq NS_W(G;D) - NS_W^-(G;D)$.*

Proposition 2.21. *If $f : V(G) \to W$ is such that $NS_W^-(G;D) = NS^-(f;D)$ and $NS_W(G;D) = NS(f;D)$, then $NS_W^{sp}(G) = NS^{sp}(f;D) = NS(f;D) - NS^-(f;D)$.*

CHAPTER 3

GENERAL RESULTS FOR NEIGHBORHOOD SUMS

In this chapter we seek to establish general results for our measures $NS_W^-(G;D)$, $NS_W(G;D)$, and $NS_W^{sp}(G;D)$. That is, the results of this chapter apply to all graphs, or at least to a very broad class of graphs. Naturally many of these results are in the form of bounds we can establish for our measures. Most notably, we will establish that $NS_W^-(G;D) \leq NS_W(G;D)$. We will also establish results that relate these parameters when we have two distance sets that are associated in a complementary fashion. Among other things, we will use complementary distance sets to provide a proof that a graph is Σ-labeled if and only if its complement is Σ'-labeled. In Section 3.4 we establish the close relationship that exists between $NS^-(G;D)$ and $NS(G;D)$ when G is (D,r)-regular. In Section 3.5 we provide some new existence theorems for a graph to be vertex magic, and use these to answer some questions posed by Simanjuntak [19]. In this section we also establish the close relationship that exists between vertex magic labelings and fractionally efficient dominating sets.

3.1 Equality Results That Apply For All Graphs and Arbitrary Weight Sets

There are a few obvious equalities that can be stated that apply to all connected (or disconnected) graphs regardless of the weight set. The first result is a direct consequence of Definition 2.7.

Theorem 3.1. *For any graph G, $NS_W^-(G;\emptyset) = NS_W(G;\emptyset) = NS_W^{sp}(G;\emptyset) = 0$.*

Corollary 3.2. *Let graph G have diameter d. If $D \subset \mathbb{N} - \{0,1,\ldots,d\}$, then $NS_W^-(G;D) = NS_W(G;D) = NS_W^{sp}(G;D) = 0$.*

Proof. Notice that for all $v \in V(G)$ we have $N_D(v) = \emptyset$. □

If G is connected, and the distance set D is such that it includes all possible distances, then our parameters are easy to calculate. We cover this situation in the next result.

Theorem 3.3. *Let G have diameter d. If $D = \{0,1,\ldots d\}$ then*

(i) $NS_W^-(G;D) = NS_W(G;D) = \sigma_W$, and

(ii) $NS_W^{sp}(G;D) = 0$.

Proof. Since the diameter of G is d and $D = \{0,1,\ldots,d\}$, for all $v \in V(G)$ we have $N_D(v) = V(G)$. Let $f : V(G) \to W$ be an arbitrary bijection. For all $v \in V(G)$ we have that $f(N_D(v)) = f(V(G)) = \sigma_W$. Thus $NS^-(f;D) = NS(f;D) = \sigma_W$ and $NS^{sp}(f;D) = NS_W(f;D) - NS_W^-(f;D) = 0$. Since the bijection f was arbitrary, we conclude that $NS_W^-(G;D) = NS_W(G;D) = \sigma_W$ and that $NS_W^{sp}(G;D) = 0$. □

Corollary 3.4. *If graph G has diameter d, then*

(i) $NS_W^-(G;\mathbb{N}) = NS_W(G;\mathbb{N}) = \sigma_W$, and

(ii) $NS_W^{sp}(G;\mathbb{N}) = 0$.

Proof. Let $D = \{0, 1, \ldots, d\}$. For all $v \in V(G)$ we have $N_{\mathbb{N}}(v) = N_D(v)$. Therefore the result follows directly from Theorem 3.3. $\square$

Corollary 3.5. *Let graph G have diameter d. If $D = \{0, 1, \ldots d\}$, then*

(i) $NS^{-}(G;D) = NS(G;D) = \frac{n(n+1)}{2}$, and

(ii) $NS^{sp}(G;D) = 0$.

Another situation in which our parameters are easy to calculate is when $D = \{0\}$.

Theorem 3.6. *For any graph G*

(i) $NS_W^{-}(G;\{0\}) = W_{MIN}$,

(ii) $NS_W(G;\{0\}) = W_{MAX}$, and

(iii) $NS_W^{sp}(G;\{0\}) = W_{MAX} - W_{MIN}$.

Proof. For all vertices $x \in V(G)$, we have that $f(N_{\{0\}}(x)) = f(x)$. Let $f : V(G) \to W$ be an arbitrary bijection. Since f is a bijection, there exists a vertex $u \in V(G)$ such that $f(u) = W_{MIN}$. For all $w \in V(G)$ we have that $f(w) \geq f(u)$, and it follows that $NS^{-}(f;\{0\}) = f(u) = W_{MIN}$. Likewise, there exists a vertex $v \in V(G)$ such that $f(v) = W_{MAX}$. For all $w \in V(G)$ we have that $f(w) \leq f(v)$, and it follows that $NS(f;\{0\}) = f(v) = W_{MAX}$. We then have that $NS^{sp}(f;\{0\}) = NS(f;\{0\}) - NS^{-}(f;\{0\}) = W - W_{MIN}$. Since f was an arbitrary bijection, we conclude that $NS_W^{-}(G;\{0\}) = W_{MIN}$, $NS_W(G;\{0\}) = W_{MAX}$, and $NS_W^{sp}(G;\{0\}) = W - W_{MIN}$. $\square$

Corollary 3.7. *For any graph G*

(i) $NS^{-}(G;\{0\}) = 1$,

(ii) $NS(G;\{0\}) = n$, and

(iii) $NS^{sp}(G;\{0\}) = n - 1$.

A complementary result can be obtained when G is connected and D includes all distances except 0.

Theorem 3.8. *Let graph G have diameter d. If $D = \{1,2,\ldots,d\}$, then*

(i) $NS_W^-(G;D) = \sigma_W - W_{MAX}$,

(ii) $NS_W(G;D) = \sigma_W - W_{MIN}$, *and*

(iii) $NS_W^{sp}(G;D) = W - W_{MIN}$.

Proof. Let $f : V(G) \to W$ be an arbitrary bijection. Since G has diameter d and since $D = \{1,2,\ldots,d\}$, for all vertices $u \in V(G)$, we have that $N_D(u) = V(G) - \{u\}$. Hence $f(N_D(u)) = \left(\sum_{w\in V(G)} f(w)\right) - f(u) = \sigma_W - f(u)$.

Since f is a bijection, there exists a vertex $u \in V(G)$ such that $f(u) = W_{MAX}$. Then for all $x \in V(G)$, $f(N_D(x)) = \sigma_W - f(x) \geq \sigma_W - f(u) = f(N_D(u))$. That is, $NS^-(f;D) = f(N_D(u)) = \sigma_W - W_{MAX}$.

Since f is a bijection, there exists a vertex $v \in V(G)$ such that $f(v) = W_{MIN}$. Then for all $x \in V(G)$, $f(N_D(x)) = \sigma_W - f(x) \leq \sigma_W - f(v) = f(N_D(v))$. That is, $NS(f;D) = f(N_D(v)) = \sigma_W - W_{MIN}$.

By definition $NS^{sp}(f;D) = NS(f;D) - NS^-(f;D) = (\sigma_W - W_{MIN}) - (\sigma_W - W_{MAX}) = W_{MAX} - W_{MIN}$.

Since the bijection f was arbitrary, we conclude that $NS_W^-(G;D) = \sigma_W - W_{MAX}$, $NS_W(G;D) = \sigma_W - W_{MIN}$, and $NS_W^{sp}(G;D) = W_{MAX} - W_{MIN}$. $\square$

Corollary 3.9. *Let graph G have diameter d. If $D = \{1,2,\ldots,d\}$, then*

(i) $NS^-(G;D) = \frac{n(n+1)}{2} - n = \frac{n(n-1)}{2}$,

(ii) $NS(G;D) = \frac{n(n+1)}{2} - 1$, *and*

(iii) $NS^{sp}(G;D) = n-1$.

Even though the results so far are very basic, they will be sufficient to provide a complete characterization of our parameters on K_n and its complement K_n^C. We will complete this characterization in Chapter 4. When the graph G is not connected, many of the previous results will not apply. We can establish the following result that will apply to all graphs.

Theorem 3.10. *Let G be a graph. Let $d = max(d(uv) : u,v \in V(G), d(uv) < \infty\}$. Let $D \subset \{0,1,2,\ldots,d\}$ and $S \subset \{d+1, d+2, ...\}$. Let $W = \{w_1, w_2, \ldots, w_n\}$ be such that $w_i \leq w_{i+1}$ for $i \in \{1,2,\ldots,n-1\}$. Then*

(i) $NS_W^-(G;S) = 0$ and $NS_W^-(G;S \bigcup D) = NS_W^-(G;D)$;

(ii) $NS_W(G;S) = 0$ and $NS_W(G;S \bigcup D) = NS_W(G;D)$;

(iii) $NS_W^{sp}(G;S) = 0$ and $NS_W^{sp}(G;S \bigcup D) = NS_W^{sp}(G;D)$;

(iv) if $D = \{0,1,\ldots,d\}$, then $NS_W(G;D) \geq \sum_{i=1}^{d+1} w_i$.

Proof. For any vertex $v \in V(G)$ we have that $NS_S(v) = \emptyset$. If we let $f : V(G) \to W$ be an arbitrary bijection, then for all $v \in V(G)$ we have that $f(N_S(v)) = f(\emptyset) = 0$. Hence, $NS^-(f;S) = NS(f;S) = NS^{sp}(f;S) = 0$, and since f was arbitrary, we conclude that $NS_W^-(G;S) = NS_W(G;S) = NS_W^{sp}(G;S) = 0$.

For any vertex $v \in V(G)$, we have that $N_{S \bigcup D}(v) = N_S(v) \bigcup N_D(v) = N_D(v)$. It then follows that for any bijection $g : V(G) \to W$, we have $g(N_{S \bigcup D}(v)) = g(N_D(v))$. This implies that $NS^-(g;S \bigcup D) = NS^-(g;D)$, $NS(g;S \bigcup D) = NS(g;D)$, and $NS^{sp}(g;S \bigcup D) =$

$NS^{sp}(g;D)$. Since g was arbitrary we have that $NS_W^-(G;S\bigcup D) = NS_W^-(G;D)$, $NS_W(G;S\bigcup D) = NS_W(G;D)$, and $NS_W^{sp}(G;S\bigcup D) = NS_W^{sp}(G;D)$.

Assume $D = \{0,1,\ldots,d\}$. If $u,w \in V(G)$ are such that $d(u,w) = d$, then $|N_D(u)| \geq d+1$, and hence $g(N_D(u)) = \sum_{x\in N_D(u)} g(x) \geq \sum_{i=1}^{d+1} w_i$. Thus, $NS_W(G;D) \geq \sum_{i=1}^{d+1} w_i$ $\square$

3.2 Relationships For Complementary Distance Sets

In this section we use the definition of complement related families of sets provided in Sewell and Slater [18] and Chapter 1 of Haynes et al. [8] in order to discuss two related distance sets D and $D^{\#}$. In general we want D and $D^{\#}$ to be disjoint sets, whose union contain all possible distances for a graph G.

Definition 3.11 ([18])**.** Let G be a graph. Families $\mathscr{F}_1$ and $\mathscr{F}_2$ of subsets of $V(G)$ will be called *complement related* when $S \in \mathscr{F}_1$ if and only if $V(G) - S \in \mathscr{F}_2$.

Lemma 3.12. *For graph G let $f : V(G) \to W$ be a bijection. Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be complement related subsets of $V(G)$ and $S_1, S_2 \in \mathscr{F}_1$. Then*

(i) $f(S_1) + f(V(G) - S_1) = \sigma_W$,

(ii) $f(S_1) \leq f(X)$ for all $X \in \mathscr{F}_1$ if and only if $f(V(G) - S_1) \geq f(Y)$ for all $Y \in \mathscr{F}_2$,

(iii) $f(S_2) \geq f(X)$ for all $X \in \mathscr{F}_1$ if and only if $f(V(G) - S_2) \leq f(Y)$ for all $Y \in \mathscr{F}_2$,

and

(iv) $|f(S_2) - f(S_1)| \geq |f(X_2) - f(X_1)|$ for all $X_1, X_2 \in \mathscr{F}_1$ if and only if $|f(S_2) - f(S_1)| \geq |f(Y_2) - f(Y_1)|$ for all $Y_1, Y_2 \in \mathscr{F}_2$.

Proof. For any $S \in \mathscr{F}_1$ we have that $f(S) + f(V(G) - S) = \sum_{v\in S} f(v) + \sum_{v\notin S} f(v) = \sum_{v\in V(G)} f(v)$ $= \sigma_W$ which proves part i.

For part ii, first assume that $f(S_1) \leq f(X)$ for all $X \in \mathscr{F}_1$ and let $T \in \mathscr{F}_2$. For both S_1 and T, from part i we have that $f(S_1) + f(V(G) - S_1) = \sigma_W = f(T) + f(V(G) - T)$. Since $f(S_1) \leq f(V(G) - T)$ we must have that $f(V(G) - S_1) \geq f(T)$. To prove the converse, assume $f(V(G) - S_1) \geq f(Y)$ for all $Y \in \mathscr{F}_2$ and let $T \in \mathscr{F}_1$. Again we have $f(S_1) + f(V(G) - S_1) = \sigma_W = f(T) + f(V(G) - T)$. Since $f(V(G) - S_1) \geq f(V(G) - T)$ we must have that $f(S_1) \leq f(T)$. This proves part ii.

Since the families $\mathscr{F}_1$ and $\mathscr{F}_2$ are arbitrary, part iii is simply a restatement of part ii.

For part iv, first assume that $|f(S_2) - f(S_1)| \geq |f(X_2) - f(X_1)|$ for all $X_1, X_2 \in \mathscr{F}_1$ and let $T_1, T_2 \in \mathscr{F}_2$. Without loss of generality assume that $f(T_1) \geq f(T_2)$. Applying part i we can deduce that $f(V(G) - T_1) \leq f(V(G) - T_2)$ and that $f(T_1) - f(T_2) = f(V(G) - T_2) - f(V(G) - T_1)$. Hence we have $|f(S_2) - f(S_1)| \geq |f(V(G) - T_2) - f(V(G) - T_1)| = f(V(G) - T_2) - f(V(G) - T_1) = f(T_1) - f(T_2) = |f(T_1) - f(T_2)|$. Since $\mathscr{F}_1$ and $\mathscr{F}_2$ are arbitrary, the proof of the converse is identical. □

Lemma 3.12 holds for any graph. If graph G is connected, then for any $D \subset \mathbb{N}$, we can define $D^\# = \mathbb{N} - D$, and then use D and $D^\#$ to create complement related families of subsets of $V(G)$.

Theorem 3.13. *Let $D^\# = \mathbb{N} - D$ and let $f : V(G) \to W$ be a bijection. If graph G is connected then*

(i) $NS(f;D) = f(N_D(v))$ if and only if $NS^-(f;D^\#) = f(N_{D^\#}(v))$,

(ii) $NS_W(G;D) = NS(f;D)$ if and only if $NS^-_W(G;D^\#) = NS^-(f;D^\#)$,

(iii) $NS_W(G;D) + NS_W^-(G;D^\#) = \sigma_W$*, and*

(iv) $NS_W^{sp}(G;D) = NS_W^{sp}(G;D^\#)$.

Proof. Let $\mathscr{F}_1 = \{N_D(v) : v \in V(G)\}$ and $\mathscr{F}_2 = \{N_{D^\#}(v) : v \in V(G)\}$. Since G is connected, for all $v \in V(G)$ we have that $N_D(v) = V(G) - N_{D^\#}(v)$ and $N_{D^\#}(v) = V(G) - N_D(v)$; that is, $\mathscr{F}_1$ and $\mathscr{F}_2$ are complement related families of subsets of $V(G)$. Let $f : V(G) \to W$ be an arbitrary bijection. For all $v \in V(G)$, we have from part i of Lemma 3.12 that $f(N_D(v)) + f(N_{D^\#}(v)) = f(N_D(v)) - f(V(G) - N_{D^\#}(v)) = \sigma_W$.

From part iii of Lemma 3.12, $NS(f;D) = f(N_D(v)) \geq f(N_D(u))$ for all $u \in V(G)$ if and only if $NS^-(f;D^\#) = f(N_{D^\#}(v)) \leq f(N_{D^\#}(w))$ for all $w \in V(G)$. This proves part i.

For part ii, let f be such that $NS_W(G;D) = NS(f;D)$ and let $g : V(G) \to W$ be arbitrary. We have $NS(f;D) \leq NS(g;D)$. Let $v \in V(G)$ be such that $NS(f;D) = f(N_D(v))$, and let $u \in V(G)$ be such that $NS^-(g;D^\#) = g(N_{D^\#}(u))$. Then $NS^-(g;D^\#) = g(N_{D^\#}(u)) = \sigma_W - g(N_D(u)) = \sigma_W - NS(g;D) \leq \sigma_W - NS(f;D) = \sigma_W - f(N_D(v)) = f(N_{D^\#}(v))$. Therefore $NS_W^-(G;D^\#) = NS^-(f;D^\#) = f(N_{D^\#}(v))$. The proof of the converse can be completed in a similar fashion.

For part iii let $f : V(G) \to W$ be a bijection such that $NS_W(G;D) = NS(f;D)$ and let $v \in V(G)$ be such that $NS(f;D) = f(N_D(v))$. By parts i and ii it follows that $NS_W^-(G;D^\#) = NS^-(f;D^\#) = f(N_{D^\#}(v))$. Hence $NS_W(G;D) + NS_W^-(G;D^\#) = f(N_D(v)) + f(N_{D^\#}(v)) = \sigma_W$.

Finally, let f be such that $NS_W^{sp}(G;D) = NS^{sp}(f;D)$. Hence there exists $u, v \in V(G)$ such that $NS_W^{sp}(G;D) = NS^{sp}(f;D) = |f(N_D(u)) - f(N_D(v))|$; that is, for all $x, y \in V(G)$, $|f(N_D(u)) - f(N_D(v))| \geq |f(N_D(x)) - f(N_D(y))|$. Thus, by part iv of Lemma 3.12,

$|f(N_D(u)) - f(N_D(v))| \geq |f(N_{D^\#}(x)) - f(N_{D^\#}(y))|$ for all $x, y \in V(G)$. Since $|f(N_D(u)) - f(N_D(v))| = |(f(N_D(u)) - f(V(G))) - (f(N_D(v)) - f(V(G)))|$ $= |f(N_{D^\#}(u)) - f(N_{D^\#}(v))|$, we have that $NS_W^{sp}(G; D^\#) = NS^{sp}(f; D^\#)$ $= NS^{sp}(f; D) = NS_W^{sp}(G; D)$. □

Corollary 3.14. *Let graph G be connected and $D^\# = \mathbb{N} - D$. Let $f : V(G) \to [n]$ be a bijection. Then*

(i) $NS(G; D) + NS^-(G; D^\#) = \frac{n(n+1)}{2}$, and

(ii) $NS^{sp}(G; D) = NS^{sp}(G; D^\#)$.

The following corollary to Theorem 3.13 appeared in O'Neal and Slater [13].

Corollary 3.15 ([13]). *Let graph G be any connected graph, and let $D^\# = \mathbb{N} - D$. G is D-vertex magic if and only if G is $D^\#$-vertex magic.*

Example 3.16. Recall graph H from Example 2.16. Letting $D = \{0, 1\}$ and $D^\# = \mathbb{N} - D$ we have that $NS^{sp}[H] = NS^{sp}(H; D) = NS^{sp}(H; D^\#) = NS^{sp}(H; \{2\}) = 4$ as shown.

One of the key objectives of this section is to establish a similar result between any graph and its complement. We would like to have a result that applies to all graphs and their complements, even those that are not connected. We can achieve this result by recalling that for any graph G, either G or G^C will be connected. By choosing appropriate distance sets, we can then achieve the desired result. A proof of the next theorem can also be argued directly, without the notion of distance sets, as was done for the case where $W = [n]$ in O'Neal and Slater [12].

Theorem 3.17. *Let G be any graph and G^C its complement. Let $f : V(G) \to W$ be a bijection and define $g : V(G^C) = V(G) \to W$ by $g(v) = f(v)$. Then*

(i) $NS[f] = f(N[v])$ if and only $NS^-(g) = g(N(v))$,

(ii) $NS_W[G] = NS[f]$ if and only if $NS_W^-(G^C) = NS^-(g)$,

(iii) $NS_W[G] + NS_W^-(G^C) = \sigma_W = NS_W(G) + NS_W^-[G^C]$, and

(iv) $NS_W^{sp}[G] = NS_W^{sp}(G^C)$.

Proof. For any graph G, either G or G^C is connected, so assume without loss of generality that G is connected. Let $D = \{0,1\}$ and $D^\# = \mathbb{N} - D$. We can now apply Theorem 3.13 (after making appropriate notational changes) to conclude that

(i) $NS[f] = f(N[v])$ if and only if $NS^-(f;D^\#) = f(N_{D^\#}(v))$,

(ii) $NS_W[G] = NS[f]$ if and only if $NS_W^-(G;D^\#) = NS^-(f;D^\#)$,

(iii) $NS_W[G] + NS_W^-(G;D^\#) = \sigma_W$, and

(iv) $NS_W^{sp}[G] = NS_W^{sp}(G;D^\#)$.

Notice for every vertex $v \in V(G) = V(G^C)$ that the $D^\#$-neighborhood of v in G is the same set as the open neighborhood of v in G^C. Hence $f(N_{D^\#}(v)) = g(N(v))$. Since both v and f are arbitrary, it follows that $NS^-(f;D^\#) = NS^-(g)$, $NS_W^-(G;D^\#) = NS_W^-(G^C)$, and $NS_W^{sp}(G;D^\#) = NS_W^{sp}(G^C)$. Substituting we have that

(i) $NS[f] = f(N[v])$ if and only $NS^-(g) = g(N(v))$,

(ii) $NS_W[G] = NS[f]$ if and only if $NS_W^-(G^C) = NS^-(g)$,

(iii) $NS_W[G] + NS_W^-(G^C) = \sigma_W$, and

(iv) $NS_W^{sp}[G] = NS_W^{sp}(G^C)$.

To see that $\sigma_W = NS_W(G) + NS_W^-[G^C]$, notice that we could set $D = \{1\}$ and $D^\# = \mathbb{N} - \{1\}$ and repeat the above arguments. □

Corollary 3.18 ([12]). *Let graph $G = (V,E)$ have complement G^C. Then $NS[G] + NS^-(G^C) = \frac{n(n+1)}{2} = NS(G) + NS^-[G^C]$.*

The following corollary appeared in Beena [4]. Because of this result, the set of all Σ-labeled graphs is simply the set of complements of all $\Sigma^{'}$-labeled graphs. Hence all Σ-labeled graphs of order 10 or less can be formed by taking the complements of the $\Sigma^{'}$-labeled graphs listed in Appendix A.

Corollary 3.19 ([4]). *Graph G is Σ-labeled if and only if G^C is $\Sigma^{'}$-labeled.*

3.3 Bounds on Neighborhood Sum Parameters for a Graph

In this section we focus on creating bounds for our parameters. In Section 3.3.1 we look at bounds that can be established based on the degree sequence of the graph. In Section 3.3.2 we extend the notion of fractional domination and packing to arbitrary distances and then establish the key result that $NS_W^-(G;D) \leq NS_W(G;D)$. In Section 3.3.3 we consider the effect that changes in the weight set W have on our parameters. In Section 3.3.4 we look at bounds that can be determined when two graphs are combined into a single larger graph.

3.3.1 Bounds Based on Number of Edges

The second part of Theorem 3.20 appeared in Schnieder and Slater [17] for the case where $W = [n]$ and $D = \{0,1\}$ or $D = \{1\}$. We state the result for arbitrary W and D;

however, the proof is a straightforward extension of the proof provided by Schneider and Slater.

Theorem 3.20. *Let graph G have vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. Let $W = \{w_1, w_2, \ldots, w_n\}$ be such that for all $i \in \{1, 2, \ldots, n-1\}$ we have that $w_i \le w_{i+1}$. Let $d_i = deg_D(v_i)$ and assume without loss of generality that $(d_1, d_2, \ldots, d_n)$ is a non-increasing sequence. Let $f : V(G) \to W$ be an arbitrary bijection. Then*

(i) $\sum_{i=1}^{n} f(N_D(v_i)) \le w_n d_1 + w_{n-1} d_2 + \cdots + w_1 d_n$ *and* $NS^-_W(G;D) \le \sum_{i=1}^{n} \frac{w_{n-i+1} d_i}{n}$*, and*

(ii) $\sum_{i=1}^{n} f(N_D(v_i)) \ge w_1 d_1 + w_2 d_2 + \cdots + w_n d_n$ *and* $NS_W(G;D) \ge \sum_{i=1}^{n} \frac{w_i d_i}{n}$.

Proof. Let $g : V(G) \to W$ be a bijection that maximizes $\sum_{i=1}^{n} g(N_D(v_i))$. Notice that $g(v_i)$ appears as a summand in $g(N_D(x))$ if and only if $x \in N_D(v_i)$. Hence, $g(v_i)$ appears as a summand for $d_i = deg_D(v_i)$ vertices, and so $\sum_{i=1}^{n} g(v_i) d_i = \sum_{i=1}^{n} g(N_D(v_i))$. If $g(v_k) \le g(v_{k+1})$ let $h : V(G) \to W$ be the bijection with $h(v_k) = g(v_{k+1})$, $h(v_{k+1}) = g(v_k)$, and $h(v_j) = g(v_j)$ when $j \notin \{k, k+1\}$. By assumption, $0 \le \sum_{i=1}^{n} g(N_D(v_i)) - \sum_{i=1}^{n} h(N_D(v_i)) = g(v_k)d_k + g(v_{k+1})d_{k+1} - h(v_k)d_k - h(v_{k+1})d_{k+1} = g(v_k)d_k + g(v_{k+1})d_{k+1} - g(v_{k+1})d_k - g(v_k)d_{k+1} = (g(v_{k+1}) - g(v_k))(d_{k+1} - d_k)$. Since $g(v_{k+1}) \ge g(v_k)$ and $d_k \ge d_{k+1}$, we must have that either $g(v_k) = g(v_{k+1})$ or $d_k = d_{k+1}$, and in either case we can assume that $g(v_k) = w_{n+1-k}$. Hence $\sum_{i=1}^{n} f(N_D(v_i)) \le \sum_{i=1}^{n} g(N_D(v_i)) = \sum_{i=1}^{n} g(v_i) d_i \le \sum_{i=1}^{n} w_{n+1-i} d_i$. Now for the arbitrary bijection f, the average value of $f(N_D(v_i))$ is $\sum_{i=1}^{n} \frac{f(v_i) d_i}{n} \le \sum_{i=1}^{n} \frac{w_{n+1-i} d_i}{n}$, and it follows that $NS^-(f;D) \le \sum_{i=1}^{n} \frac{w_{n+1-i} d_i}{n}$. Since f was an arbitrary bijection we have that $NS^-_W(G;D) \le \sum_{i=1}^{n} \frac{w_{n+1-i} d_i}{n}$.

Next let $g : V(G) \to W$ be a bijection that minimizes $\sum_{i=1}^{n} g(N_D(v_i))$. Notice again that $g(v_i)$ appears as a summand in $g(N_D(x))$ if and only if $x \in N_D(v_i)$. Hence, $g(v_i)$ appears as a

summand for $d_i = deg_D(v_i)$ vertices, and so $\sum_{i=1}^{n} g(v_i)d_i = \sum_{i=1}^{n} g(N_D(v_i))$. If $g(v_k) \geq g(v_{k+1})$ let $h : V(G) \to W$ be the bijection with $h(v_k) = g(v_{k+1})$, $h(v_{k+1}) = g(v_k)$, and $h(v_j) = g(v_j)$ when $j \notin \{k, k+1\}$. By assumption, $0 \leq \sum_{i=1}^{n} h(N_D(v_i)) - \sum_{i=1}^{n} g(N_D(v_i)) = h(v_k)d_k + h(v_{k+1})d_{k+1} - g(v_k)d_k - g(v_{k+1})d_{k+1} = g(v_{k+1})d_k + g(v_k)d_{k+1} - g(v_k)d_k - g(v_{k+1})d_{k+1} = (g(v_k) - g(v_{k+1}))(d_{k+1} - d_k)$. Since $g(v_k) \geq g(v_{k+1})$ and $d_k \geq d_{k+1}$, we must have that either $g(v_k) = g(v_{k+1})$ or $d_k = d_{k+1}$, and in either case we can assume that $g(v_k) = w_k$. Hence $\sum_{i=1}^{n} f(N_D(v_i)) \geq \sum_{i=1}^{n} g(N_D(v_i)) = \sum_{i=1}^{n} g(v_i)d_i \geq \sum_{i=1}^{n} w_i d_i$. Now for the arbitrary bijection f, the average value of $f(N_D(v_i))$ is $\sum_{i=1}^{n} \frac{f(v_i)d_i}{n} \geq \sum_{i=1}^{n} \frac{w_i d_i}{n}$, and it follows that $NS(f;D) \geq \sum_{i=1}^{n} \frac{w_i d_i}{n}$. Since f was an arbitrary bijection we have that $NS_W(G;D) \geq \sum_{i=1}^{n} \frac{w_i d_i}{n}$. $\square$

Corollary 3.21. *If graph G is (D,r) regular then*

(i) $NS_W^-(G;D) \leq \sum_{i=1}^{n} \frac{rw_i}{n}$, *and*

(ii) $NS_W(G;D) \geq \sum_{i=1}^{n} \frac{rw_i}{n}$.

The following corollary to Theorem 3.20 appeared in O'Neal and Slater [12].

Corollary 3.22 ([12]). *Let graph G have vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ with non-increasing degree sequence $(d_1, d_2, \ldots, d_n)$, and let $f : V(G) \to [n]$ be a bijection. Then*

(i) $NS^-[G] \leq \left\lfloor \sum_{i=1}^{n} \frac{i(d_{n+1-i}+1)}{n} \right\rfloor = \left\lfloor \sum_{i=1}^{n} \frac{(n+1)(d_{n+1-i}+1)}{n} \right\rfloor$,

(ii) $NS^-(G) \leq \left\lfloor \sum_{i=1}^{n} \frac{id_{n+1-i}}{n} \right\rfloor = \left\lfloor \sum_{i=1}^{n} \frac{(n+1)d_{n+1-i}}{2} \right\rfloor$,

(iii) $NS[G] \geq \left\lceil \sum_{i=1}^{n} \frac{i(d_i+1)}{n} \right\rceil = \left\lceil \sum_{i=1}^{n} \frac{(n+1)(d_i+1)}{2} \right\rceil$, *and*

(iv) $NS(G) \geq \left\lceil \sum_{i=1}^{n} \frac{id_i}{n} \right\rceil = \left\lceil \sum_{i=1}^{n} \frac{(n+1)d_i}{2} \right\rceil$.

Proof. Since $W = [n]$, all neighborhood sums will be integer valued. In particular, $NS^-[D]$, $NS^-(G)$, $NS[G]$, and $NS(G)$ will be integer valued. With this observation, the result follows directly from Theorem 3.20. $\square$

Corollary 3.23 ([12]). *If graph G be is regular of degree r, then*

(i) $NS^{-}[G] \leq \left\lfloor \frac{(r+1)(n+1)}{2} \right\rfloor$,

(ii) $NS^{-}(G) \leq \left\lfloor \frac{r(n+1)}{2} \right\rfloor$,

(iii) $NS[G] \geq \left\lceil \frac{(r+1)(n+1)}{2} \right\rceil$, *and*

(iv) $NS(G) \geq \left\lceil \frac{r(n+1)}{2} \right\rceil$.

A simple consequence of Corollary 3.21 is that for any (D,r) regular graph G we have $NS_W^{-}(G;D) \leq NS_W(G;D)$. When $W = [n]$ and G is r regular, it follows that $NS^{-}[G] \leq NS[G]$ and $NS^{-}(G) \leq NS(G)$. These results are stated formally in the following three corollaries. However, the proof of Theorem 3.20 suggests that when a graph H is not (D,r)-regular, that by placing the larger labels on the lower degree (and then higher degree) vertices we might be able to have an example where $NS^{-}(H;D) > NS(G;D)$. This turns out not to be possible as we will show in Section 3.3.2.

Corollary 3.24. *If graph G is (D,r) regular, then $NS_W^{-}(G;D) \leq NS_W(G;D)$, and if G is D-vertex magic, then its D-vertex magic constant is unique.*

Corollary 3.25. *If graph G is r regular, then $NS^{-}[G] \leq NS[G]$, and if G is $\{0,1\}$-vertex magic (Σ'-labeled), then its $\{0,1\}$-vertex magic constant is unique.*

Corollary 3.26. *If graph G is r regular, then $NS^{-}(G) \leq NS(G)$, and if G is $\{1\}$-vertex magic (Σ-labeled), then its $\{1\}$-vertex magic constant is unique.*

We can also use $\Delta_D(G)$ $(\delta_D(G))$ to determine an upper (lower) bound for $NS_W(G;D)$ $\big($respectively, $NS_W^{-}(G;D)\big)$.

Theorem 3.27. *Let $W = \{w_1, w_2, \ldots, w_n\}$ be such that for all $1 \leq i \leq n-1$ we have $w_i \leq w_{i+1}$. Then for any graph G*

(i) $NS^-_W(G;D) \geq \sum_{i=1}^{\delta_D(G)} w_i$*, and*

(ii) $NS_W(G;D) \leq \sum_{i=n-\Delta_D(G)+1}^{n} w_i$.

Proof. Let $f : V(G) \to W$ be an arbitrary bijection and let $v \in V(G)$. Since $deg_D(v) \geq \delta_D(G)$, $f(N_D(v))$ must be at least the sum of the $\delta_D(G)$ smallest weights from W. Hence, $f(N_D(v)) \geq \sum_{i=1}^{\delta_D(G)} w_i$. Since both the vertex v and the bijection f were arbitrary, it follows that $NS^-_W(G;D) \geq \sum_{i=1}^{\delta_D(G)} w_i$.

Similarly, since $deg_D(v) \leq \Delta_D(G)$, $f(N_D(v))$ cannot exceed the sum of the $\Delta_D(G)$ largest weights from W. Hence, $f(N_D(v)) \leq \sum_{i=n-\Delta_D(G)+1}^{n} w_i$. Since both the vertex v and the bijection f were arbitrary, it follows that $NS_W(G;D) \leq \sum_{i=n-\Delta_D(G)+1}^{n} w_i$. □

Corollary 3.28. *For any graph G*

(i) $NS^-(G;D) \geq \frac{\delta_D(G)(\delta_D(G)+1)}{2}$*, and*

(ii) $NS(G;D) \leq \frac{\Delta_D(G)(2n-\Delta_D(G)+1)}{2}$.

Proof. The first part of this corollary follows directly from Theorem 3.27. From Theorem 3.27 we have that $NS(G;D) \leq \sum_{i=n-\Delta_D(G)+1}^{n} w_i = \sum_{i=n-\Delta_D(G)+1}^{n} i = (n-\Delta_D(G))\Delta_D(G) + \frac{\Delta_D(G)(\Delta_D(G)+1)}{2} = \frac{2n\Delta_D(G)-\Delta_D^2(G)+\triangle_D(G)}{2} = \frac{\Delta_D(2n-\Delta_D(G)+1)}{2}$. □

Corollary 3.29. *For any graph G*

(i) $NS^-[G] \geq \frac{(\delta(G)+1)(\delta(G)+2)}{2}$,

(ii) $NS^-(G) \geq \frac{\delta(G)(\delta(G)+1)}{2}$,

(iii) $NS[G] \leq \frac{(\Delta(G)+1)(2n-\triangle(G))}{2}$*, and*

(iv) $NS(G) \leq \frac{\Delta(G)(2n-\Delta(G)+1)}{2}$.

3.3.2 Bounds Based on the Fractional Domination Number

In this section we will establish bounds that are based on the fractional domination and packing numbers for the graph. In order to state these results in their most general form, we extend the notion of fractional domination and packing to arbitrary distances. The extended notion of fractional domination and packing are then used to establish bounds for the minimax and maximin neighborhood sums. In particular, we will show that for any graph G, weight set W, and distance set D, that $NS_W^-(G;D) \leq NS_W(G;D)$. An immediate and important consequence of this result is that vertex magic constants, when they exist, are unique. Many of the results of this section can be found in O'Neal and Slater [15] for the case where $W = [n]$.

3.3.2.1 Extension of Fractional Domination and Packing to Arbitrary Distance Sets

In Chapter 1 we provided the definitions of dominating and packing functions. We now extend these definitions for arbitrary neighborhoods.

Definition 3.30 ([15]). A function $g : V(G) \to \mathbb{N}$ is said to be a *D-neighborhood dominating function* if for every $v \in V(G)$, $\sum\limits_{u \in N_D(v)} g(u) \geq 1$.

Definition 3.31 ([15]). The *D-neighborhood domination number* of G, denoted by $\gamma(G;D)$, is defined as $\gamma(G;D) = min\{\sum\limits_{v \in V(G)} g(v) : g$ is a D-neighborhood dominating function$\}$.

Notice that when $D = \{0,1\}$ these definitions reduce to the usual notion of domination. By convention we will shorten our notation to $\gamma(G) = \gamma(G;\{0,1\})$. Also notice that when $D = \{1\}$, these definitions reduce to the notion of open (total) domination. By convention we will shorten our notation to $\gamma^o(G) = \gamma(G;\{1\})$.

If $g : V(G) \to \mathbb{N}$ is a D-neighborhood dominating function of G such that $\sum_{v \in V(G)} g(v) = \gamma(G;D)$, then we will say that g is a $\gamma(G;D)$ function.

For domination using closed neighborhoods, we are ensured that $\gamma(G) \leq n$ because, by assigning a 1 to each vertex, every vertex dominates itself. More generally, if $0 \in D$, then we will be ensured that $\gamma(G;D) \leq n$. However, when $0 \notin D$, we are not guaranteed that there exists a function $f : V(G) \to \mathbb{N}$ that is a D-neighborhood dominating function of G. In fact, G has a D-neighborhood dominating function f if and only if, for every $u \in V(G)$, there exists a $v \in V(G)$ such that $d(u,v) \in D$. For the case where there does not exist a D-neighborhood dominating function for G, we define $\gamma(G;D) = \infty$.

Definition 3.32 ([15])**.** A function $g : V(G) \to \mathbb{R}^+$ is said to be a *D-neighborhood fractional dominating function* if for every $v \in V(G)$, $\sum_{u \in N_D(v)} g(u) \geq 1$.

Definition 3.33 ([15])**.** The *D-neighborhood fractional domination number* of G, denoted by $\gamma_f(G;D)$, is defined as $\gamma_f(G;D) = min\{\sum_{v \in V(G)} g(v) : g$ is a D-neighborhood fractional dominating function$\}$.

When $D = \{0,1\}$ these definitions reduce to the usual notion of fractional domination, and we write $\gamma_f(G) = \gamma_f(G;\{0,1\})$. When $D = \{1\}$, these definitions reduce to the notion of open (total) fractional domination, and we write $\gamma_f^o(G) = \gamma_f(G;\{1\})$.

If $g : V(G) \to \mathbb{R}^+$ is a D-neighborhood fractional dominating function for G such that $\sum_{v \in V(G)} g(v) = \gamma_f(G;D)$, then we will say that g is a $\gamma_f(G;D)$ function. If G and D are such that there does not exist a D-neighborhood fractional dominating function for G, then we define $\gamma_f(G;D) = \infty$.

Definition 3.34 ([15]). A function $g : V(G) \to \mathbb{N}$ is said to be a *D-neighborhood packing function* if for every $v \in V(G)$, $\sum\limits_{u \in N_D(v)} g(u) \leq 1$.

Definition 3.35 ([15]). The *D-neighborhood packing number* of G, denoted by $\rho(G;D)$, is defined as $\rho(G;D) = max\{\sum\limits_{v \in V(G)} g(v) : g$ is a D-neighborhood packing function$\}$.

When $D = \{0,1\}$ these definitions reduce to the usual notion of packing, and we write $\rho(G) = \rho(G;\{0,1\})$. When $D = \{1\}$, these definitions reduce to the notion of open (total) packing, and we write $\rho^o(G) = \rho(G;\{1\})$.

In the traditional sense of packing where we are interested in closed neighborhoods, we are guaranteed that $\rho(G) \leq n$, because no vertex can be assigned a number greater than 1. More generally, when $0 \in D$, we are ensured that $\rho(G;D) \leq n$ for the same reason. However, when $0 \notin D$, it is possible that the packing number (as we have defined it) can be unbounded. For example, if G has an isolated vertex v and $D = \{1\}$, then we can assign a number as large as we want to v. In fact, $\rho(G;D)$ is unbounded if and only if there exists a $u \in V(G)$ such that, for every $v \in V(G)$, $d(u,v) \notin D$. When this situation occurs, we will define $\rho(G;D) = \infty$.

Definition 3.36 ([15]). A function $g : V(G) \to \mathbb{R}^+$ is said to be a *D-neighborhood fractional packing function* if for every $v \in V(G)$, $\sum\limits_{u \in N_D(v)} g(u) \leq 1$.

Definition 3.37 ([15]). The *D-neighborhood fractional packing number* of G, denoted by $\rho_f(G;D)$, is defined as $\rho_f(G;D) = max\{\sum\limits_{v \in V(G)} g(v) : g$ is a D-neighborhood fractional packing function$\}$.

When $D = \{0,1\}$ these definitions reduce to the usual notion of fractional packing, and we write $\rho_f(G) = \rho_f(G;\{0,1\})$. When $D = \{1\}$ these definitions reduce to the

$$\gamma(G;D) = \text{MIN} \quad \sum_{i=1}^{n} X_i \quad \text{subject to} \quad A_D X \geq 1, \quad X_i \in \mathbb{N}$$

$$\gamma_f(G;D) = \text{MIN} \quad \sum_{i=1}^{n} X_i \quad \text{subject to} \quad A_D X \geq 1, \quad X_i \geq 0$$

Figure 3.1: Programs for $\gamma(G;D)$ and $\gamma_f(G;D)$

$$\rho(G;D) = \text{MAX} \quad \sum_{i=1}^{n} X_i \quad \text{subject to} \quad A_D X \leq 1, \quad X_i \in \mathbb{N}$$

$$\rho_f(G;D) = \text{MAX} \quad \sum_{i=1}^{n} X_i \quad \text{subject to} \quad A_D X \leq 1, \quad X_i \geq 0$$

Figure 3.2: Programs for $\rho(G;D)$ and $\rho_f(G;D)$

notion of open (total) fractional packing, and we write $\rho_f^o(G) = \rho_f(G;\{1\})$. When the D-neighborhood fractional packing number of the graph G is unbounded, we will define $\rho_f(G;D) = \infty$.

For a graph G, let X be a $n \times 1$ vector of values assigned to the vertices of G. The problem of determining the parameters $\gamma(G;D)$, $\gamma_f(G;D)$, $\rho(G;D)$, and $\rho_f(G;D)$ can be formulated as integer and linear programs as shown in Figures 3.1 and 3.2.

Since any solution to the D-neighborhood dominating problem is also a solution to the D-neighborhood fractional dominating problem, it is clear that $\gamma_f(G;D) \leq \gamma(G;D)$. Similarly, since any solution to the D-neighborhood packing problem is also a solution to the D-neighborhood fractional packing problem, we have $\rho(G;D) \leq \rho_f(G;D)$.

Because A_D is symmetric, the D-neighborhood fractional dominating problem and the D-neighborhood fractional packing problems are duals of each other. Notice also that the D-neighborhood fractional packing number always has a (possibly unbounded) solu-

tion. By the Fundamental Theorem of Duality for linear programming [3], exactly one of the following will hold:

(i) Both the D-neighborhood fractional domination problem and the D-neighborhood fractional packing problem have optimal solutions and $\rho_f(G;D) = \gamma_f(G;D) \in \mathbb{R}^+$.

(ii) The D-neighborhood fractional domination problem is infeasible and the D-neighborhood fractional packing problem is unbounded. In this case, we have defined $\rho_f(G;D) = \gamma_f(G;D) = \infty$.

Hence we will write $\rho_f(G;D) = \gamma_f(G;D)$ with the understanding that the common value may not be finite. We summarize these results in the following theorem.

Theorem 3.38 ([15])**.** *For any graph G,* $\rho(G;D) \leq \rho_f(G;D) = \gamma_f(G;D) \leq \gamma(G;D)$.

We can also extend the notion of efficient domination to arbitrary neighborhoods.

Definition 3.39 ([15])**.** A function $g: V(G) \to \mathbb{N}$ is said to be a *D-neighborhood efficient dominating function* if g is both a D-neighborhood dominating function of G and a D-neighborhood packing of G.

Definition 3.40 ([15])**.** A function $g: V(G) \to \mathbb{R}^+$ is said to be a *D-neighborhood efficient fractional dominating function* if g is both a D-neighborhood fractional dominating function of G and a D-neighborhood fractional packing of G.

3.3.2.2 Uniqueness of D-Vertex Magic Constants

We first create a lower bound for $NS_W(G;D)$ based on the $\gamma_f(G;D)$.

Lemma 3.41 ([15]). *Let $x_1, x_2, \ldots, x_n$ be non-negative numbers such that $\sum_{i=1}^{n} x_i = a > 0$ and let $y_1, y_2, \ldots, y_n \in \mathbb{R}$. Let $\sum_{i=1}^{n} x_i y_i = c$. Then there exists $j, k \in \{1, 2, \ldots, n\}$ such that $y_j \leq \frac{c}{a} \leq y_k$.*

Proof. Let $j, k \in \{1, 2, \ldots, n\}$ be such that for all $i \in \{1, 2, \ldots n\}$, $y_j \leq y_i \leq y_k$. Then $y_j = y_j \sum\limits_{i=1}^{n} \frac{x_i}{a} \leq \sum\limits_{i=1}^{n} \frac{x_i y_i}{a} = \frac{c}{a} \leq y_k \sum\limits_{i=1}^{n} \frac{x_i}{a} = y_k$. $\square$

Lemma 3.42 ([15]). *If $g : V(G) \to W$ is any bijection, and $h : V(G) \to \mathbb{R}^+$ is a $\gamma_f(G;D)$ function, then $\sum\limits_{v \in V(G)} \sum\limits_{u \in N_D(v)} h(v)g(u) \geq \sigma_W$.*

Proof. Notice that since h is a $\gamma_f(G;D)$ function, for all $u \in V(G)$, $\sum\limits_{v \in N_D(u)} h(v) \geq 1$. Then $\sum\limits_{v \in V(G)} \sum\limits_{u \in N_D(v)} h(v)g(u) = \sum\limits_{u \in V(G)} \sum\limits_{v \in N_D(u)} h(v)g(u) = \sum\limits_{u \in V(G)} g(u) \sum\limits_{v \in N_D(u)} h(v) \geq \sum\limits_{u \in V(G)} g(u) = \sigma_W$. $\square$

Theorem 3.43 ([15]). *For any graph G, $NS_W(G;D) \geq \frac{\sigma_W}{\gamma_f(G;D)}$.*

Proof. Let $g : V(G) \to W$ be any bijection. If there does not exist a $\gamma_f(G;D)$ function, then by definition $\gamma_f(G;D) = \infty$. In this case, there exists a vertex $u \in V(G)$ such that $N_D(u) = \varnothing$. Hence, $NS(g;D) \geq g(N_D(u)) = 0$. If $\gamma_f(G;D) \in \mathbb{R}^+$, let $h : V(G) \to \mathbb{R}^+$ be a $\gamma_f(G;D)$ function. Let $S = \{v \in V(G) : h(v) \neq 0\}$. Then $\sum\limits_{v \in S} h(v)g(N_D(v)) = \sum\limits_{v \in S} h(v) \sum\limits_{u \in N_D(v)} g(u) = \sum\limits_{v \in V(G)} h(v) \sum\limits_{u \in N_D(v)} g(u) = \sum\limits_{v \in V(G)} \sum\limits_{u \in N_D(v)} h(v)g(u) \geq \sigma_W$ by Lemma 3.42. Notice that for all $v \in V(G)$ we have $h(v) \geq 0$ and that $\sum\limits_{v \in V(G)} h(v) = \gamma_f(G;D) > 0$. Thus by Lemma 3.41 there exists a $v \in S$ such that $g(N_D(v)) \geq \frac{\sigma_W}{\gamma(G;D)}$. Since g was an arbitrary bijection, it follows that $NS_W(G) \geq \frac{\sigma_W}{\gamma(G;D)}$. $\square$

Corollary 3.44 ([15]). *For any graph G, $NS[G] \geq \left\lceil \frac{n(n+1)}{2\gamma_f(G)} \right\rceil$ and $NS(G) \geq \left\lceil \frac{n(n+1)}{2\gamma_f^o(G)} \right\rceil$.*

Proof. In the case that $W = [n]$, all neighborhood sums will be integer valued. With this note, the result follows directly from Theorem 3.43. $\square$

Next we create an upper bound for $NS_W^-(G;D)$ based on $\rho_f(G;D)$.

Lemma 3.45 ([15]). *If $g : V(G) \to W$ is any bijection, and $h : V(G) \to \mathbb{R}^+$ is a $\rho_f(G;D)$ function, then $\sum_{v \in V(G)} \sum_{u \in N_D(v)} h(v)g(u) \leq \sigma_W$.*

Proof. Notice that since h is a $\rho_f(G;D)$ function, for all $u \in V(G)$, $\sum_{v \in N_D(u)} h(v) \leq 1$. Then $\sum_{v \in V(G)} \sum_{u \in N_D(v)} h(v)g(u) = \sum_{u \in V(G)} \sum_{v \in N_D(u)} h(v)g(u) = \sum_{u \in V(G)} g(u) \sum_{v \in N_D(u)} h(v) \leq \sum_{u \in V(G)} g(u) = \sigma_W$. $\square$

Theorem 3.46 ([15]). *For any graph G, $NS_W^-(G;D) \leq \frac{\sigma_W}{\rho_f(G;D)}$.*

Proof. Let $g : V(G) \to W$ be any bijection. If $\rho_f(G;D) = \infty$, then there exist a vertex $u \in V(G)$ such that $N_D(u) = \emptyset$. Then $NS^-(g;D) \leq g(N_D(u)) = 0 \leq \frac{\sigma_W}{\rho_f(G;D)}$. If $\rho_f(G;D) \in \mathbb{R}^+$, let $h : V(G) \to \mathbb{R}^+$ be a $\rho_f(G;D)$ function. Let $S = \{v \in V(G) : h(v) \neq 0\}$. Then $\sum_{v \in S} h(v)g(N_D(v)) = \sum_{v \in S} h(v) \sum_{u \in N_D(v)} g(u) = \sum_{v \in V(G)} h(v) \sum_{u \in N_D(v)} g(u) = \sum_{v \in V(G)} \sum_{u \in N_D(v)} h(v)g(u) \leq \sigma_W$ by Lemma 3.42. Notice that for all $v \in V(G)$ we have $h(v) \geq 0$ and that $\sum_{v \in V(G)} h(v) = \rho_f(G;D) > 0$. Thus by Lemma 3.41 there exists a $v \in S$ such that $g(N_D(v)) \leq \frac{\sigma_W}{\rho(G;D)}$. Since g was an arbitrary bijection, it follows that $NS_W^-(G;D) \leq \frac{\sigma_W}{\rho_f(G;D)}$. $\square$

Corollary 3.47 ([15]). *For any graph G, $NS^-[G] \leq \left\lfloor \frac{n(n+1)}{2\rho_f(G)} \right\rfloor$ and $NS^-(G) \leq \left\lfloor \frac{n(n+1)}{2\rho_f^o(G)} \right\rfloor$.*

Proof. In the case that $W = [n]$, all neighborhood sums will be integer valued. With this note, the result follows directly from Theorem 3.46. $\square$

Theorem 3.48 ([15])**.** *For any graph G,* $NS_W^-(G;D) \le \frac{\sigma_W}{\rho_f(G;D)} = \frac{\sigma_W}{\gamma_f(G;D)} \le NS_W(G;D)$.

Proof. From Theorem 3.38 we know that for any graph G we have that $\rho_f(G;D) = \gamma_f(G;D)$. Therefore from Theorems 3.43 and 3.46 it follows that $NS_W^-(G;D) \le \frac{\sigma_W}{\rho_f(G)} = \frac{\sigma_W}{\gamma_f(G;D)} \le NS_W(G;D)$. □

Corollary 3.49 ([15])**.** *For any graph G,* $NS^-[G] \le NS[G]$ *and* $NS^-(G) \le NS(G)$.

Theorem 3.50. *If graph G is D-vertex magic, then its unique D-vertex magic constant is* $c = \frac{n(n+1)}{2\gamma_f(G;D)}$.

Proof. Let $f : V(G) \to [n]$ be a vertex magic labeling for G. By Theorems 3.43, 3.46 and 3.48 we have $NS^-(f;D) \le NS^-(G;D) \le \frac{n(n+1)}{2\gamma_f(G;D)} \le NS(G;D) \le NS(f;D)$. Since G is D-vertex magic, its vertex magic constant $c = NS^-(f;D) = NS(f;D)$. Therefore, $c = \frac{n(n+1)}{2\gamma_f(G;D)}$. □

Corollary 3.51 ([15])**.** *If graph G is* Σ'*-labeled, then* $NS^-[G] = \frac{n(n+1)}{2\rho_f(G)} = \frac{n(n+1)}{2\gamma_f(G)} = NS[G]$.

Corollary 3.52 ([15])**.** *If graph G is* Σ*-labeled, then* $NS^-(G) = \frac{n(n+1)}{2\rho_f^o(G)} = \frac{n(n+1)}{2\gamma_f^o(G)} = NS(G)$.

Example 3.53. Recall from Examples 2.10 and 2.10 that we had $NS^-[H] = 9$ and $NS[H] = 11$. As shown in Grinstead and Slater [7], $\rho_f(H) = \gamma_f(H) = \frac{3}{2}$. Notice that $\frac{n(n+1)}{2\gamma_f(H)} = \frac{5(6)}{2(\frac{3}{2})} = 10$. Recall also from Examples 2.10 and 2.10 that we had $NS^-(H) = 7$ and $NS(H) = 8$. Consider the function $f : V(H) \to \mathbb{R}^+$ where $f(v_1) = f(v_2) = f(v_4) = 0$ and $f(v_3) = f(v_5) = 1$. It is easy to see that f is an efficient open dominating function for H. Hence, $\rho_f^o(H) = \gamma_f^o(G) = 2$. Notice that $\frac{n(n+1)}{2\gamma_f^o(H)} = \frac{5(6)}{2(2)} = 7.5$.

Notice that if $W \subset \mathbb{Z}$, where W may be a multiset, then all neighborhood sums will be integer valued. Given such a restriction on W, we can state the following corollaries that provide a necessary condition for the spread of the neighborhood sums to be zero.

Corollary 3.54. *Let $W \subset \mathbb{Z}$ be a multiset. $NS_W^{sp}(G;D) = 0$ only if there exists an integer k such that $k\gamma_f(G;D) = \sigma_W$.*

Corollary 3.55. *Graph G is D-vertex magic only if there exists an integer k such that $k\gamma_f(G;D) = \frac{n(n+1)}{2}$.*

Corollary 3.56. *Graph G is Σ'-labeled only if there exists an integer k such that $k\gamma_f(G) = \frac{n(n+1)}{2}$.*

Corollary 3.57. *Graph G is Σ-labeled only if there exists an integer k such that $k\gamma_f^o(G) = \frac{n(n+1)}{2}$.*

Since we can easily calculate the values of $\gamma_f(G;D) = \rho_f(G;D)$ for the case when G is (D,r)-regular, we can state specifically what the bounds for $NS_W^-(G;D)$ and $NS_W(G;D)$ will be.

Theorem 3.58. *If graph G is (D,r) regular, then $NS_W^-(G;D) \leq \frac{r\sigma_W}{n} \leq NS_W(G;D)$.*

Proof. Define the function $h : V(G) \to [0,1]$ by $f(v) = \frac{1}{r}$ for all $v \in V(G)$. Since G is (D,r) regular, for all $v \in V(G)$ we have that $h(N_D(v)) = \sum\limits_{u \in N_D(v)} h(u) = \sum\limits_{u \in N_D(v)} \frac{1}{r} = 1$. That is, h is a $\gamma_f(G;D)$ function and a $\rho_f(G;D)$ function. Furthermore, since G has order n, $\gamma_f(G;D) = \sum\limits_{v \in V(G)} h(v) = \sum\limits_{v \in V(G)} \frac{1}{r} = \frac{n}{r}$. Applying Theorems 3.43 and 3.46 we have that $NS_W^-(G;D) \leq \frac{r\sigma_W}{n} \leq NS_W(G;D)$. $\square$

Corollary 3.59 ([15]). *If graph G is r regular, then* $NS^{-}[G] \leq \left\lfloor \frac{(r+1)(n+1)}{2} \right\rfloor \leq \left\lceil \frac{(r+1)(n+1)}{2} \right\rceil$ $\leq NS[G]$ *and* $NS^{-}(G) \leq \left\lfloor \frac{r(n+1)}{2} \right\rfloor \leq \left\lceil \frac{r(n+1)}{2} \right\rceil \leq NS(G)$.

Notice that Corollary 3.59 is simply a restatement of Corollary 3.23 that was achieved using the degree sequence bound.

3.3.3 Bounds Based on Change In Weight Set

It is clear that the parameters $NS_W(G;D)$, $NS^{-}_W(G;D)$, and $NS^{sp}_W(G;D)$ are affected by the structure of the graph, the set of distances D under consideration, and the set of values in W that are used to label the graph. In this section we look at how changes in the weight set affect the parameters.

The first result we present simply establishes that larger labels will produce larger values of the parameters.

Theorem 3.60. *Let* $W = \{w_1, w_2, \ldots, w_n\} \subset \mathbb{R}$ *be a multiset such that for all* $i \in \{1, 2, \ldots, n-1\}$ *we have* $w_i \leq w_{i+1}$. *Let* $S = \{s_1, s_2, \ldots, s_n\} \subset \mathbb{R}$ *be a multiset such that* $w_i \leq s_i$ *for all* $i \in \{1, 2, \ldots, n-1\}$. *Then for any graph G*

(i) $NS^{-}_W(G;D) \leq NS^{-}_S(G;D)$, *and*

(ii) $NS_W(G;D) \leq NS_S(G;D)$.

Proof. Let $g : V(G) \to W$ be the bijection such that $NS^{-}_W(G;D) = NS^{-}(g;D)$ and without loss of generality assume that $V(G) = \{v_1, v_2, \ldots, v_n\}$ is such that $g(v_i) = w_i$. Define the bijection $g^{\#} : V(G) \to S$ by $g^{\#}(v_i) = s_i$. Then for all $v \in V(G)$ we have that $g^{\#}(N_D(v)) \geq g(N_D(v))$. Hence, $NS^{-}_S(G;D) \geq NS^{-}(g^{\#};D) \geq NS^{-}(g;D) = NS^{-}_W(G;D)$. Therefore part i is proved.

For part ii, let $f : V(G) \to S$ be the bijection such that $NS_S(G;D) = NS(f;D)$ and without loss of generality assume that $V(G) = \{v_1, v_2, \dots, v_n\}$ is such that $f(v_i) = s_i$. Define the bijection $f^\# : V(G) \to W$ by $f^\#(v_i) = w_i$. Then for all $v \in V(G)$ we have that $f(N_D(v)) \geq f^\#(N_D(v))$. Hence, $NS_W(G;D) \leq NS(f^\#;D) \leq NS(f;D) = NS_S(G;D)$. Therefore part ii is proved. $\square$

When one weight set is a multiple of the other, then we can state the next stronger result.

Theorem 3.61. *Let G be a graph. Let $W = \{w_1, w_2, \dots\} \subset \mathbb{R}$ be a multiset, $c \geq 0$ be a constant, and define the multiset $S = \{cw_1, cw_2, \dots, cw_n\}$. For any bijection $f : V(G) \to W$ define the bijection $g : V(G) \to S$ by $g(v) = cf(v)$. Then*

(i) $NS^-(g;D) = cNS^-(f;D)$, $NS(g;D) = cNS(f;D)$, and $NS^{sp}(g;D) = cNS^{sp}(f;D)$;

(ii) $NS_S^-(G;D) = cNS_W^-(G;D)$;

(iii) $NS_S(G;D) = cNS_W(G;D)$; and

(iv) $NS_S^{sp}(G;D) = cNS_W^{sp}(G;D)$.

Proof. For any vertex $v \in V(G)$ we have that $g(N_D(v)) = \sum_{u \in N_D(v)} g(u) = \sum_{u \in N_D(v)} cf(u) = cf(N_D(v))$. Since this result holds for all vertices, and since c is non-negative, it follows that $NS^-(g;D) = cNS^-(f;D)$ and $NS(g;D) = cNS(f;D)$, and hence that $NS^{sp}(g;D) = cNS^{sp}(f;D)$. Parts ii, iii, and iv then follow since the bijection f was arbitrary. $\square$

Next we consider the effect that negating the weights has on the parameters.

Theorem 3.62. *Let G be any graph. Let $W^+ = \{w_1, w_2, \dots, w_n\} \subset \mathbb{R}$ be a multiset and define $W^- = \{-w_1, -w_2, \dots, -w_n\}$. Let $f : V(G) \to W^+$ be a bijection such that*

$NS_{W^+}(G;D) = NS(f;D)$. Define the bijection $f^{\#} : V(G) \to W^-$ by $f^{\#} = -f$. Then $NS_{W^+}(G;D) = NS(f;D) = -NS^-(f^{\#};D) = -NS^-_{W^-}(G;D)$.

Proof. Let $v \in V(G)$ be such that $NS_{W^+}(G;D) = NS(f;D) = f(N_D(v))$. Let $u \in V(G)$ be arbitrary. Then $f^{\#}(N_D(u)) = \sum_{w \in N_D(u)} f^{\#}(w) = -\sum_{w \in N_D(u)} f(w) \geq -\sum_{w \in N_D(v)} f(w) = \sum_{w \in N_D(v)} f^{\#}(w) = f^{\#}(N_D(v))$. Hence, $NS^-(f^{\#};D) = f^{\#}(N_D(v)) = -\sum_{w \in N_D(v)} f(w)$ $= -NS_{W^+}(G;D)$.

Now let $g^{\#} : V(G) \to W^-$ be an arbitrary bijection and define the bijection $g : V(G) \to W^+$ by $g = -g^{\#}$. Let $x \in V(G)$ be such that $NS^-(g^{\#};D) = g^{\#}(N_D(x))$. Then $g^{\#}(N_D(x)) = \sum_{w \in N_D(x)} g^{\#}(w) = -\sum_{w \in N_D(x)} g(w) = -g(N_D(x)) \geq -NS_W(G;D) = -f(N_D(v)) = -\sum_{w \in N_D(v)} f(w) = \sum_{w \in N_D(v)} f^{\#}(w) = f^{\#}(N_D(v))$. Hence, $NS^-(g^{\#};D) \geq NS^-(f^{\#};D)$. Since g was arbitrary, we have that $NS^-_{W^-}(G;D) = NS^-(f^{\#};D)$.

Therefore $NS_{W^+}(G;D) = -NS^-_{W^-}(G;D)$. □

Next we consider the effect that adding a constant to each weight has on the parameters. In general we will not be able to say what the result will be since the D-neighborhoods of the vertices will have different numbers of elements. However, when G is (D,r)-regular, we can establish the following result.

Theorem 3.63. *Let $W = \{w_1, w_2, \ldots, w_n\}$ be a multiset. Let $s \in \mathbb{R}$ and define the multiset $S = \{w_1 + s, w_2 + s, \ldots, w_n + s\}$. Let $f : V(G) \to W$ be an arbitrary bijection and define the bijection $g : V(G) \to S$ by $g = f + s$. If graph G is (D,r) regular, then*

(i) $NS^-(g;D) = rs + NS^-(f;D)$,

(ii) $NS(g;D) = rs + NS(f;D)$,

(iii) $NS^{sp}(g;D) = NS^{sp}(f;D)$,

(iv) $NS_S^-(G;D) = rs + NS_W^-(G;D)$,

(v) $NS_S(G;D) = rs + NS_W(G;D)$, *and*

(vi) $NS_S^{sp}(G;D) = NS_W^{sp}(G;D)$.

Proof. For any vertex $v \in V(G)$ we have that $g(N_D(v)) = \sum_{u \in N_D(v)} g(u) = \sum_{u \in N_D(v)} (f(u)+s) = rs + \sum_{u \in N_D(v)} f(u) = rs + f(N_D(u))$. Since this result is true for the arbitrary bijection f and for every vertex, each of the results follows. □

3.3.4 Bounds on Neighborhood Sum Parameters for Graph Combinations

In the previous sections we have examined how changes in the distance set D or weight set W can affect the parameters $NS_W^-(G;D)$, $NS_W(G;D)$, and $NS_W^{sp}(G;D)$. In this section we explore what occurs with these parameters when we combine two graphs G_1 and G_2 in various ways.

Theorem 3.64. *Let $W \subset \mathbb{R}^+$. Let G be any graph and H be a graph formed by adding an edge to G. Let $k \in \mathbb{N} - \{0\}$.*

(i) If $D = \{0\}$, then $NS_W^-(G;D) = NS_W^-(H;D)$ and $NS_W(G;D) = NS_W(H;D)$.

(ii) If $D = \{0,1,\ldots,k\}$ or $D = \{1,2,\ldots,k\}$, then $NS_W^-(G;D) \leq NS_W^-(H;D)$ and $NS_W(G;D) \leq NS_W(H;D)$.

(iii) If $D = \{k,k+1,\ldots\}$, then $NS_W^-(G;D) \geq NS_W^-(H;D)$ and $NS_W(G;D) \geq NS_W(H;D)$.

Proof. Notice that for any vertices $u,v \in V(G) = V(H)$, the distance between u and v on G cannot exceed the distance between u and v on H.

Part i follows directly from Theorem 3.6.

In part ii, for all vertices $u \in V(G) = V(H)$, we have that the D-neighborhood of v on G is a subset of the D-neighborhood of v on H. Let $f : V(G) \to W$ be any bijection. Define the bijection $g : V(H) \to W$ by $g(v) = f(v)$. Then for all $v \in V(G) = V(H)$, $f(N_D(v)) = \sum_{u \in N_D(v)} f(u) \leq \sum_{u \in N_D(v)} g(u) = g(N_D(v))$, where the neighborhoods on the left side of the inequality are from G and the neighborhoods from the right side of the inequality are from H. Since this relationship is true for all bijections, it follows that $NS_W^-(G;D) \leq NS_W^-(H)$ and $NS_W(G;D) \leq NS_W(H;D)$.

In part iii, for all vertices $u \in V(G) = V(H)$, we have that the D-neighborhood of v on H is a subset of the D-neighborhood of v on G. Let $g : V(H) \to W$ be any bijection. Define the bijection $f : V(G) \to W$ by $f(v) = g(v)$. Then for all $v \in V(G) = V(H)$, $g(N_D(v)) = \sum_{u \in N_D(v)} g(u) \leq \sum_{u \in N_D(v)} f(u) = f(N_D(v))$, where the neighborhoods on the left side of the inequality are from H and the neighborhoods from the right side of the inequality are from G. Since this relationship is true for all bijections, it follows that $NS_W^-(G;D) \geq NS_W^-(H;D)$ and $NS_W(G;D) \geq NS_W(H;D)$. $\square$

Corollary 3.65. *Let $W \subset \mathbb{R}^+$. Let G be a spanning subgraph of graph H and let $k \in \mathbb{N} - \{0\}$.*

(i) If $D = \{0\}$, then $NS_W^-(G;D) = NS_W^-(H;D)$ and $NS_W(G;D) = NS_W(H;D)$.

(ii) If $D = \{0, 1, \ldots, k\}$ or $D = \{1, 2, \ldots, k\}$, then $NS_W^-(G;D) \leq NS_W^-(H;D)$ and $NS_W(G;D) \leq NS_W(H;D)$.

(iii) If $D = \{k, k+1, \ldots\}$, then $NS_W^-(G;D) \geq NS_W^-(H;D)$ and $NS_W(G;D) \geq NS_W(H;D)$.

Proof. The result follows by repeatedly adding edges to G to form H, applying Theorem 3.64 at each step. □

Corollary 3.66. *If G is a spanning subgraph of graph H, then*

(i) $NS^-[G] \leq NS^-[H]$,

(ii) $NS[G] \leq NS[H]$,

(iii) $NS^-(G) \leq NS^-(H)$,

(iv) $NS(G) \leq NS(H)$.

The next type of graph combination we look at is the union of two graphs. In order to develop our result we will need the next three lemmas.

Lemma 3.67. *Let a,b,x,y be positive real numbers. Then* $max\left\{\left(\frac{a+b}{a}\right)x, \left(\frac{a+b}{b}\right)y\right\} \geq x+y$.

Proof. Without loss of generality, assume that $y \geq x$, and hence, $y = cx$ for some $c \geq 1$. Then we want to prove that $max\left\{\left(\frac{a+b}{a}\right)x, \left(\frac{a+b}{b}\right)cx\right\} \geq x+cx$. If $\left(\frac{a+b}{a}\right)x < x+cx$, then $b/a < c$, and hence, $a/b > c$. In this case, $\left(\frac{a+b}{b}\right)cx = \left(\frac{a}{b}\right)cx + cx \geq c^2x + cx \geq x + cx$ since $c^2 \geq 1$. Therefore, $\left(\frac{a+b}{b}\right)y \geq x+y$. □

Lemma 3.68. *Let a,b be positive real numbers and define $c_1 = \frac{a+b}{a}$ and $c_2 = \frac{a+b}{b}$. For any positive integers n_1,n_2 let $S = \{c_1, 2c_1, \ldots, n_1c_1, c_2, 2c_2, \ldots, n_2c_2\}$. Then for all $i \in \{1,2,\ldots,n_1+n_2\}$ there exists at least $(n_1+n_2)-i+1$ elements of S that are greater than or equal to i.*

Proof. The proof will be by induction on i.

Base Case $(i = 1)$: Notice that c_1 and c_2 are both greater than 1. Hence all n_1+n_2 elements of S are greater than or equal to 1.

Induction: We will assume that the result holds for $i = k-1$, and we want to show that this implies that the result holds for $i = k$. So by this induction hypothesis, there exists integers k_1 and k_2 such that each element of $S_1 = \{k_1c_1, (k_1+1)c_1, \dots, n_1c_1, kc_2, (k_2+1)c_2, \dots, n_2c_2\}$ is greater than or equal to $k-1$. Further we have that $|S_1| = (n_1 - k_1 + 1) + (n_2 - k_2 + 1) = n_1 + n_2 - (k_1 + k_2) + 2 \geq n_1 + n_2 - k + 2$. Hence, $k_1 + k_2 \leq k$. Now if $k_1 + k_2 \leq k - 1 < k$, then $|S_1| \geq n_1 + n_2 - k + 1$ and the result is proved. If $k_1 + k_2 = k$, then by Lemma 3.67, $max\{c_1k_1, c_2k_2\} \geq k_1 + k_2 = k$. In this case we can form the set $S_2 = S_1 \bigcup max\{k_1c_1, k_2c_2\} \subset S$, where $|S_2| = n_1 + n_2 - k + 1$ and each element of S_2 is greater than or equal to k, and in this case the result is also proved. $\square$

Corollary 3.69. *Let a, b be positive real numbers and define $c_1 = \frac{a+b}{a}$ and $c_2 = \frac{a+b}{b}$. For any positive integers n_1, n_2 let $S = \{-c_1, -2c_1, \dots, -n_1c_1, -c_2, -2c_2, \dots, -n_2c_2\}$. Then for all $i \in \{1, 2, \dots n_1 + n_2\}$ there exists at least $(n_1 + n_2) - i + 1$ elements of S that are less than or equal to $-i$.*

Proof. Take $S^+ = \{c_1, 2c_1, \dots, n_1c_1, c_2, 2c_2, \dots, n_2c_2\}$. Then by Lemma 3.68 there exists at least $(n_1 + n_2) - i + 1$ elements of S^+ that are greater than or equal to i. There exists $(n_1 + n_2) - i + 1$ corresponding elements of S that are then less than or equal to $-i$. $\square$

Theorem 3.70. *For any graphs G and H*

(i) $NS^-(G \bigcup H; D) \leq NS^-(G; D) + NS^-(H; D)$, and

(ii) $NS(G \bigcup H; D) \leq NS(G; D) + NS(H; D)$.

Proof. Assume that $|V(G)| = n_1$ and $|V(H)| = n_2$.

For part i, we let $[-n_1] = \{-1, -2, \dots, -n_1\}$, $[-n_2] = \{-1, -2, \dots, -n_2\}$, and $[-n_1 - n_2] = \{-1, -2, \dots, -n_1 - n_2\}$. From Theorem 3.62 we have

(i) $NS_{[-n_1-n_2]}(G\bigcup H;D) = -NS^{-}_{[n_1+n_2]}(G\bigcup H;D) = -NS^{-}(G\bigcup H;D)$,

(ii) $NS_{[-n_1]}(G;D) = -NS^{-}_{[n_1]}(G;D) = -NS^{-}(G;D)$, and

(iii) $NS_{[-n_2]}(H;D) = -NS^{-}_{[n_2]}(H;D) = -NS^{-}(H;D)$.

We will show that $-NS_{[-n_1-n_2]}(G\bigcup H;D) \geq -NS_{[-n_1]}(G;D) - NS_{[-n_2]}(H;D)$; from this result it will follow that $NS^{-}(G\bigcup H;D) \leq NS^{-}(G;D) + NS^{-}(H;D)$.

Define $c_1 = \frac{NS_{[-n_1]}(G;D)+NS_{[-n_2]}(H;D)}{NS_{[-n_1]}(G;D)}$ and $c_2 = \frac{NS_{[-n_1]}(G;D)+NS_{[-n_2]}(H;D)}{NS_{[-n_2]}(H;D)}$. Let $g : V(G) \to [-n_1]$ be such that $NS_{[-n_1]}(G;D) = NS(g;D)$ and let $h : V(H) \to [-n_2]$ be such that $NS_{[-n_2]}(H;D) = NS(h;D)$. Let $W = \{-c_1, -2c_2, \ldots, -n_1c_1, -c_2, -2c_2, \ldots, -n_2c_2\}$ and define the bijection $f : V(G)\bigcup V(H) \to W$ by $f(v) = c_1g(v)$ if $v \in V(G)$ and $f(v) = c_2h(v)$ if $v \in V(H)$. Then $NS(f;D) = max\{c_1NS_{[-n_1]}(G;D), c_2NS_{[-n_2]}(H;D)\} = NS_{[-n_1]}(G;D) + NS_{[-n_2]}(H;D)$. By Corollary 3.69, W is such that for all $i \in \{1,2,\ldots,n_1+n_2\}$ there exists at least n_1+n_2-i+1 elements of W that are less than or equal to $-i$. That is, if we denote the set W as $W = \{w_1, w_2, \ldots, w_{n_1+n_2}\}$, where $w_i \leq w_{i+1}$, then for all $i \in \{1,2,\ldots,n_1+n_2\}$, $w_i \leq -i$. Therefore, by Theorem 3.60, $-NS_{[-n_1-n_2]}(G\bigcup H;D) \geq -NS_W(G\bigcup H;D) \geq -NS(f;D) = -NS_{[-n_1]}(G;D) - NS_{[-n_2]}(H;D)$. Therefore the result that $NS^{-}(G\bigcup H;D) \leq NS^{-}(G;D) + NS^{-}(H;D)$ follows from the discussion in the previous paragraph.

For part ii, define $c_1 = \frac{NS(G;D)+NS(H;D)}{NS(G;D)}$ and $c_2 = \frac{NS(G;D)+NS(H;D)}{NS(H;D)}$. Let $g : V(G) \to [n_1]$ be such that $NS(G;D) = NS(g;D)$ and let $h : V(H) \to [n_2]$ be such that $NS(H;D) = NS(h;D)$. Let $W = \{c_1, 2c_1, \ldots, n_1c_1, c_2, 2c_2, \ldots, n_2c_2\}$ and define the bijection $f : V(G)\bigcup V(H) \to W$ by $f(v) = c_1g(v)$ if $v \in V(G)$ and $f(v) = c_2h(v)$ if $v \in V(H)$. Then $NS(f;D) = max\{c_1NS(G;D), c_2NS(H;D)\} = NS(G;D) + NS(H;D)$. By Lemma 3.68, W is such that for all $i \in \{1,2,\ldots n_1+n_2\}$ there exists at least n_1+n_2-i+1 elements of

W greater than or equal to i. That is, if we denote the set W as $W = \{w_1, w_2, \ldots w_{n_1+n_2}\}$, where $w_i \leq w_{i+1}$, then for all $i \in \{1, 2, \ldots, n_1 + n_2\}$, $w_i \geq i$. Therefore, by Theorem 3.60, $NS(G \bigcup H; D) \leq NS_W(G \bigcup H; D) \leq NS(f; D) = NS(G; D) + NS(H; D)$. □

Corollary 3.71. *Let G and H be arbitrary graphs. Then*

(i) $NS^-[G \bigcup H] \leq NS^-[G] + NS^-[H]$*,*

(ii) $NS[G \bigcup H] \leq NS[G] + NS[H]$*,*

(iii) $NS^-(G \bigcup H) \leq NS^-(G) + NS^-(H)$*, and*

(iv) $NS(G \bigcup H) \leq NS(G) + NS(H)$*.*

The bound in Theorem 3.70 is tight. For example, it will be tight whenever $D = \{0\}$. In general there will be no similar relationship for the spread parameters. As will be shown in Chapter 4, $NS^{sp}[K_3] = NS^{sp}[3K_3] = NS^{sp}[K_4] = NS^{sp}[2K_4] = 0$ and $NS^{sp}[2K_3] = 1$. So we have that $1 = NS^{sp}[2K_3] > NS^{sp}[K_3] + NS^{sp}[K_3] = 0$, $0 = NS^{sp}[3K_3] < NS^{sp}[2K_3] + NS^{sp}[K_3] = 1$, and $0 = NS^{sp}[2K_4] = NS^{sp}[K_4] + NS^{sp}[K_4]$.

We next look at the join of two graphs. Again, we will need a lemma to establish our main result.

Lemma 3.72. *Let G and H be arbitrary graphs. If* $1 \notin D$*, then*

(i) $NS^-(G + H; D) = NS^-(G \bigcup H; D)$*, and*

(ii) $NS(G + H; D) = NS(G \bigcup H; D)$*.*

Proof. Let $|V(G)| = n_1$ and $|V(H)| = n_2$. Let $f : V(G + H) \to [n_1 + n_2]$ be an arbitrary bijection and define the bijection $g : V(G \bigcup H) \to [n_1 + n_2]$ by $g(v) = f(v)$ for all $v \in V(G) \bigcup V(H)$. Notice that for all $u \in V(G)$ and for all $v \in V(H)$ that $uv \notin E(G \bigcup H)$ and that in the graph $G + H$ that $d(uv) = 1 \notin D$. Hence, for all vertices $w \in V(G) \bigcup V(H)$, the

D-neighborhood of w in $G+H$ contains exactly the same vertices as the D-neighborhood of w in $G \bigcup H$. Hence, $f(N_D(w)) = g(N_D(w))$. Since the bijection f and the vertex w were arbitrary, both results i and ii follow. $\square$

Theorem 3.73. *Let G and H be arbitrary graphs. If $1 \in D$, then*

(i) $NS^-(G;D) + NS^-(H;D) \leq NS^-(G+H;D)$, and

(ii) $NS(G;D) + NS(H;D) \leq NS(G+H;D)$.

If $1 \notin D$, then

(iii) $NS^-(G+H;D) \leq NS^-(G;D) + NS^-(H;D)$, and

(iv) $NS(G+H;D) \leq NS(G;D) + NS(H;D)$.

Proof. Let $|V(G)| = n_1$ and $|V(H)| = n_2$. We will denote the vertices of G as $V(G) = \{v_{1,1}, v_{1,2}, \ldots, v_{1,n_1}\}$ and the vertices of H as $V(H) = \{v_{2,1}, v_{2,2}, \ldots, v_{2,n_2}\}$. Recall that $G+H$ has vertex set $V(G+H) = V(G) \bigcup V(H) = \{v_{1,1}, \ldots v_{1,n_1}, v_{2,1}, \ldots, v_{2,n_2}\}$ and edge set $E(G+K) = E(G) \bigcup E(H) \bigcup \{uv : u \in V(G) \,\text{and}\, v \in V(H)\}$.

We first prove part i. We have that $1 \in D$. Let $g : V(G) \to [n_1]$ be a bijection such that $NS^-(G;D) = NS^-(g;D)$. Let $h : V(H) \to [n_2]$ be a bijection such that $NS^-(H;D) = NS^-(h;D)$. Let $W = [n_1] \bigcup [n_2]$. Define the bijection $f : V(G+H) \to W$ by $f(v) = g(v)$ for all $v \in V(G)$, and $f(v) = h(v)$ for all $v \in V(H)$. Let $u \in V(G+H)$ be such that $NS_W(G+H;D) = f(N_D(u))$. Then if $u \in V(G)$ we have $NS^-(G+H;D) \geq NS_W^-(G+H;D) = f(N_D(u)) = g(N_D(u)) + \sum_{i=1}^{n_2} f(v_{2,i}) \geq NS^-(G;D) + NS^-(H;D)$. Similarly, if $u \in V(H)$ we have $NS^-(G+H;D) \geq NS_W^-(G+H;D) = f(N_D(u)) = h(N_D(u)) + \sum_{i=1}^{n_1} f(v_{1,i}) \geq NS^-(G;D) + NS^-(H;D)$. Therefore, $NS^-(G;D) + NS^-(H;D) \leq NS^-(G+H;D)$.

For part ii assume that $1 \in D$. Let $f: V(G+H) \to [n_1+n_2]$ be a bijection such that $NS(G+H;D) = NS(f;D)$. Let $W_1 = \{f(v_{1,1}), f(v_{1,2}), \ldots, f(v_{1,n_1})\}$ and let $W_2 = \{f(v_{2,1}), f(v_{2,2}), \ldots, f(v_{2,n_2})\}$. Define the bijection $g: V(G) \to W_1$ by $g(v) = f(v)$ for all $v \in V(G)$. Define the bijection $h: V(H) \to W_2$ by $h(v) = f(v)$ for all $v \in V(H)$. Notice that if $u \in V(G)$ and $v \in V(H)$, then in the graph $G+H$ we have $d(u,v) = 1 \in D$. Thus for all $v \in V(G)$ we have $f(N_D(v)) = g(N_D(v)) + \sum_{i=1}^{n_2} f(v_{2,i})$. Similarly, for all $v \in V(H)$ we have $f(N_D(v)) = h(N_D(v)) + \sum_{i=1}^{n_1} f(v_{1,i})$. Let $u \in V(G+H)$ be the vertex such that $f(N_D(v)) = NS(f;D) = NS(G+H;D)$. If $u \in V(G)$, we have that $f(N_D(v)) = g(N_D(v)) + \sum_{i=1}^{n_2} f(v_{2,i}) \geq NS_{W_1}(G;D) + NS_{W_2}(H;D) \geq NS(G;D) + NS(H;D)$. Similarly, if $u \in V(H)$ we have that $f(N_D(v)) = h(N_D(v)) + \sum_{i=1}^{n_1} f(v_{1,i}) \geq NS_{W_1}(G;D) + NS_{W_2}(H;D) \geq NS(G;D) + NS(H;D)$. Therefore, $NS(G;D) + NS(H;D) \leq NS(G+H;D)$.

For parts iii and iv, we have that $1 \notin D$. From Lemma 3.72 we then have that $NS^-(G+H;D) = NS^-(G \bigcup H;D)$ and that $NS(G+H;D) = NS(G \bigcup H;D)$. Therefore, the results from iii and iv follow directly from Theorem 3.70. □

Corollary 3.74. *Let G and H be arbitrary graphs. Then*

(i) $NS^-[G] + NS^-[H] \leq NS^-[G+H]$,

(ii) $NS[G] + NS[H] \leq NS[G+H]$,

(iii) $NS^-(G) + NS^-(H) \leq NS^-(G+H)$, *and*

(iv) $NS(G) + NS(H) \leq NS(G+H)$.

For any graphs G and H, it is clear from the definitions that $G \times H$ is a spanning subgraph of $G * H$. Applying Corollary 3.66 we get the following result.

Corollary 3.75. *Let G have order n_1 and H have order n_2. Let $W = \{w_1, w_2, \ldots, w_{n_1 n_2}\}$ be a multiset of non-negative numbers and let $k \in \mathbb{N} - \{0\}$.*

(i) If $D = \{0\}$, then $NS^-_W(G \times H; D) = NS^-_W(G * H; D)$ and $NS_W(G \times H; D) = NS_W(G * H; D)$.

(ii) If $D = \{0, 1, \ldots k\}$ or $D = \{1, 2, \ldots, k\}$, then $NS^-_W(G \times H; D) \le NS^-_W(G * H; D)$ and $NS_W(G \times H; D) \le NS_W(G * H; D)$.

(iii) If $D = \{k, k+1, \ldots\}$, then $NS^-_W(G \times H; D) \ge NS^-_W(G * H; D)$ and $NS_W(G \times H; D) \ge NS_W(G * H; D)$.

For arbitrary graphs G and H, neighborhood set $D \subset \mathbb{N}$, and weight set W, we would like to establish a relationship between the product $NS_W(G; D) NS_W(H; D)$ and $NS_W(G \times H; D)$ or $NS_W(G * H; D)$. For example, we would like to establish that $NS[G]NS[H] \le NS[G * H]$. However, as Example 3.76 shows, such a result does not hold.

Example 3.76. In Chapter 4 we show that $NS[K_2] = 3$, $NS[K_2^C] = 2$, $NS[2K_2] = 5$, $NS[C_4] = 9$, and $NS[K_4] = 10$. Now $2K_2 = K_2 * K_2^C$ and we have $NS[K_2]NS[K_2^C] = 6 > 5 = NS[2K_2]$. On the other hand, $C_4 = K_2 \times K_2$ and $K_4 = K_2 * K_2$. In this case we have that $NS[K_2]NS[K_2] = NS[C_4] = 9 < 10 = NS[K_4]$.

In Example 3.77 we show that a result of the form $NS^-[G]NS^-[H] \le NS^-[G * H]$ would not hold either.

Example 3.77. The $k \times j$ torus, denoted by $T_{k \times j}$, can be defined as $T_{k \times j} = C_k \times C_j$. In Theorem 4.58 we show that $NS^-[C_5] = 8$ and $NS^-[C_6] = 10$. Figure 3.3 shows that $T_{5 \times 5}$ is Σ'-labeled with $\{0, 1\}$-magic constant 65. Hence $NS^-[T_{5 \times 5}] = 65 > 64 = NS^-[C_5]NS^-[C_5]$.

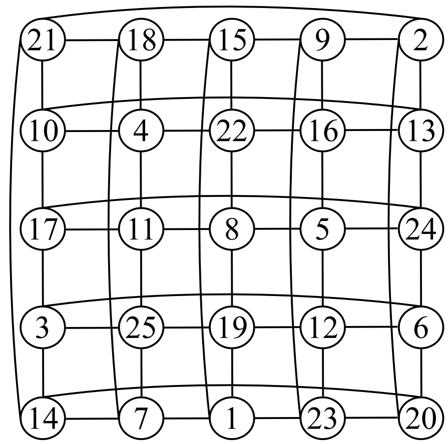

Figure 3.3: $\Sigma^{'}$-labeling of $T_{5\times5}$

Since $T_{k\times j}$ is 4-regular and has order kj, we know from Corollary 3.59 that $NS^{-}[T_{6\times6}] \leq \left\lfloor \frac{5(37)}{2} \right\rfloor = 92$. Hence $NS^{-}[T_{5\times5}] \leq 92 < 100 = NS^{-}[C_6]NS^{-}[C_6]$.

3.4 Relationship Between Parameters on (D,r)-Regular Graphs

When W has a specific structure and G is (D,r)-regular, we can establish a strong relationship between $NS_W^{-}(G;D)$ and $NS_W(G;D)$. By strong relationship, we mean that once we know one of the parameters, we can easily calculate the other. Moreover, if we know a bijection $f : V(G) \to W$ such that $NS_W(G;D) = NS(f;D)$, it is a simple matter to determine a bijection $g : V(G) \to W$ such that $NS_W^{-}(G;D) = NS^{-}(g;D)$, and vice-versa. The particular structure of W that we will require will include the case when $W = [n]$.

Theorem 3.78. *Let $c,s \in \mathbb{R}$ and define $W = \{c+s, 2c+s, \ldots, nc+s\}$. If graph G is (D,r) regular, then $NS_W^{-}(G;D) + NS_W(G;D) = r[c(n+1)+2s]$.*

Proof. Let $h : V(G) \to W$ be an arbitrary bijection and define the function $h^{\#} = (n+1)c + 2s - h$. We first establish that $h^{\#}$ is a bijection between $V(G)$ and W. Let $kc+s \in W$, then

$k \in [n]$. Since h is a bijection, there exists a $v \in V(G)$ such that $h(v) = (n-k+1)c+s$. For this v we have $h^{\#}(v) = (n+1)c+2s-h(v) = (n+1)c+2s-(n-k+1)c-s = kc+s$. Hence $h^{\#}$ is an injection from $V(G)$ to W. Since $V(G)$ and W have the same (finite) cardinality, it follows that $h^{\#} : V(G) \to W$ is a bijection.

Now notice also that for all $v \in V(G)$, $h(v)+h^{\#}(v) = (n+1)c+2s$. Then since G is (D,r)-regular, $h(N_D(v)) + h^{\#}(N_D(v)) = \sum_{u \in N_D(v)} \left(h(u)+h^{\#}(u)\right) = \sum_{u \in N_D(v)} [(n+1)\,c+2s] = r\,[c\,(n+1)+2s]$. In particular if vertex $w \in V(G)$ is such that $h(N_D(w))$ is as large as possible, that is $NS(h;D) = h(N_D(w))$, then $h^{\#}(N_D(w))$ is as small as possible, that is $NS^{-}(h^{\#};D) = h^{\#}(N_D(w))$. But since h is an arbitrary bijection, $NS_W(G;D) = NS(h;D) = h(N_D(w))$ if and only if $NS^{-}_{W}(G;D) = NS(h^{\#};D) = h^{\#}(N_D(w))$. Therefore, $NS^{-}_{W}(G;D) + NS_W(G;D) = r[c(n+1)+2s]$. □

Corollary 3.79. *If graph G is (D,r) regular, then $NS^{-}(G;D)+NS(G;D) = r(n+1)$.*

Proof. This result follows directly from Theorem 3.78 by taking $c = 1$ and $s = 0$. □

Corollary 3.80. *If graph G is r regular, then*

(i) $NS^{-}[G]+NS[G] = (r+1)(n+1)$, and

(ii) $NS^{-}(G)+NS(G) = r(n+1)$.

A large part of Chapter 4 will focus on determining exact values for our parameters for cycles and complete bipartite graphs. Corollary 3.80 cuts the amount of work in half. Surprisingly, when we consider competitive games in Chapter 5, the game values will be related in a very similar fashion. The similar result for games will be established in Theorem 5.13.

3.5 Existence Theorems for a Graph to be Vertex Magic

Much of the research in the area of neighborhood sums has focused on determining whether specific graphs or graph families are D-vertex magic for some distance set D. The results of this section provide some tools to exclude some graphs from being D-vertex magic. In the first result we, use the distance D adjacency matrix to exclude some graphs from being D-vertex magic.

Theorem 3.81. *Let $W = \{w_1, w_2, \dots w_n\} \subset \mathbb{R}$ be a multiset for which there exists $i, j \in \{1, 2, \dots n\}$ such that $w_i \neq w_j$. If graph G is (D, r)-regular and A_D^{-1} exists, then $NS_W^{sp}(G; D) >$* 0.

Proof. Notice that if $NS_W^{sp}(G; D) = 0$, then there exists a vector $x \in \mathbb{R}^n$, where x is a permutation of the vector $[w_1, w_2, \dots, w_n]^T$, and a constant c, such that $A_D x = c\overrightarrow{1}$, where $\overrightarrow{1} \in \mathbb{R}^n$ is an all ones vector. Since G is (D, r)-regular, the vector $y \in \mathbb{R}^n$, where each element of y is $\frac{c}{r}$ is such that $A_D y = c\overrightarrow{1}$. If A_D^{-1} exists, then y is the unique solution to $A_D x = c\overrightarrow{1}$. Since there exists $i, j \in \{1, 2, \dots n\}$ where $w_i \neq w_j$, there cannot exist a solution vector x that is a permutation of the vector $[w_1, w_2, \dots, w_n]^T$. Hence $NS_W^{sp}(G; D) > 0$. $\square$

Corollary 3.82 ([13]). *If graph G is (D, r)-regular and A_D^{-1} exists, then G is not D-vertex magic.*

Corollary 3.83 ([13]). *If graph G is regular and A^{-1} exists, then G is not $\{1\}$-vertex magic; that is, G is not Σ-labeled.*

Corollary 3.84 ([13]). *If graph G is regular and N^{-1} exists, then G is not $\{0, 1\}$-vertex magic, that is, G is not Σ'-labeled.*

In her presentation at the 2010 IWOGL Conference [19], Rinovia Simanjuntak presented several open problems that asked whether some specific graphs were $\{2\}$-vertex magic. The results of this section combined with those from Section 3.2 can be used to answer several of the questions posed. In the next example we demonstrate one of the results. The complete set of results can be found in O'Neal and Slater [13].

Example 3.85. Consider the graph G shown in Figure 3.4 which has order 15 and diameter 2. G is regular of degree 4. We ask if there is a set $D \subset \{0,1,2\}$ such that G is D-vertex magic. Developing an ad hoc argument to show that no such D could exist would likely be very tedious. Consider this alternative argument that no such D other than $D = \varnothing$ or $D = \{0,1,2\}$ can exist.

Since we are considering whether G is D-vertex magic, our label set is $W = [15]$. Thus by Theorem 3.6, $NS^{sp}(G;\{0\}) = 15 - 1 = 14$; it then follows from Corollary 3.14 that $NS^{sp}(G;\{1,2\} = 14$. We have that $det(A) = 1280 \neq 0$ so A^{-1} exists. Hence by Theorem 3.81, $NS^{sp}(G) > 0$, and then by Corollary 3.14 it follows that $NS^{sp}(G;\{0,2\}) > 0$. Similarly, $det(N) = 6400 \neq 0$, so N^{-1} exists. Hence by Theorem 3.81, $NS^{sp}[G] > 0$, and then by Corollary 3.14 it follows that $NS^{sp}(G;\{2\}) > 0$. Therefore, other than the trivial case where $D = \varnothing$ or $D = \{0,1,2\}$, there does not exists a set $D \subset \{0,1,2\}$ such that G is D-vertex magic.

Miller et al. [11] showed that an r-regular graph where r is odd cannot be $\{1\}$-vertex magic. The idea behind the proof is simply that $\frac{rn(n+1)}{2n} = \frac{r(n+1)}{2}$ will not be an integer. Notice that r odd implies that n is even. This simple observation is extended for arbitrary D and W in the next two theorems.

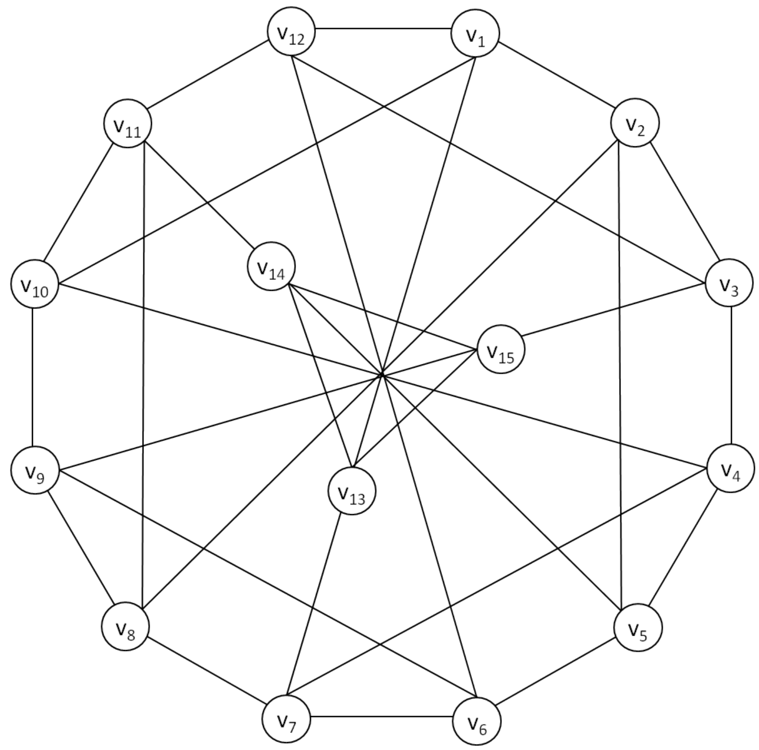

Figure 3.4: Graph G of order 15 and diameter 2

Theorem 3.86. *Let $W \subset \mathbb{Z}$. If graph G is (D,r)-regular, then $NS_W^{sp}(G;D) = 0$ only if n divides $r \sum_{i=1}^{n} w_i$.*

Proof. If $NS_W^{sp}(G;D) = 0$, then there exists a bijection $f : V(G) \to W$ such that for every $v \in V(G)$ we have $f(N_D(v)) = \sum_{u \in N_D(v)} f(u) = \frac{r}{n} \sum_{i=1}^{n} w_i$. Since $f(N_D(v))$ is integer valued, n must divide $r \sum_{i=1}^{n} w_i$. $\square$

Theorem 3.87 ([13])**.** *If graph G has even order and is (D,r)-regular with r odd, then G is not D-vertex magic.*

Proof. If G is D-vertex magic, then there exists a bijection $f : V(G) \to [n]$ such that for every $v \in V(G)$ we have $f(N_D(v)) = \sum_{u \in N_D(v)} f(u) = \frac{n(n+1)}{2} \times \frac{r}{n}$. Since $n+1$ and r are both

odd, this sum is not an integer, which is a contraction. Hence, no such bijection f exists, and therefore G is not D-vertex magic. $\square$

Corollary 3.88. *If graph G has order n and is (D,r)-regular with r odd, and if $0 \notin D$, then G is not D-vertex magic.*

Proof. Since $0 \notin D$, all of the elements on the main diagonal of A_D are 0. Since A_D is symmetric, there are an even number of non-zero entries in A_D, thus nr is even. Since r is odd, we must have n even. Hence the result follows from Theorem 3.87. $\square$

Corollary 3.89 ([13])**.** *There does not exist a regular graph of even order that is both $\{1\}$-vertex magic and $\{0,1\}$-vertex magic. That is, there does not exist a regular graph of even order that is both Σ-labeled and Σ'-labeled.*

Proof. Notice that if graph G is r-regular, then it is $(\{1\},r)$-regular and $(\{0,1\},r+1)$-regular. Since either r or $r+1$ is odd, G cannot be both $\{1\}$-vertex magic and $\{0,1\}$-vertex magic. $\square$

For two sets S and T, we denote the symmetric difference of S and T by $S \triangle T$. The next theorem is from Beena [4]. We will extend it for the arbitrary distance set D when W is restricted being a set.

Theorem 3.90 ([4])**.** *Let u and v be vertices of a Σ' labeled graph G. Then $|N[u] \triangle N[v]| = 0$ or ≥ 3.*

Theorem 3.91. *Let G be a graph of order n and let $D \subset \mathbb{N}$. Let $W = \{w_1, w_2, \ldots w_n\}$ be a set. If there exists $u, v \in V(G)$ such that $1 \leq |N_D(u) \triangle N_D(v)| \leq 2$ then $NS_W^{sp}(G;D) > 0$.*

Proof. Let $f : V(G) \to W$ be an arbitrary bijection. If $N_D(u)$ is a proper subset of $N_D(v)$ (or vice versa), then clearly $f(N_D(v)) \neq f(N_D(u))$. Hence we cannot have $|N_D(u) \triangle N_D(v)| = 1$. If $|N_D(u) \triangle N_D(v)| = 2$, then there must exist a vertex $w \in N_D(u) - N_D(v)$ and a vertex $x \in N_D(v) - N_D(u)$. Since W is a set, we have that $f(w) \neq f(x)$, thus $f(N_D(u)) \neq f(N_D(v))$. Therefore it follows that $NS^{sp}_W(G;D) > 0$. □

Corollary 3.92. *Let u and v be vertices of a Σ-labeled graph. Then* $|N(u) \triangle N(v)| = 0$ *or* $|N(u) \triangle N(v)| \geq 3$.

We next state a necessary and sufficient condition for $NS^{sp}_W(G;D) = 0$ in terms of fractionally efficient domination functions.

Theorem 3.93 ([15]). *Let G be a graph and* $D \subset \mathbb{N}$ *be such that* $\gamma(G;D) \in \mathbb{R}^+$. *There exists a multiset* $W = \{w_1, w_2, \ldots, w_n\} \subset \mathbb{R}$ *of non-negative elements with at least one non-zero element such that* $NS^{sp}_W(G;D) = 0$ *if and only if there exists a function* $f : V(G) \to \mathbb{R}^+$ *that is a D-neighborhood fractionally efficient dominating function of G.*

Proof. Since $\gamma(G;D) \in \mathbb{R}^+$, for all $u \in V(G)$ there exists a $v \in V(G)$ such that $d(u,v) \in V(G)$.

Assume there exists a multiset $W = \{w_1, w_2, \ldots, w_n\} \subset \mathbb{R}$ of non-negative numbers, not all zero, such that $NS^{sp}_W(G;D) = 0$. Then there exists a bijection $g : V(G) \to W$ where $NS^{sp}(g;D) = NS^{sp}_W(G;D) = 0$. It follows that $NS(g;D) = NS_W(G;D) = NS^-_W(G;D) = NS^-(g;D) > W_{MAX} > 0$. Let $S = \left\{\frac{w_1}{NS_W(G;D)}, \frac{w_2}{NS_W(G;D)}, \ldots, \frac{w_n}{NS_W(G;D)}\right\}$; then for all $i \in \{1, 2, \ldots, n\}$ we have that $\frac{w_i}{NS_W(G;D)} \in [0,1]$. Define the bijection $f : V(G) \to S$ by $f(v) = \frac{g(v)}{NS_W(G)}$. For all $v \in V(G)$ we have that $f(N_D(v)) = \sum_{u \in N_D(v)} f(u) = \sum_{u \in N_D(v)} \frac{g(u)}{NS_W(G)} = 1$. Therefore, f is a D-neighborhood fractionally efficient dominating function of G.

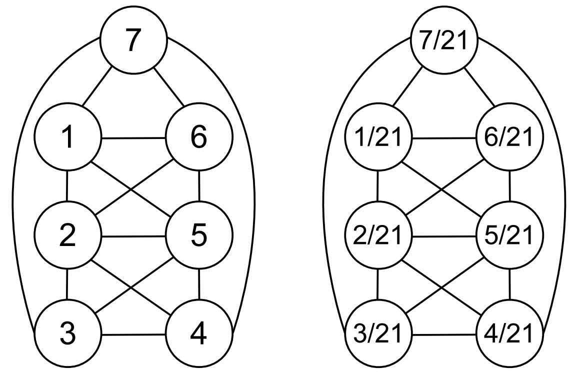

Figure 3.5: Σ'-labelings and fractionally efficient dominating functions

To prove the converse, let $f : V(G) \to \mathbb{R}^+$ be a D-neighborhood fractionally efficient dominating function of G. Let $W = \{f(v) : v \in V(G)\}$. Clearly not every element of W can be zero. Define the bijection $g : V(G) \to W$ by $g(v) = f(v)$. Then for all $v \in V(G)$ we have $g(N_D(v)) = \sum_{u \in N_D(v)} g(v) = \sum_{u \in N_D(v)} f(v) = 1$. Hence, $NS(g;D) = NS^-(g;D)$ and $NS^{sp}(g;D) = 0$. Therefore, $NS^{sp}_W(G;D) = 0$. □

In Theorem 3.93 we must have the qualification that a D-neighborhood dominating function exists. Consider the graph K_n^C which is the graph of n isolated vertices. For any bijection $f : V(K_n^C) \to W$, we have that $NS^-(f) = NS(f) = 0$. Hence $NS^{sp}(K_n^C) = 0$ and K_n^C is Σ-labeled. However $V(K_n^C)$ is not a dominating set for K_n^C, so no open dominating function can exist, much less an efficient dominating function. If $0 \in D$, then we know that a D-neighborhood dominating function will exist. Hence, in the following corollaries that are limited to the closed neighborhood case, we can drop the requirement that $\gamma(G;D) \in \mathbb{R}^+$.

Example 3.94. Figure 3.5 shows a Σ'-labeling for a graph G and the $\{0,1\}$-neighborhood fractionally efficient dominating function constructed as in Theorem 3.93.

Corollary 3.95 ([15]). *For any graph G, there exists a multiset $W = \{w_1, w_2, \ldots, w_n\} \subset \mathbb{R}$ of non-negative elements with at least one non-zero element such that $NS^{sp}_W[G] = 0$ if and only if there exists a function $f : V(G) \to \mathbb{R}^+$ that is a fractionally efficient dominating function of G.*

Proof. Notice that for any graph G we will have $\gamma(G) \leq n$. Hence the result will follow from Theorem 3.93. □

Corollary 3.96 ([15]). *Let G be a graph such $\delta(G) > 0$. There exists a multiset $W = \{w_1, w_2, \ldots, w_n\} \subset \mathbb{R}$ of non-negative elements with at least one non-zero element such that $NS^{sp}_W(G) = 0$ if and only if there exists a function $f : V(G) \to \mathbb{R}^+$ that is a fractionally efficient open dominating function of G.*

Proof. Since $\delta(G) > 0$, we are ensured that $\gamma^o(G) \in \mathbb{R}^+$. Hence the result follows from Theorem 3.93. □

Corollary 3.97 ([15]). *Let G be a graph and $D \subset \mathbb{N}$ be such that $\gamma(G; D) \in \mathbb{R}^+$. Graph G is D-vertex magic only if there exists a function $f : V(G) \to \mathbb{R}^+$ that is a D-neighborhood fractionally efficient dominating function of G.*

Corollary 3.98 ([15]). *Let G be a graph such $\delta(G) > 0$. Graph G is Σ-labeled only if there exists a function $f : V(G) \to \mathbb{R}^+$ that is a fractionally efficient open dominating function of G.*

Corollary 3.99 ([15]). *Graph G is Σ'-labeled only if there exists a function $f : V(G) \to \mathbb{R}^+$ that is a fractionally efficient dominating function of G.*

We can now combine our results for complementary distance with the results of this section to establish the next theorem.

Theorem 3.100 ([15]). *Let G be a connected graph and let $D^{\#} = \mathbb{N} - D$. If $\gamma(G;D) \in \mathbb{R}^+$ and $\gamma(G;D^{\#}) \in \mathbb{R}^+$ then there exists a D-neighborhood fractionally efficient dominating function of G if and only if there exists a $D^{\#}$-neighborhood fractionally efficient dominating function of G.*

Proof. Since $\gamma(G;D) \in \mathbb{R}^+$, if G has a D-neighborhood fractionally efficient dominating function, then by Theorem 3.93, there exists a multiset $W = \{w_1, w_2, \ldots, w_n\} \subset \mathbb{R}$ of non-negative elements with at least one non-zero element such that $NS_W^{sp}(G;D) = 0$. Since G is connected, by Theorem 3.13 we have that $NS_W^{sp}(G;D) = NS_W^{sp}(G;D^{\#})$. Thus $NS_W^{sp}(G;D^{\#}) = 0$. Therefore, since $\gamma(G;D^{\#}) \in \mathbb{R}^+$, by Theorem 3.93, there exists a $D^{\#}$-neighborhood fractionally efficient dominating function of G. $\square$

We can now establish a similar result for any graph G and its complement G^C. Theorem 3.101 is interesting because from the statement of the theorem, it has nothing to do with graph labeling, yet graph our graph labeling results make the proof very straightforward.

Theorem 3.101 ([15]). *Let G with radius at least two. There exists a fractionally efficient dominating function of G if and only if there exists a fractionally efficient open dominating function of G^C.*

Proof. Since G has radius at least two, there does not exist a vertex with degree $n-1$. Hence G^C has no isolated vertices and $\gamma^o(G) \in \mathbb{R}^+$. Note also that for any graph G we have that $\gamma(G) \in \mathbb{R}^+$.

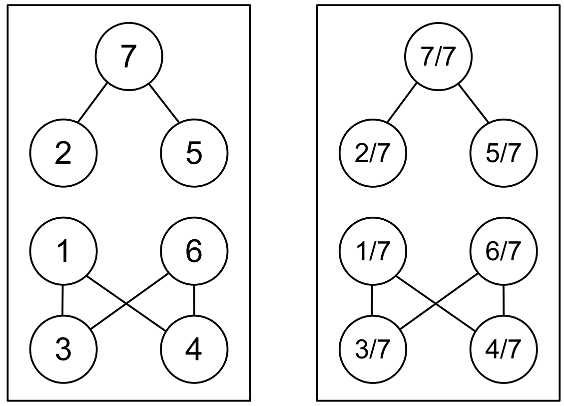

Figure 3.6: Σ-labelings and fractionally efficient open dominating functions

If G has a fractionally efficient dominating function, then by Theorem 3.93, there exists a multiset $W = \{w_1, w_2, \ldots, w_n\} \subset \mathbb{R}$ of non-negative elements with at least one non-zero element such that $NS_W^{sp}[G] = 0$. By Theorem 3.17 we have that $NS_W^{sp}[G] = NS_W^{sp}(G^C)$. Thus $NS_W^{sp}(G^C) = 0$. Therefore, since $\gamma^o(G;D) \in \mathbb{R}^+$, by Theorem 3.93, there exists a fractionally efficient open dominating function of G. □

If we dropped the requirement that $rad(G) \geq 2$, Theorem 3.101 would not hold. Consider the case where $G = P_3$. We can efficiently dominate G by selecting the vertex of degree two. However, $G^C = K_1 \bigcup K_2$, for which no open dominating function exists.

Example 3.102. Figure 3.6 demonstrates the complement of the graph from Figure 3.5, which is Σ-labeled, and the corresponding fractionally efficient open dominating function.

CHAPTER 4

NEIGHBORHOOD SUMS FOR SPECIFIC FAMILIES OF GRAPHS

In this chapter we will determine the values of $NS^-_W(G;D)$, $NS_W(G;D)$, and $NS^{sp}_W(G;D)$ for some common graphs. Usually we will restrict the set of labels to be $W = [n]$ and only consider the cases where $D = \{0,1\}$ or $D = \{1\}$. Even with these restrictions there will be some cases that remain open. In Section 4.1 we state the results for the complete graph K_n and its complement K_n^C; these results will follow directly from the results of Section 3.1. In Section 4.2 we provide results for graphs whose maximum degree is one. This first two sections provide examples of graphs of every order that are Σ'-labeled, and hence, their complements provides examples of graphs for all orders which are Σ-labeled. In Section 4.3 we look at the union of cycles of the same order, and in Section 4.4 we consider complete bipartite graphs. In Section 4.5 we determine the values of the parameters for a limited number of cases when the graph G is a $2 \times q$ torus, or equivalently $G = P_2 \times C_q$.

4.1 Complete Graphs and Their Complements

It is a trivial matter to determine the minimax, maximin, and spread values for the neighborhood sums on complete graphs. In this section we state these results for completeness.

Theorem 4.1. *For the complete graph K_n*

(i) $NS_W^-(K_n;\{0\}) = W_{MIN}$, $NS_W(K_n;\{0\}) = W_{MAX}$, and $NS_W^{sp}(K_n;\{0\}) = W_{MAX} - W_{MIN}$;

(ii) $NS_W^-[K_n] = NS_W[K_n] = \sigma_W$, and $NS_W^{sp}[K_n] = 0$;

(iii) $NS_W^-(K_n) = \sigma_W - W_{MAX}$, $NS_W(K_n) = \sigma_W - W_{MIN}$, and $NS_W^{sp}(K_n) = W_{MAX} - W_{MIN}$.

Proof. The results from part i follow from Theorem 3.6. Since the diameter of a complete graph is 1, part ii follows directly fromTheorem 3.3. Since the diameter of a compete graph is 1, part iii follows directly from Theorem 3.8. □

Corollary 4.2. *For the complete graph K_n*

(i) $NS^-(K_n;\{0\}) = 1$, $NS(K_n;\{0\}) = n$, and $NS^{sp}(K_n;\{0\}) = n-1$;

(ii) $NS^-[K_n] = NS[K_n] = \frac{n(n+1)}{2}$, and $NS^{sp}[K_n] = 0$;

(iii) $NS^-(K_n) = \frac{n(n+1)}{2} - n = \frac{n(n-1)}{2}$, $NS(K_n) = \frac{n(n+1)}{2} - 1$, and $NS^{sp}(K_n) = n-1$.

Theorem 4.3. *For the complete graph K_n*

(i) $NS_W^-(K_n^C;\{0\}) = W_{MIN}$, $NS_W(K_n^C;\{0\}) = W_{MAX}$, and $NS_W^{sp}(K_n^C;\{0\}) = W_{MAX} - W_{MIN}$;

(ii) $NS^{-}_{W}[K_n^C] = W_{MIN}$, $NS_W[K_n^C] = W_{MAX}$, *and* $NS^{sp}_{W}[K_n^C] = W_{MAX} - W_{MIN}$*;*

(iii)$NS^{-}_{W}(K_n^C) = NS_W(K_n^C) = NS^{sp}_{W}(K_n^C) = 0$.

Proof. The results from part i follow from Theorem 3.6. Part ii follows directly from part i and Theorem 3.10. Part iii follows directly from Corollarys 3.18 and 4.2. □

Corollary 4.4. *For the complete graph* K_n

(i) $NS^{-}(K_n^C;\{0\}) = 1$, $NS(K_n^C;\{0\}) = n$, *and* $NS^{sp}(K_n^C;\{0\}) = n-1$*;*

(ii) $NS^{-}[K_n^C] = 1$, $NS[K_n^C] = n$, *and* $NS^{sp}[K_n^C] = n-1$*;*

(iii)$NS^{-}(K_n^C) = NS(K_n^C) = NS^{sp}(K_n^C) = 0$.

Notice that $K_{n_1}^C \bigcup K_{n_2}^C = K_{n_1+n_2}^C$. From Corollary 4.4 we have that $NS[K_{n_1}^C \bigcup K_{n_2}^C] = NS[K_{n_1+n_2}^C] = n_1 + n_2 = NS[K_{n_1}^C] + NS[K_{n_2}^C]$. From Corollary 3.71 we would have concluded that $NS[K_{n_1}^C \bigcup K_{n_2}^C] \leq NS[K_{n_1}^C] + NS[K_{n_2}^C] = n_1 + n_2$. So the bounds from Corollary 3.71 are tight.

As shown in part iii of Corollary 4.2, K_n is Σ'-labeled for all n. Thus as shown in part iii of Corollary 4.4, K_n^C is Σ-labeled for all n.

4.2 Graphs With $\Delta(G) = 1$ and Their Complements

We will first consider the case where $\delta(G) = 1$. If graph G is such that $\Delta(G) = \delta(G) = 1$, then the order n of G is even and G is the union of $\frac{n}{2}$ K_2's. That is, $G = \frac{n}{2}K_2$. Notice that in this case that $G^C = K_{t_1,t_2,\ldots t_{\frac{n}{2}}}$, where $t_1 = t_2 = \cdots = t_{\frac{n}{2}} = 2$. The corollaries for the complementary results in this section will follow by applying Theorem 3.17.

Theorem 4.5. *If* $G = pK_2$*, then*

(i) $NS_W^-(G) = W_{MIN}$*,* $NS_W(G) = W_{MAX}$*, and* $NS_W^{sp}(G) = W_{MAX} - W_{MIN}$*; and*

(ii) $NS_W^-[G] \leq \frac{2\sigma_W}{n} \leq NS_W[G]$*.*

Proof. For part i, let $f : V(G) \to W$ be an arbitrary bijection. Let $v \in V(G)$ be the vertex such that $f(v) = W_{MIN}$. Since $G = pK_2$, there exists a vertex $u \in V(G)$ such that $f(N(u)) = f(v) = W_{MIN}$. Since for all $w \in V(G)$, $N(w) \neq \varnothing$, we must have that $f(N(w)) \geq W_{MIN}$. Hence $NS_W^-[f] = W_{MIN}$. Similarly, let $v \in V(G)$ be the vertex such that $f(v) = W_{MAX}$. Since $G = pK_2$, there exists a vertex $u \in V(G)$ such that $f(N(u)) = f(v) = W_{MAX}$. Since for all $w \in V(G)$ we have that $|N(w)| = 1$, it follows that $f(N(w)) \leq W_{MAX}$. Hence $NS[f] = W_{MAX}$. We then have that $NS^{sp}[f] = NS[f] - NS^-[f] = W_{MAX} - W_{MIN}$. Since f was an arbitrary bijection, we conclude that $NS_W^-(G) = W_{MIN}$, $NS_W(G) = W_{MAX}$, and $NS_W^{sp}(G) = W_{MAX} - W_{MIN}$.

For part ii, consider that since G is $(\{0,1\},2)$-regular. Hence by Theorem 3.58 we have that $NS_W^-[G] \leq \frac{2\sigma_W}{n} \leq NS_W[G]$. □

Corollary 4.6. *If* $G = K_{t_1,t_2,\ldots,t_p}$*, where* $t_1 = t_2 = \cdots = t_p = 2$ *then*

(i) $NS_W^-(G) \leq \frac{\sigma_W(n-2)}{n} \leq NS_W(G)$*; and*

(ii) $NS_W^-[G] = \sigma_W - W_{MAX}$*,* $NS_W[G] = \sigma_W - W_{MIN}$*, and* $NS_W^{sp}[G] = W_{MAX} - W_{MIN}$*.*

Theorem 4.7. *If* $G = pK_2$*, then*

(i) $NS^-(G) = 1$*,* $NS(G) = n$*, and* $NS^{sp}(G) = n-1$*; and*

(ii) $NS^-[G] = NS[G] = n+1$ *and* $NS^{sp}[G] = 0$*.*

Proof. Part i follows as a direct consequence of Theorem 4.5. For part ii, let $V(G) = \{v_{i,1}, v_{i,2} : 1 \leq i \leq p\}$, where for all i, $v_{i,1}v_{i,2} \in E(G)$. Define the bijection $f : V(G) \to [n]$

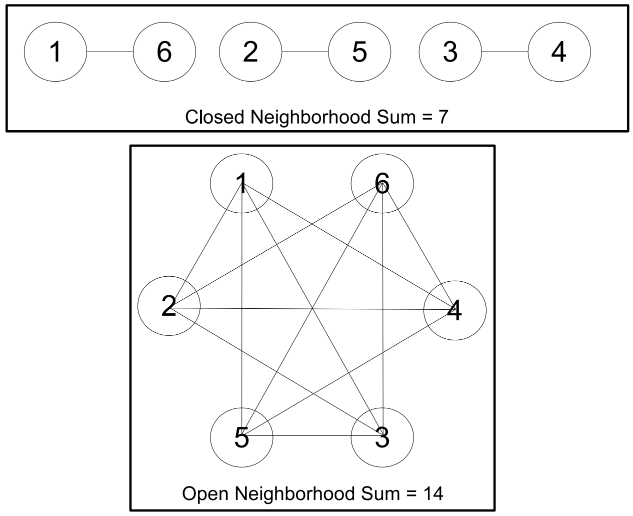

Figure 4.1: Σ'-labeling of $3K_2$ and Σ-labeling of $K_{2,2,2}$

by $f(v_{i,1}) = i$ and $f(v_{i,2}) = n+1-i$. Then for all $u \in V(G)$ we have that $f(N[u]) = n+1$. Hence, G is Σ'-labeled and $NS^-[G] = NS[G] = n+1$ and $NS^{sp}[G] = 0$. □

Corollary 4.8. *If $G = K_{p_1,p_2,\ldots,p_t}$, where $p_1 = p_2 = \cdots = p_t = 2$ then*

(i) $NS^-(G) = NS(G) = \frac{(n+1)(n-2)}{2}$ and $NS^{sp}(G) = 0$; and

(ii) $NS^-[G] = \frac{n(n-1)}{2}$, $NS[G] = \frac{(n+2)(n-1)}{2}$, and $NS^{sp}[G] = n-1$.

Example 4.9. Figure 4.1 shows the Σ'-labeling of $3K_2$ as constructed in the proof of Theorem 4.7 and the Σ labeling of its complement $K_{2,2,2}$.

When graph G is such that $\Delta(G) = 1$ and $\delta(G) = 0$, then we have that $G = pK_2 \bigcup qK_1$, where $2p+q = n$ and $1 \leq q < n$. In this case we will have that $G^C = K_{t_1,\ldots,t_q,t_{q+1},\ldots,t_{p+q}}$, where $1 = t_1 = \cdots = t_q$ and $2 = t_{q+1} = \cdots = t_{p+q}$.

Theorem 4.10. *Let $W = \{w_1, w_2, \ldots, w_n\}$, where $w_1 \leq w_2 \leq \cdots \leq w_n$. Let $G = pK_2 \bigcup qK_1$, where $2p+q = n$ and $1 \leq q < n$. Then*

(i) $NS^-_W(G) = min\{w_{q+1}, 0\}$, $NS_W(G) = max\{w_{2p}, 0\}$*;and*

(ii) $NS^-_W[G] \le \frac{\sigma_W}{p+q} \le NS_W[G]$.

Proof. We first prove part i. Let $f : V(G) \to W$ be an arbitrary bijection. Since G has $q \ge 1$ vertices of degree 0, there exists a vertex $v \in V(G)$ such that $f(N(v)) = f(\varnothing) = 0$. Since G has $2p \ge 2$ vertices of degree 1, there exists a vertex $u \in V(G)$ such that $f(N(u)) \le w_{q+1}$. Hence, $NS^-_W(G) \le min\{f(N(u)), f(N(v))\} = min\{w_{q+1}, 0\}$. Similarly, since G has $2p \ge 2$ vertices of degree 1, there exists a vertex $w \in V(G)$ such that $f(N(w)) \ge w_{2p}$. Hence, $NS_W(G) \ge max\{f(N(w)), f(N(v))\} = max\{w_{2p}, 0\}$. To achieve the bound shown for $NS^-_W(G)$, let $g : V(G) \to W$ be a bijection that assigns the smallest q weights, $w_1, w_2, \ldots, w_q$ to the isolated vertices. We then have $NS^-(g) = min\{w_{q+1}, 0\}$ and therefore $NS^-_W(G) = min\{w_{q+1}, 0\}$. To achieve the bound shown for $NS_W(G)$, let $h : V(G) \to W$ be a bijection that assigns the largest q weights $w_{2p+1}, w_{2p+2}, \ldots, w_{2p+q}$ to the isolated vertices. We then have $NS(h) = max\{w_{2p}, 0\}$, and therefore $NS_W(G) = max\{w_{2p}, 0\}$.

For part ii, notice that $\rho_f(G) = \gamma_f(G) = p + q$. Then the result that $NS^-_W[G] \le \frac{\sigma_W}{p+q} \le NS_W[G]$ follows directly from Theorems 3.43 and 3.46. $\square$

Corollary 4.11. *Let* $W = \{w_1, w_2, \ldots, w_n\} \subset \mathbb{R}$*, where* $w_1 \le w_2 \le \cdots \le w_n$*. Let* $G = K_{t_1,\ldots,t_q,t_{q+1},\ldots,t_{p+q}}$*, where* $2p + q = n$ *and* $1 \le q < n$*. Then*

(i) $NS^-_W(G) \le \frac{\sigma_W(p+q-1)}{p+q} \le NS_W(G)$*; and*

(ii) $NS^-_W[G] = \sigma_W - max\left\{w_{2p}, 0\right\}$, $NS[G] = \sigma_W - min\left\{w_{q+1}, 0\right\}$.

Theorem 4.12. *Let* $G = pK_2 \bigcup qK_1$*, where* $2p + q = n$ *and* $q \ge 1$*. Then*

(i) $NS^-(G) = 0$*,* $NS(G) = 2p$*, and* $NS^{sp}(G) = 2p$*; and*

(ii) $NS^-[G] = n + 1 - q$*,* $NS[G] = n$*, and* $NS^{sp}[G] = q - 1$.

Proof. For part i, let $f : V(G) \to [n]$ be a bijection where the largest q weights $w_{2p+1}, w_{2p+2}, \ldots, w_{2p+q}$ are placed on the isolated vertices. We then have that $NS^{-}(f) = 0$, $NS(f) = w_{2p}$, and $NS^{sp}(f) = w_{2p}$. The result that $NS^{-}(G) = 0$ and $NS(G) = 2p$ then follows from Theorem 4.10. Since f is such that $NS^{-}(G) = NS^{-}(f)$ and $NS(G) = NS(f)$, we also have that $NS^{sp}(G) = NS^{sp}(f) = NS(f) - NS^{-}(f) = 2p$.

For part ii, first consider that the maximum label of n must be counted in at least one closed neighborhood, hence $NS[G] \geq n$. Also consider that since there are q vertices of degree 0, there exists a vertex $v \in V(G)$ such that $|N[v]| = 1$ and $f(N[v]) = f(v) \leq 2p+1$. Hence, $NS^{-}[G] \leq 2p+1$. Let $V(G) = \{v_{i,1}, v_{i,2} : 1 \leq i \leq p, v_{i,1}v_{i,2} \in E(G)\} \bigcup \{v_{i,1} : p+1 \leq i \leq p+q, deg(v_{i,1}) = 0\}$. Define the bijection $g : V(G) \to [n]$ by $f(v_{i,1}) = i$, $f(v_{i,2}) = 2p+1-i$ for $1 \leq i \leq p$ and $f(v_{i,1}) = p+i$ for $p+1 \leq i \leq p+q$. If $deg(v) = 1$, we have that $f(N[v]) = 2p+1$. If $deg(v) = 0$, we have that $2p+1 \leq f(N[v]) \leq 2p+q = n$. Hence, $NS^{-}[f] = 2p+1$ and $NS[f] = n$. Therefore, $NS^{-}[G] = 2p+1 = n+1-q$ and $NS[G] = n$. Since $NS^{-}[G] = NS^{-}[f]$ and $NS[G] = NS[f]$ we conclude that $NS^{sp}[G] = NS^{sp}[f] = NS[f] - NS^{-}[f] = q-1$. $\square$

Corollary 4.13. *Let $G = K_{t_1,\ldots,t_q,t_{q+1},\ldots,t_{p+q}}$, where $2p+q = n$ and $q \geq 1$. Then*

(i) $NS^{-}(G) = \frac{n(n-1)}{2}$, $NS(G) = \frac{n(n-1)}{2} + q - 1$, and $NS^{sp}(G) = q-1$; and

(ii) $NS^{-}[G] = \frac{n(n+1)}{2} - 2p$, $NS[G] = \frac{n(n+1)}{2}$, and $NS^{sp}[G] = 2p$.

Theorems 4.7 and 4.12 establish that a graph G with $\Delta(G) = 1$ is Σ'-labeled graph if and only if it has at most one isolated vertex. This result is not new and can be found in Beena [4]. Corollarys 4.8 and 4.13 establish the complementary Σ-labeled graphs. Using these results, Σ' and Σ-labeled graphs of all orders can be created. The new contribution of

these results is the determination of the maximin, minimax, and spread values for the cases where the graphs are not Σ' or Σ-labeled.

4.3 2-Regular Graphs

Theorem 4.7 showed that all 1-regular graphs, which must have even order, will be Σ'-labeled. This problem was easy to solve because of the simple characterization of 1-regular graphs. In this section, we look at some special cases of 2-regular graphs. If G is a 2-regular graph, then G is the union of one or more cycles. We will limit our attention to the case where G is the union of cycles of the same order. Throughout this section we will denote the vertices and edges of the cycle C_n by $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(C_n) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\}$ respectively. When graph $G = kC_t = C_t^{(1)} \bigcup C_t^{(2)} \bigcup \cdots \bigcup C_t^{(k)}$ is the union of k cycles each of order t we will denote the vertices of G by $V(G) = \bigcup_{i=1}^{k} V(C_t^{(i)}) = \bigcup_{i=1}^{k} \{v_{i,1}, v_{i,2}, \ldots, v_{i,t}\}$. We will denote the edges of G by $E(G) = \bigcup_{i=1}^{k} E(C_t^{(i)}) = \bigcup_{i=1}^{k} \{v_{i,1}v_{i,2}, \ldots, v_{i,t-1}v_{i,t}, v_{i,t}v_{i,1}\}$. In general, we will only consider the case where $W = [n]$, and $D = \{1\}$ or $D = \{1,2\}$. Even given these restrictions, we will only be able to present a partial set of results.

4.3.1 Open Neighborhood Sums

Theorem 4.14 was presented in O'Neal and Slater [12] and is stated here without proof. The more general result on the union of cycles of equal order in Theorem 4.29 can also be taken as a proof of Theorem 4.14. The results of this subsection appear in O'Neal and Slater [14].

Theorem 4.14 ([12]). *If $n \equiv 0,1,3 (mod 4)$ and $n \neq 4$, then $NS^{-}(C_n) = n$, $NS(C_n) = n+2$, and $NS^{sp}(C_n) = 2$; if $n \equiv 2(mod 4)$, then $NS^{-}(C_n) = n-1$, $NS(C_n) = n+3$, and $NS^{sp}(C_n) = 4$. Cycle C_4 is Σ-labeled and $NS^{-}(C_4) = NS(C_4) = 5$.*

We first establish that, as long as W is restricted to being a set (not a multiset), then the only cycle that can have an open neighborhood spread of zero is the C_4. The result that the union four cycles is the only Σ-labeled 2-regular graph is not a new result and was presented by Arumugam in [2]. The results here allow for a more generalized label set W, and is not limited to the case where $W = [n]$.

Lemma 4.15. *Let $G = kC_t$ and W be a set. If $NS^{sp}_W(G) = 0$, then $t = 4$.*

Proof. Let $C_t^{(i)}$ be an arbitrary cycle that is a component of G. Without loss of generality, we can assume that the components of G are indexed such that $i = 1$. If $t = 3$, then for any bijection $f : V(G) \to W$ we have that $f(N(v_{1,1})) = f(v_{1,2}) + f(v_{1,3}) \neq f(v_{1,1}) + f(v_{1,3}) = f(N(v_{1,2}))$. So when $n = 3$, there does not exist a bijection such that $NS^{sp}[f] = 0$. If $t \geq 4$ and if there exists a bijection $f : V(G) \to W$ such that $NS^{sp}(f) = 0$, then $f(N(v_{1,2})) = f(N(v_{1,4}))$, which implies that $f(v_{1,1}) + f(v_{1,3}) = f(v_{1,3}) + f(v_{1,5})$. As f is a bijection we conclude that $v_{1,1} = v_{1,5}$ and hence $t = 4$. Therefore all of the components of G are C_4's. □

Theorem 4.16. *Let $G = kC_t$ and $W = \{w_1, w_2, \ldots, w_n\}$, where $w_1 < w_2 < \cdots < w_n$. $NS^{sp}_W(G) = 0$ if and only if $t = 4$ and $w_1 + w_n = w_2 + w_{n-1} = \cdots = w_{2k} + w_{2k+1}$.*

Proof. Assume $NS^{sp}_W(G) = 0$. By Lemma 4.15 we have that $t = 4$. Let $f : V(G) \to W$ be a bijection such that $NS^{sp}(f) = 0$. Without loss of generality assume that $f(v_{1,1}) = w_n$.

Assume that $f(v_{1,3}) > w_1$; then we have $f(N(v_{1,2})) > w_1 + w_n$. Let $v \in V(G)$ be such that $f(v) = w_1$ and let $u \in N(v)$. We then have that $f(N(u)) < w_1 + w_n$, which is a contradiction. Thus we must have $f(v_{1,3}) = w_1$. Now let $x \in V(G)$ be such that $f(x) = w_{n-1}$, $w \in N(x)$, and $N(w) = \{x, y\}$. We must then have $f(N(w) = f(x) + f(y) = w_{n-1} + f(y) = w_1 + w_n$. If $f(y) > w_2$, then the label w_2 cannot be paired with a second label big enough to create open neighborhood sums of $w_1 + w_n$; hence, $f(y) = w_2$. It then follows that we must have $w_1 + w_n = w_2 + w_{n-1}$. Repeating this argument we get that $w_1 + w_n = w_2 + w_{n-1} = \cdots = w_{2k} + w_{2k+1}$.

Next assume that $t = 4$ and $w_1 + w_n = w_2 + w_{n-1} = \cdots = w_{2k} + w_{2k+1}$. Define the bijection $g : V(G) \to W$ by $f(v_{i,1}) = w_{2i-1}$, $f(v_{i,2}) = w_{2i}$, $f(v_{i,3}) = w_{n-2i+2}$, and $f(v_{i,4}) = w_{n-2i+1}$ for all $1 \leq i \leq k$. If $j \in \{2, 4\}$, then for all $1 \leq i \leq k$ we have that $f(N(v_{i,j})) = w_{2i-1} + w_{n-2i+2} = w_1 + w_n$. If $j \in \{1, 3\}$, then for all $1 \leq i \leq k$ we have that $f(N(v_{i,j})) = w_{2i} + w_{n-2i+1} = w_1 + w_n$. Therefore $NS_W^{sp}(G) = 0$. □

Corollary 4.17. *Let $G = kC_t$ and $W = \{w_1, w_2, \ldots, w_n\}$, where $w_1 < w_2 < \cdots w_n$. If $NS_W^{sp}(G) = 0$, then $NS_W^-(G) = NS_W(G) = w_1 + w_n$.*

Corollary 4.18 ([14])**.** *Let $G = kC_t$. $NS^{sp}(G) = 0$ if and only if $t = 4$.*

Corollary 4.19 ([14])**.** *Let $G = kC_4$. $NS^-(G) = NS(G) = 4k + 1 = n + 1$.*

Example 4.20. Figure 4.2 demonstrates a Σ-labeling for $3C_4$.

A simple YES/NO classification of Σ-labeled 2-regular graphs would be complete at this point. However, we can develop a complete characterization of the maximin, minimax, and spread of the open neighborhood sums for all 2-regular graphs that are the union of

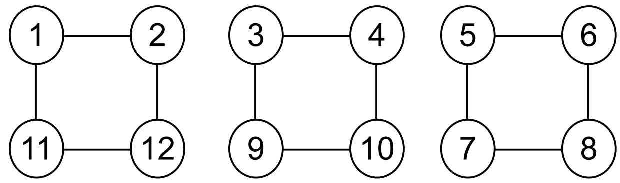

Figure 4.2: Σ-labeling for $3C_4$

cycles of equal order. We work out these results by cases based on the base 4 modularity of t.

Theorem 4.21 ([14]). *If $G = kC_t$ and $t \equiv 1(mod 4)$, then $NS^-(G) = kt - k + 1 = n - k + 1$, $NS(G) = kt + k + 1 = n + k + 1$, and $NS^{sp}(G) = 2k$.*

Proof. Since t is odd, $\left\lfloor \frac{t}{2} \right\rfloor$ labels can be placed on a cycle of order t with no vertex having labels assigned to both elements in its open neighborhood. However, if $\left\lfloor \frac{t}{2} \right\rfloor + 1$ labels are placed on a cycle of order t, then there must exist a vertex having labels assigned to both elements in its open neighborhood. So for the cycles $C_t^{(1)}, \ldots, C_t^{(k)}$, we can place $k\left\lfloor \frac{t}{2} \right\rfloor$ labels on G before some vertex of G has labels assigned to both elements in its open neighborhood. Hence, for t odd, $NS(G) \geq \left(kt - k\left\lfloor \frac{t}{2} \right\rfloor + 1\right) + \left(kt - k\left\lfloor \frac{t}{2} \right\rfloor\right) = 2kt - 2k\left\lfloor \frac{t}{2} \right\rfloor + 1 = 2kt - k(t-1) + 1 = kt + k + 1$. Applying Corollary 3.80 we have that $NS^-(G) \leq kt - k + 1$. Applying Proposition 2.20 we have that $NS^{sp}(G) \geq 2k$.

Define the bijection $f : V(G) \to [kt]$ as follows:

(i) if $j \equiv 0(\text{mod} 4)$ set $f(v_{i,j}) = k\left(\left\lceil \frac{t}{2} \right\rceil - \frac{j}{2}\right) - i + 1$;

(ii) if $j \equiv 1(\text{mod} 4)$ set $f(v_{i,j}) = k\left(t - \left\lfloor \frac{j}{2} \right\rfloor\right) - i + 1$;

(iii) if $j \equiv 2(\text{mod} 4)$ set $f(v_{i,j}) = k\left(\left\lceil \frac{t}{2} \right\rceil + \frac{j}{2}\right) - k + i$;

(iv) else if $j \equiv 3(\text{mod} 4)$ set $f(v_{i,j}) = k\left\lceil \frac{j}{2} \right\rceil - k + i$.

Fix $i \in \{1, \ldots, k\}$. If $j \equiv 0 (\mathrm{mod}\, 4)$, then $f(N(v_{i,j})) = f(v_{i,j-1}) + f(v_{i,j+1}) = \left(k\left\lceil\frac{j-1}{2}\right\rceil - k + i\right) + \left(k\left(t - \left\lfloor\frac{j+1}{2}\right\rfloor\right) - i + 1\right) = kt - k + 1$.

Next consider the case where $j \equiv 1 (\mathrm{mod}\ 4)$. If $j = 1$, then $f(N(v_{i,1})) = f(v_{i,t}) + f(v_{i,2}) = \left(k\left(t - \left\lfloor\frac{t}{2}\right\rfloor\right) - i + 1\right) + \left(k\left(\left\lceil\frac{t}{2}\right\rceil + 1\right) - k + i\right) = kt + k + 1$. If $1 < j < t$, then $f(N(v_{i,j})) = f(v_{i,j-1}) + f(v_{i,j+1}) = \left(k\left(\left\lceil\frac{t}{2}\right\rceil - \frac{j-1}{2}\right) - i + 1\right) + \left(k\left(\left\lceil\frac{t}{2}\right\rceil + \frac{j+1}{2}\right) - k + i\right) = kt + k + 1$. If $j = t$, then $f(N(v_{i,t})) = f(v_{i,t-1}) + f(v_{i,1}) = \left(k\left(\left\lceil\frac{t}{2}\right\rceil - \frac{t-1}{2}\right) - i + 1\right) + \left(k\left(t - \left\lfloor\frac{1}{2}\right\rfloor\right) - i + 1\right) = kt + k - 2i + 2$. Notice that since $1 \le i \le k$, we have that $kt - k + 2 \le f(N(v_{i,t})) \le kt + k$.

If $j \equiv 2 (\mathrm{mod} 4)$, then $f(N(v_{i,j})) = f(v_{i,j-1}) + f(v_{i,j+1}) = \left(k\left(t - \left\lfloor\frac{j-1}{2}\right\rfloor\right) - i + 1\right) + \left(k\left\lceil\frac{j+1}{2}\right\rceil - k + i\right) = kt + k + 1$.

If $j \equiv 3 (\mathrm{mod} 4)$, then $f(N(v_{i,j})) = f(v_{i,j-1}) + f(v_{i,j+1}) = \left(k\left(\left\lceil\frac{t}{2}\right\rceil + \frac{j-1}{2}\right) - k + i\right) + \left(k\left(\left\lceil\frac{t}{2}\right\rceil - \frac{j+1}{2}\right) - i + 1\right) = kt - k + 1$.

For the bijection f we have that $NS^-(f) = kt - k + 1$, $NS(f) = kt + k + 1$, and $NS^{sp}(f) = 2k$. Hence $NS^-(G) \ge kt - k + 1$, $NS(G) \le kt + k + 1$, and $NS^{sp}(G) \le 2k$. Using the result from the first paragraph we conclude that $NS^-(G) = kt - k + 1$, $NS(G) = kt + k + 1$, and $NS^{sp}(G) = 2k$. □

Example 4.22. Figure 4.3 demonstrates that $NS^-(3C_5) = kt - k + 1 = 13$, $NS(3C_5) = kt + k + 1 = 19$, and $NS^{sp}(3C_5) = 2k = 6$.

Theorem 4.23 ([14]). *If $G = kC_t$ and $t \equiv 3 (mod 4)$, then $NS^-(G) = kt - k + 1 = n - k + 1$, $NS(G) = kt + k + 1 = n + k + 1$, and $NS^{sp}(G) = 2k$.*

Proof. Since t is odd, $\left\lfloor\frac{t}{2}\right\rfloor$ labels can be placed on a cycle of order t with no vertex having labels assigned to both elements in its open neighborhood. However, if $\left\lfloor\frac{t}{2}\right\rfloor + 1$ labels

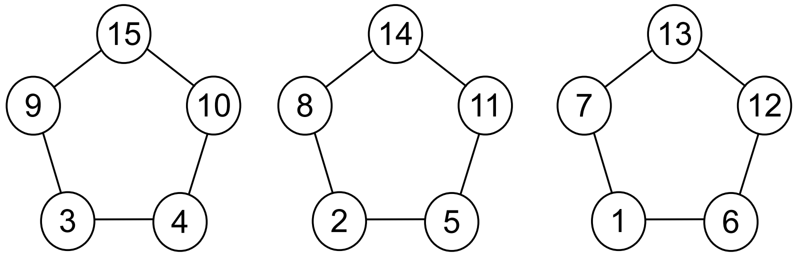

Figure 4.3: $NS^{-}(3C_5) = 13$, $NS(3C_5) = 19$, and $NS^{sp}(3C_5) = 6$

are placed on a cycle of order t, then there must exist a vertex having labels assigned to both elements in its open neighborhood. So for the cycles $C_t^{(1)}, \ldots, C_t^{(k)}$, we can place $k\left\lfloor \frac{t}{2} \right\rfloor$ labels on G before some vertex of G has labels assigned to both elements in its open neighborhood. Hence, for t odd, $NS(G) \geq \left(kt - k\left\lfloor \frac{t}{2} \right\rfloor + 1\right) + \left(kt - k\left\lfloor \frac{t}{2} \right\rfloor\right) = 2kt - 2k\left\lfloor \frac{t}{2} \right\rfloor + 1 = 2kt - k(t-1) + 1 = kt + k + 1$. Applying Corollary 3.80 we have that $NS^{-}(G) \leq kt - k + 1$. Applying Proposition 2.20 we have that $NS^{sp}(G) \geq 2k$.

Define the bijection $f : V(G) \to [kt]$ as follows:

(i) if $j \equiv 0 (\bmod 4)$ set $f(v_{i,j}) = k\left(\left\lceil \frac{t}{2} \right\rceil + \frac{j}{2}\right) - k + i$;

(ii) if $j \equiv 1 (\bmod 4)$ set $f(v_{i,j}) = k\left(t - \left\lfloor \frac{j}{2} \right\rfloor\right) - i + 1$;

(iii) if $j \equiv 2 (\bmod 4)$ set $f(v_{i,j}) = k\left(\left\lceil \frac{t}{2} \right\rceil - \frac{j}{2}\right) - i + 1$;

(iv) else if $j \equiv 3 (\bmod 4)$ set $f(v_{i,j}) = k\left\lceil \frac{j}{2} \right\rceil - k + i$.

Fix $i \in \{1, \ldots k\}$. If $j \equiv 0 (\bmod 4)$, then $f(N(v_{i,j})) = f(v_{i,j-1}) + f(v_{i,j+1}) = \left(k\left\lceil \frac{j-1}{2} \right\rceil - k + i\right) + \left(k\left(t - \left\lfloor \frac{j+1}{2} \right\rfloor\right) - i + 1\right) = kt - k + 1$.

Next consider the case where $j \equiv 1 (\bmod 4)$. If $j = 1$, then $f(N(v_{i,1})) = f(v_{i,t}) + f(v_{i,2}) = \left(k\left\lceil \frac{t}{2} \right\rceil - k + i\right) + \left(k\left(\left\lceil \frac{t}{2} \right\rceil - \frac{2}{2}\right) - i + 1\right) = kt - k + 1$. If $j > 1$, then $f(N(v_{i,j})) = f(v_{i,j-1}) + f(v_{i,j+1}) = \left(k\left(\left\lceil \frac{t}{2} \right\rceil + \frac{j-1}{2}\right) - k + i\right) + \left(k\left(\left\lceil \frac{t}{2} \right\rceil - \frac{j+1}{2}\right) - i + 1\right) = kt - k + 1$.

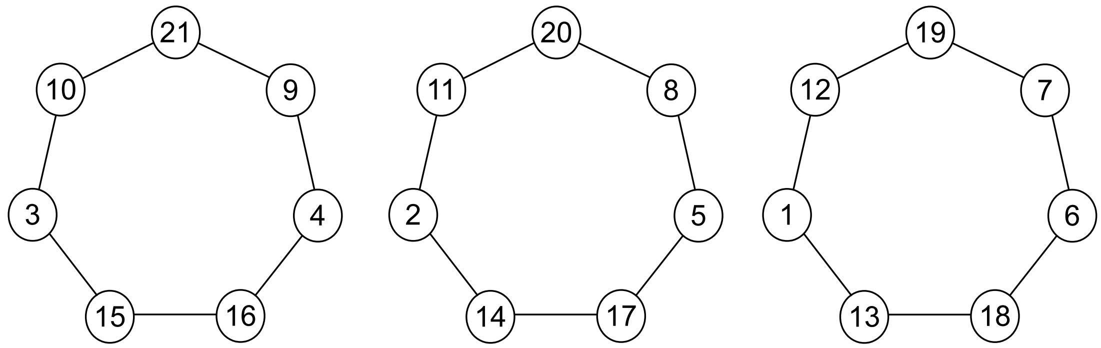

Figure 4.4: $NS^-(3C_7) = 19$, $NS(3C_7) = 25$, and $NS^{sp}(3C_7) = 6$

If $j \equiv 2(\text{mod} 4)$, then $f(N(v_{i,j})) = f(v_{i,j-1}) + f(v_{i,j+1}) = \left(k\left(t - \left\lfloor \frac{j-1}{2} \right\rfloor\right) - i + 1\right) + \left(k\left\lceil \frac{j+1}{2} \right\rceil - k + i\right) = kt + k + 1$.

If $j \equiv 3(\text{mod } 4)$ and $j < t$, then $f(N(v_{i,j})) = f(v_{i,j-1}) + f(v_{i,j+1}) = \left(k\left(\left\lceil \frac{t}{2} \right\rceil - \frac{j-1}{2}\right) - i + 1\right) + \left(k\left(\left\lceil \frac{t}{2} \right\rceil + \frac{j+1}{2}\right) - k + i\right) = kt + k + 1$. If $j = t$, then $f(N(v_{i,j})) = f(v_{i,t-1}) + f(v_{i,1}) = \left(k\left(\left\lceil \frac{t}{2} \right\rceil - \frac{t-1}{2}\right) - i + 1\right) + \left(k\left(t - \left\lfloor \frac{1}{2} \right\rfloor\right) - i + 1\right) = kt + k - 2i + 2$. Since $1 \le i \le k$, we have that $kt - k + 2 \le kt + k - 2i + 2 \le kt + k$.

For the bijection f we have that $NS^-(f) = kt - k + 1$, $NS(f) = kt + k + 1$, and $NS^{sp}(f) = 2k$. Hence $NS^-(G) \ge kt - k + 1$, $NS(G) \le kt + k + 1$, and $NS^{sp}(G) \le 2k$. Using the result from the first paragraph we conclude that $NS^-(G) = kt - k + 1$, $NS(G) = kt + k + 1$, and $NS^{sp}(G) = 2k$. □

Example 4.24. Figure 4.4 demonstrates that $NS^-(3C_7) = kt - k + 1 = 19$, $NS(3C_7) = kt + k + 1 = 25$, and $NS^{sp}(3C_7) = 2k = 6$.

Theorem 4.25 ([14])**.** *If $G = kC_t$ and $t \equiv 2(mod 4)$, then $NS^-(G) = kt - 2k + 1$, $NS(G) = kt + 2k + 1$, and $NS^{sp}(G) = 4k$.*

Proof. Notice that $\frac{t}{2}-1$ labels can be placed on a cycle of order t with no vertex having labels assigned to both elements in its open neighborhood. However, if $\frac{t}{2}$ labels are placed on a cycle of order t, then there must exist a vertex having labels assigned to both elements in its open neighborhood. So for the cycles $C_t^{(1)},\ldots,C_t^{(k)}$, we can place $k\left(\frac{t}{2}-1\right)$ labels before some vertex of G has labels assigned to both elements in its open neighborhood. Hence, for $t \equiv 2 \pmod 4$, $NS(G) \geq \left(kt-k\left(\frac{t}{2}-1\right)+1\right)+\left(kt-k\left(\frac{t}{2}-1\right)\right) = kt+2k+1$. Applying Corollary 3.80 we have that $NS^{-}(G) \leq kt-2k+1$. Applying Proposition 2.20 we have that $NS^{sp}(G) \geq 4k$.

Define the bijection $f: V(G) \to [kt]$ as follows:

(i) if $j \equiv 0 \pmod 4$ set $f(v_{i,j}) = k(t-j+1)-i+1$;

(ii) if $j \equiv 1 \pmod 4$ set $f(v_{i,j}) = kj-i+1$;

(ii) if $j \equiv 2 \pmod 4$ set $f(v_{i,j}) = kj-k+i$;

(iv) else if $j \equiv 3 \pmod 4$ set $f(v_{i,j}) = k(t-j)+i$.

Fix $i \in \{1,\ldots k\}$. If $j \equiv 0 \pmod 4$, then $f(N(v_{i,j})) = f(v_{i,j-1})+f(v_{i,j+1}) = (k(t-(j-1))+i)+(k(j+1)-i+1) = kt+2k+1$.

Next consider the case where $j \equiv 1 \pmod 4$. If $j=1$, then $f(N(v_{i,j})) = f(v_{i,t})+f(v_{i,2}) = (kt-k+i)+(2k-k+i) = kt+2i$. Since $1 \leq i \leq k$, we have that $kt+2 \leq kt+2i \leq kt+2k$. If $j>1$, then $f(N(v_{i,j})) = f(v_{i,j-1})+f(v_{i,j+1}) = (k(t-(j-1)+1)-i+1)+(k(j+1)-k+i) = kt+2k+1$.

Now consider the case where $j \equiv 2 \pmod 4$. If $j<t$, then $f(N(v_{i,j})) = f(v_{i,j-1})+f(v_{i,j+1}) = (k(j-1)-i+1)+(k(t-(j+1))+i) = kt-2k+1$. If $j=t$, then $f(N(v_{i,j})) = f(v_{i,t-1})+f(v_{i,1}) = (k(t-1)-i+1)+(k-i+1) = kt-2i+2$. Since $1 \leq i \leq k$, we have that $kt-2k+2 \leq kt-2i+1 \leq kt$.

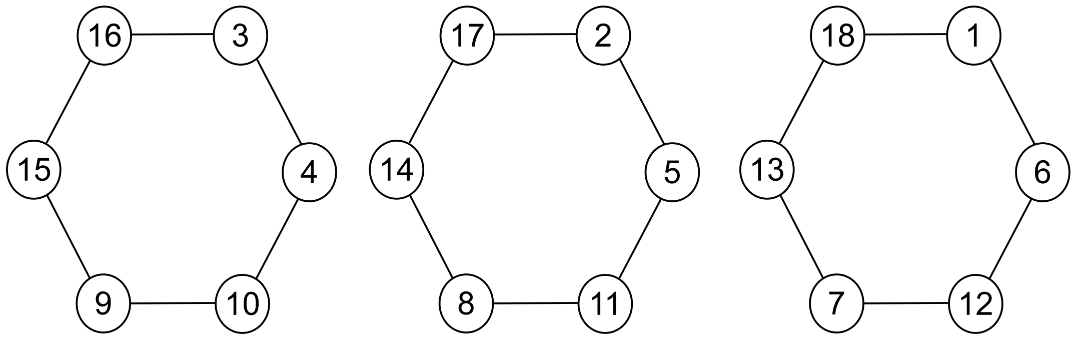

Figure 4.5: $NS^{-}(3C_6) = 13$, $NS(3C_6) = 25$, and $NS^{sp}(3C_6) = 12$

If $j \equiv 3(\text{mod } 4)$, then $f(N(v_{i,j})) = f(v_{i,j-1}) + f(v_{i,j+1}) = (k(j-1)-k+i)+(k(t-(j+1)+1)-i+1) = kt-2k+1$.

For the bijection f we have that $NS^{-}(f) = kt-2k+1$, $NS(f) = kt+2k+1$, and $NS^{sp}(f) = 4k$. Hence $NS^{-}(G) \geq kt-2k+1$, $NS(G) \leq kt+2k+1$, and $NS^{sp}(G) \leq 4k$. Using the result from the first paragraph we conclude that $NS^{-}(G) = kt-2k+1$, $NS(G) = kt+2k+1$, and $NS^{sp}(G) = 4k$. □

Example 4.26. Figure 4.5 demonstrates that $NS^{-}(3C_6) = kt-2k+1 = 13$, $NS(3C_6) = kt+2k+1 = 25$, and $NS^{sp}(3C_6) = 4k = 12$.

Theorem 4.27 ([14]). *If $G = kC_t$, $t \equiv 0(mod 4)$, and $t > 4$, then $NS^{-}(G) = kt$, $NS(G) = kt+2$, and $NS^{sp}(G) = 2$.*

Proof. Notice that $\frac{t}{2}$ labels can be placed on a cycle of order t with no vertex having labels assigned to both elements in its open neighborhood. However, if $\frac{t}{2}+1$ labels are placed on a cycle of order t, then there must exist two vertices that have labels assigned to both elements in their open neighborhoods. Since $t > 4$, there must be at least three distinct labels in these two open neighborhoods. Hence, $NS(G) \geq \left(kn - k\frac{t}{2} + 2\right) + \left(kt - k\frac{t}{2}\right) =$

$kt+2$. Applying Corollary 3.80 we have that $NS^{-}(G) \leq kt$. Applying Proposition 2.20 we have that $NS^{sp}(G) \geq 2$.

Define the function $h : V(G) \to \{0, \frac{t}{4}, (2)\frac{t}{4}, (3)\frac{t}{4}, \ldots, (2k-1)\frac{t}{4}\}$ by $h(v_{i,j}) = \frac{(i-1)t}{2} + \frac{(1+(-1)^{j})t}{8}$. Note that for any value of i the following equalities hold:

(i) $h(v_{i,1}) = h(v_{i,t-1})$ since t is even;

(ii) $h(v_{i,2}) = h(v_{i,t})$ since t is even; and

(iii) for any $1 < j < t$, $h(v_{i,j-1}) = h(v_{i,j+1})$ since $j-1$ and $j+1$ are either both even or both odd.

Next define the function $g : V(G) \to [kt]$ as follows:

(i) $j \equiv 1,2 \pmod 4$ and $1 \leq j \leq 4$ set $g(v_{i,j}) = \left\lceil \frac{j}{2} \right\rceil$,

(ii) $j \equiv 1,2 \pmod 4$ and $4 < j \leq 4\left\lceil \frac{t+4}{8} \right\rceil$ set $g(v_{i,j}) = \left\lceil \frac{j}{2} \right\rceil - 1$,

(iii) $j \equiv 1,2 \pmod 4$ and $4\left\lceil \frac{t+4}{8} \right\rceil < j \leq t$ set $g(v_{i,j}) = \frac{t}{2} - \left\lceil \frac{j}{2} \right\rceil + 2$,

(iv) $j \equiv 0,3 \pmod 4$ and $1 \leq j \leq 4$ set $g(v_{i,j}) = kt$,

(v) $j \equiv 0,3 \pmod 4$ and $4 < j \leq 4\left\lceil \frac{t}{8} \right\rceil$ set $g(v_{i,j}) = kt - \left\lceil \frac{j}{2} \right\rceil + 2$, and

(vi) $j \equiv 0,3 \pmod 4$ and $4\left\lceil \frac{t}{8} \right\rceil < j \leq t$ set $g(v_{i,j}) = kt - \frac{t}{2} + \left\lceil \frac{j}{2} \right\rceil - 1$.

Then we define the bijection $f : V(G) \to [kt]$ as

(i) $f(v_{i,j}) = g(v_{i,j}) + h(v_{i,j})$ when $j \equiv 1,2 \pmod 4$, and

(ii) $f(v_{i,j}) = g(v_{i,j}) - h(v_{i,j})$ when $j \equiv 0,3 \pmod 4$.

We fix a value of $i \in \{1, \ldots, k\}$ and consider the value of $f(N(v_{i,j}))$ for $1 \leq j \leq t$ on a case by case basis.

CASE A $(j \equiv 1 (\bmod 4))$

CASE A.1 $(j = 1)$: $f(N(v_{i,j})) = f(v_{i,t}) + f(v_{i,2}) = \left(kt - \frac{t}{2} + \left\lceil \frac{t}{2} \right\rceil - 1\right) + \left(\left\lceil \frac{2}{2} \right\rceil\right) = kt$.

CASE A.2 $\left(4 < j \leq 4\left\lceil \frac{t}{8} \right\rceil\right)$: $f(N(v_{i,j})) = \left(kt - \left\lceil \frac{j-1}{2} \right\rceil + 2\right) + \left(\left\lceil \frac{j+1}{2} \right\rceil - 1\right) = kt + 2$.

CASE A.3 $\left(4\left\lceil \frac{t}{8} \right\rceil < j \leq 4\left\lceil \frac{t+4}{8} \right\rceil\right)$: If $t \equiv 4 (\bmod 8)$, then $4\left\lceil \frac{t}{8} \right\rceil = 4\left\lceil \frac{t+4}{8} \right\rceil$ and so no such j exists. If $t \equiv 0 (\bmod 8)$, since $j \equiv 1 (\bmod 4)$, we have $j - 1 = 4\left\lceil \frac{t}{8} \right\rceil$. Hence $j + 1 = 4\left\lceil \frac{t}{8} \right\rceil + 2 < 4\left\lceil \frac{t+4}{8} \right\rceil$. It follows that $f(N(v_{i,j})) = \left(kt - \left\lceil \frac{j-1}{2} \right\rceil + 2\right) + \left(\left\lceil \frac{j+1}{2} \right\rceil - 1\right) = kt + 2$.

CASE A.4 $\left(4\left\lceil \frac{t+4}{8} \right\rceil < j \leq t\right)$: If $t \equiv 4$ (Mod 4)and $j = 4\left\lceil \frac{t+4}{8} \right\rceil + 1$, then $f(N(v_{i,j})) = \left(kt - \left\lceil \frac{j-1}{2} \right\rceil + 2\right) + \left(\frac{t}{2} - \left\lceil \frac{j+1}{2} \right\rceil + 2\right) = kt + 1$. Otherwise, $f(N(v_{i,j})) = \left(kt - \frac{t}{2} + \left\lceil \frac{j-1}{2} \right\rceil - 1\right) + \left(\frac{t}{2} - \left\lceil \frac{j+1}{2} \right\rceil + 2\right) = kt$.

CASE B $(j \equiv 2 (\bmod 4))$

CASE B.1 $(j = 2)$: $f(N(v_{i,j})) = f(v_{i,1}) + f(v_{i,3}) = \left(\left\lceil \frac{1}{2} \right\rceil\right) + (kt) = kt + 1$.

CASE B.2 $\left(4 < j \leq 4\left\lceil \frac{t}{8} \right\rceil\right)$: $f(N(v_{i,j})) = \left(\left\lceil \frac{j-1}{2} \right\rceil - 1\right) + \left(kt - \left\lceil \frac{j+1}{2} \right\rceil + 2\right) = kt$.

CASE B.3 $\left(4\left\lceil \frac{t}{8} \right\rceil < j \leq 4\left\lceil \frac{t+4}{8} \right\rceil\right)$: If $t \equiv 4 (\bmod 8)$, then $4\left\lceil \frac{t}{8} \right\rceil = 4\left\lceil \frac{t+4}{8} \right\rceil$ and no such j exists. If $t \equiv 0 (\bmod 8)$, since $j \equiv 2 (\bmod 4)$, we have $j - 1 = 4\left\lceil \frac{t}{8} \right\rceil + 1$. In this case $j + 1 = 4\left\lceil \frac{t}{8} \right\rceil + 3 < 4\left\lceil \frac{t+4}{8} \right\rceil$. Furthermore, $\left\lceil \frac{j-1}{2} \right\rceil + \left\lceil \frac{j+1}{2} \right\rceil = j + 1 = 4\left\lceil \frac{t}{8} \right\rceil + 3 = \frac{t}{2} + 3$. Hence $f(N(v_{i,j})) = \left(\left\lceil \frac{j-1}{2} \right\rceil - 1\right) + \left(kt - \frac{t}{2} + \left\lceil \frac{j+1}{2} \right\rceil - 1\right) = kt + 1$.

CASE B.4 $\left(4\left\lceil \frac{t+4}{8} \right\rceil < j \leq t\right)$: $f(N(v_{i,j})) = \left(\frac{t}{2} - \left\lceil \frac{j-1}{2} \right\rceil + 2\right) + \left(kt - \frac{t}{2} + \left\lceil \frac{j+1}{2} \right\rceil - 1\right) = kt + 2$.

CASE C $(j \equiv 3 (\bmod 4))$

CASE C.1 $(j = 3)$: $f(N(v_{i,j})) = f(v_{i,1}) + f(v_{i,4}) = \left(\left\lceil \frac{1}{2} \right\rceil\right) + (kt) = kt + 1$.

CASE C.2 $\left(4 < j \leq 4\left\lceil \frac{t}{8} \right\rceil\right)$: $f(N(v_{i,j})) = \left(\left\lceil \frac{j-1}{2} \right\rceil - 1\right) + \left(kt - \left\lceil \frac{j+1}{2} \right\rceil + 2\right) = kt$.

CASE C.3 $\left(4\left\lceil \frac{t}{8} \right\rceil < j \leq 4\left\lceil \frac{t+4}{8} \right\rceil\right)$: If $t \equiv 4 (\bmod 8)$, then $4\left\lceil \frac{t}{8} \right\rceil = 4\left\lceil \frac{t+4}{8} \right\rceil$ and no such j exists. If $t \equiv 0 (\bmod 8)$, since $j \equiv 3 (\bmod 4)$, we have $j - 1 = 4\left\lceil \frac{t}{8} \right\rceil + 2$. In this case $j+1 = 4\left\lceil \frac{t}{8} \right\rceil + 4 = 4\left\lceil \frac{t+4}{8} \right\rceil$. Hence $f(N(v_{i,j})) = \left(\left\lceil \frac{j-1}{2} \right\rceil - 1\right) + \left(kt - \left\lceil \frac{j+1}{2} \right\rceil + 2\right) = kt + 1$.

CASE C.4 $\left(4\left\lceil \frac{t+4}{8} \right\rceil < j \leq t\right)$: $f(N(v_{i,j})) = \left(\frac{t}{2} - \left\lceil \frac{j-1}{2} \right\rceil + 2\right) + \left(kt - \frac{t}{2} + \left\lceil \frac{j+1}{2} \right\rceil - 1\right) = kt + 2$.

CASE D $(j \equiv 0 (\bmod 4))$

CASE D.1 $(j = 4)$: $f(N(v_{i,j})) = f(v_{i,3}) + f(v_{i,5}) = (kt) + \left(\left\lceil \frac{5}{2} \right\rceil - 1\right) = kt + 2$.

CASE D.2 $\left(4 < j < 4\left\lceil \frac{t}{8} \right\rceil\right)$: $f(N(v_{i,j})) = \left(kt - \left\lceil \frac{j-1}{2} \right\rceil + 2\right) + \left(\left\lceil \frac{j+1}{2} \right\rceil - 1\right) = kt + 2$.

CASE D.3 $\left(j = 4\left\lceil \frac{t}{8} \right\rceil\right)$: If $t \equiv 4 (\bmod 8)$, then $4\left\lceil \frac{t}{8} \right\rceil = 4\left\lceil \frac{t+4}{8} \right\rceil$ and $\left\lceil \frac{j-1}{2} \right\rceil + \left\lceil \frac{j+1}{2} \right\rceil = j+1 = 4\left\lceil \frac{t}{8} \right\rceil + 1 = \frac{t}{2} + 3$. In this case $f(N(v_{i,j})) = \left(kt - \left\lceil \frac{j-1}{2} \right\rceil + 2\right) + \left(\frac{t}{2} - \left\lceil \frac{j+1}{2} \right\rceil + 2\right) = kt + 1$. If $t \equiv 0 (\bmod 8)$, then $j + 1 < 4\left\lceil \frac{t+4}{8} \right\rceil$. In this case $f(N(v_{i,j})) = \left(kt - \left\lceil \frac{j-1}{2} \right\rceil + 2\right) + \left(\left\lceil \frac{j+1}{2} \right\rceil - 1\right) = kt + 2$.

CASE D.4 $\left(4\left\lceil \frac{t}{8} \right\rceil < j \leq 4\left\lceil \frac{t+4}{8} \right\rceil\right)$: In this case $j = 4\left\lceil \frac{t+4}{8} \right\rceil$. Thus $f(N(v_{i,j})) = \left(kt - \frac{t}{2} + \left\lceil \frac{j-1}{2} \right\rceil - 1\right) + \left(\frac{t}{2} - \left\lceil \frac{j+1}{2} \right\rceil + 2\right) = kt$.

CASE D.5 $\left(4\left\lceil \frac{t+4}{8} \right\rceil < j \leq t\right)$: $f(N(v_{i,j})) = \left(kt - \frac{t}{2} + \left\lceil \frac{j-1}{2} \right\rceil - 1\right) + \left(\frac{t}{2} - \left\lceil \frac{j+1}{2} \right\rceil + 2\right) = kt$.

For the bijection f we have that $NS^-(f) = kt$, $NS(f) = kt + 2$, and $NS^{sp}(f) = 2$. Hence $NS^-(G) \geq kt$, $NS(G) \leq kt + 2$, and $NS^{sp}(G) \leq 2$. Using the result from the first paragraph we conclude that $NS^-(G) = kt$, $NS(G) = kt + 2$, and $NS^{sp}(G) = 2$. □

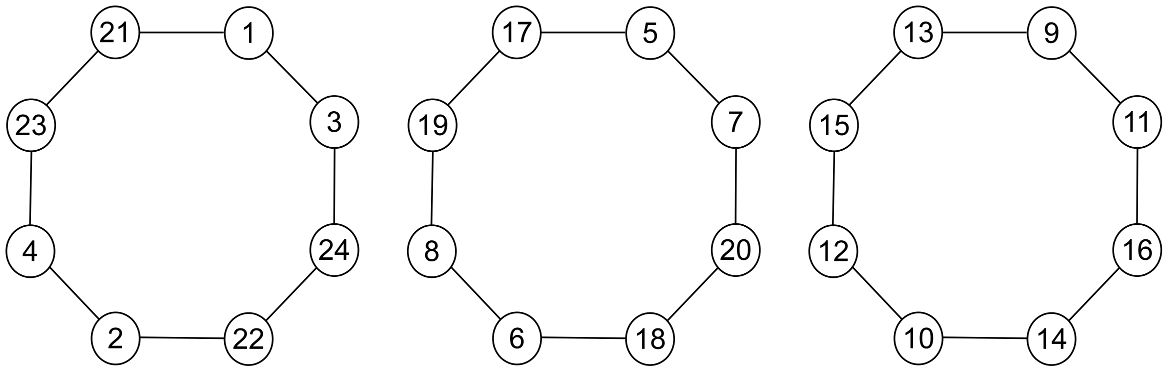

Figure 4.6: $NS^{-}(3C_8) = 24$, $NS(3C_8) = 26$, and $NS^{sp}(3C_8) = 2$

Example 4.28. Figure 4.6 demonstrates that $NS^{-}(3C_8) = kt = 24$, $NS(3C_8) = kt + 2 = 26$, and $NS^{sp}(3C_8) = 2$.

We summarize the results for the open neighborhood sums on the union of cycles of equal order in the following theorem.

Theorem 4.29. *If kC_t is the union of k cycles of order t, then*

(i) If $t \equiv 1,3 (mod\, 4)$, then $NS^{-}(kC_t) = kt - k + 1$, $NS(kC_t) = kt + k + 1$, and $NS^{sp}(kC_t) = 2k$;

(ii) If $t \equiv 2 (mod\, 4)$, then $NS^{-}(kC_t) = kt - 2k + 1$, $NS(kC_t) = kt + 2k + 1$, and $NS^{sp}(kC_t) = 4k$;

(iii) If $t > 4$ and $t \equiv 0 (mod\, 4)$, then $NS^{-}(kC_t) = kt$, $NS(kC_t) = kt + 2$, and $NS^{sp}(kC_t) = 2$; and

(iv) $NS^{-}(kC_4) = NS(kC_4) = 4k + 1$ and $NS^{sp}(kC_4) = 0$.

4.3.2 Closed Neighborhood Sums

In this subsection we seek to determine the values $NS^{-}[kC_t]$, $NS[kC_t]$, and $NS^{sp}[kC_t]$. Unlike the open neighborhood case, except for the case where $t \in \{3,6\}$, we know the exact value of these parameters only when $k = 1$. Moreover when $k = 1$, $t \equiv 3(\text{Mod } 4)$ and $t \geq 21$ we only know a bound for the parameters.

As a starting point, for any 2-regular graph G, we have from Corollary 3.59 that $NS^{-}[G] \leq \left\lfloor \frac{3(n+1)}{2} \right\rfloor \leq \left\lceil \frac{3(n+1)}{2} \right\rceil \leq NS[G]$. From Corollary 3.80 we also know that $NS^{-}[G] + NS[G] = 3(n+1)$. The later result means that any result we determine for $NS[G]$, can be used to determine a complementary result for $NS^{-}[G]$.

Also the previous work of Anstee, Ferguson, and Griggs [1] is very useful to our efforts. In this work the authors developed labelings for cycles where any consecutive three sums had low discrepancy from an average value. Consecutive three sums is equivalent to looking at closed neighborhood sums, and the average value used for a cycle of order n is $\frac{3(n+1)}{2}$, which was the basis for the bound in Corollary 3.59. Anstee, Ferguson, and Griggs showed that for any cycle of order at least six, there exists labelings such that no consecutive three sums (closed neighborhood sum) deviated from the average by more than two. For cycles of even order, since the average is not an integer, the deviation from the average must actually be no more than 1.5. The following theorem is simply a restatement of their result.

Theorem 4.30 ([1]). *For any cycle C_n where $n \geq 6$, $NS^{sp}[C_n] \leq 4$ when n is odd, and $NS^{sp}[C_n] \leq 3$ when n is even.*

As a consequence of this result, we can establish lower and upper bounds for $NS^{-}[C_n]$ and $NS[C_n]$ respectively when $n \geq 6$.

Theorem 4.31. *For any cycle C_n where $n \geq 6$: $NS^{-}[C_n] \geq \left\lfloor \frac{3(n+1)}{2} \right\rfloor - 1$ and $NS[C_n] \leq \left\lceil \frac{3(n+1)}{2} \right\rceil + 1$ for even n; and $NS^{-}[C_n] \geq \left\lfloor \frac{3(n+1)}{2} \right\rfloor - 2$ and $NS[C_n] \leq \left\lceil \frac{3(n+1)}{2} \right\rceil + 2$ for odd n.*

Proof. This result follows immediately from Theorem 4.30 and the facts that $NS^{sp}[C_n] \geq NS[C_n] - NS^{-}[C_n]$ and $NS[C_n] + NS^{-}[C_n] = 3(n+1)$. $\square$

We are interested in determining $NS^{-}[kC_t]$, $NS[kC_t]$, and $NS^{sp}[kC_t]$ for all values of k and t. Theorem 4.31 will be very useful in the case where $k = 1$. We first establish the result that the only Σ'-labeled 2-regular graphs are the unions of an odd number of C_3's.

Lemma 4.32. *Let $W = \{w_1, w_2, \ldots, w_n\}$ be any set. $NS^{sp}_W[C_n] = 0$ if and only if $n = 3$.*

Proof. Let $f : V(C_n) \to W$ be an arbitrary bijection. If $n = 3$, then $f(N[v_1]) = f(N[v_2]) = f(N[v_3]) = f(v_1) + f(v_2) + f(v_3)$, and hence, $NS^{sp}_W[C_3] = NS^{sp}[f] = 0$. If $n > 3$, since W is a set and f is a bijection, we have $f(N[v_2]) = f(v_1) + f(v_2) + f(v_3) \neq f(v_2) + f(v_3) + f(v_4) = f(N[v_3])$. Therefore, $NS^{sp}_W[C_n] > 0$. $\square$

Theorem 4.33. *Let G be a 2-regular graph. $NS^{sp}[G] = 0$ if and only if $G = kC_3$ where k is odd.*

Proof. Let $G = \bigcup_{i=1}^{k} C_{t_i}^{(i)}$ be a union of cycles with vertex set $V(C_{t_i}^{(i)}) = \{v_{i,1}, \ldots, v_{i,t_i}\}$ and with edge set $E(C_{t_i}^{(i)}) = \{v_{i,1}v_{i,2}, \ldots, v_{i,t_i-1}v_{i,t_i}, v_{i,t_i}v_{i,1}\}$. Let $t_i \in \{3, 4, \ldots\}$ indicate the order of each cycle in the union.

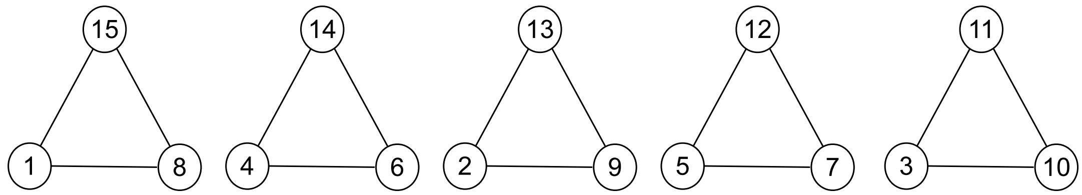

Figure 4.7: $NS^{-}[5C_3] = NS[5C_3] = 24$ and $NS^{sp}[5C_3] = 0$

If there exists a $j \in \{1,2,\ldots,k\}$ such that $t_j \geq 4$, then by Lemma 4.32, there exists vertices $u,v \in V(C_{t_j}^{(j)})$ such that $f(N[u]) \neq f(N[v])$. Thus $NS^{sp}[G] = 0$ only if $t_i = 3$ for all $i \in \{1,2,\ldots,k\}$.

If G is the union of an even number of C_3's, that is, k is even, then by Corollary 3.59, $NS^{-}[G] \leq \left\lfloor \frac{3(3k+1)}{2} \right\rfloor < \left\lceil \frac{3(3k+1)}{2} \right\rceil \leq NS[G]$. Since $NS^{-}[G] < NS[G]$, and since all neighborhood sums are integer valued, by Proposition 2.20, $NS^{sp}[G] \geq 1$.

Finally, assume that $G = kC_3$ and that k is odd. For $i = 1,3,\ldots,k$ set $f(v_{i,1}) = 3k - i + 1$, $f(v_{i,2}) = k + \left\lfloor \frac{i-1}{2} \right\rfloor + \frac{k+1}{2}$, and $f(v_{i,3}) = 1 + \left\lceil \frac{i-1}{2} \right\rceil$. For $i = 2,4,\ldots,k-1$ set $f(v_{i,1}) = 3k - i + 1$, $f(v_{i,2}) = k + \left\lfloor \frac{i-1}{2} \right\rfloor + 1$, and $f(v_{i,3}) = \frac{k+1}{2} + \left\lceil \frac{i-1}{2} \right\rceil$. For any value of i and for any vertex $v \in V(C_3^{(i)})$ we have $f(N[v]) = 3k - i + 1 + k + \left\lfloor \frac{i-1}{2} \right\rfloor + \frac{k+1}{2} + 1 + \left\lceil \frac{i-1}{2} \right\rceil = \frac{9k}{2} + \frac{3}{2}$. Therefore, $NS^{sp}[kC_3] = 0$. $\square$

Corollary 4.34. *If k is odd, then $NS^{-}[kC_3] = NS[kC_3] = \frac{9k+3}{2}$.*

Example 4.35. Figure 4.7 demonstrates that $NS^{-}[5C_3] = NS[5C_3] = \frac{9k+3}{2} = 24$ and $NS^{sp}[5C_3] = 0$.

While kC_3 will not be Σ'-labeled when k is even, the next theorem does establish the exact value of the parameters $NS^{-}[kC_3]$, $NS[kC_3]$, and $NS^{sp}[kC_3]$ when k is even.

Theorem 4.36. *If k is even, then $NS^{-}[kC_3] = \frac{9k+2}{2}$, $NS[kC_3] = \frac{9k+4}{2}$, and $NS^{sp}[kC_3] = 1$.*

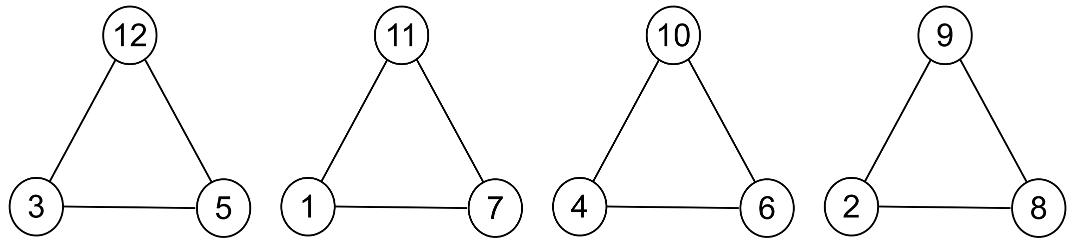

Figure 4.8: $NS^{-}[4C_3] = 19$, $NS[4C_3] = 20$, and $NS^{sp}[4C_3] = 1$

Proof. By Theorem 4.33, $NS^{sp}[kC_3] \geq 1$. If there exists a bijection $f : V(kC_3) \to [3k]$ such that $NS^{sp}[f] = 1$, then not only will we have that $NS^{sp}[kC_3] = 1$, but also that $NS^{-}[kC_3] = NS^{-}[f]$ and that $NS[kC_3] = NS[f]$.

Define the bijection $f : V(kC_3) \to [3k]$ as follows: For $i = 1, 3, \ldots, k-1$ set $f(v_{i,1}) = 3k - i + 1$, $f(v_{i,2}) = k + \lceil \frac{i-1}{2} \rceil + 1$, and $f(v_{i,3}) = \frac{k}{2} + \lfloor \frac{i-1}{2} \rfloor + 1$. For $i = 2, 4, \ldots, k$ set $f(v_{i,1}) = 3k - i + 1$, $f(v_{i,2}) = k + \lceil \frac{i-1}{2} \rceil + \frac{k}{2}$, and $f(v_{i,3}) = \lfloor \frac{i-1}{2} \rfloor + 1$. For any odd value of i and for any $v \in V(C_3^{(i)})$ we have $f(N[v]) = 3k - i + 1 + k + \lceil \frac{i-1}{2} \rceil + 1 + \frac{k}{2} + \lfloor \frac{i-1}{2} \rfloor + 1 = \frac{9k+4}{2}$. For any even value of i and for any $v \in V(C_3^{(i)})$ we have $f(N[v]) = 3k - i + 1 + k + \lceil \frac{i-1}{2} \rceil + \frac{k}{2} + \lfloor \frac{i-1}{2} \rfloor + 1 = \frac{9k+2}{2}$. Therefore, $NS^{-}[kC_3] = NS^{-}[f] = \frac{9k+2}{2}$, $NS[kC_3] = NS[f] = \frac{9k+4}{2}$, and $NS^{sp}[kC_3] = NS^{sp}[f] = 1$. $\square$

Example 4.37. Figure 4.8 demonstrates that $NS^{-}[4C_3] = \frac{9k+2}{2} = 19$, $NS[4C_3] = \frac{9k+4}{2} = 20$, and $NS^{sp}[4C_3] = 1$.

Corollary 4.38. *Let G be a 2-regular graph that is not the union of C_3's. Then $NS^{sp}[G] \geq 1$.*

We next consider the values of our parameters on the four cycle and the five cycle. We develop the results based on the general label multiset W to highlight the influence that the structure of the graph has on the results.

Theorem 4.39. *Let $W = \{w_1, w_2, w_3, w_4\}$ be such that $w_1 \le w_2 \le w_3 \le w_4$. Then $NS^-_W[C_4] = w_1 + w_2 + w_3$, $NS_W[C_4] = w_2 + w_3 + w_4$, and $NS^{sp}_W[C_4] = w_4 - w_1$.*

Proof. Let $f : V(C_4) \to W$ be an arbitrary bijection. There exists a vertex $u \in V(C_4)$ such that $f(N[u]) = w_1 + w_2 + w_3$, and since all vertices $w \in V(C_4)$ have closed neighborhoods with cardinality 3, $f(N[u]) \le f(N[w])$. Similarly, there exists a vertex $v \in V(C_4)$ such that $f(N[v]) = w_2 + w_3 + w_4$, and since all vertices $w \in V(C_4)$ have closed neighborhoods with cardinality of 3, $f(N[v]) \ge f(N[w])$. Hence $NS^-[f] = w_1 + w_2 + w_3$, $NS[f] = w_2 + w_3 + w_4$, and $NS^{sp}[f] = NS[f] - NS^-[f] = w_4 - w_1$. Since f was an arbitrary bijection, $NS^-_W[C_4] = w_1 + w_2 + w_3$, $NS_W[C_4] = w_2 + w_3 + w_4$, and $NS^{sp}_W[C_4] = w_4 - w_1$. □

Corollary 4.40. $NS^-[C_4] = 6$, $NS[C_4] = 9$, *and* $NS^{sp}[C_4] = 3$.

Theorem 4.41. *Let $W = \{w_1, w_2, w_3, w_4, w_5\}$ be such that $w_1 \le w_2 \le w_3 \le w_4 \le w_5$. Then $NS^-_W[C_5] \le w_1 + w_2 + w_5$, $NS_W[C_5] \ge w_1 + w_4 + w_5$, and $NS^{sp}_W[C_5] \ge w_4 - w_2$.*

Proof. Let $f : V(C_5) \to W$ be an arbitrary bijection. Since $d(f^{-1}(w_1), f^{-1}(w_2)) \le 1$, there exists a vertex $u \in V(C_5)$ such that $f(N[u]) \le w_1 + w_2 + w_5$. Similarly, since $d(f^{-1}(w_4), f^{-1}(w_5)) \le 1$, there exists a vertex $v \in V(C_5)$ such that $f(N[v]) \ge w_1 + w_4 + w_5$. Hence $NS^-[f] \le w_1 + w_2 + w_5$, $NS[f] \ge w_1 + w_4 + w_5$, and $NS^{sp}[f] = NS[f] - NS^-[f] \ge= w_4 - w_2$. Since f was an arbitrary bijection, $NS^-_W[C_5] \le w_1 + w_2 + w_5$, $NS_W[C_5] \ge w_1 + w_4 + w_5$, and $NS^{sp}_W[C_5] \ge w_4 - w_2$. □

Theorem 4.42. $NS^-[C_5] = 8$, $NS[C_5] = 10$, *and* $NS^{sp}[C_5] = 2$.

Proof. Define the bijection $f : V(C_5) \to [5]$ by $f(v_1) = 5$, $f(v_2) = 2$, $f(v_3) = 3$, $f(v_4) = 4$, and $f(v_5) = 1$. Then we have $f(N[v_1]) = 8$, $f(N[v_2]) = 10$, $f(N[v_3]) = 9$, $f(N[v_4]) = 8$, and

$f(N[v_5]) = 10$. Hence $NS^-[f] = 8$, $NS[f] = 10$, and $NS^{sp}[f] = 2$. Applying Theorem 4.41, we conclude that $NS^-[C_5] = 8$, $NS[C_5] = 10$, and $NS^{sp}[C_5] = 2$. $\square$

We now look at the union of C_6's, and like the case for the union of C_3's, we can completely characterize $NS^-[kC_6]$, $NS[kC_6]$, and $NS^{sp}[kC_6]$. We first establish the result for the case where k is odd, and then for the more complicated case when k is even.

Theorem 4.43. *If k is odd, then $NS^-[kC_6] = 9k+1$, $NS[kC_6] = 9k+2$, and $NS^{sp}[kC_6] = 1$.*

Proof. From Corollary 3.59 we know that $NS^-[kC_6] \leq \left\lfloor \frac{3(6k+1)}{2} \right\rfloor = 9k+1$ and that $NS[kC_6] \geq \left\lceil \frac{3(6k+1)}{2} \right\rceil = 9k+2$. From Proposition 2.20 we then know that $NS^{sp}[kC_6] \geq NS[kC_6] - NS^-[kC_6] = 1$. Define the bijection $f : V(kC_6) \to [6k]$ as follows:

(i) set $f(v_{i,1}) = 2i-1$;

(ii) if i is odd set $f(v_{i,2}) = 6k-i+1$, else set $f(v_{i,2}) = 5k-i+1$;

(iii) if i is odd set $f(v_{i,3}) = 3k-i+1$, else set $f(v_{i,3}) = 4k-i+1$;

(iv) set $f(v_{i,4}) = 2i$;

(v) if i is odd set $f(v_{i,5}) = 6k-i$, else set $f(v_{i,5}) = 5k-i$; and

(vi) if i is odd set $f(v_{i,6}) = 3k-i+2$, else set $f(v_{i,6}) = 4k-i+2$.

Then for any $i \in \{1,2,\ldots,k\}$ we have that $f(N[v_{i,1}]) = f(N[v_{i,3}]) = f(N[v_{i,5}] = 9k+2$ and that $f(N[v_{i,2}]) = f(N[v_{i,4}]) = f(N[v_{i,6}]) = 9k+1$. Hence, $NS^-[kC_6] \geq 9k+1$, $NS[kC_6] \leq 9k+2$, and $NS^{sp}[kC_6] \leq 1$. Therefore, we conclude that $NS^-[kC_6] = 9k+1$, $NS[kC_6] = 9k+2$, and $NS^{sp}[kC_6] = 1$. $\square$

Example 4.44. Figure 4.9 demonstrates that $NS^-[3C_6] = 9k+1 = 28$, $NS[3C_6] = 9k+2 = 29$, and $NS^{sp}[kC_6] = 1$.

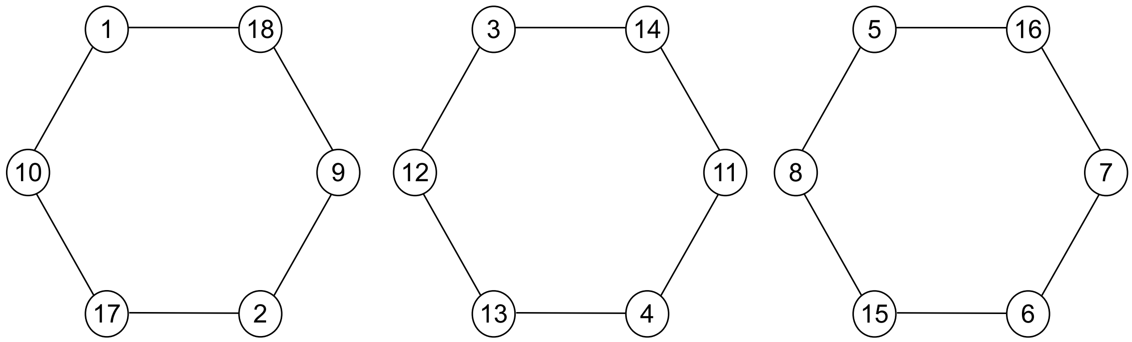

Figure 4.9: $NS^{-}[3C_6] = 28$, $NS[3C_6] = 29$, and $NS^{sp}[3C_6] = 1$

Corollary 4.45. *$NS^{-}[C_6] = 10$, $NS[C_6] = 11$, and $NS^{sp}[C_6] = 1$.*

It is going to turn out that when k is even, that $NS^{sp}[kC_6] = 3$. In order to establish this we will prove a series of lemmas. Lemma 4.46 will show that on any cycle of order at least four, as long as W is a set, that two vertices where the minimax closed neighborhood sum is achieved cannot be adjacent. We use this first lemma to establish in Lemma 4.47 the result that if a cycle of even order achieves the minimax lower bound from Corollary 3.59, then exactly half of the vertices will have the bound value as their closed neighborhood sum, and the other half will have a closed neighborhood sum of one less than the bound, and that these vertices will alternate. Using Lemma 4.47, we then determine in Lemma 4.48 how the labels on kC_6 must be arranged in order for the bound from Corollary 3.59 to be achieved. Finally in Theorem 4.49 we show why such an arrangement is not possible, and then show a labeling that will achieve the new improved lower bound for $NS[kC_6]$ when k is even.

Lemma 4.46. *Let W be any set, $n > 3$, $f : V(C_n) \to W$ be any bijection, and let $v \in V(C_n)$ such that $f(N[v]) = NS[f]$. If $u \in N(v)$, then $f(N[u]) < NS[f]$.*

Proof. Denote $N(v) = \{v_1, v_2\}$ and denote $N(v_1) = \{u, v\}$. Then $f(N[v]) = NS[f] = f(v_1) + f(v) + f(v_2)$ and $f(N[v_1]) = f(u) + f(v_1) + f(v)$. Since f is a bijection, W is a set, and $n > 3$, we have that $f(u) \neq f(v_2)$. Since $f(N[v]) = NS[f]$ is the maximum value of a closed neighborhood sum, we must have $f(u) < f(v_2)$. Therefore $f(NS[v_1]) < NS[f]$. Applying a similar argument we can show that $f(NS[v_2]) < NS[f]$. $\square$

Lemma 4.47. *Let G be a 2-regular graph with even order, and let $f : V(G) \to [n]$ be a bijection such that $NS[f] = \left\lceil \frac{3(n+1)}{2} \right\rceil = \frac{3n+4}{2}$. For $v \in V(G)$ and for all $u \in N(v)$ we have*

(i) if $f(N[v]) = NS[f]$, then $f(N[u]) = NS[f] - 1$; and

(ii) if $f(N[v]) < NS[f]$, then $f(N[v]) = NS[f] - 1$ and $f(N[u]) = NS[f]$.

Proof. Notice that for any 2-regular graph, every weight is assigned to a vertex that is a member of exactly three closed neighborhoods. Hence, the total of all closed neighborhood sums is $3f(V(G)) = \frac{3n(n+1)}{2}$.

From Lemma 4.46 we know that at most half of the vertices of G have closed neighborhood sums of $NS[f] = \frac{3n+4}{2}$. If exactly half of the vertices of G have closed neighborhood sums of $NS[f] = \frac{3n+4}{2}$, and the other half have closed neighborhood sums of $NS[f] - 1 = \frac{3n+2}{2}$, then we have $3f(V(G)) = \frac{n}{2}\left(\frac{3n+4}{2}\right) + \frac{n}{2}\left(\frac{3n+2}{2}\right) = \frac{3n(n+1)}{2}$. However, if more than half of the vertices have closed neighborhood sums less than or equal $NS[f] - 1$, then $3f(V(G)) < \frac{3n(n+1)}{2}$, which would be contradiction. Hence, exactly half of the vertices have closed neighborhood sums of $NS[f]$ and the other half have closed neighborhood sums equal to $NS[f] - 1$. From Lemma 4.46 we know that the vertices with closed neighborhood sums of $NS[f]$ must alternate with those having closed neighborhood sums of $NS[f] - 1$, which proves the result. $\square$

Lemma 4.48. *If k is even and $f : V(kC_6) \to [6k]$ is a bijection such that $NS[f] = \left\lceil \frac{3(6k+1)}{2} \right\rceil = 9k+2$, then for any $w \in \{1,3,\ldots,6k-1\}$:*

(i) $f^{-1}(w)$ and $f^{-1}(w+1)$ are elements of the same C_6,

(ii) $f(N[f^{-1}(w)]) = NS[f]$,

(iii) $f(N[f^{-1}(w+1)]) = NS[f]-1$, and

(iv) $N[f^{-1}(w)] \bigcap N[f^{-1}(w+1)] = \varnothing$.

Proof. Notice that since $6k$ is even, by Lemma 4.47, half of the vertices have closed neighborhood sums of $NS[f]$ and the other half have closed neighborhood sums of $NS[f]-1$ and these vertices alternate. The proof will be by induction on w.

Base Case ($w = 1$): Assume without loss of generality that $f(v_{1,1}) = 1$. Notice that $f(N[v_{1,2}]) = f(v_{1,1}) + f(v_{1,2}) + f(v_{1,3})$ and $f(N[v_{1,3}]) = f(v_{1,2}) + f(v_{1,3}) + f(v_{1,4})$. It follows that $f(N[v_{1,3}]) - f(N[v_{1,2}]) = f(v_{1,4}) - f(v_{1,1})$. Since $f(v_{1,1}) = 1$, we have that $f(v_{1,4}) > f(v_{1,1})$ and hence $f(N[v_{1,3}]) > f(N[v_{1,2}])$. But then by the note in the first paragraph, $f(N[v_{1,3}]) = NS[f]$ and $f(N[v_{1,2}]) = NS[f]-1$. Hence $f(v_{1,4}) = 2$, which proves i and iv. Also from the note in the first paragraph we have that $f(N[v_{1,1}]) = NS[f]$ and $f(N[v_{1,4}]) = NS[f]-1$, which proves ii and iii.

Induction: Let $w \in \{3,5,\ldots,6k-1\}$. Assume that the result holds for all labels less than w. Without loss of generality that $f(v_{1,1}) = w$. Notice that $f(N[v_{1,2}]) = f(v_{1,1}) + f(v_{1,2}) + f(v_{1,3})$ and $f(N[v_{1,3}]) = f(v_{1,2}) + f(v_{1,3}) + f(v_{1,4})$. It follows that $f(N[v_{1,3}]) - f(N[v_{1,2}]) = f(v_{1,4}) - f(v_{1,1})$. By the induction hypothesis, and since $f(v_{1,1}) = w$, we must have $f(v_{1,4}) > f(v_{1,1})$. Thus $f(N[v_{1,3}]) > f(N[v_{1,2}])$. By the note in the first paragraph, $f(N[v_{1,3}]) = NS[f]$ and $f(N[v_{1,2}]) = NS[f]-1$. Hence $f(v_{1,4}) = w+1$, which proves

i and iv. It also follows by the note in the first paragraph that $f(N[v_{1,1}]) = NS[f]$ and $f(N[v_{1,4}]) = NS[f]-1$, which proves ii and iii. $\square$

Theorem 4.49. *For k even,* $NS^-[kC_6] = 9k$*,* $NS[kC_6] = 9k+3$*, and* $NS^{sp}[kC_6] = 3$.

Proof. Assume to the contrary that there exists a bijection $f : V(kC_6) \to [6k]$ such that $NS[f] = \left\lceil \frac{3(6k+1)}{2} \right\rceil = 9k+2$. Let $v \in V(kC_6)$ be such that $f(N[v]) = 9k+2$, and let $N(v) = \{u,w\}$. By Lemma 4.48 $f(v)$ must be odd. By Lemma 4.47 $f(N[u]) = f(N[w]) = NS[f]-1$, and then by Lemma 4.48 $f(u)$ and $f(w)$ must be even. Hence $f(N[v]) = f(u)+f(v)+f(w)$ is odd. But since k is even, $f(N[v]) = NS[f] = 9k+2$ is even, which is a contradiction. Hence, no such bijection f exists when k is even, and then we have that $NS[kC_6] \geq 9k+3$. By Corollary 3.80 it follows that $NS^-[kC_6] \leq 9k$, and then by Proposition 2.20 that $NS^{sp}[kC_6] \geq NS[kC_6] - NS^-[kC_6] = 3$.

Define the bijection $g : V(kC_6) \to [6k]$ as follows:

(i) set $g(v_{i,1}) = 2i-1$;

(ii) if i is odd set $g(v_{i,2}) = 5k-i+1$, else set $g(v_{i,2}) = 6k-i+2$;

(iii) if i is odd set $g(v_{i,3}) = 4k-i$, else set $g(v_{i,3}) = 3k-i+1$;

(iv) set $g(v_{i,4}) = 2i$;

(v) if i is odd set $g(v_{i,5}) = 5k-i$, else set $g(v_{i,5}) = 6k-i+1$; and

(vi) if i is odd set $g(v_{i,6}) = 4k-i+1$, else set $g(v_{i,6}) = 3k-i+2$.

Then for any $i \in \{1,3,\ldots,k-1\}$ we have that $f(N[v_{i,1}]) = f(N[v_{i,3}]) = f(N[v_{i,5}]) = 9k+1$ and that $f(N[v_{i,2}]) = f(N[v_{i,4}]) = f(N[v_{i,6}]) = 9k$. For any $i \in \{2,4,\ldots,k\}$ we have that $f(N[v_{i,1}]) = f(N[v_{i,3}]) = f(N[v_{i,5}]) = 9k+3$ and that $f(N[v_{i,2}]) = f(N[v_{i,4}]) =$

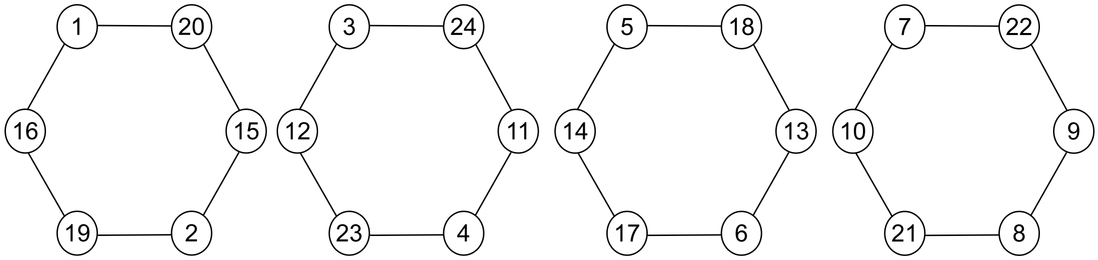

Figure 4.10: $NS^{-}[4C_6] = 36$, $NS[4C_6] = 39$, and $NS^{sp}[4C_6] = 3$

$f(N[v_{i,6}]) = 9k+2$. Hence, $NS^{-}[kC_6] \geq 9k$, $NS[kC_6] \leq 9k+3$, and $NS^{sp}[kC_6] \leq 3$. Therefore, we conclude that $NS^{-}[kC_6] = 9k$, $NS[kC_6] = 9k+3$, and $NS^{sp}[kC_6] = 3$. $\square$

Example 4.50. Figure 4.10 demonstrates that $NS^{-}[4C_6] = 9k = 36$, $NS[4C_6] = 9k+3 = 39$, and $NS^{sp}[4C_6] = 3$.

We now establish an improved lower bound for $NS[kC_t]$ for all $t \notin \{3,6\}$. We will use this result to then establish an improved lower bound for $NS^{sp}[kC_t]$. For most cases of t, we can then argue that these new bounds are optimal by connecting our results to those established by Anstee [1].

Theorem 4.51. *If $t \notin \{3,6\}$, then $NS^{-}[kC_t] \leq \left\lfloor \frac{3(kt+1)}{2} \right\rfloor - 1$ and $NS[kC_t] \geq \left\lceil \frac{3(kt+1)}{2} \right\rceil + 1$.*

Proof. Notice that since each vertex of kC_t is a member of exactly three closed neighborhoods, for any bijection $g : V(kC_t) \to [kt]$, the sum of all closed neighborhood sums is $\sum_{x \in V(kC_t)} g(N[x]) = 3 \sum_{x \in V(kC_t)} g(x) = \frac{3kt(kt+1)}{2}$. In each case we will show that $NS[kC_t] \geq \left\lceil \frac{3(kt+1)}{2} \right\rceil + 1$, and the result that $NS^{-}[kC_t] \leq \left\lfloor \frac{3(kt+1)}{2} \right\rfloor - 1$ will follow from Corollary 3.80.

CASE A ($t = 4$): Let $f : V(kC_t) \to [kt]$ be a bijection such that $NS[f] = NS[kC_t]$ and assume to the contrary that $NS[f] = \left\lceil \frac{3(kt+1)}{2} \right\rceil = \frac{3kt+4}{2}$. Let $S = \{v \in V(kC_t) : f(N[v]) = NS[f]\}$. From Lemma 4.47 we know that $|S| \leq \frac{kt}{2}$. Applying Theorem 4.39, we have for

all $i \in \{1,2,\ldots,k\}$ that there exists vertices $u,v \in V(C_4^{(i)})$ such that $f(N[u]) \geq f(N[v])+3$. That is, $NS^{sp}[f] \geq 3$, and there exists a vertex $w \in V(kC_t)-S$ such that $f(N[w]) \leq NS[f]-3$. Summing all closed neighborhood sums we have $\sum\limits_{x\in V(kC_t)} f(N[x]) = \sum\limits_{x\in S} f(N[x]) + \sum\limits_{x\in V(kC_t)-S-w} f(N[x]) + f(N[w]) \leq \left(\frac{kt}{2}\right) NS[f] + \left(\frac{kt}{2}-1\right)(NS[f]-1) + NS[f] - 3 = \frac{3kt(tk+1)}{2} - 2 < \frac{3kt(kt+1)}{2}$, which is a contradiction. Therefore, $NS[kC_t] = NS[f] \geq \frac{3kt+4}{2} + 1 = \left\lceil \frac{3(kt+1)}{2} \right\rceil + 1$.

CASE B ($t = 5$ and k even): Let $f : V(kC_t) \to [kt]$ be a bijection such that $NS[f] = NS[kC_t]$ and assume to the contrary that $NS[f] = \left\lceil \frac{3(kt+1)}{2} \right\rceil = \frac{3kt+4}{2}$. Let $S = \{v \in V(kC_t) : f(N[v]) = NS[f]\}$. From Lemma 4.47 we know that $|S| \leq \frac{kt}{2}$. Applying Theorem 4.39, we have for all $i \in \{1,2,\ldots,k\}$ that there exists vertices $u,v \in V(C_5^{(i)})$ such that $f(N[u]) \geq f(N[v])+2$. That is, $NS^{sp}[f] \geq 2$, and there exists a vertex $w \in V(kC_t)-S$ such that $f(N[w]) \leq NS[f]-2$. Summing all closed neighborhood sums we have $\sum\limits_{x\in V(kC_t)} f(N[x]) = \sum\limits_{x\in S} f(N[x]) + \sum\limits_{x\in V(kC_t)-S-w} f(N[x]) + f(N[w]) \leq \left(\frac{kt}{2}\right) NS[f] + \left(\frac{kt}{2}-1\right)(NS[f]-1) + NS[f] - 2 = \frac{3kt(tk+1)}{2} - 1 < \frac{3kt(kt+1)}{2}$, which is a contradiction. Therefore, $NS[kC_t] = NS[f] \geq \frac{3kt+4}{2} + 1 = \left\lceil \frac{3(kt+1)}{2} \right\rceil + 1$.

CASE C (kt odd): Let $f : V(kC_t) \to [kt]$ be a bijection such that $NS[f] = NS[kC_t]$ and assume to the contrary that $NS[f] = \left\lceil \frac{3(kt+1)}{2} \right\rceil = \frac{3(kt+1)}{2}$. Let $S = \{v \in V(G) : f(N[v]) = NS[f]\}$. From Lemma 4.47 we know that $|S| \leq \left\lfloor \frac{kt}{2} \right\rfloor$. Summing all closed neighborhood sums we have $\sum\limits_{x\in V(kC_t)} f(N[x]) = \sum\limits_{x\in S} f(N[x]) + \sum\limits_{x\in V(kC_t)-S} f(N[x])) \leq \left\lfloor \frac{kt}{2} \right\rfloor NS[f] + \left\lceil \frac{kt}{2} \right\rceil (NS[f]-1) = \frac{3kt(tk+1)}{2} - \left\lceil \frac{kt}{2} \right\rceil < \frac{3kt(kt+1)}{2}$, which is a contradiction. Therefore, $NS[kC_t] = NS[f] \geq \frac{3kt+4}{2} + 1 = \left\lceil \frac{3(kt+1)}{2} \right\rceil + 1$.

CASE D (kt even and $t \geq 6$): Let $f : V(kC_t) \to [kt]$ be a bijection such that $NS[f] = \left\lceil \frac{3(kt+1)}{2} \right\rceil$. Without loss of generality, we can assume that $f(N[v_{1,3}]) = NS[f]$. Since $t \geq 6$, we can assume that $v_{1,t}, v_{1,1}, \ldots, v_{1,5}$ are distinct and that $v_{1,1}, v_{1,2}, \ldots, v_{1,6}$ are distinct. By Lemma 4.47, since kt is even and $f(N[v_{1,3}]) = NS[f]$, we can deduce that $f(N[v_{1,2}]) = f(N[v_{1,4}]) = f(N[v_{1,4}]) = NS[f] - 1$ and that $f(N[v_{1,1}]) = f(N[v_{1,3}]) = f(N[v_{1,5}]) = NS[f]$. Thus $f(N[v_{1,2}]) = f(v_{1,1}) + f(v_{1,2}) + f(v_{1,3}) = f(v_{1,3}) + f(v_{1,4}) + f(v_{1,5}) = f(N[v_{1,4}])$ and it follows that $f(v_{1,1}) + f(v_{1,2}) = f(v_{1,4}) + f(v_{1,5})$. Also we have that $f(N[v_{1,1}]) = f(v_{1,t}) + f(v_{1,1}) + f(v_{1,2}) = f(v_{1,4}) + f(v_{1,5}) + f(v_{1,6}) = f(N[v_{1,5}])$. Substituting we get that $f(v_{1,t}) + f(v_{1,1}) + f(v_{1,2}) = f(v_{1,1}) + f(v_{1,2}) + f(v_{1,6})$. Hence $f(v_{1,t}) = f(v_{1,1})$. Since f is a bijection, we conclude that $v_{1,t} = v_{1,1}$ and that $t = 6$. Therefore if kt is even and $t > 6$, then $NS[kC_t] \geq \left\lceil \frac{3(kt+1)}{2} \right\rceil + 1$. $\square$

Corollary 4.52. *Let $t \notin \{3,6\}$. If kt is odd, then $NS^{sp}[kC_t] \geq 2$. If kt is even, then $NS^{sp}[kC_t] \geq 3$.*

Corollary 4.53. *$NS^{sp}[C_n] \geq 2$ when $n \neq 3$ and n is odd. $NS^{sp}[C_n] \geq 3$ when $n \neq 6$ and n is even.*

We can now connect the results from Anstee [1] that were stated in Theorem 4.31 with the result in Corollary 4.53 and establish values for $NS^{-}[C_t]$, $NS[C_t]$, and $NS^{sp}[C_t]$ for the case when t is even.

Theorem 4.54. *For $n \geq 8$ and n even, $NS^{-}[C_n] = \frac{3n}{2}$, $NS[C_n] = \frac{3n+6}{2}$, and $NS^{sp}[C_n] = 3$.*

Proof. For all even $n \geq 8$, from Theorem 4.51 we have that $NS^{-}[C_n] \leq \frac{3n}{2}$ and that $NS[C_n] \geq \frac{3n+6}{2}$. From Corollary 4.53 we have that $NS^{sp}[C_n] \geq 3$. From Theorem 4.31 we know that $NS^{-}[C_n] \geq \frac{3n}{2}$, $NS[C_n] \leq \frac{3n+6}{2}$, and $NS^{sp}[C_n] \leq 3$. $\square$

For the case when $t \equiv 1(\text{Mod } 6)$ or $t \equiv 5(\text{Mod } 6)$ we first argue an improved lower bound for $NS[C_t]$ which will show that the bound of Theorem 4.31 is optimal.

Theorem 4.55. *If $n \equiv 1(Mod\, 6)$, then $NS^-[C_n] = \frac{3(n+1)}{2} - 2$, $NS[C_n] = \frac{3(n+1)}{2} + 2$, and $NS^{sp}[C_n] = 4$.*

Proof. From Theorem 4.31 we know that $NS^-[C_n] \geq \frac{3(n+1)}{2} - 2$, $NS[C_n] \leq \frac{3(n+1)}{2} + 2$, and $NS^{sp}[C_n] \leq 4$.

Let $f : V(C_n) \to [n]$ be an arbitrary bijection. Assume without loss of generality that $f(v_n) = 1$. Then we have that $\sum_{i=1}^{n-1} f(v_i) = \frac{n(n+1)}{2} - 1 = \sum_{i=1}^{\frac{n-1}{3}} f(N[v_{3i-1}])$. The average of the closed neighborhood sums from the set $\{f(N[v_2]), f(N[v_5]), \ldots, f(N[v_{n-2}])\}$ is $\frac{n(n+1)-2}{2} \times \frac{3}{n-1} = \frac{3(n+1)+3}{2}$. Since this average value is not an integer, one of the closed neighborhood sums from the set must be at least $\frac{3(n+1)}{2} + 2$.

Since f was an arbitrary bijection, we have that $NS[C_n] \geq \frac{3(n+1)}{2} + 2$. From Corollary 3.80 it follows that $NS^-[C_n] \leq \frac{3(n+1)}{2} - 2$. From Proposition 2.20 it follows that $NS^{sp}[C_n] \geq 4$. $\square$

Theorem 4.56. *If $n \equiv 5(Mod 6)$ and $n > 5$, then $NS^-[C_n] = \frac{3(n+1)}{2} - 2$, $NS[C_n] = \frac{3(n+1)}{2} + 2$, and $NS^{sp}[C_n] = 4$.*

Proof. From Theorem 4.31 we know that $NS^-[C_n] \geq \frac{3(n+1)}{2} - 2$, $NS[C_n] \leq \frac{3(n+1)}{2} + 2$, and $NS^{sp}[C_n] \leq 4$.

Let $f : V(C_n) \to [n]$ be an arbitrary bijection. Assume without loss of generality that $f(v_1) = n$. The we have that $f(v_1) + \sum_{i=1}^{n} f(v_i) = \frac{n(n+1)}{2} + n = \sum_{i=1}^{\frac{n+1}{3}} f(N[v_{3i-1}])$. The average of the closed neighborhood sums from the set $\{f(N[v_2]), f(N[v_5]), \ldots, f(N[v_n])\}$

is $\frac{n(n+1)+2n}{2} \times \frac{3}{n+1} = \frac{3(n+1)}{2} + \frac{3n}{n+1} - \frac{3}{2}$. Since $n > 5$ we have that $\frac{3n}{n+1} - \frac{3}{2} > 1$. Hence, one of the closed neighborhood sums from the set must be at least $\frac{3(n+1)}{2} + 2$.

Since f was an arbitrary bijection, we have that $NS[C_n] \geq \frac{3(n+1)}{2} + 2$. From Corollary 3.80 it follows that $NS^{-}[C_n] \leq \frac{3(n+1)}{2} - 2$. From Proposition 2.20 it follows that $NS^{sp}[C_n] \geq 4$. □

When $n \equiv 3(\text{Mod } 6)$, the constructions provided in Anstee [1] achieve a closed neighborhood spread of four. One would have hoped that either an argument could be made similar to Theorems 4.55 and 4.56 that this was the optimum value, or that a construction could be found to show that the lower bound of Corollary 4.53 was the optimum value. Theorem 4.57 will show that when $n \in \{9, 15\}$, the bounds from Corollary 4.53 are indeed achieved. However, for the cases when $n \geq 21$, the problem remains open.

Theorem 4.57. *If $n \in \{9, 15\}$, then $NS^{-}[C_n] = \frac{3(n+1)}{2} - 1$, $NS[C_n] = \frac{3(n+1)}{2} + 1$, and $NS^{sp}[C_n] = 2$.*

Proof. From Theorem 4.51 we have that $NS^{-}[C_n] \leq \frac{3(n+1)}{2} - 1$ and that $NS[C_n] \geq \frac{3(n+1)}{2} + 1$. From Corollary 4.53 we have that $NS^{sp}[C_n] \geq 2$. Figure 4.11 demonstrates bijections that achieve these bounds when $n \in \{9, 15\}$.

□

Theorem 4.58 summarizes the results we have shown for the closed neighborhood sums of cycles.

Theorem 4.58. *For cycle C_n:*

(i) $NS^{-}[C_3] = NS[C_3] = 6$ and $NS^{sp}[C_3] = 0$;

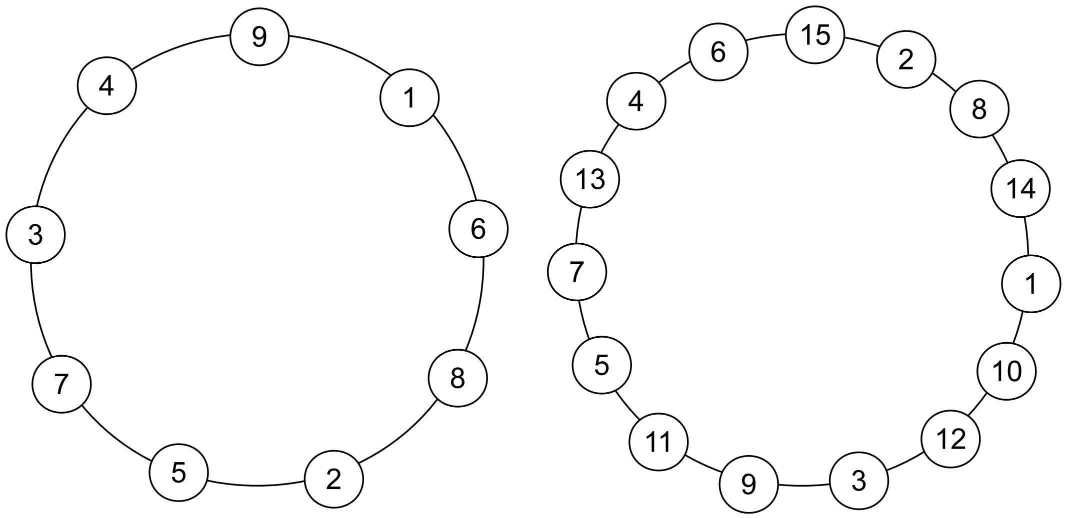

Figure 4.11: $NS^{sp}[C_9] = NS^{sp}[C_{15}] = 2$

(ii) if $n \in \{5, 9, 15\}$*, then* $NS^-[C_n] = \frac{3(n+1)}{2} - 1$*,* $NS[C_n] = \frac{3(n+1)}{2} + 1$*, and* $NS^{sp}[C_n] = 2$*;*

(iii) $NS^-[C_6] = 10$*,* $NS[C_6] = 11$*, and* $NS^{sp}[C_6] = 1$*;*

(iv) if $n \neq 6$ *and n is even, then* $NS^-[C_n] = \frac{3n}{2}$*,* $NS[C_n] = \frac{3n+6}{2}$*, and* $NS^{sp}[C_n] = 3$*;*

(v) if $n \geq 7$ *and* $n \equiv 1, 5 (Mod\, 6)$*, then* $NS^-[C_n] = \frac{3(n+1)}{2} - 2$*,* $NS[C_n] = \frac{3(n+1)}{2} + 2$*, and* $NS^{sp}[C_n] = 4$*; and*

(vi) if $n \geq 21$ *and* $n \equiv 3 (Mod\, 6)$*, then* $\frac{3(n+1)}{2} - 2 \leq NS^-[C_n] \leq \frac{3(n+1)}{2} - 1$*,* $\frac{3(n+1)}{2} + 1 \leq NS[C_n] \leq \frac{3(n+1)}{2} + 2$*, and* $NS^{sp}[C_n] \in \{2, 3, 4\}$.

4.4 Unions of Complete Graphs and Complete Mutlipartite Graphs

In this section we present some partial results for the values of the maximin, minimax, and spread parameters when the underlying graph is a union of complete graphs. When the union contains more than two components, we will require that each of each

component of the graph have the same order. If the union only contains two components, we consider components of all orders. We will only consider the closed neighborhood case. Since the complement of such a graph is a complete multipartite graph, we obtain the complementary open neighborhood results as well.

In this section we let kK_t be the union of k complete graphs of order t. We will denote the vertices as $V(kK_t) = \bigcup_{i=1}^{k} V(K_t^{(i)}) = \bigcup_{i=1}^{k} \{v_{i,1}, v_{i,2}, \ldots, v_{i,t}\}$, where for all $i \in \{1,2,\ldots,k\}$, $v_{i,p}v_{i,q} \in E(kK_t)$ for all $p,q \in \{1,2,\ldots,t\}$ when $p \neq q$.

When $t = 3$, $K_3 = C_3$, and our result follows directly from the result we established for cycles.

Theorem 4.59. *If k is even, then $NS^-[kK_3] = \frac{9k+2}{2}$, $NS[kK_3] = \frac{9k+4}{2}$, and $NS^{sp}[kK_3] = 1$. If k is odd, $NS^-[kK_3] = NS[kK_3] = \frac{9k+3}{2}$, and $NS^{sp}[kK_3] = 0$.*

Proof. Since $K_3 = C_3$ this result is a restatement of Theorem 4.33, Corollary 4.34, and Theorem 4.36. $\square$

Theorem 4.60. *If k is odd or t is even, then $NS^-[kK_t] = NS[kK_t] = \frac{t(kt+1)}{2}$ and $NS^{sp}[G] = 0$. If k is even and $t \geq 3$ is odd, then $NS^-[kK_t] = \frac{t(kt+1)-1}{2}$, $NS[kK_t] = \frac{t(kt+1)+1}{2}$, and $NS^{sp}[kK_t] = 1$.*

Proof. We will complete the proof by looking the possible cases for k and t.

CASE A (t even): The proof will be by induction on k. If $k = 1$, then $NS^-[kK_t] = NS[G] = \frac{t(t+1)}{2}$ and $NS^{sp}[G] = 0$ as shown in Corollary 4.2. It we demonstrate a bijection $g : V(G) \to [kt]$ where $NS^{sp}[g] = 0$, then it will follow from Proposition 2.20 and Corollary 3.23 that $NS^-[kK_t] = NS[kK_t] = \frac{t(kt+1)}{2}$. We assume that for $H = (k-1)K_t$ we have that $NS^{sp}[H] = 0$; we want to show that this implies that when $G = kK_t$ we have

$NS^{sp}[G]=0$. For H there exists a bijection $h: V(H) \to [(k-1)n]$ such that $NS^{sp}[h]=0$. Let $W=\{\frac{t}{2}+1, \frac{t}{2}+2, \ldots, kt-\frac{t}{2}\}$ and form the bijection $f: V(H) \to W$ by $f=h+\frac{t}{2}$. Now form the bijection $g: V(G) \to [kt]$ as follows. Set $g(v_{i,j})=f(v_{i,j})$ for $i \in \{1,2,\ldots,k-1\}$. For $j \le \frac{t}{2}$ take $g(v_{k,j})=j$ and for $\frac{t}{2}<j \le t$ take $g(v_{k,j})=kt-(t-j)$. For $1 \le i \le k-1$ and for $1 \le j \le t$ we have $f(N[v_{i,j}])=\frac{t^2}{2}+\frac{t((k-1)t+1)}{2}=\frac{t(kt+1)}{2}$. For $i=k$ and for $1 \le j \le t$ we have $f(N[v_{k,j}])=\frac{\frac{t}{2}\left(\frac{t}{2}+1\right)}{2}+\frac{t(kt-t)}{2}+\frac{t^2}{4}+\frac{\frac{t}{2}\left(\frac{t}{2}+1\right)}{2}=\frac{t(kt+1)}{2}$. Therefore, $NS^-[G]=NS[G]=\frac{t(kt+1)}{2}$ and $NS^{sp}[G]=0$.

CASE B (k odd): Since the case for even t was proved in Case A, we only need to prove this case when both k and $t \ge 3$ are odd. Let k be an arbitrary odd positive integer; we will complete the proof by induction on t. In Theorem 4.59 we proved the result for all odd k when $t=3$. Let $H=\bigcup_{i=1}^{k} K_{t-2}^{(i)}$. By our induction hypothesis, there exists a bijection $h: V(H) \to [k(t-2)]$ such that $NS^{sp}[h]=0$. Let $G=\bigcup_{i=1}^{k} K_t^{(i)}$. Form the bijection $g: V(G) \to [kt]$ as follows. For $i \in \{1,2,\ldots,k\}$ and $j \in \{1,2,\ldots,t-2\}$ let $g(v_{i,j})=h(v_{i,j})$. Set $g(v_{i,t-1})=k(t-2)+i$, and $g(v_{i,t})=kt-i+1$. Then for every $v \in V(G)$ we have that $g(N[v])=NS[h]+k(t-2)+i+kt-i+1=\frac{(t-2)(k(t-2)+1)}{2}+2kt-2k+1=\frac{t(kt+1)}{2}$. Therefore, $NS^-[G]=NS[G]=\frac{t(kt+1)}{2}$ and $NS^{sp}[G]=0$.

CASE C (k even, t odd): If $NS^{sp}[kK_t]=0$, then each closed neighborhood sum must equal $\frac{kt(kt+1)}{2} \times \frac{t}{kt}=\frac{t(kt+1)}{2}$. But since k is even and t is odd, we have that $kt+1$ is odd, and hence, $t(kt+1)$ is also odd. But this implies that $\frac{t(kt+1)}{2}$ is not an integer, which is a contradiction. Thus $NS^{sp}[kK_t] \ge 1$. Let k be an arbitrary even positive integer; we will complete the proof by induction on t. In Theorem 4.59 we proved the result for all even k when $t=3$. Let $H=kK_{t-2}$. By our induction hypothesis, there exists a bijection

$h: V(H) \to [k(t-2)]$ such that $NS^{sp}[h] = 1$. Let $G = kK_t$. Form the bijection $g: V(G) \to [kt]$ as follows. For $i \in \{1, 2, \ldots, k\}$ and $j \in \{1, 2, \ldots, t-2\}$ let $g(v_{i,j}) = h(v_{i,j})$. Set $g(v_{i,t-1}) = k(t-2) + i$. Set $g(v_{i,t}) = kt - i + 1$. Then for every $v \in V(G)$ we have that $g(N[v]) \le NS[h] + k(t-2) + i + kt - i + 1 = \frac{(t-2)(k(t-2)+1)+1}{2} + 2kt - 2k + 1 = \frac{t(kt+1)+1}{2}$. Similarly, for every $v \in V(G)$ we have that $g(N[v]) \ge NS^-[h] + k(t-2) + i + kt - i + 1 = \frac{(t-2)(k(t-2)+1)-1}{2} + 2kt - 2k + 1 = \frac{t(kt+1)-1}{2}$. Therefore, $NS^-[G] = \frac{t(kt+1)-1}{2}$, $NS[G] = \frac{t(kt+1)+1}{2}$, and $NS^{sp}[G] = 1$. □

Corollary 4.61. *Let $G = K_{t_1, t_2, \ldots, t_k}$, where $t_1 = t_2 = \cdots = t_k = t$.*

(i) If k is odd or t is even, then $NS^-(G) = NS(G) = \frac{t(k-1)(kt+1)}{2}$ and $NS^{sp}(G) = 0$.

(ii) If k is even and $t \ge 3$ is odd, then $NS^-(G) = \frac{t(k-1)(kt+1)-1}{2}$, $NS(G) = \frac{t(k-1)(kt+1)+1}{2}$, and $NS^{sp}(G) = 1$.

Proof. Notice that for any values of k and t we have from Corollary 3.18 that $NS[kK_t] + NS^-(G) = NS^-[kK_t] + NS(G) = \frac{kt(kt+1)}{2}$. Therefore, the result follows directly from Theorem 4.60. □

Theorem 4.62. *For the complete bipartite graph $K_{p,q}$, where $p \ge q$*

1) If $pq + \frac{q(q+1)}{2} < \frac{p(p+1)}{2}$, then $NS^-(K_{p,q}) = pq + \frac{q(q+1)}{2}$ and $NS(K_{p,q}) = \frac{p(p+1)}{2}$;

2) If $pq + \frac{q(q+1)}{2} \ge \frac{p(p+1)}{2}$, then $NS^-(K_{p,q}) = \left\lfloor \frac{(p+q)(p+q+1)}{4} \right\rfloor$ and $NS(K_{p,q}) = \left\lceil \frac{(p+q)(p+q+1)}{4} \right\rceil$;

3) $NS^{sp}(K_{p,q}) = NS(K_{p,q}) - NS^-(K_{p,q})$.

Proof. Let $V_1 = \{v_{1,1}, v_{1,2}, \ldots, v_{1,p}\}$ and $V_2 = \{v_{2,1}, v_{2,2}, \ldots, v_{2,q}\}$ be the partite sets of $K_{p,q}$.

Notice that the minimum possible sum of weights assigned to the vertices of V_1 is $\frac{p(p+1)}{2}$ and the maximum possible sum of weights assigned to the vertices of V_2 is $pq +$

$\frac{q(q+1)}{2}$. Hence for the case where $pq+\frac{q(q+1)}{2}<\frac{p(p+1)}{2}$, we have that $NS^-(K_{p,q})\leq pq+\frac{q(q+1)}{2}$ and $NS(K_{p,q})\geq\frac{p(p+1)}{2}$. Let $f:V(K_{p,q})\to[p+q]$ be the bijection such that for $1\leq i\leq p$, $f(v_{1,i})=i$ and for $1\leq k\leq q$, $f(v_{2,k})=p+k$. Then for all $v\in V_1$ we have $f(N(v))=\sum_{u\in V_2}f(u)=pq+\frac{q(q+1)}{2}$. For all $v\in V_2$ we have $f(N(v))=\sum_{u\in V_1}f(u)=\frac{p(p+1)}{2}$. Hence $NS^-(f)=pq+\frac{q(q+1)}{2}$ and $NS(f)=\frac{p(p+1)}{2}$. Therefore $NS^-(K_{p,q})=pq+\frac{q(q+1)}{2}$ and $NS(K_{p,q})=\frac{p(p+1)}{2}$. Since f is such that $NS^-(K_{p,q})=NS^-(f)$ and $NS(K_{p,q})=NS(f)$, it follows that $NS^{sp}(K_{p,q})=NS(K_{p,q})-NS^-(K_{p,q})$.

Next consider the case where $pq+\frac{q(q+1)}{2}\geq\frac{p(p+1)}{2}$. Since for any bijection $f:V(K_{p,q})\to[p+q]$ we have that $f(N(v_{1,1}))+f(N(v_{2,1}))=\frac{(p+q)(p+q+1)}{2}$, it follows that $NS^-(f)\leq\left\lfloor\frac{(p+q)(p+q+1)}{4}\right\rfloor$ and $NS(f)\geq\left\lceil\frac{(p+q)(p+q+1)}{4}\right\rceil$. Therefore we have $NS^-(K_{p,q})\leq\left\lfloor\frac{(p+q)(p+q+1)}{4}\right\rfloor$ and $NS(K_{p,q})\geq\left\lceil\frac{(p+q)(p+q+1)}{4}\right\rceil$.

Notice that for any $g:V(K_{p,q})\to[p+q]$, if $g(N(v_{2,1}))\leq g(N(v_{1,1}))\leq g(N(v_{2,1}))+1$, then $2g(N(v_{2,1}))\leq g(N(v_{1,1}))+g(N(v_{2,1}))=\frac{(p+q)(p+q+1)}{2}\leq 2g(N(v_{2,1}))$. It would follow that $g(N(v_{2,1}))\leq\frac{(p+q)(p+q+1)}{4}\leq g(N(v_{2,1}))+\frac{1}{2}$ and hence that $g(N(v_{2,1}))=\left\lfloor\frac{(p+q)(p+q+1)}{4}\right\rfloor$. Since $g(N(v_{1,1}))+g(N(v_{2,1}))=\frac{(p+q)(p+q+1)}{2}$, it would also follow that $g(N(v_{1,1}))=\left\lceil\frac{(p+q)(p+q+1)}{4}\right\rceil$.

Next we iteratively construct a bijection that achieves the desired bounds. Let $f_0:V(K_{p,q})\to[p+q]$ be defined by $f_0(v_{1,i})=i$ for $1\leq i\leq p$ and $f_0(v_{2,k})=p+k$ for $1\leq k\leq q$. Then $f_0(N(v_{1,1}))=pq+\frac{q(q+1)}{2}\geq\frac{p(p+1)}{2}=f_0(N(v_{2,1}))$. If $f_0(N(v_{1,1}))\leq f_0(N(v_{2,1}))+1$, then by the previous paragraph we have that $f_0(N(v_{1,1}))=\left\lceil\frac{(p+q)(p+q+1)}{4}\right\rceil$ and $f_0(N(v_{2,1}))=\left\lfloor\frac{(p+q)(p+q+1)}{4}\right\rfloor$. If $f_0(N(v_{1,1}))>f_0(N(v_{2,1}))+1$, then construct the bijection $f_1:V(K_{p,q})\to[p+q]$ as follows. Let $u\in V_1$ be such that for all $w\in V_1$ where

$f_0(w) > f_0(u)$ there does not exist a $v \in V_2$ such that $f_0(v) > f_0(w)$; that is, $f_0(u)$ is the largest label assigned a vertex in V_1 that is less than a label assigned to a vertex in V_2. In this case, there exists a vertex $v \in V_2$ such that $f_0(v) = f_0(u) + 1$. We then set $f_1(u) = f_0(v)$, $f_1(v) = f_0(u)$, and for all $w \in V(K_{p,q}) - \{u, v\}$ we set $f_1(w) = f_0(w)$. Then $f_1(N(v_{1,1})) = f_0(N(v_{1,1})) - 1$, $f(N(v_{2,1})) = f_0(N(v_{2,1})) + 1$, and we have $f_1(N(v_{1,1})) \geq f_1(N(v_{2,1}))$. If $f_1(N(v_{1,1})) \leq f_1(N(v_{2,1})) + 1$, then by the previous paragraph we have that $f_1(N(v_{1,1})) = \left\lceil \frac{(p+q)(p+q+1)}{4} \right\rceil$ and $f_1(N(v_{2,1})) = \left\lfloor \frac{(p+q)(p+q+1)}{4} \right\rfloor$. If $f_1(N(v_{1,1})) > f_1(N(v_{2,1})) + 1$, then we construct a bijection f_2 in like fashion. Since $p \geq q$, there exists a bijection f_k such that $f_k(N(v_{1,1})) \leq f_k(N(v_{2,1})) + 1$, and for this bijection we will have $NS(f_k) = f_k(N(v_{1,1})) = \left\lceil \frac{(p+q)(p+q+1)}{4} \right\rceil$ and $NS^-(f_k) = f_k(N(v_{2,1})) = \left\lfloor \frac{(p+q)(p+q+1)}{4} \right\rfloor$. Therefore, $NS^-(K_{p,q}) = \left\lfloor \frac{(p+q)(p+q+1)}{4} \right\rfloor$ and $NS(K_{p,q}) = \left\lceil \frac{(p+q)(p+q+1)}{4} \right\rceil$. Since the same bijection achieves $NS^-(K_{p,q})$ and $NS(K_{p,q})$, we again conclude that $NS^{sp}(K_{p,q}) = NS(K_{p,q}) - NS^-(K_{p,q})$. $\square$

Corollary 4.63. *For $G = K_p \bigcup K_q$, where $p \geq q$*

1) If $pq + \frac{q(q+1)}{2} < \frac{p(p+1)}{2}$, then $NS^-[G] = pq + \frac{q(q+1)}{2}$ and $NS[G] = \frac{p(p+1)}{2}$;

2) If $pq + \frac{q(q+1)}{2} \geq \frac{p(p+1)}{2}$, then $NS^-[G] = \left\lfloor \frac{(p+q)(p+q+1)}{4} \right\rfloor$ and $NS[G] = \left\lceil \frac{(p+q)(p+q+1)}{4} \right\rceil$;

3) $NS^{sp}[G] = NS[G] - NS^-[G]$.

4.5 $2 \times q$ Torus

The $2 \times q$ Torus will be denoted by $T_{2\times q}$ and the vertices will be identified as shown in Figure 4.12. The graph $T_{2\times q}$ is 3-regular. From Corollary 3.59 we have that $NS^-(T_{2\times q}) \leq 3q + 1 < 3q + 2 \leq NS(T_{2\times q})$ and that $NS^-[T_{2\times q}] \leq 4q + 2 \leq NS[T_{2\times q}]$. It

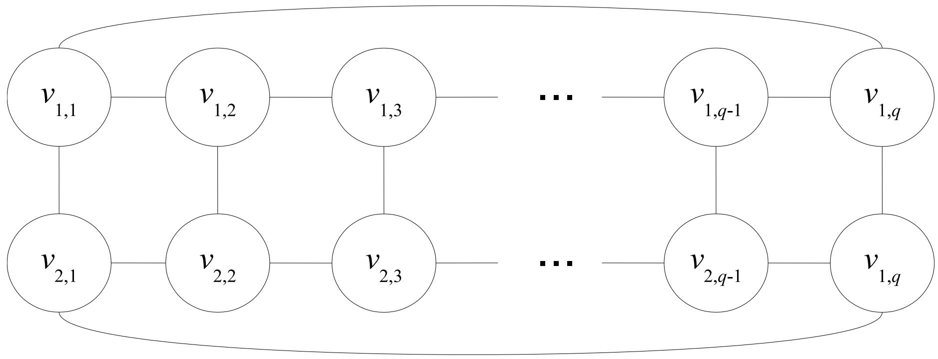

Figure 4.12: Torus $T_{2\times q}$

then follows from Proposition 2.20 that $NS^{sp}(T_{2\times k}) \geq 1$. When q is odd, we can solve the open neighborhood case by relating the problem for $T_{2\times q}$ to closed neighborhood results for the cycle C_{2q}. When q is even, the open neighborhood case remains an open problem. For all q, we present the values of $NS^{-}[T_{2\times q}]$ and $NS[T_{2\times q}]$.

Theorem 4.64. *If q is odd, then $NS^{-}(T_{2\times q}) = NS^{-}[C_{2q}]$, $NS(T_{2\times q}) = NS[C_{2q}]$, and $NS^{sp}(T_{2\times q}) = NS^{sp}[C_{2q}]$. Hence, for $q = 3$, $NS^{-}(T_{2\times 3}) = 10$, $NS(T_{2\times 3}) = 11$, and $NS^{sp}(T_{2\times 3}) = 1$. For $q > 3$, $NS^{-}(T_{2\times q}) = 3q$, $NS(T_{2\times q}) = 3q+3$, and $NS^{sp}(T_{2\times q}) = 3$.*

Proof. Let $f : V(C_{2q}) \to [2q]$ be any bijection. Define the bijection $g : V(T_{2\times q}) \to [2q]$ by $g(v_{i,j}) = f(v_{j+(\frac{q}{2})(1+(-1)^{i+j-1})})$, where $i \in \{1,2\}$and $1 \leq j \leq q$. For example, the bijection shown for the C_{10} in Figure 4.13 would produce the associated bijection for the $T_{2\times 5}$.

With g defined in this manner, the set of closed neighborhood sums for f is the same as the set of open neighborhood sums for g. For example, in the case shown in Figure 4.13, $f(N[v_1]) = f(v_{10}) + f(v_1) + f(v_2) = g(v_{2,5}) + g(v_{1,1}) + g(v_{2,2}) = g(N(v_{2,1}))$. In general, $g(N(v_{i,j})) = f(N[j + (\frac{q}{2})(1+(-1)^{i+j})])$. Hence it follows that $NS(g) = NS[f]$, $NS^{-}(g) =$

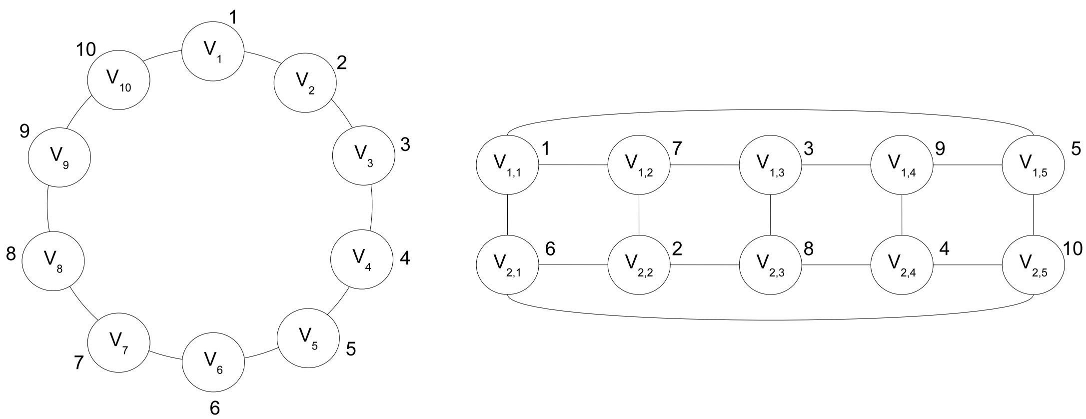

Figure 4.13: Mapping of C_{2q} to $T_{2\times q}$

$NS^{-}[f]$, and $NS^{sp}(g) = NS^{sp}[f]$. Since this result holds for any such bijection f, it follows that $NS^{-}(T_{2\times q}) = NS^{-}[C_{2q}]$, $NS(T_{2\times q}) = NS[C_{2q}]$, and $NS^{sp}(T_{2\times q}) = NS^{sp}[C_{2q}]$. $\square$

Next we look at the closed neighborhood values, which will require a case by case analysis based on the value of q. Notice that if $q = 2$, then $T_{2\times 2} = C_4$, so we can begin by considering values $q \geq 3$. First we handle the special case where $q = 3$.

Theorem 4.65. *$NS^{-}[T_{2\times 3}] = 12$, $NS[T_{2\times 3}] = 16$, and $NS^{sp}[T_{2\times 3}] = 4$.*

Proof. Let $f : V(T_{2\times 3}) \to [6]$ be any bijection. Without loss of generality, assume that $f(v_{1,1}) = 1$ and that $f(v_{1,2}) < f(v_{1,3})$. If $f(v_{1,2}) \leq 4$, then $f(N[v_{2,3}]) = 21 - f(v_{1,1}) - f(v_{1,2}) \geq 16$. If $f(v_{1,2}) > 4$, then $f(v_{1,2}) = 5$ and $f(v_{1,3}) = 6$. In this later case, assume $f(v_{2,j}) = 4$, then $f(N[v_{1,j}]) = f(v_{1,1}) + f(v_{1,2}) + f(v_{1,3}) + f(v_{2,j}) = 1+5+6+4 = 16$. Therefore $NS[T_{2\times 3}] \geq 16$. By Corollary 3.80 it follows that $NS^{-}[T_{2\times 3}] \leq 12$ and by Proposition 2.20 that $NS^{sp}[T_{2\times 3}] \geq 4$.

Now consider the bijection $g : V(T_{2\times 3}) \to [6]$, where $g(v_{1,1}) = 1$, $g(v_{1,2}) = 5$, $g(v_{1,3}) = 4$, $g(v_{2,1}) = 6$, $g(v_{2,2}) = 2$, and $g(v_{2,3}) = 3$. For this bijection we have that

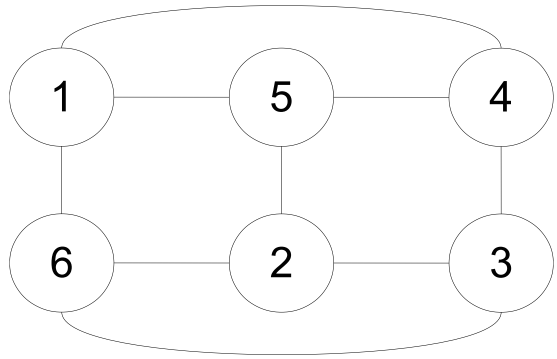

Figure 4.14: $NS^{-}[T_{2\times 3}] = 12$, $NS[T_{2\times 3}] = 16$, and $NS^{sp}[T_{2\times 3}] = 4$

$NS^{-}[g] = 12$,$NS[g] = 16$, and $NS^{sp}[g] = 4$. Therefore $NS^{-}[T_{2\times 3}] = 12$, $NS[T_{2\times 3}] = 16$, and $NS^{sp}[T_{2\times 3}] = 4$. The bijection g is shown in Figure 4.14. □

In Lemma 4.66 we show that $NS^{sp}[T_{2\times q}] \neq 1$, and then in Lemma 4.67 that, when $q \geq 4$, we have that $NS^{sp}[T_{2\times q}] \geq 2$. Lemma 4.67 will allow us in Lemma 4.68 to establish improved bounds for $NS^{-}[T_{2\times q}]$ and $NS[T_{2\times q}]$ which will be shown by construction to be optimum.

Lemma 4.66. *If $NS^{sp}[T_{2\times q}] > 0$, then $NS^{sp}[T_{2\times q}] \geq 2$.*

Proof. Notice that for any bijection $f : V(T_{2\times q}) \to [2q]$ all neighborhood sum values are integer valued. Hence if $NS^{sp}[T_{2\times q}] > 0$, for any bijection f, we have that $NS^{-}[f] < 4q + 2 < NS[f]$. But this implies that $NS^{sp}[f] > NS[f] - NS^{-}[f] \geq 2$. □

Lemma 4.67. *For $q \geq 4$ $NS^{sp}[T_{2\times q}] \geq 2$.*

Proof. Let $f : V(T_{2\times q}) \to [2q]$ be an arbitrary bijection. We will show that $NS^{sp}[T_{2\times q}] > 0$ and it will follow from Lemma 4.66 that $NS^{sp}[T_{2\times q}] \geq 2$. Notice that if $NS^{sp}[T_{2\times k}] = 0$, then $f(N[v_{1,2}]) = f(v_{1,1}) + f(v_{1,2}) + f(v_{1,3}) + f(v_{2,2}) = f(v_{2,2}) + f(v_{2,3}) + f(v_{2,4}) + f(v_{1,3}) = f(N[v_{2,3}])$. Hence, $f(v_{1,1}) + f(v_{1,2}) = f(v_{2,3}) + f(v_{2,3})$. More generally we have

(i) $f(v_{1,i})+f(v_{1,i+1})=f(v_{2,i+2})+f(v_{2,i+3})$ for all $i\in\{1,2,\ldots,q-3\}$,

(ii) $f(v_{1,q-2})+f(v_{1,q-1})=f(v_{2,q})+f(v_{2,1})$,

(iii) $f(v_{1,q-1})+f(v_{1,q})=f(v_{2,1})+f(v_{2,2})$, and

(iv) $f(v_{1,q})+f(v_{1,1})=f(v_{2,2})+f(v_{2,3})$.

Adding all q of these equations we get that $2\sum_{i=1}^{q} f(v_{1,i}) = 2\sum_{i=1}^{q} f(v_{2,i})$, or that $\sum_{i=1}^{q} f(v_{1,i}) = \sum_{i=1}^{q} f(v_{2,i})$. Since $\sum_{i=1}^{q} f(v_{1,i})+f(v_{2,i}) = \frac{2q(2q+1)}{2} = q(2q+1)$, it follows that if $NS^{sp}[T_{2\times q}]=0$, then $\sum_{i=1}^{q} f(v_{1,i}) = \sum_{i=1}^{q} f(v_{2,i}) = \frac{q(2q+1)}{2}$.

Now if q is odd, then $q(2q+1)$ is also odd, and then $\frac{q(2q+1)}{2}$ is not an integer. Since all labels are integers, this is a contradiction. Hence, if q is odd, then $NS^{sp}[T_{2\times q}]\geq 2$.

Now consider the case where q is even. If $NS^{sp}[T_{2\times q}]=0$ and $q\equiv 0(\mathrm{Mod}\,4)$, then from the note in the first paragraph, we can establish that there exist constants c_1 and c_2 such that

(i) $c_1=f(v_{1,1})+f(v_{1,2})=f(v_{2,3})+f(v_{2,4})=f(v_{1,5})+f(v_{1,6})=f(v_{2,7})+f(v_{2,8})= \cdots = f(v_{1,q-3})+f(v_{1,q-2})=f(v_{2,q-1})+f(v_{2,q})$ and

(ii) $c_2=f(v_{2,1})+f(v_{2,2})=f(v_{1,3})+f(v_{1,4})=f(v_{2,5})+f(v_{2,6})=f(v_{1,7})+f(v_{1,8})= \cdots = f(v_{2,q-3})+f(v_{2,q-2})=f(v_{1,q-1})+f(v_{1,q})$.

If $NS^{sp}[T_{2\times q}]=0$ and $q\equiv 2(\mathrm{Mod}\,4)$, then in a similar manner, we can establish that

(i) $c_1=f(v_{1,1})+f(v_{1,2})=f(v_{2,3})+f(v_{2,4})=f(v_{1,5})+f(v_{1,6})=f(v_{2,7})+f(v_{2,8})= \cdots = f(v_{1,q-5})+f(v_{1,q-4})=f(v_{2,q-3})+f(v_{2,q-2})=f(v_{1,q-1})+f(v_{1,q})$ and

(ii) $c_2=f(v_{2,1})+f(v_{2,2})=f(v_{1,3})+f(v_{1,4})=f(v_{2,5})+f(v_{2,6})=f(v_{1,7})+f(v_{1,8})= \cdots = f(v_{2,q-5})+f(v_{2,q-4})=f(v_{1,q-3})+f(v_{1,q-2})=f(v_{2,q-1})+f(v_{2,q})$.

In either case it then follows that $\frac{qc_1}{2}+\frac{qc_2}{2}=\frac{2q(2q+1)}{2}$ and hence $c_1+c_2=4q+2$. In particular we have that $f(v_{1,1})+f(v_{1,2})+f(v_{2,1})+f(v_{2,2})=4q+2$. Now since

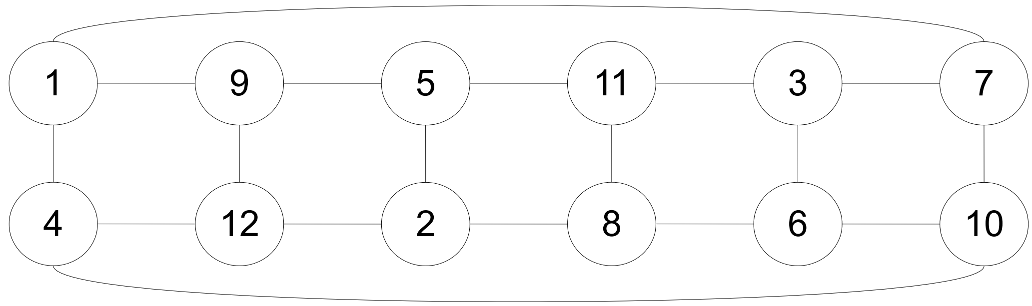

Figure 4.15: $NS[T_{2\times 6}] = 27$

$NS^{sp}[T_{2\times q}] = 0$, we know from Corollary 3.59 that $f(N[v_{1,2}]) = f(v_{1,1}) + f(v_{1,2}) + f(v_{1,3}) + f(v_{2,2}) = 4q+2$. But this implies that $f(v_{2,1}) = f(v_{1,3})$, which is a contradiction. Therefore, $NS^{sp}[T_{2\times q}] \geq 2$. $\square$

Lemma 4.68. *If $q \geq 4$, then $NS^{-}[T_{2\times q}] \leq 4q+1$ and $NS[T_{2\times q}] \geq 4q+3$.*

Proof. Since from Lemma 4.67 we know that $NS^{sp}[T_{2\times q}] \geq 2$, we have that $NS^{-}[T_{2\times q}] < 4q+2 < NS[T_{2\times q}]$. Therefore, $NS^{-}[T_{2\times q}] \leq 4q+1$ and $NS[T_{2\times q}] \geq 4q+3$. $\square$

We now need to handle the special case where $q = 6$.

Theorem 4.69. *$NS^{-}[T_{2\times 6}] = 25$, $NS[T_{2\times 6}] = 27$.*

Proof. From Lemma 4.68 we have that $NS[T_{2\times 6}] \geq 4q+3 = 27$. The bijection demonstrated in Figure 4.15 demonstrates that $NS[T_{2\times 6}] \leq 27$. Therefore $NS[T_{2\times 6}] = 27$. It then follows from Corollary 3.80 that $NS^{-}[T_{2\times 6}] = 8q+4-NS[T_{2\times 6}] = 4q+1 = 25$. $\square$

We now handle all cases when $q \notin \{3,6\}$ in the following series of theorems.

Theorem 4.70. *If $q \equiv 0 (Mod\, 4)$, then $NS^{-}[T_{2\times q}] = 4q+1$, $NS[T_{2\times q}] = 4q+3$, and $NS^{sp}[T_{2\times q}] = 2$.*

Proof. If we can demonstrate a bijection $f : V(T_{2\times q}) \to [2q]$ such that $NS^{-}[f] = 4q+1$, $NS[f] = 4q+3$, and $NS^{sp}[T_{2\times q}] = 2$, then the result will follow from Lemmas 4.67 and 4.68. For $i \in \{1,2,\ldots,q\}$ define the bijection $f : V(T_{2\times q}) \to [2q]$ as follows:

(i) If $i = 1$, then set $f(v_{1,1}) = 1$ and $f(v_{2,1}) = \frac{q}{2}+1$.

(ii) If $i \equiv 1(\text{Mod}\,4)$ and $1 < i \leq \frac{q}{2}+1$, then set $f(v_{1,i}) = i-1$ and $f(v_{2,i}) = \frac{q}{2}+i-1$.

(iii) If $i \equiv 3(\text{Mod}\,4)$ and $1 < i \leq \frac{q}{2}+1$, then set $f(v_{1,i}) = \frac{q}{2}+i-1$ and $f(v_{2,i}) = i-1$.

(iv) If $i \equiv 1(\text{Mod}\,4)$ and $i > \frac{q}{2}+1$, then set $f(v_{1,i}) = q-i+2$ and $f(v_{2,i}) = \frac{3q}{2}-i+2$.

(v) If $i \equiv 3(\text{Mod}\,4)$ and $i > \frac{q}{2}+1$, then set $f(v_{1,i}) = \frac{3q}{2}-i+2$ and $f(v_{2,i}) = q-i+2$.

(vi) If $i \equiv 2(\text{Mod}\,4)$ and $i \leq \frac{q}{2}$, then set $f(v_{1,i}) = \frac{3q}{2}-i+2$ and $f(v_{2,i}) = 2q-i+2$.

(vii) If $i \equiv 0(\text{Mod}\,4)$ and $i \leq \frac{q}{2}$, then set $f(v_{1,i}) = 2q-i+2$ and $f(v_{2,i}) = \frac{3q}{2}-i+2$.

(viii) If $i \equiv 2(\text{Mod}\,4)$ and $i > \frac{q}{2}$, then set $f(v_{1,i}) = \frac{q}{2}+i-1$ and $f(v_{2,i}) = q+i-1$,

(ix) Else if $i \equiv 0(\text{Mod}\,4)$ and $i > \frac{q}{2}$, then set $f(v_{1,i}) = q+i-1$ and $f(v_{2,i}) = \frac{q}{2}+i-1$.

Notice that if $i \equiv 0,3(\text{Mod}\,4)$, then $f(v_{1,i}) - f(v_{2,i}) = \frac{q}{2}$; and if $i \equiv 1,2(\text{Mod}\,4)$, then $f(v_{1,i}) - f(v_{2,i}) = \frac{-q}{2}$. Thus for all $i \in \{1,2,\ldots q\}$ we have that $f(N[v_{1,i}]) = f(N[v_{2,i}])$. We now consider the values of $f(N[v_{1,i}])$ for all $i \in \{1,2,\ldots,q\}$ on a case by case basis.

CASE A ($i \equiv 1(\text{Mod}\,4)$)

CASE A.1 ($i = 1$): $f(N[v_{1,1}]) = f(v_{1,q}) + f(v_{1,1}) + f(v_{1,2}) + f(v_{2,1}) = (q+q-1) + 1 + \left(\frac{3q}{2}-2+2\right) + \left(\frac{q}{2}+1\right) = 4q+1$.

CASE A.2 $\left(1 < i \leq \frac{q}{2}\right)$: $f(N[v_{1,i}]) = f(v_{1,i-1}) + f(v_{1,i}) + f(v_{1,i+1}) + f(v_{2,i}) = (2q-(i-1)+2) + (i-1) + \left(\frac{3q}{2}-(i+1)+2\right) + \left(\frac{q}{2}+i-1\right) = 4q+2$.

CASE A.3 $\left(i = \frac{q}{2}+1\right)$: $f(N[v_{1,i}]) = f(v_{1,i-1}) + f(v_{1,i}) + f(v_{1,i+1}) + f(v_{2,i}) = (2q-(i-1)+2) + (i-1) + \left(\frac{q}{2}+(i+1)-1\right) + \left(\frac{q}{2}+i-1\right) = 3q+2i+1 = 4q+3$.

CASE A.4 $\left(i > \frac{q}{2}+1\right)$: $f(N[v_{1,i}]) = f(v_{1,i-1}) + f(v_{1,i}) + f(v_{1,i+1}) + f(v_{2,i}) = (q+(i-1)-1)+(q-i+2)+\left(\frac{q}{2}+(i+1)-1\right)+\left(\frac{3q}{2}-i+2\right) = 4q+2.$

CASE B $(i \equiv 2(\mathrm{Mod}\,4))$

CASE B.1 $(i=2)$: $f(N[v_{1,2}]) = f(v_{1,1}) + f(v_{1,2}) + f(v_{1,3}) + f(v_{2,2}) = 1+\left(\frac{3q}{2}-2+2\right)+\left(\frac{q}{2}+3-1\right)+(2q-2+2) = 4q+3.$

CASE B.2 $\left(2 < i \leq \frac{q}{2}\right)$: $f(N[v_{1,i}]) = f(v_{1,i-1}) + f(v_{1,i}) + f(v_{1,i+1}) + f(v_{2,i}) = ((i-1)-1)+\left(\frac{3q}{2}-i+2\right)+\left(\frac{q}{2}+(i+1)-1\right)+(2q-i+2) = 4q+2.$

CASE B.3 $\left(i = \frac{q}{2}+2\right)$: $f(N[v_{1,i}]) = f(v_{1,i-1}) + f(v_{1,i}) + f(v_{1,i+1}) + f(v_{2,i}) = ((i-1)-1)+\left(\frac{q}{2}+i-1\right)+\left(\frac{3q}{2}-(i+1)+2\right)+(q+i-1) = 3q+2i-3 = 4q+1.$

CASE B.4 $\left(i > \frac{q}{2}+2\right)$: $f(N[v_{1,i}]) = f(v_{1,i-1}) + f(v_{1,i}) + f(v_{1,i+1}) + f(v_{2,i}) = (q-(i-1)+2)+\left(\frac{q}{2}+i-1\right)+\left(\frac{3q}{2}-(i+1)+2\right)+(q+i-1) = 4q+2.$

CASE C $(i \equiv 3(\mathrm{Mod}\,4))$

CASE C.1 $\left(i \leq \frac{q}{2}-1\right)$: $f(N[v_{1,i}]) = f(v_{1,i-1}) + f(v_{1,i}) + f(v_{1,i+1}) + f(v_{2,i}) = \left(\frac{3q}{2}-(i-1)+2\right)+\left(\frac{q}{2}+i-1\right)+(2q-(i+1)+2)+(i-1) = 4q+2.$

CASE C.2 $\left(i = \frac{q}{2}+1\right)$: $f(N[v_{1,i}]) = f(v_{1,i-1}) + f(v_{1,i}) + f(v_{1,i+1}) + f(v_{2,i}) = \left(\frac{3q}{2}-(i-1)+2\right)+\left(\frac{q}{2}+i-1\right)+(q+(i+1)-1)+(i-1) = 3q+2i+1 = 4q+3.$

CASE C.3 $\left(i > \frac{q}{2}+1\right)$: $f(N[v_{1,i}]) = f(v_{1,i-1}) + f(v_{1,i}) + f(v_{1,i+1}) + f(v_{2,i}) = \left(\frac{q}{2}+(i-1)-1\right)+\left(\frac{3q}{2}-i+2\right)+(q+(i+1)-1)+(q-i+2) = 4q+2.$

CASE D $(i \equiv 0(\mathrm{Mod}\,4))$

CASE D.1 $\left(i \leq \frac{q}{2}\right)$: $f(N[v_{1,i}]) = f(v_{1,i-1}) + f(v_{1,i}) + f(v_{1,i+1}) + f(v_{2,i}) = \left(\frac{q}{2}+(i-1)-1\right)+(2q-i+2)+((i+1)-1)+\left(\frac{3q}{2}-i+2\right) = 4q+2.$

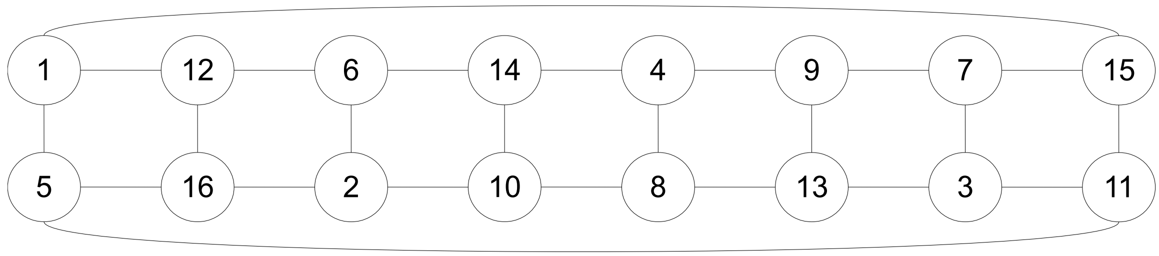

Figure 4.16: $NS^{-}[T_{2\times 8}] = 33$, $NS[T_{2\times 8}] = 35$, and $NS^{sp}[T_{2\times 8}] = 2$

CASE D.2 $\left(i = \frac{q}{2}+2\right)$: $f(N[v_{1,i}]) = f(v_{1,i-1}) + f(v_{1,i}) + f(v_{1,i+1}) + f(v_{2,i}) = \left(\frac{q}{2}+(i-1)-1\right)+(q+i-1)+(q-(i+1)+2)+\left(\frac{q}{2}+i-1\right) = 3q+2i-2 = 4q+1.$

CASE D.3 $\left(i > \frac{q}{2}+2\right)$: $f(N[v_{1,i}]) = f(v_{1,i-1}) + f(v_{1,i}) + f(v_{1,i+1}) + f(v_{2,i}) = \left(\frac{3q}{2}-(i-1)+2\right)+(q+i-1)+(q-(i+1)+2)+\left(\frac{q}{2}+i-1\right) = 4q+2.$

For the bijection f, we have that $NS^{-}[f] = 4q+1$, $NS[f] = 4q+3$, and $NS^{sp}[f] = 2$ which proves the result. □

Example 4.71. The bijection shown in Figure 4.16 demonstrates that $NS^{-}[T_{2\times 8}] = 4q+1 = 33$, $NS[T_{2\times 8}] = 4q+3 = 35$, and $NS^{sp}[T_{2\times q}] = 2$.

Theorem 4.72. *If $q \equiv 1 (Mod\, 4)$, then $NS^{-}[T_{2\times q}] = 4q+1$ and $NS[T_{2\times q}] = 4q+3$.*

Proof. From Lemma 4.68 we know that $NS[T_{2\times q}] \geq 4q+3$. If we can demonstrate a bijection $f : V(T_{2\times q}) \to [2q]$ such that $NS[f] = 4q+3$, then it will follow that $NS[T_{2\times q}] = 4q+3$. From Corollary 3.80 we will then have that $NS^{-}[T_{2\times q}] = 8q+4-NS[T_{2\times q}] = 4q+1$. For $i \in \{1,2,\ldots,q\}$ define the bijection $f : V(T_{2\times q}) \to [2q]$ as follows:

(i) If $i = 1$, then set $f(v_{1,1}) = 1$ and $f(v_{2,1}) = 2$.

(ii) If $i = 2$, then set $f(v_{1,2}) = 2q$ and $f(v_{2,2}) = 2q-2$.

(iii) If $i = 3$, then set $f(v_{1,3}) = 4$ and $f(v_{2,3}) = 3$.

(iv) If $i \equiv 1(\text{Mod}\,4)$ and $5 \leq i < q$, then set $f(v_{1,i}) = 2i-3$ and $f(v_{2,i}) = 2i-2$.

(v) If $i \equiv 3(\text{Mod}\,4)$ and $5 \leq i < q$, then set $f(v_{1,i}) = 2i-2$ and $f(v_{2,i}) = 2i-3$.

(vi) If $i \equiv 1(\text{Mod}\,4)$ and $i = q$, then set $f(v_{1,i}) = 2q-3$ and $f(v_{2,i}) = 2q-1$.

(vii) If $i \equiv 2(\text{Mod}\,4)$ and $2 < i$, then set $f(v_{1,i}) = 2q-2i+3$ and $f(v_{2,i}) = 2q-2i+4$.

(viii) Else if $i \equiv 0(\text{Mod}\,4)$ and $2 < i$, then set $f(v_{1,i}) = 2q-2i+4$ and $f(v_{2,i}) = 2q-2i+3$.

Notice that if $i \equiv 0,3(\text{Mod}\,4)$, then $f(v_{1,i}) - f(v_{2,i}) = 1$. If $i \notin \{2,q\}$ and if $i \equiv 1,2(\text{Mod}\,4)$, then $f(v_{1,i}) - f(v_{2,i}) = -1$. Thus for all $i \in \{2,4,5,\ldots q-2,q\}$ we have that $f(N[v_{1,i}]) = f(N[v_{2,i}])$. We now consider the values of $f(N[v_{1,i}])$ for all $i \in \{1,2,\ldots,q\}$ on a case by case basis.

CASE A ($i \equiv 1(\text{Mod}\,4)$)

CASE A.1 ($i = 1$):

$f(N[v_{1,1}]) = f(v_{1,q}) + f(v_{1,1}) + f(v_{1,2}) + f(v_{2,1}) = (2q-3)+1+(2q)+2 = 4q$.

$f(N[v_{2,1}]) = f(v_{2,q}) + f(v_{2,1}) + f(v_{2,2}) + f(v_{1,1}) = (2q-1)+2+(2q-2)+1 = 4q$.

CASE A.2 ($1 < i < q$): $f(N[v_{1,i}]) = f(v_{1,i-1}) + f(v_{1,i}) + f(v_{1,i+1}) + f(v_{2,i}) = (2q-2(i-1)+4)+(2i-3)+(2q-2(i+1)+3)+(2i-2) = 4q+2$.

CASE A.3 ($i = q$): $f(N[v_{1,q}]) = f(v_{1,q-1}) + f(v_{1,q}) + f(v_{1,1}) + f(v_{2,q}) = (2q-2(q-1)+4)+(2q-3)+1+(2q-1) = 4q+3$.

CASE B ($i \equiv 2(\text{Mod}\,4)$)

CASE B.1 $(i=2)$: $f(N[v_{1,2}]) = f(v_{1,1}) + f(v_{1,2}) + f(v_{1,3}) + f(v_{2,2}) = 1+2q+4+(2q-2) = 4q+3$.

CASE B.2 $(2<i)$: $f(N[v_{1,i}]) = f(v_{1,i-1}) + f(v_{1,i}) + f(v_{1,i+1}) + f(v_{2,i}) = (2(i-1)-3)+(2q-2i+3)+(2(i+1)-2)+(2q-2i+4) = 4q+2$.

CASE C $(i \equiv 3(\text{Mod}\,4))$

CASE C.1 $(i=3)$:

$f(N[v_{1,3}]) = f(v_{1,2})+f(v_{1,3})+f(v_{1,4})+f(v_{2,3}) = 2q+4+(2q-8+4)+3 = 4q+3$.

$f(N[v_{2,3}]) = f(v_{2,2})+f(v_{2,3})+f(v_{2,4})+f(v_{1,3}) = (2q-2)+3+(2q-8+3)+4 = 4q$.

CASE C.2 $(3<i)$: $f(N[v_{1,i}]) = f(v_{1,i-1}) + f(v_{1,i}) + f(v_{1,i+1}) + f(v_{2,i}) = (2q-2(i-1)+3)+(2i-2)+(2q-2(i+1)+4)+(2i-3) = 4q+2$.

CASE D $i \equiv 0(\text{Mod}\,4)$

CASE D.1 $(i=4$ and $q \neq 5)$: $f(N[v_{1,4}]) = f(v_{1,3})+f(v_{1,4})+f(v_{1,5})+f(v_{2,4}) = 4+(2q-8+4)+(2(4+1)-3)+(2q-8+3) = 4q+2$.

CASE D.2 $(4<i<q-1)$: $f(N[v_{1,i}]) = f(v_{1,i-1})+f(v_{1,i})+f(v_{1,i+1})+f(v_{2,i}) = (2(i-1)-2)+(2q-2i+4)+(2(i+1)-3)+(2q-2i+3) = 4q+2$.

CASE D.3 $(i=q-1)$:

$f(N[v_{1,q-1}]) = f(v_{1,q-2}) + f(v_{1,q-1}) + f(v_{1,q}) + f(v_{2,q-1}) = (2(q-2)-2)+(2q-2(q-1)+4)+(2q-3)+(2q-2(q-1)+3) = 4q+2$.

$f(N[v_{2,q-1}]) = f(v_{2,q-2}) + f(v_{2,q-1}) + f(v_{2,q}) + f(v_{1,q-1}) = (2(q-2)-3)+(2q-2(q-1)+4)+(2q-1)+(2q-2(q-1)+3) = 4q+3$.

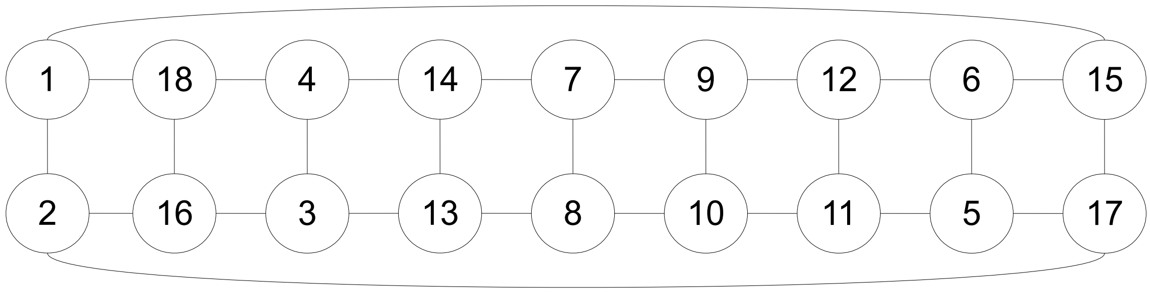

Figure 4.17: $NS[T_{2\times 9}] = 39$

For the bijection f we have that $NS[f] = 4q+3$ which proves the result. $\square$

Example 4.73. The bijection shown in Figure 4.17 demonstrates that $NS[T_{2\times 9}] = 4q+3 = 39$.

Theorem 4.74. *If* $q \equiv 2(Mod 4)$ *and* $q > 6$*, then* $NS^{-}[T_{2\times q}] = 4q+1$ *and* $NS[T_{2\times q}] = 4q+3$.

Proof. From Lemma 4.68 we know that $NS[T_{2\times q}] \geq 4q+3$. If we can demonstrate a bijection $f : V(T_{2\times q}) \to [2q]$ such that $NS[f] = 4q+3$, then it will follow that $NS[T_{2\times q}] = 4q+3$. From Corollary 3.80 we will then have that $NS^{-}[T_{2\times q}] = 8q+4-NS[T_{2\times q}] = 4q+1$. For $i \in \{1,2,\ldots,q\}$ define the bijection $f : V(T_{2\times q}) \to [2q]$ as follows:

(i) If $i = 1$, then set $f(v_{1,1}) = 1$ and $f(v_{2,1}) = 2$.

(ii) If $i = 2$, then set $f(v_{1,2}) = 2q-2$ and $f(v_{2,2}) = 2q$.

(iii) If $i = 3$, then set $f(v_{1,3}) = 4$ and $f(v_{2,3}) = 3$.

(iv) If $i \equiv 1(\text{Mod}\,4)$ and $5 \leq i < \frac{q}{2}$, then set $f(v_{1,i}) = 2i-3$ and $f(v_{2,i}) = 2i-2$.

(v) If $i \equiv 3(\text{Mod}\,4)$ and $5 \leq i < \frac{q}{2}$, then set $f(v_{1,i}) = 2i-2$ and $f(v_{2,i}) = 2i-3$.

(vi) If $i \equiv 1(\text{Mod}\,4)$ and $i = \frac{q}{2}$, then set $f(v_{1,\frac{q}{2}}) = q-3$ and $f(v_{2,\frac{q}{2}}) = q-1$.

(vii) If $i \equiv 3(\text{Mod}\,4)$ and $i = \frac{q}{2}$, then set $f(v_{1,\frac{q}{2}}) = q-1$ and $f(v_{2,\frac{q}{2}}) = q-3$.

(viii) If $i \equiv 1(\text{Mod}\,4)$ and $i = \frac{q}{2}+2$, then set $f(v_{1,\frac{q}{2}+1}) = q-2$ and $f(v_{2,\frac{q}{2}+1}) = q$.

(ix) If $i \equiv 3(\text{Mod}\,4)$ and $i = \frac{q}{2}+2$, then set $f(v_{1,\frac{q}{2}+1}) = q$ and $f(v_{2,\frac{q}{2}+1}) = q-2$.

(x) If $i \equiv 1(\text{Mod}\,4)$ and $i > \frac{q}{2}+2$, then set $f(v_{1,i}) = 2q-2i+4$ and $f(v_{2,i}) = 2q-2i+3$.

(xi) If $i \equiv 3(\text{Mod}\,4)$ and $i > \frac{q}{2}+2$, then set $f(v_{1,i}) = 2q-2i+3$ and $f(v_{2,i}) = 2q-2i+4$.

(xii) If $i \equiv 2(\text{Mod}\,4)$ and $4 \leq i \leq \frac{q}{2}+1$, then set $f(v_{1,i}) = 2q-2i+4$ and $f(v_{2,i}) = 2q-2i+3$.

(xiii) If $i \equiv 0(\text{Mod}\,4)$ and $4 \leq i \leq \frac{q}{2}+1$, then set $f(v_{1,i}) = 2q-2i+3$ and $f(v_{2,i}) = 2q-2i+4$.

(xiv) If $i \equiv 2(\text{Mod}\,4)$ and $\frac{q}{2}+1 < i < q$, then set $f(v_{1,i}) = 2i-2$ and $f(v_{2,i}) = 2i-3$.

(xv) If $i \equiv 0(\text{Mod}\,4)$ and $i > \frac{q}{2}+1$, then set $f(v_{1,i}) = 2i-3$ and $f(v_{2,i}) = 2i-2$.

(xvi) Else if $i = q$, then set $f(v_{1,q}) = 2q-1$ and $f(v_{2,q}) = 2q-3$.

We now consider the values of $f(N[v_{1,i}])$ for all $i \in \{1,2,\ldots,q\}$ on a case by case basis.

CASE A $(i \equiv 1(\text{Mod}\,4))$

CASE A.1 $(i = 1)$:

$f(N[v_{1,1}]) = f(v_{1,q}) + f(v_{1,1}) + f(v_{1,2}) + f(v_{2,1}) = (2q-1)+1+(2q-2)+2 = 4q$.

$f(N[v_{2,1}]) = f(v_{2,q}) + f(v_{2,1}) + f(v_{2,2}) + f(v_{1,1}) = (2q-3)+2+2q+1 = 4q$.

CASE A.2 $\left(1 < i \leq \frac{q}{2}-2\right)$:

$f(N[v_{1,i}]) = f(v_{1,i-1}) + f(v_{1,i}) + f(v_{1,i+1}) + f(v_{2,i}) = (2q-2(i-1)+3)+(2i-3)+(2q-2(i+1)+4)+(2i-2) = 4q+2$.

$f(N[v_{2,i}]) = f(v_{2,i-1}) + f(v_{2,i}) + f(v_{2,i+1}) + f(v_{1,i}) = (2q - 2(i-1) + 4) + (2i-2) + (2q - 2(i+1) + 3) + (2i-3) = 4q + 2.$

CASE A.3 $\left(i = \frac{q}{2}\right)$:

$f(N[v_{1,i}]) = f(v_{1,i-1}) + f(v_{1,i}) + f(v_{1,i+1}) + f(v_{2,i}) = (2q - 2(i-1) + 3) + (q-3) + (2q - 2(i+1) + 4) + (q-1) = 6q - 4i + 3 = 4q + 3.$

$f(N[v_{2,i}]) = f(v_{2,i-1}) + f(v_{2,i}) + f(v_{2,i+1}) + f(v_{1,i}) = (2q - 2(i-1) + 4) + (q-1) + (2q - 2(i+1) + 3) + (q-3) = 6q - 4i + 3 = 4q + 3.$

CASE A.4 $\left(i = \frac{q}{2} + 2\right)$:

$f(N[v_{1,i}]) = f(v_{1,i-1}) + f(v_{1,i}) + f(v_{1,i+1}) + f(v_{2,i}) = (2q - 2(i-1) + 3) + (q-2) + (2(i+1) - 2) + q = 4q + 3.$

$f(N[v_{2,i}]) = f(v_{2,i-1}) + f(v_{2,i}) + f(v_{2,i+1}) + f(v_{1,i}) = (2q - 2(i-1) + 4) + q + (2(i+1) - 3) + (q-2) = 4q + 3.$

CASE A.5 $\left(\frac{q}{2} + 2 < i < q - 1\right)$:

$f(N[v_{1,i}]) = f(v_{1,i-1}) + f(v_{1,i}) + f(v_{1,i+1}) + f(v_{2,i}) = (2(i-1) - 3) + (2q - 2i + 4) + (2(i+1) - 2) + (2q - 2i + 3) = 4q + 2.$

$f(N[v_{2,i}]) = f(v_{2,i-1}) + f(v_{2,i}) + f(v_{2,i+1}) + f(v_{1,i}) = (2(i-1) - 2) + (2q - 2i + 3) + (2(i+1) - 3) + (2q - 2i + 4) = 4q + 2.$

CASE A.6 $(i = q - 1)$:

$f(N[v_{1,q-1}]) = f(v_{1,q-2}) + f(v_{1,q-1}) + f(v_{1,q}) + f(v_{2,q-1}) = (2(q-2) - 3) + (2q - 2(q-1) + 4) + (2q-1) + (2q - 2(q-1) + 3) = 4q + 3.$

$f(N[v_{2,i}]) = f(v_{2,i-1}) + f(v_{2,i}) + f(v_{2,i+1}) + f(v_{1,i}) = (2(q-2) - 2) + (2q - 2(q-1) + 3) + (2q-3) + (2q - 2(q-1) + 4) = 4q + 2.$

CASE B $(i \equiv 2(\text{Mod}\,4))$

CASE B.1 $(i = 2)$:

$f(N[v_{1,2}]) = f(v_{1,1}) + f(v_{1,2}) + f(v_{1,3}) + f(v_{2,2}) = 1 + (2q-2) + 4 + 2q = 4q + 3.$

$f(N[v_{2,2}]) = f(v_{2,1}) + f(v_{2,2}) + f(v_{2,3}) + f(v_{1,2}) = 2 + 2q + 3 + (2q-2) = 4q + 3.$

CASE B.2 $\left(2 < i < \frac{q}{2} - 1\right)$:

$f(N[v_{1,i}]) = f(v_{1,i-1}) + f(v_{1,i}) + f(v_{1,i+1}) + f(v_{2,i}) = (2(i-1)-3) + (2q-2i+4) + (2(i+1)-2) + (2q-2i+3) = 4q + 2.$

$f(N[v_{2,i}]) = f(v_{2,i-1}) + f(v_{2,i}) + f(v_{2,i+1}) + f(v_{1,i}) = (2(i-1)-2) + (2q-2i+3) + (2(i+1)-3) + (2q-2i+4) = 4q + 2.$

CASE B.3 $\left(i = \frac{q}{2} - 1\right)$:

$f(N[v_{1,i}]) = f(v_{1,i-1}) + f(v_{1,i}) + f(v_{1,i+1}) + f(v_{2,i}) = (2(i-1)-3) + (2q-2i+4) + (q-1) + (2q-2i+3) = 5q - 2i + 2 = 4q + 3.$

$f(N[v_{2,i}]) = f(v_{2,i-1}) + f(v_{2,i}) + f(v_{2,i+1}) + f(v_{1,i}) = (2(i-1)-2) + (2q-2i+3) + (q-3) + (2q-2i+4) = 5q - 2i + 1 = 4q + 2.$

CASE B.3 $\left(i = \frac{q}{2} + 1\right)$:

$f(N[v_{1,i}]) = f(v_{1,i-1}) + f(v_{1,i}) + f(v_{1,i+1}) + f(v_{2,i}) = (q-3) + (2q-2i+4) + q + (2q-2i+3) = 6q - 4i + 4 = 4q.$

$f(N[v_{2,i}]) = f(v_{2,i-1}) + f(v_{2,i}) + f(v_{2,i+1}) + f(v_{1,i}) = (q-1) + (2q-2i+3) + (q-2) + (2q-2i+4) = 6q - 4i + 4 = 4q.$

CASE B.5 $\left(i = \frac{q}{2} + 3\right)$:

$f(N[v_{1,i}]) = f(v_{1,i-1}) + f(v_{1,i}) + f(v_{1,i+1}) + f(v_{2,i}) = (q-2) + (2i-2) + (2q-2(i+1)+3) + (2i-3) = 3q + 2i - 6 = 4q.$

$f(N[v_{2,i}]) = f(v_{2,i-1}) + f(v_{2,i}) + f(v_{2,i+1}) + f(v_{1,i}) = q + (2i-3) + (2q-2(i+1)+4) + (2i-2) = 3q+2i-3 = 4q+3.$

CASE B.6 $\left(\frac{q}{2}+3 < i < q\right)$:

$f(N[v_{1,i}]) = f(v_{1,i-1}) + f(v_{1,i}) + f(v_{1,i+1}) + f(v_{2,i}) = (2q-2(i-1)+4) + (2i-2) + (2q-2(i+1)+3) + (2i-3) = 4q+2.$

$f(N[v_{2,i}]) = f(v_{2,i-1}) + f(v_{2,i}) + f(v_{2,i+1}) + f(v_{1,i}) = (2q-2(i-1)+3) + (2i-3) + (2q-2(i+1)+4) + (2i-2) = 4q+2.$

CASE B.7 $(i = q)$:

$f(N[v_{1,q}]) = f(v_{1,q-1}) + f(v_{1,q}) + f(v_{1,1}) + f(v_{2,q}) = (2q-2(q-1)+4) + (2q-1) + 1 + (2q-3) = 4q+3.$

$f(N[v_{2,q}]) = f(v_{2,q-1}) + f(v_{2,q}) + f(v_{2,1}) + f(v_{1,q}) = (2q-2(q-1)+3) + (2q-3) + 2 + (2q-1) = 4q+3.$

CASE C $(i \equiv 3 (\mathrm{Mod}\, 4))$

CASE C.1 $(i = 3)$:

$f(N[v_{1,3}]) = f(v_{1,2}) + f(v_{1,3}) + f(v_{1,4}) + f(v_{2,3}) = (2q-2) + 4 + (2q-8+3) + 3 = 4q.$

$f(N[v_{2,3}]) = f(v_{2,2}) + f(v_{2,3}) + f(v_{2,4}) + f(v_{1,3}) = 2q+3+(2q-8+4)+4 = 4q+3.$

CASE C.2 $\left(3 < i < \frac{q}{2}-2\right)$:

$f(N[v_{1,i}]) = f(v_{1,i-1}) + f(v_{1,i}) + f(v_{1,i+1}) + f(v_{2,i}) = (2q-2(i-1)+4) + (2i-2) + (2q-2(i+1)+3) + (2i-3) = 4q+2.$

$f(N[v_{2,i}]) = f(v_{2,i-1}) + f(v_{2,i}) + f(v_{2,i+1}) + f(v_{1,i}) = (2q-2(i-1)+3) + (2i-3) + (2q-2(i+1)+4) + (2i-2) = 4q+2.$

CASE C.3 $\left(i = \frac{q}{2}\right)$:

$f(N[v_{1,i}]) = f(v_{1,i-1}) + f(v_{1,i}) + f(v_{1,i+1}) + f(v_{2,i}) = (2q-2(i-1)+4) + (q-1) + (2q-2(i+1)+3) + (q-3) = 6q-4i+3 = 4q+3.$

$f(N[v_{2,i}]) = f(v_{2,i-1}) + f(v_{2,i}) + f(v_{2,i+1}) + f(v_{1,i}) = (2q-2(i-1)+3) + (q-3) + (2q-2(i+1)+4) + (q-1) = 6q-4i+3 = 4q+3.$

CASE C.4 $\left(i = \frac{q}{2}+2\right)$:

$f(N[v_{1,i}]) = f(v_{1,i-1}) + f(v_{1,i}) + f(v_{1,i+1}) + f(v_{2,i}) = (2q-2(i-1)+4) + q + (2(i+1)-3) + (q-2) = 4q+3.$

$f(N[v_{2,i}]) = f(v_{2,i-1}) + f(v_{2,i}) + f(v_{2,i+1}) + f(v_{1,i}) = (2q-2(i-1)+3) + (q-2) + (2(i+1)-2) + q = 4q+3.$

CASE C.5 $\left(i > \frac{q}{2}+2\right)$:

$f(N[v_{1,i}]) = f(v_{1,i-1}) + f(v_{1,i}) + f(v_{1,i+1}) + f(v_{2,i}) = (2(i-1)-2) + (2q-2i+3) + (2(i+1)-3) + (2q-2i+4) = 4q+2.$

$f(N[v_{2,i}]) = f(v_{2,i-1}) + f(v_{2,i}) + f(v_{2,i+1}) + f(v_{1,i}) = (2(i-1)-3) + (2q-2i+4) + (2(i+1)-2) + (2q-2i+3) = 4q+2.$

CASE D $(i \equiv 0 (\mathrm{Mod}\, 4))$

CASE D.1 $(i = 4)$:

$f(N[v_{1,4}] = f(v_{1,3}) + f(v_{1,4}) + f(v_{1,5}) + f(v_{2,4}) = 4 + (2q-8+3) + (10-3) + (2q-8+4) = 4q+2.$

$f(N[v_{2,4}]) = f(v_{2,3}) + f(v_{2,4}) + f(v_{2,5}) + f(v_{1,4}) = 3 + (2q-8+4) + (10-2) + (2q-8+3) = 4q+2.$

CASE D.2 $\left(4 < i < \frac{q}{2} - 1\right)$:

$f(N[v_{1,i}]) = f(v_{1,i-1}) + f(v_{1,i}) + f(v_{1,i+1}) + f(v_{2,i}) = (2(i-1)-2) + (2q-2i+3) + (2(i+1)-3) + (2q-2i+4) = 4q+2.$

$f(N[v_{2,i}]) = f(v_{2,i-1}) + f(v_{2,i}) + f(v_{2,i+1}) + f(v_{1,i}) = (2(i-1)-3) + (2q-2i+4) + (2(i+1)-2) + (2q-2i+3) = 4q+2.$

CASE D.3 $\left(i = \frac{q}{2} - 1\right)$:

$f(N[v_{1,i}]) = f(v_{1,i-1}) + f(v_{1,i}) + f(v_{1,i+1}) + f(v_{2,i}) = (2(i-1)-2) + (2q-2i+3) + (q-3) + (2q-2i+4) = 5q-2i = 4q+2.$

$f(N[v_{2,i}]) = f(v_{2,i-1}) + f(v_{2,i}) + f(v_{2,i+1}) + f(v_{1,i}) = (2(i-1)-3) + (2q-2i+4) + (q-1) + (2q-2i+3) = 5q-2i+1 = 4q+3.$

CASE D.4 $\left(i = \frac{q}{2} + 1\right)$:

$f(N[v_{1,i}]) = f(v_{1,i-1}) + f(v_{1,i}) + f(v_{1,i+1}) + f(v_{2,i}) = (q-1) + (2q-2i+3) + (q-2) + (2q-2i+4) = 6q-4i+4 = 4q.$

$f(N[v_{2,i}]) = f(v_{2,i-1}) + f(v_{2,i}) + f(v_{2,i+1}) + f(v_{1,i}) = (q-3) + (2q-2i+4) + q + (2q-2i+3) = 6q-4i+4 = 4q.$

CASE D.5 $\left(i = \frac{q}{2} + 3\right)$:

$f(N[v_{1,i}]) = f(v_{1,i-1}) + f(v_{1,i}) + f(v_{1,i+1}) + f(v_{2,i}) = q + (2i-3) + (2q-2(i+1)+4) + (2i-2) = 3q+2i-3 = 4q+3.$

$f(N[v_{2,i}]) = f(v_{2,i-1}) + f(v_{2,i}) + f(v_{2,i+1}) + f(v_{1,i}) = (q-2) + (2i-2) + (2q-2(i+1)+3) + (2i-3) = 3q+2i-6 = 4q.$

CASE D.6 $\left(i > \frac{q}{2} + 3\right)$:

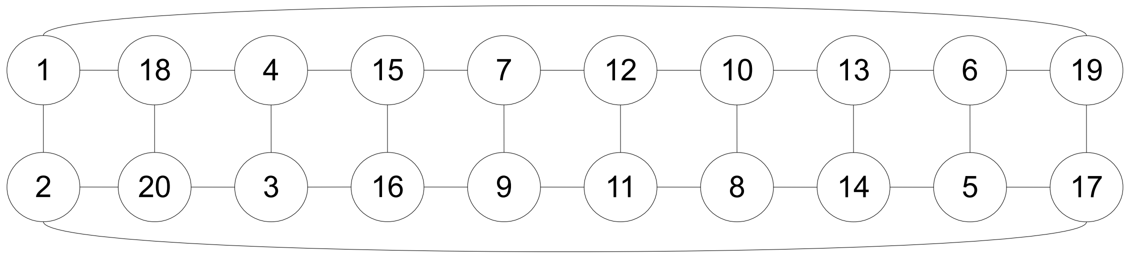

Figure 4.18: $NS[T_{2\times 10}] = 43$

$f(N[v_{1,i}]) = f(v_{1,i-1}) + f(v_{1,i}) + f(v_{1,i+1}) + f(v_{2,i}) = (2q-2(i-1)+3) + (2i-3) + (2q-2(i+1)+4) + (2i-2) = 4q+2.$

$f(N[v_{2,i}]) = f(v_{2,i-1}) + f(v_{2,i}) + f(v_{2,i+1}) + f(v_{1,i}) = (2q-2(i-1)+4) + (2i-2) + (2q-2(i+1)+3) + (2i-3) = 4q+2.$

For the bijection f we have that $NS[f] = 4q+3$ which proves the result. □

Example 4.75. The bijection shown in Figure 4.18 demonstrates that $NS[T_{2\times 10}] = 4q+3 = 43$.

Theorem 4.76. *If $q \equiv 3(Mod 4)$ and $q > 3$, then $NS^{-}[T_{2\times q}] = 4q+1$ and $NS[T_{2\times q}] = 4q+3$.*

Proof. From Lemma 4.68 we know that $NS[T_{2\times q}] \geq 4q+3$. If we can demonstrate a bijection $f : V(T_{2\times q}) \to [2q]$ such that $NS[f] = 4q+3$, then it will follow that $NS[T_{2\times q}] = 4q+3$. From Corollary 3.80 we will then have that $NS^{-}[T_{2\times q}] = 8q+4-NS[T_{2\times q}] = 4q+1$. For $i \in \{1,2,\ldots,q\}$ define the bijection $f : V(T_{2\times q}) \to [2q]$ as follows:

(i) If $i = 1$, then set $f(v_{1,1}) = 1$ and $f(v_{2,1}) = 2$.

(ii) If $i = 2$, then set $f(v_{1,2}) = 2q-2$ and $f(v_{2,2}) = 2q$.

(iii) If $i = 3$, then set $f(v_{1,3}) = 4$ and $f(v_{2,3}) = 3$.

(iv) If $i \equiv 1(\text{Mod}\,4)$ and $5 \leq i < q$, then set $f(v_{1,i}) = 2i-3$ and $f(v_{2,i}) = 2i-2$.

(v) If $i \equiv 3(\mathrm{Mod}\,4)$ and $5 \leq i < q$, then set $f(v_{1,i}) = 2i-2$ and $f(v_{2,i}) = 2i-3$.

(vi) If $i \equiv 3(\mathrm{Mod}\,4)$ and $i = q$, then set $f(v_{1,i}) = 2q-1$ and $f(v_{2,i}) = 2q-3$.

(vii) If $i \equiv 2(\mathrm{Mod}\,4)$ and $2 < i$, then set $f(v_{1,i}) = 2q-2i+4$ and $f(v_{2,i}) = 2q-2i+3$.

(viii) Else if $i \equiv 0(\mathrm{Mod}\,4)$ and $2 < i$, then set $f(v_{1,i}) = 2q-2i+3$ and $f(v_{2,i}) = 2q-2i+4$.

Notice that if $i \equiv 0,1(\mathrm{Mod}\,4)$, then $f(v_{1,i}) - f(v_{2,i}) = -1$. If $i \notin \{2,q\}$ and if $i \equiv 2,3(\mathrm{Mod}\,4)$, then $f(v_{1,i}) - f(v_{2,i}) = 1$. Thus for all $i \in \{2,4,5,\ldots q-2,q\}$ we have that $f(N[v_{1,i}]) = f(N[v_{2,i}])$. We now consider the values of $f(N[v_{1,i}])$ for all $i \in \{1,2,\ldots,q\}$ on a case by case basis.

CASE A ($i \equiv 1(\mathrm{Mod}\,4)$)

CASE A.1 ($i = 1$):

$f(N[v_{1,1}]) = f(v_{1,q}) + f(v_{1,1}) + f(v_{1,2}) + f(v_{2,1}) = (2q-1)+1+(2q-2)+2 = 4q$.

$f(N[v_{2,1}]) = f(v_{2,q}) + f(v_{2,1}) + f(v_{2,2}) + f(v_{1,1}) = (2q-3)+2+(2q)+1 = 4q$.

CASE A.2 ($1 < i$): $f(N[v_{1,i}]) = f(v_{1,i-1}) + f(v_{1,i}) + f(v_{1,i+1}) + f(v_{2,i}) = (2q-2(i-1)+3)+(2i-3)+(2q-2(i+1)+4)+(2i-2) = 4q+2$.

CASE B ($i \equiv 2(\mathrm{Mod}\,4)$)

CASE B.1 ($i = 2$): $f(N[v_{1,2}]) = f(v_{1,1}) + f(v_{1,2}) + f(v_{1,3}) + f(v_{2,2}) = 1+(2q-2)+4+2q = 4q+3$.

CASE B.2 ($2 < i < q-1$): $f(N[v_{1,i}]) = f(v_{1,i-1}) + f(v_{1,i}) + f(v_{1,i+1}) + f(v_{2,i}) = (2(i-1)-3)+(2q-2i+4)+(2(i+1)-2)+(2q-2i+3) = 4q+2$.

CASE B.3 $(i = q-1)$:

$f(N[v_{1,q-1}]) = f(v_{1,q-2}) + f(v_{1,q-1}) + f(v_{1,q}) + f(v_{2,q-1}) = (2(i-1)-3) + (2q-2i+4) + (2q-1) + (2q-2i+3) = 6q-2i+1 = 4q+3.$

$f(N[v_{2,q-1}]) = f(v_{2,q-2}) + f(v_{2,q-1}) + f(v_{2,q}) + f(v_{1,q-1}) = (2(i-1)-2) + (2q-2i+3) + (2q-3) + (2q-2i+4) = 6q-2i = 4q+2.$

CASE C $(i \equiv 3(\mathrm{Mod}\,4))$

CASE C.1 $(i = 3)$:

$f(N[v_{1,3}]) = f(v_{1,2}) + f(v_{1,3}) + f(v_{1,4}) + f(v_{2,3}) = (2q-2) + 4 + (2q-8+3) + 3 = 4q.$

$f(N[v_{2,3}]) = f(v_{2,2}) + f(v_{2,3}) + f(v_{2,4}) + f(v_{1,3}) = 2q+3+(2q-8+4)+4 = 4q+3.$

CASE C.2 $(3 < i < q)$: $f(N[v_{1,i}]) = f(v_{1,i-1}) + f(v_{1,i}) + f(v_{1,i+1}) + f(v_{2,i}) = (2q-2(i-1)+4) + (2i-2) + (2q-2(i+1)+3) + (2i-3) = 4q+2.$

CASE C.3 $(i = q)$: $f(N[v_{1,q}]) = f(v_{1,q-1}) + f(v_{1,q}) + f(v_{1,1}) + f(v_{2,q}) = (2q-2(i-1)+4) + (2q-1) + 1 + (2q-3) = 6q-2i+3 = 4q+3.$

CASE D $i \equiv 0(\mathrm{Mod}\,4)$

CASE D.1 $(i = 4)$: $f(N[v_{1,4}]) = f(v_{1,3}) + f(v_{1,4}) + f(v_{1,5}) + f(v_{2,4}) = 4 + (2q-8+3) + (2(4+1)-3) + (2q-8+4) = 4q+2.$

CASE D.2 $(4 < i)$: $f(N[v_{1,i}]) = f(v_{1,i-1}) + f(v_{1,i}) + f(v_{1,i+1}) + f(v_{2,i}) = (2(i-1)-2) + (2q-2i+4) + (2(i+1)-3) + (2q-2i+3) = 4q+2.$

For the bijection f we have that $NS[f] = 4q+3$ which proves the result. $\square$

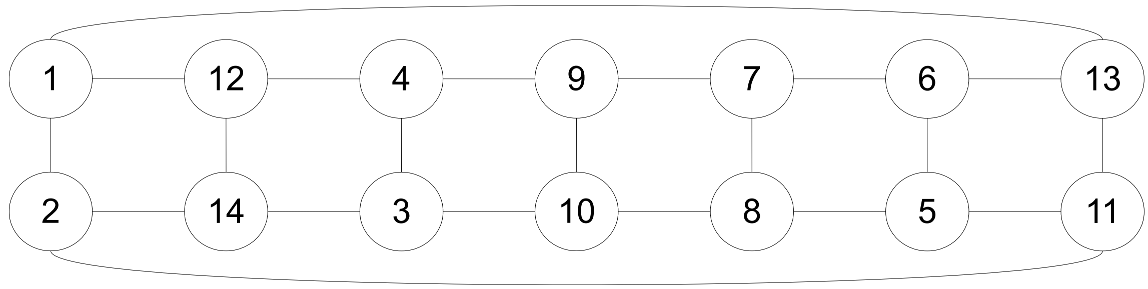

Figure 4.19: $NS[T_{2\times 7}] = 31$

Example 4.77. The bijection shown in Figure 4.18 demonstrates that $NS[T_{2\times 7}] = 4q+3 = 31$.

We can conclude from the proofs of Theorems 4.72, 4.74 and 4.76 that when $q \not\equiv 0 (\text{Mod } 4)$ we will have $NS^{sp}[T_{2\times q}] \in \{2,3\}$, however the exact value remains an open question.

CHAPTER 5

COMPETITIVE GAMES FOR NEIGHBORHOOD SUMS

5.1 Introduction

In previous chapters we have focused on determining parameters of a graph G that are based on the set of possible neighborhood sum values when we consider all possible bijections $f : V(G) \to W$. In this chapter we focus our attention on the set of possible neighborhood sum values when the bijection f is a labeling that is the result of a competitive game played by two rational players, Maximizer and Minimizer. Maximizer will have the objective of making an outcome as big as possible, while Minimizer will have the opposite objective of making an outcome as small as possible.

For example, consider the following game played on the graph C_5. The values $\{1,2,3,4,5\}$ are available for play on a vertex; each value can only be played once on a single vertex. The first player, say Maximizer, chooses any label and places it on an open vertex (one that is unlabeled). Minimizer plays next, choosing any one of the remaining labels and places it on an unlabeled vertex. Maximizer then plays, choosing any one of the labels that has not been played, and places it on an unlabeled vertex. The players take turns in this fashion, choosing a label that has not been played and placing it on an unlabeled vertex. When all labels have been placed, and thus all vertices have a label, the

game ends. For purposes of this example, the value being competed is the maximum value of a closed neighborhood sum. That is, Maximizer's objective is to make the maximum closed neighborhood sum as large as possible, and Minimizer's objective is to make the maximum closed neighborhood sum as small as possible. Since we assume both players play rationally, this game has an outcome that we call the value of the game. It is this value of the game, that is determined by the underlying graph and by the parameters of the game, that we seek to find.

For the game just described, we claim that the value of the game is 12. Notice that 12 is the maximum possible outcome of the game, and the only way that the game can have an outcome of 12 is if the game ends with one of the vertices having the labels 3, 4 and 5 in its closed neighborhood. To show that the value of the game is 12, we need to demonstrate a strategy for Maximizer that always achieves 12. In order to achieve this outcome, Maximizer should adopt the following strategy:

(i) On his first move, play the value 5 on any vertex.

(ii) If Minimizer plays the 1 or 2 on her first move, Maximizer plays the 2 or 1, respectively, on the open vertex adjacent to Minimizer's move. This ensures that there will be a vertex that has the 3, 4, and 5 in its closed neighborhood.

(iii) If Minimizer plays a 3 or 4 on her first move, it must be played within distance two of the 5. If the 3 or 4 is played adjacent to the 5, then Maximizer then plays the 4 or 3, respectively, adjacent to the 5. If the 3 or 4 is played at distance two from the 5, then Maximizer plays the 4 or 3, respectively, such that it is adjacent to the 5 and Minimizer's placement. Either of these cases will lead to the a vertex having the 3, 4, and 5 in its closed neighborhood.

Now consider a game with slightly different parameters. Maximizer, instead of having the option of choosing any label on each of his turns, must place the largest remaining label. Likewise, Minimizer on her turns must play the smallest remaining label. Otherwise the game remains the same. Maximizer still plays first and has the objective to make the maximum closed neighborhood sum as large as possible. Minimizer seeks to make the maximum closed neighborhood sum as small as possible.

For this game we claim that the value of the game is 11. To prove this we need to demonstrate a strategy for Maximizer that ensures he can always achieve at least 11, and we need to demonstrate a strategy for Minimizer such that she can ensure the outcome of the game is no more than 11. To achieve at least 11, Maximizer's strategy is as follows:

(i) On his first move, play the 5 on any vertex.

(ii) After Minimizer plays the 1 on her first move, place the 4 adjacent to the 5. Then either the vertex labeled 4, or the vertex labeled 5, will have closed neighborhood sum of at least 11 at the end of the game.

Minimizer's strategy to ensure that the value of the game is no more than 11 is as follows:

(i) After Maximizer places the 5 on his first move, place the 1 adjacent to the 5. Notice that there will be two unlabeled vertices at distance two from the vertex labeled with the 1.

(ii) After Maximizer places the 4 on his second move, place the 2 on an open vertex at distance two from the vertex labeled with the 1. This ensures that the 1 and 2 are not adjacent, and hence no vertex will have the 3, 4, and 5 in its closed neighborhood at the end of the game. That is, the value of the game will be no more than 11.

We now formalize these ideas.

Definition 5.1. The *neighborhood sum game* is a two player competitive game played between the players Maximizer and Minimizer. An instance of the game is characterized by the following parameters:

(i) The graph G on which the game is played;

(ii) First player: either Maximizer (denoted by X) or Minimizer (denoted by N);

(ii) Game outcome: NS denoting the maximum neighborhood sum, NS^{-} denoting the minimum neighborhood sum, or NS^{sp} denoting the difference between the maximum and minimum neighborhood sums;

(iii) Neighborhood type: denoted by a distance set D, where typically $D = \{0,1\}$ or $D = \{1\}$, but more generally D can be any subset of $\mathbb{N}$;

(iv) Label set: typically the set $[n]$ but more generally any multiset W with cardinality n;

(v) Play options for Maximizer: $X(A)$ meaning labels can be selected in any order, $X(B)$ meaning the smallest (or bottom) remaining label must be played at each turn, or $X(T)$ meaning the largest (or top) remaining label must be played at each turn;

(vi) Play options for Minimizer: $N(A)$ meaning labels can be selected in any order, $N(B)$ meaning the smallest (or bottom) remaining label must be played at each turn, or $N(T)$ meaning the largest (or top) remaining label must be played at each turn.

The rules of the game are as follows:

(i) The graph G, label set W, and the neighborhood type D are determined prior to the start of the game, as is the first player, either X or N.

(ii) The label selection options are chosen for Maximizer, either $X(A)$, $X(B)$, or $X(T)$.

(iii) The label selection options are chosen for Minimizer, either $N(A)$, $N(B)$, or $N(T)$.

(iv) The first player selects a label from the remaining labels of W and places it on an unlabeled vertex. Labels must be selected according the options for the first player.

(v) The second player selects a label from the remaining labels of W and places it on an unlabeled vertex. Labels must be selected according to the options for the second player.

(vi) Players alternate turns until all labels from W have been placed on a vertex.

(viii) Once all labels have been placed, the game value is calculated.

Maximizer's objective is to make the game outcome as large as possible. Minimizer's objective is to make the game outcome as small as possible.

We will denote an instance of a game as the six tuple Γ, where the elements of Γ are

(i) The graph G on which the game is being played;

(ii) The outcome of the game, where NS will indicate the maximum neighborhood sum, NS^- will indicate the minimum neighborhood sum, and NS^{sp}will indicate the spread of the neighborhood sums;

(iii) The weight set W used to play the game;

(iv) The neighborhood set D;

(v) An identification of the first player and his labeling options, for example $X(A)$ would signify Maximizer plays first and can choose from any remaining label, or $N(B)$ would signify Minimizer plays first and must use the smallest remaining label;

(vi) An identification of the second player and his labeling options, for example, $X(T)$ would signify that Maximizer plays second and must use the smallest remaining label.

For example, if $\Gamma = (G, NS, W, D, X(T), N(B))$, then we are referring to the maximum D-neighborhood sum game played on graph G with weight set W where Maximizer plays first and must play the largest remaining label, and Minimizer plays second and must play the smallest remaining label.

Definition 5.2. The *value of a neighborhood sum game* is the outcome of the game assuming both players play rationally.

We will denote the value of a game in a manner such that the parameters of the game that is being considered is evident. For example, $NS_{W,X(A),N(A)}(G;D)$ is the value of the game played on graph G where the maximum W valued, D-neighborhood sum is being considered and where Maximizer plays first with the option to choose any available label, followed by Minimizer who can also choose any available label. We will normally consider games with label set $W = [n]$ and thus drop the W from the notation. We will also adopt our previous conventions for indicating the type of neighborhood when $D = \{0,1\}$ or $D = \{1\}$. For example, $NS^{-}_{N(B),X(T)}[C_n]$ is the game played on the cycle of order n, using labels $[n] = \{1,2,\ldots,n\}$, where we are considering the minimum closed neighborhood

sum, and where Minimizer plays first using labels from smallest to largest, followed by Maximizer using labels from largest to smallest.

Notice that once we fix the graph G on which the game is being played, the label set W, the distance set D defining the neighborhoods under consideration, and the game outcome being considered (one of NS, NS^-, or NS^{sp}) there are still 18 possible games. Specifically we have two options for which player plays first, three labeling options for Maximizer, and three labeling options for Minimizer. In order to make the statements of the results more concise, we will use $X(*)$ to mean any labeling option for Maximizer and $N(*)$ to mean any labeling option for Minimizer. So if we say a result is true for $NS_{X(*)N(*)}[G]$, we mean it is true for all nine label selection options were Maximizer plays first.

When the context of the game under consideration is clear, when describing the strategies that each player will follow, we will adopt the following notation. $X_t = (x, v_i)$ will signify that on his t'th move, Maximizer places label x on vertex v_i. Likewise $N_t = (x, v_i)$ will signify that on her t'th move Minimizer places label x on vertex v_i. $X_t = (*, v_i)$ will signify that on his t'th move, Maximizer places any allowable value on vertex v_i; a similar notation will be adopted for Minimizer. Similarly, $X_t = (x, *)$ will signify that on his t'th move Maximizer places the label x on one of the remaining vertices. $X_t = (*, *)$ would then signify that Maximizer makes any legal move at turn t; a similar notation will be followed for Minimizer. If multiple games are being considered, or if the context of the game being discussed is not clear, the above notation will be supplemented. For example, $X_t(\Gamma) = (x, v_i)$ would signify that Maximizer places label x on vertex v_i while playing the game Γ.

5.2 Basic Results

We begin with some simple results that connect this chapter on the neighborhood sum games to our previous results.

Theorem 5.3. *For any graph G, label set W, and distance set D we have*

(i) $NS^{-}_{W,X(*),N(*)}(G;D) \leq NS^{-}_{W}(G;D)$ *and* $NS^{-}_{W,N(*),X(*)}(G;D) \leq NS^{-}_{W}(G;D)$*;*

(ii) $NS_{W,X(*),N(*)}(G;D) \geq NS_{W}(G;D)$ *and* $NS_{W,N(*),X(*)}(G;D) \geq NS_{W}(G;D)$*; and*

(iii) $NS^{sp}_{W,X(*),N(*)}(G;D) \geq NS^{sp}_{W}(G;D)$ *and* $NS^{sp}_{W,N(*),X(*)}(G;D) \geq NS^{sp}_{W}(G;D)$*.*

Proof. For an arbitrary neighborhood sum game, let $f : V(G) \to W$ be the bijection defined by the placement of the labels at the end of the game. Then we have that $NS^{-}(f;D) \leq NS^{-}_{W}(G;D)$, $NS(f;D) \geq NS_{W}(G;D)$, and $NS^{sp}(f;D) \geq NS^{sp}_{W}(G;D)$. Therefore, each result in the theorem holds. $\square$

Corollary 5.4. *For any graph G we have*

(i) $NS^{-}_{X(*),N(*)}(G) \leq NS^{-}(G)$, $NS^{-}_{N(*),X(*)}(G) \leq NS^{-}(G)$, $NS^{-}_{X(*),N(*)}[G] \leq NS^{-}(G)$, *and* $NS^{-}_{N(*),X(*)}[G] \leq NS^{-}[G]$*;*

(ii) $NS_{X(*),N(*)}(G) \geq NS(G)$, $NS_{N(*),X(*)}(G) \geq NS(G)$, $NS_{X(*),N(*)}[G] \geq NS[G]$, *and* $NS_{N(*),X(*)}[G] \geq NS[G]$*; and*

(iii) $NS^{sp}_{X(*),N(*)}(G) \geq NS^{sp}(G)$, $NS^{sp}_{N(*),X(*)}(G) \geq NS^{sp}(G)$, $NS^{sp}_{X(*),N(*)}[G] \geq NS^{sp}(G)$, *and* $NS^{sp}_{N(*),X(*)}[G] \geq NS^{sp}[G]$*.*

In the remaining theorems in this section, it will be convenient to speak of labeling options in abstract terms, and we will use variables to do so. For example, a labeling option x can take on any value from the set $\{A, B, T\}$ of labeling options. Here A represents the

option to select labels in any order, B represents the labeling option where the smallest available label must be played at each turn, and T represents the labeling option where the largest available label must be played at each turn. If $x = B$, then we will on occasion use $-x$ to represent T. Likewise if $x = T$, then $-x$ will represent B. If $x = A$, then $-x = A$. This convention will allow us to avoid repeating what is essentially the same result in order to include all the allowable labeling options.

It is clear that a player cannot do as well when his labeling options are limited to choosing labels smallest to largest, or limited to choosing labels largest to smallest, as when he is free to choose any label. Hence we can establish inequalities such as $NS^{-}_{W,X(*),N(A)}(G;D) \geq NS^{-}_{W,X(A),N(A)}(G;D)$ and $NS_{W,X(*),N(A)}(G;D) \geq NS_{W,X(A),N(A)}(G;D)$. That is, when all other parameters of the game are held constant, Maximizer can never do better than when his option is to select labels in any order he wishes. Similar results hold for Minimizer and for other combinations of labeling options. We state this result in the following theorem.

Theorem 5.5. *Let $x \in \{A, B, T\}$ be a fixed labeling option. Then*

(i) $NS_{W,X(*),N(x)}(G;D) \leq NS_{W,X(A),N(x)}(G;D)$ and

$NS_{W,N(x),X(*)}(G;D) \leq NS_{W,N(x),X(A)}(G;D)$;

(ii) $NS_{W,X(x),N(*)}(G;D) \geq NS_{W,X(x),N(A)}(G;D)$ and

$NS_{W,N(*),X(x)}(G;D) \geq NS_{W,N(A),X(x)}(G;D)$;

(iii) $NS^{-}_{W,X(*),N(x)}(G;D) \leq NS_{W,X(A),N(x)}(G;D)$ and

$NS^{-}_{W,N(x),X(*)}(G;D) \leq NS_{W,N(x),X(A)}(G;D)$;

(iv) $NS^{-}_{W,X(x),N(*)}(G;D) \geq NS_{W,X(x),N(A)}(G;D)$ and

$NS^{-}_{W,N(*),X(x)}(G;D) \geq NS_{W,N(A),X(x)}(G;D)$;

(v) $NS^{sp}_{W,X(*),N(x)}(G;D) \leq NS^{sp}_{W,X(A),N(x)}(G;D)$ *and*

$NS^{sp}_{W,N(x),X(*)}(G;D) \leq NS^{sp}_{W,N(x),X(A)}(G;D)$; *and*

(vi) $NS^{sp}_{W,X(x),N(*)}(G;D) \geq NS_{W,X(x),N(A)}(G;D)$ *and*

$NS^{sp}_{W,N(*),X(x)}(G;D) \geq NS_{W,N(A),X(x)}(G;D)$.

Later in this chapter we will determine many of the values for the minimax game played on cycles and complete bipartite graphs. We will consider both the open and closed neighborhood cases for cycles and the closed neighborhood cases for the complete bipartite graph. Hence, all the graphs we will consider will be regular. If we can establish a result for our competitive games on graphs that is similar to Theorem 3.78, then the values of the maximin versions of the games on (D,r)-regular graphs will be easily determined once we know the values for the minimax versions of the games. As it turns out, a very similar result exists and will be shown in Theorem 5.13. In order to lay the ground work for that we result, we present several preliminary results that show the effects that changes in the weight set W have on the game values.

Theorem 5.6. *Let G be any graph. Let $W^+ = \{w_1, w_2, \ldots, w_n\} \subset \mathbb{R}$ be a multiset, where $w_1 \leq w_2 \leq \cdots \leq w_n$ and $W^- = \{-w_1, -w_2, \ldots, -w_n\}$. Let $x, y \in \{A, B, T\}$ be labeling options. Then*

(i) $NS_{W^+,X(x),N(y)}(G;D) = -NS^{-}_{W^-,N(-x),X(-y)}(G;D)$ *and*

(ii) $NS_{W^+,N(x),X(y)}(G;D) = -NS^{-}_{W^-,X(-x),N(-y)}(G;D)$.

Proof. Let $\Gamma_1 = (G, NS, W^+, D, X(x), N(y))$ and $\Gamma_2 = (G, NS^-, W^-, D, N(-x), X(-y))$. When considering the game Γ_1, there exists a strategy that Maximizer can employ to ensure

that the outcome of the game Γ_1 is at least $NS_{W^+,X(x),N(y)}(G;D)$. Notice that this strategy ensures the minimum value of the outcome of the game, regardless of how Minimizer plays. We now want to show that there is a strategy that Minimizer can employ on game Γ_2 such that the outcome of the game is no more than $-NS_{W^+,X(x),N(y)}(G;D)$. Minimizer's strategy in order to achieve this will be to apply Maximizer's strategy from game Γ_1 to game Γ_2. Specifically, if on any particular move in game Γ_1 Maximizer would have labeled vertex v with weight w_i, then on the same move in game Γ_2 Minimizer will label vertex v with weight $-w_i$.

For game Γ_1, let w_{i_1} be the first weight played by Maximizer, let w_{i_2} be the first weight played by Minimizer, let w_{i_3} be the second weight played by Maximizer, and so on until Minimizer plays the final weight of w_{i_n}. Following this convention, let $a = (w_{i_1}, w_{i_2}, \ldots, w_{i_n})$ be any sequence of weights from W^+ that can result from playing game Γ_1. Similarly let $b = \left(w_{j_i}, w_{j_2}, \ldots, w_{j_n}\right)$ be any sequence of weights from W^- that can result from playing game Γ_2. We need to establish a one-to-one correspondence between the possible sequences of play from game Γ_1 and those from game Γ_2. This correspondence will be provided by $a = -b$.

If $x = A$, then $-x = A$. On his first move of game Γ_1, Maximizer would have been able to choose w_{i_1} to be any value from W^+. Likewise, on her first move in game Γ_2, Minimizer can choose any value, in particular she can choose w_{j_1} to be $-w_{i_1}$. If $x = B$, then on his first move of game Γ_1, Maximizer would have had to select the smallest label, that is $w_{i_1} = w_1$. Since $-x = T$, on her first move in game Γ_2, Minimizer will be forced to select the largest label, and hence $w_{j_1} = -w_1$. Similarly, if $x = T$, then on his first move of game Γ_1, Maximizer would have had to select the largest label, that is, $w_{i_1} = w_n$.

Since $-x = B$, on her first move in game Γ_2, Minimizer will be forced to select labels the smallest label which means $w_{j_1} = -w_n$. This means that w_{i_1} can have value w if and only if w_{j_1} can have value $-w$. Hence if after Maximizer makes his first move in game Γ_1, the set of weights remaining for Minimizer to choose from is $W_1^+ = \{w_1^{(1)}, w_2^{(1)}, \ldots, w_{n-1}^{(1)}\}$, where $w_1^{(1)} \leq w_2^{(1)} \leq \cdots \leq w_{n-1}^{(1)}$, then after Minimizer's first move in game Γ_2, the remaining set of weights for Maximizer to choose from is $W_1^- = \{-w_1^{(1)}, -w_2^{(1)}, \ldots, -w_{n-1}^{(1)}\}$.

If $y = A$, then $-y = A$. In this case, on her first move in game Γ_1, Minimizer would have been able to choose w_{i_2} to be any value in W_1^+. Likewise, on his first move in game Γ_2, Maximizer can choose w_{j_2} to be any value from W_1^-. If $y = B$, then on her first move of game Γ_1, Minimizer would have selected the smallest label from set W_1^+, that is, $w_{i_2} = w_1^{(1)}$. Since $-x = T$, on his first move of game Γ_2, Maximizer will have to play the largest available label, that is, $w_{j_2} = -w_1^{(1)}$. If $y = T$, then on her first move of game Γ_1, Minimizer would have selected the largest label from set W_1^+, that is, $w_{i_2} = w_{n-1}^{(1)}$. Since $-x = B$, on his first move of game Γ_2, Maximizer will have to play the smallest available label, that is, $w_{j_2} = -w_{n-1}^{(1)}$. Thus, as with the first move, w_{i_2} can have value w if and only if w_{j_2} can have value $-w$. Repeating this line of argument for all n placements of labels, one can see that $a = (w_{i_1}, w_{i_2}, \ldots, w_{i_n})$ is a sequence of weights from W^+ that can result from playing game Γ_1 if and only if $b = -a$ is a sequence of weights from W^- that can result from playing game Γ_2.

Next let v_1 be the first vertex labeled by Maximizer in game Γ_1, let v_2 be the first vertex labeled by Minimizer in game Γ_1, let v_3 be the second vertex labeled by Maximizer in game Γ_1, and so on, until Minimizer labels vertex v_n. Following this convention, we can let $c = (v_1, v_2, \ldots, v_n)$ be a sequence that represents the order in which vertices are labeled

in game Γ_1. Clearly since there is no restriction on the order in which vertices are labeled, c can be any permutation of the vertices from G. Moreover, c is also a valid sequence in which vertices can be labeled in game Γ_2. Hence, if Maximizer labeled vertex v_i on his k'th turn of game Γ_1, then Minimizer can label vertex v_i on her k'th turn in game Γ_2.

Now Maximizer's strategy for game Γ_1 can be summarized as follows:

(i) Set $X_1 = (w_{k_1}, v_1)$, that is, label vertex v_1 with weight w_{k_1}.

(ii) On his turn i, as a result of previous moves by Minimizer, set $X_i = (w_{k_i}, v_i)$, that is, label vertex v_i with weight w_{k_i}.

Regardless of how Minimizer plays in game Γ_1, this strategy will result in there being a vertex with D-neighborhood sum of at least $NS_{W^+,X(x),N(y)}(G;D)$. However, since we can assume that Minimizer plays rationally, no vertex will have D-neighborhood sum that exceeds this value. Hence, there will exist a vertex $u \in V(G)$ with labels in its D-neighborhood that sum to exactly $NS_{W^+,X(x),N(y)}(G;D)$, and no other vertex will have D-neighborhood sum in excess of this value.

Minimizer will then follow the following strategy for game Γ_2.

(i) Set $N_1 = (-w_{k_1}, v_1)$, that is, since Maximizer labeled vertex v_1 with weight w_{k_1} on his first turn in game Γ_1, label the same vertex with weight $-w_{k_1}$.

(ii) On turn i, as a result of the previous moves by Maximizer, set $N_i = (-w_{k_i}, v_i)$, that is, since Maximizer labeled vertex v_i with weight w_{k_i}on his i'th turn, label the same vertex with weight $-w_{k_i}$.

This strategy will result in there being a vertex $u \in V(G)$ with labels in its D-neighborhood that sum to no more than $-NS_{W^+,X(x),N(y)}(G;D)$. However, since we can assume that Maximizer plays rationally, no vertex will have D-neighborhood sum less than

this value. Therefore, it follows that $NS^{-}_{W^-,N(-x),X(-y)}(G;D) = -NS_{W^+,X(x),N(y)}(G;D)$ or $NS_{W^+,X(x),N(y)}(G;D) = -NS^{-}_{W^-,N(-x),X(-y)}(G;D)$.

The proof can be repeated with the roles of Minimizer and Maximizer switched to show that $NS_{W^+,N(x),X(y)}(G;D) = -NS^{-}_{W^-,X(-x),N(-y)}(G;D)$. $\square$

Corollary 5.7. *Let $W^+ = \{w_1, w_2, \ldots, w_n\} \subset \mathbb{R}$ be a multiset, where $w_1 \leq w_2 \leq \cdots \leq w_n$ and $W^- = \{-w_1, -w_2, \ldots, -w_n\}$. Let $x, y \in \{A, B, T\}$ be labeling options. Then*

(i) $NS_{W^+,X(x),N(y)}(G) = -NS^{-}_{W^-,N(-x),X(-y)}(G)$,

(ii) $NS_{W^+,X(x),N(y)}[G] = -NS^{-}_{W^-,N(-x),X(-y)}[G]$,

(iii) $NS_{W^+,N(x),X(y)}(G) = -NS^{-}_{W^-,X(-x),N(-y)}(G)$, and

(iv) $NS_{W^+,N(x),X(y)}[G] = -NS^{-}_{W^-,X(-x),N(-y)}[G]$.

Example 5.8. To illustrate Corollary 5.7, consider the closed neighborhood sum game played on $G = C_5$, where $W^+ = [5] = \{1,2,3,4,5\}$ and $W^- = \{-1,-2,-3,-4,-5\}$.

For the game $\Gamma_1 = (C_5, NS, W^+, \{0,1\}, X(A), N(B))$, Maximizer can set $X_1 = (3, v_1)$. Since Minimizer must play the 1 on her first move, Maximizer can play the 2 adjacent to the 1 on his second move. In this manner, there will be a closed neighborhood that contains the 3, 4, and 5. Since this is the maximum possible value for a closed neighborhood sum, we conclude that $NS_{W^+,X(A),N(B)}[C_5] = 12$.

For the game $\Gamma_2 = (C_5, NS, W^-, \{0,1\}, N(A), X(T))$, Minimizer can set $N_1 = (-3, v_1)$. Since Maximizer must play the -1 on his first move, Minimizer can play the -2 adjacent to the -1 on her second move. In this manner, there will be a closed neighbor-

hood that contains the -3, -4, and -5. Since this is the minimum possible value for a closed neighborhood sum, we conclude that $NS_{W^-,N(A),X(T)}[C_5] = -12 = -NS_{W^+,X(A),N(B)}[C_5]$.

Theorem 5.9. *Let $W = \{w_1, w_2, \ldots, w_n\}$ be a multiset. Let $s \in \mathbb{R}$ and define the multiset $S = \{w_1+s, w_2+s, \ldots, w_n+s\}$. Let $x, y \in \{A, B, T\}$ be labeling options. If graph G is (D,r)-regular, then*

(i) $NS_{S,X(x),N(y)}(G;D) = rs + NS_{W,X(x),N(y)}(G;D)$,

(ii) $NS_{S,N(x),X(y)}(G;D) = rs + NS_{W,N(x),X(y)}(G;D)$,

(iii) $NS^-_{S,X(x),N(y)}(G;D) = rs + NS_{W,X(x),N(y)}(G;D)$,

(iv) $NS^-_{S,N(x),X(y)}(G;D) = rs + NS_{W,N(x),X(y)}(G;D)$,

(v) $NS^{sp}_{S,X(x),N(y)}(G;D) = NS^{sp}_{W,X(x),N(y)}(G;D)$, and

(vi)$NS^{sp}_{S,N(x),X(y)}(G;D) = NS^{sp}_{W,N(x),X(y)}(G;D)$.

Proof. Let $\Gamma_1 = (G, NS, W, D, X(x), N(y))$ and $\Gamma_2 = (G, NS, S, D, X(x), N(y))$. When considering the game Γ_1, there exists a strategy where that Maximizer can employ to ensure that the outcome of the game Γ_1 is at least $NS_{W,X(x),N(y)}(G;D)$. Notice that this strategy ensures the minimum value of the outcome of the game, regardless of how Minimizer plays. Similarly, for game Γ_1, there is a strategy that Minimizer can employ to ensure that the outcome of the game is no more than $NS_{W,X(x),N(y)}(G;D)$, regardless of how Maximizer plays. We want to show that there is a strategy that Maximizer can employ on game Γ_2 such that the outcome of the game is at least $rs + NS_{W,X(x),N(y)}(G;D)$, and that this strategy works regardless of how Minimizer plays. Also we need to show that there is a strategy that Minimizer can employ on game Γ_2 that ensures that the outcome of the game is no more than $rs + NS_{W,X(x),N(y)}(G;D)$, and that this strategy works regardless of how Maximizer

plays. For both Maximizer and Minimizer, they can adapt their strategy from game Γ_1 for use in game Γ_2 by playing $w_i + s$ in game Γ_2 in any situation where they would have played w_i in game Γ_1.

First consider the sequence in which weights from W is played in game Γ_1. Since both players retain their game Γ_1 labeling option in game Γ_2, $a = (w_{i_1}, w_{i_2}, \ldots, w_{i_n})$ is a possible sequence in which weights can be played in Γ_1 if and only if $b = (w_{i_1} + s, w_{i_2} + s, \ldots, w_{i_n} + s)$ is a sequence in which the weights can be played in Γ_2. Additionally, since the game does not place any restrictions on the way in which vertices can be labeled, any permutation of the vertices from G can represent the sequence in which vertices are labeled in either game.

The strategy that Maximizer follows in game Γ_1 can be summarized as follows:

(i) Set $X_1 = (w_{i_1}, v_1)$.

(ii) At turn k, depending on how Minimizer has played, set $X_i = (w_{i_k}, v_k)$.

Maximizer's strategy will ensure that the outcome of the game is at least $NS_{W,X(x),N(y)}(G;D)$. On the other hand, since we can assume Minimizer will play rationally, we know that the outcome of the game will be no more than $NS_{W,X(x),N(y)}(G;D)$. Hence, there will exist a vertex $u \in V(G)$ such that the sum of the labels in the $N_D(u)$ will sum to exactly $NS_{W,X(x),N(y)}(G;D)$, and for all $w \in V(G)$, the sum of the labels in $N_D(w)$ does not exceed $NS_{W,X(x),N(y)}(G;D)$.

During game Γ_2, Maximizer can then adopt the following strategy:

(i) Set $X_1 = (w_{i_1} + s, v_1)$. That is, since on his first move in game Γ_1, Maximizer labeled vertex v_1 with weight w_{i_1}, during game Γ_2, he will label vertex v_1 with weight $w_{i_1} + s$.

(ii) At each turn k, depending on how Minimizer has played, set $X_i = (w_{i_k} + s, v_k)$. That is, since on his k'th move in game Γ_1, Maximizer labeled vertex v_k with weight w_{i_k}, during game Γ_2, he will label vertex v_k with weight $w_{i_k} + s$.

Since graph G is (D,r)-regular, for any $w \in V(G)$, the sum of the labels in $N_D(w)$ after game Γ_2 will be rs plus the sum of the labels in $N_D(w)$ after game Γ_1. This will be true in particular for $u \in V(G)$. Hence, in game Γ_2, $N_D(u) \geq rs + NS_{W,X(x),N(y)}(G;D)$. Furthermore, since Minimizer will play rationally, and can adopt her strategy from game Γ_1 as well, for all $w \in V(G)$, the sum of the labels in $N_D(w)$ does not exceed $rs + NS_{W,X(x),N(y)}(G;D)$. Therefore, $NS_{S,X(x),N(y)}(G;D) = rs + NS_{W,X(x),N(y)}(G;D)$.

The proofs for each of the other cases can be completed in a similar fashion. Notice that when proving parts v and vi, that since all D-neighborhood sums in game Γ_2 have values of rs plus their D-neighborhood sum value from game Γ_1, this is true in particular for the vertices with minimum and maximum D-neighborhood sum. Hence, the spread values will be the same in both games as the rs terms cancel each other out. $\square$

Corollary 5.10. *Let $W = \{w_1, w_2, \ldots, w_n\}$ be a multiset. Let $s \in \mathbb{R}$ and define the multiset $S = \{w_1 + s, w_2 + s, \ldots, w_n + s\}$. Let $x, y \in \{A, B, T\}$ be labeling options. If graph G is r-regular, then*

(i) $NS_{S,X(x),N(y)}(G) = rs + NS_{W,X(x),N(y)}(G)$,

(ii) $NS_{S,N(x),X(y)}(G) = rs + NS_{W,N(x),X(y)}(G)$,

(iii) $NS^{-}_{S,X(x),N(y)}(G) = rs + NS_{W,X(x),N(y)}(G)$,

(iv) $NS^{-}_{S,N(x),X(y)}(G) = rs + NS_{W,N(x),X(y)}(G)$,

(v) $NS^{sp}_{S,X(x),N(y)}(G) = NS^{sp}_{W,X(x),N(y)}(G)$, *and*

(vi) $NS^{sp}_{S,N(x),X(y)}(G) = NS^{sp}_{W,N(x),X(y)}(G)$.

Corollary 5.11. *Let* $W = \{w_1, w_2, \ldots, w_n\}$ *be a multiset. Let* $s \in \mathbb{R}$ *and define the multiset* $S = \{w_1 + s, w_2 + s, \ldots, w_n + s\}$. *Let* $x, y \in \{A, B, T\}$ *be labeling options. If graph* G *is r-regular, then*

(i) $NS_{S,X(x),N(y)}[G] = (r+1)s + NS_{W,X(x),N(y)}[G]$,

(ii) $NS_{S,N(x),X(y)}[G] = (r+1)s + NS_{W,N(x),X(y)}[G]$,

(iii) $NS^{-}_{S,X(x),N(y)}[G] = (r+1)s + NS_{W,X(x),N(y)}[G]$,

(iv) $NS^{-}_{S,N(x),X(y)}(G) = (r+1)s + NS_{W,N(x),X(y)}[G]$,

(v) $NS^{sp}_{S,X(x),N(y)}[G] = NS^{sp}_{W,X(x),N(y)}[G]$, *and*

(vi) $NS^{sp}_{S,N(x),X(y)}[G] = NS^{sp}_{W,N(x),X(y)}[G]$.

Example 5.12. To demonstrate Corollary 5.11, consider again the game $\Gamma_1 = (C_5, NS, [5], \{0,1\}, X(A), N(B))$. That is $W = \{1,2,3,4,5\}$. Notice that C_5 is 2-regular. Define $S = \{1+10, 2+10, 3+10, 4+10, 5+10\} = \{11,12,13,14,15\}$ and let $\Gamma_2 = (C_5, NS, S, \{0,1\}, X(A), N(B))$. From Example 5.8, we know that $NS_{W,X(A),N(B)}[C_5] = 12$. During game Γ_2, Maximizer will adopt his strategy from game Γ_1 and set $X_1 = (13, v_1)$. Since Minimizer must play the 11 on her first move, on his second move, Maximizer will place the 12 adjacent to where Minimizer played. By following this strategy, there will exist a vertex that has labels 13, 14, and 15 in its closed neighborhood. Since this is the maximum possible closed neighborhood sum in game Γ_2, we know that $NS_{S,X(A),N(B)}[C_5] = 13 + 14 + 15 = (2+1)10 + 12 = (r+1)s + NS_{W,X(A),N(B)}[C_5]$, where $r = 2$ and $s = 10$.

Theorem 5.13. *Let $c, s \in \mathbb{R}$ and define $W = \{c+s, 2c+s, \ldots, nc+s\}$. Let $x, y \in \{A, B, T\}$ be labeling options. If graph G is $(D;r)$-regular, then*

(i)$NS_{W,X(x),N(y)}(G;D) + NS^{-}_{W,N(-x),X(-y)}(G;D) = r[c(n+1)+2s]$, and

(ii) $NS_{W,N(x),X(y)}(G;D) + NS^{-}_{W,X(-x),N(-y)}(G;D) = r[c(n+1)+2s]$.

Proof. Define $W^{-} = \{-(c+s), -(2c+s), \ldots, -(nc+s)\}$. From Theorem 5.6 we know that $NS^{-}_{W^{-},N(-x),X(-y)}(G;D) = -NS_{W,X(x),N(y)}(G;D)$. Let $t = c(n+1)+2s$ and define $W^{+} = \{-(c+s)+t, -(2c+s)+t, \ldots, -(nc+s)+t\}$. From Theorem 5.9 we know that $NS^{-}_{W^{+},N(-x),X(-y)}(G;D) = rt + NS^{-}_{W^{-},N(-x),X(-y)}(G;D) = rt - NS_{W,X(x),N(y)}(G;D)$. Hence, $NS_{W,X(x),N(y)}(G;D) + NS^{-}_{W^{+},N(-x),X(-y)}(G;D) = rt = r[c(n+1)+2s]$. However, $W^{+} = \{-(c+s)+t, -(2c+s)+t, \ldots, -(nc+s)+t\}$ $=\{-(nc+s)+t, -((n-1)c+s)+t, \ldots, -(c+s)+t\} = \{-(nc+s)+c(n+1)+2s, -((n-1)c+s)+c(n+1)+2s, \ldots, -(c+s)+c(n+1)+2s\} = \{c+s, 2c+s, \ldots, nc+s\}$. That is, $W^{+} = W$. Therefore, $NS_{W,X(x),N(y)}(G;D) + NS^{-}_{W,N(-x),X(-y)}(G;D) = r[c(n+1)+2s]$.

The proof for part ii can be completed in a similar fashion. □

Corollary 5.14. *Let $x, y \in \{A, B, T\}$ be labeling options. If graph G is r regular, then*

(i)$NS_{X(x),N(y)}(G) + NS^{-}_{N(-x),X(-y)}(G) = r(n+1)$, and

(ii) $NS_{N(x),X(y)}(G) + NS^{-}_{X(-x),N(-y)}(G) = r(n+1)$.

Corollary 5.15. *Let $x, y \in \{A, B, T\}$ be labeling options. If graph G is r regular, then*

(i)$NS_{X(x),N(y)}[G] + NS^{-}_{N(-x),X(-y)}[G] = (r+1)(n+1)$, and

(ii) $NS_{N(x),X(y)}[G] + NS^{-}_{X(-x),N(-y)}[G] = (r+1)(n+1)$.

Corollary 5.16. *Let $x, y \in \{A, B, T\}$ be labeling options. For cycle C_n we have*

(i) $NS_{X(x),N(y)}(C_n) + NS^{-}_{N(-x),X(-y)}(C_n) = 2n+2$,

(ii) $NS_{N(x),X(y)}(C_n) + NS^-_{X(-x),N(-y)}(C_n) = 2n+2$*,*

(iii)$NS_{X(x),N(y)}[C_n] + NS^-_{N(-x),X(-y)}[C_n] = 3n+3$*, and*

(iv) $NS_{N(x),X(y)}[C_n] + NS^-_{X(-x),N(-y)}[C_n] = 3n+3$*.*

Corollary 5.17. *Let* $x, y \in \{A, B, T\}$ *be labeling options. For the complete bipartite graph* $K_{t,t}$ *we have*

(i) $NS_{X(x),N(y)}(K_{t,t}) + NS^-_{N(-x),X(-y)}(K_{t,t}) = 2t^2 + t$*,*

(ii) $NS_{N(x),X(y)}(C_n) + NS^-_{X(-x),N(-y)}(C_n) = 2t^2 + t$*.*

Example 5.18. Recall that in Example 5.8 we showed that $NS_{X(A),N(A)}[C_5] = 12$. Corollary 5.16 would then imply that $NS^-_{N(A),X(A)}[C_5] = 18 - 12 = 6$. We could argue this fact directly as follows. In order to achieve this value, which is the absolute minimum possible value for a closed neighborhood sum, Minimizer must ensure that there is a vertex that has the labels 1, 2, and 3 in its closed neighborhood. Equivalently, Minimizer can ensure that the 4 and 5 are assigned to adjacent vertices. The strategy that Minimizer should employ to achieve this outcome is as follows:

(i) Set $N_1 = (1, v_1)$.

(ii) If $X_1 = (2, *)$ or $X_1 = (3, *)$, then on her second move, Minimizer should play the 3 or the 2, respectively, in such a way that there is a vertex that has labels 1, 2, and 3 in its closed neighborhood.

(iii) If $X_1 = (4, *)$ or $X_1 = (5, *)$, then on her second move, Minimizer should play the 5 or the 4, respectively, on a vertex adjacent to Maximizer's first move.

We close this section with a few comments. First, we note that there are graphs in which there is no benefit to playing either first or second. For example, on a com-

plete graph, the order of play and the labeling options for each player make no difference; the outcome of the game is fixed no matter the strategy of the players. Second, there are graphs that give the advantage to the first player, and this advantage can be arbitrarily large. Consider a closed neighborhood sum game played on the star $K_{1,n-1}$ where the outcome is measured by the minimum closed neighborhood sum and where both players can play any label. In this case, the first player can completely determine the outcome of the game by making his first move on the center vertex. Thus we have $NS^-_{X(A),N(A)}[K_{1,n-1}] = n+1$ and $NS^-_{N(A),X(A)}[K_{1,n-1}] = 3$. Thus, if Maximizer plays first, he can achieve $NS^-[K_{1,n-1}]$, which is the maximum value across all bijections, and if Minimizer plays first, she can achieve smallest minimum value across all bijections. Third, there are graphs that give the advantage to the second player, and this advantage can also be arbitrarily large. Consider a closed neighborhood sum game played on the union of k copies of K_2 where the outcome is measured by the maximum closed neighborhood sum. In this case we will have $NS_{X(A),N(A)}[kK_2] = 2k+1$ and $NS_{N(A),X(A)}[kK_2] = 4k-1$. In this case, if Maximizer plays second, he can achieve the maximum possible closed neighborhood sum among all bijections, and if Minimizer plays second, she can achieve $NS[kK_2]$, the minimax value across all bijections.

For the remainder of this chapter we will focus our attention on determining game values for cycles and complete bipartite graphs. We will limit our attention to either open or closed neighborhoods and always use weight set $[n]$.

Table 5.1: Value of maximum open neighborhood sum game on C_n

	Order of Graph			
Scenario	3	4	$n \geq 5$, n odd	$n \geq 6$, n even
X(B),N(B)	5	6	$2n-1$	$2n-2$
X(B),N(T)	5	5	$\geq n+\lceil \frac{n}{2} \rceil$	$\geq n+\lceil \frac{n}{2} \rceil$
X(B),N(A)	5	5		
X(T),N(B)	5	5	$2n-1$	$2n-1$
X(T),N(T)	5	6	$2n-2$	$2n-2$
X(T),N(A)	5	5	$2n-2$	$2n-2$
X(A),N(B)	5	6	$2n-1$	$2n-1$
X(A),N(T)	5	6	$2n-1$	$2n-1$
X(A),N(A)	5	5	$2n-1$	$2n-2$
N(B),X(B)	5	7	$2n-2$	$2n-1$
N(B),X(T)	5	6	$2n-1$	$2n-1$
N(B),X(A)	5	7	$2n-1$	$2n-1$
N(T),X(B)	5	6	$\geq n+\lfloor \frac{n}{2} \rfloor+1$	$\geq n+\lfloor \frac{n}{2} \rfloor+1$
N(T),X(T)	5	7	$2n-1$	$2n-1$
N(T),X(A)	5	7	$2n-1$	$2n-1$
N(A),X(B)	5	6		
N(A),X(T)	5	6	$2n-2$	$2n-2$
N(A),X(A)	5	7	$2n-2$	$2n-1$

5.3 Open Neighborhood Sums on Cycles

In this section we focus on determining the values for the maximum open neighborhood sum game on cycle C_n. Once these values are determined, Corollary 5.16 can be applied to determine the corresponding values for the minimum open neighborhood sum games. Table 5.1 summarizes the results we will prove in this section. The values in the Table 5.1 that have not been filled in are open problems.

Theorem 5.19. *For cycle C_3 we have*

(i) $NS^{-}_{X(*),N(*)}(C_n) = NS^{-}_{N(*),X(*)}(C_n) = 3,$

(ii) $NS_{X(*),N(*)}(C_n) = NS_{N(*),X(*)}(C_n) = 5$*, and*

(iii) $NS^{sp}_{X(*),N(*)}(C_n) = NS^{sp}_{N(*),X(*)}(C_n) = 2$*.*

Proof. For any bijection $f : V(C_3) \to [3]$ we have $NS^{-}(f) = 3$, $NS(f) = 5$, and $NS^{sp}(f) = 2$. The results of this theorem follow directly from this fact. □

Theorem 5.20. $NS_{X(B),N(B)}(C_4) = 6$.

Proof. Without loss of generality we can assume that $X_1 = (1, v_1)$. If Minimizer sets $N_1 = (2, v_2)$, then the maximum possible open neighborhood sum will occur on vertices v_1 and v_3 and will be $2+4 = 6$. Hence Minimizer can ensure that the outcome of the game is no more than 6.

If Minimizer sets $N_1 = (2, v_2)$ or $N_1 = (2, v_4)$, then Maximizer can set $X_2 = (3, v_3)$ and the closed neighborhood sum of $v_1$and v_3 will turn out to be 6. However, if $N_1 = (2, v_3)$, then regardless of how Maximizer plays, vertices v_1 and v_3 will have open neighborhood sums of 7. Hence Maximizer can ensure that the outcome of the game is at least 6.

Therefore, the value of the game is 6; that is, $NS_{X(B),N(B)}(C_4) = 6$. □

Theorem 5.21. $NS_{X(B),N(T)}(C_4) = 5$.

Proof. If the maximum open neighborhood sum at the end of the game is 5, then the resulting labeling is a Σ-labeling. Hence we only need to demonstrate how Minimizer can ensure that 5 as is achieved, as this is the minimax open neighborhood sum value for C_4. Assume without loss of generality that $X_1 = (1, v_1)$. Minimizer then sets $N_1 = (4, v_3)$. The open neighborhood sums of vertices v_2 and v_4 will then be $1+4 = 5$, and the open neighborhood sums of vertices v_1 and v_3 will be $2+3 = 5$. □

Corollary 5.22. $NS_{X(B),N(A)}(C_4) = 5$.

Proof. Since a player can always do at least as well when they have the option of placing labels in any order as they can when their options are limited, it follows from Theorem 5.21 that $NS_{X(B),N(A)}(C_4) \leq NS_{X(B),N(T)}(C_4) = 5$. From Corollary 5.4 and Theorem 4.14 we know that $NS_{X(B),N(A)}(C_4) \geq NS(C_4) = 5$. Therefore, $NS_{X(B),N(A)}(C_4) = 5$. $\square$

Theorem 5.23. $NS_{X(T),N(B)}(C_4) = 5$.

Proof. As in Theorem 5.21 we only need to demonstrate a strategy for Minimizer whereby she can achieve a game outcome of 5. Assume without loss of generality that $X_1 = (4, v_1)$. Minimizer then sets $N_1 = (1, v_3)$. The open neighborhood sums of vertices v_2 and v_4 will then be $4 + 1 = 5$, and vertices v_1 and v_3 will have open neighborhood sums of $2 + 3 = 5$. $\square$

Corollary 5.24. $NS_{X(T),N(A)}(C_4) = 5$.

Theorem 5.25. $NS_{X(T),N(T)}(C_4) = 6$.

Proof. Without loss of generality we can assume that $X_1 = (4, v_1)$. If Minimizer sets $N_1 = (3, v_2)$, then the maximum possible open neighborhood sum will occur on vertices v_2 and v_4 and will be $2 + 4 = 6$. Notice that if Minimizer sets $N_1 = (3, v_2)$ or $N_1 = (3, v_4)$, then Maximizer can set $X_2 = (2, v_3)$ and the closed neighborhood sum of $v_2$and v_4 will turn out to be 6. However, if $N_1 = (3, v_3)$, then regardless of how Maximizer plays, vertices v_2 and v_4 will have open neighborhood sums of 7. $\square$

Theorem 5.26. $NS_{X(A),N(B)}(C_4) = 6$.

Proof. From Theorem 5.20 we can deduce that $NS_{X(A),N(B)}(C_4) \geq 6$. Thus we need to demonstrate a strategy whereby Minimizer can ensure that the outcome of the game is no more than 6. Assume without loss of generality that Maximizer's first move is onto vertex v_1. Minimizer's strategy is then as follows.

(i) If $X_1 = (1, v_1)$ or $X_1 = (2, v_1)$, then set $N_1 = (2, v_2)$ or $N_1 = (1, v_2)$ respectively. This will ensure that no vertex will have the 3 and 4 assigned vertices in its open neighborhood. Hence the maximum open neighborhood sum will be no more than 6.

(ii) If $X_1 = (3, v_1)$ or $X_1 = (4, v_1)$, then set $N_1 = (1, v_3)$. This will also ensure that no vertex will have the 3 and 4 assigned vertices in its open neighborhood. Hence the maximum open neighborhood sum will be no more than 6. □

Theorem 5.27. $NS_{X(A),N(T)}(C_4) = 6$.

Proof. From Theorem 5.25 we can deduce that $NS_{X(A),N(T)}(C_4) \geq 6$. Thus we need to demonstrate a strategy whereby Minimizer can ensure that the outcome of the game is no more than 6. Assume without loss of generality that Maximizer's first move is onto vertex v_1. Minimizer's strategy is then as follows.

(i) If $X_1 = (1, v_1)$ or $X_1 = (2, v_1)$, then set $N_1 = (4, v_3)$. This will ensure that no vertex will have the 3 and 4 assigned vertices in its open neighborhood. Hence the maximum open neighborhood sum will be no more than 6.

(ii) If $X_1 = (3, v_1)$ or $X_1 = (4, v_1)$, then set $N_1 = (4, v_2)$ or $N_1 = (3, v_2)$ respectively. This will also ensure that no vertex will have the 3 and 4 assigned vertices in its open neighborhood. Hence the maximum open neighborhood sum will be no more than 6. □

Theorem 5.28. $NS_{X(A),N(A)}(C_4) = 5$.

Proof. As in Theorem 5.21 we only need to demonstrate a strategy for Minimizer whereby she can achieve a game outcome of 5. Assume without loss of generality that Maximizer's first move is onto vertex v_1. Minimizer's strategy is then as follows.

(i) If $X_1 = (1, v_1)$ set $N_1 = (4, v_3)$ and all vertices will have open neighborhood sums of 5.

(ii) If $X_1 = (2, v_1)$ set $N_1 = (3, v_3)$ and all vertices will have open neighborhood sums of 5.

(iii) If $X_1 = (3, v_1)$ set $N_1 = (2, v_3)$ and all vertices will have open neighborhood sums of 5.

(iv) If $X_1 = (4, v_1)$ set $N_1 = (1, v_3)$ and all vertices will have open neighborhood sums of 5. $\square$

Next we consider the maximum open neighborhood sum game played on C_4 where Minimizer plays first. Notice that the maximum possible value of an open neighborhood sum will be 7. Hence, when proving that the outcome of a game is 7, we only need to demonstrate that Maximizer can achieve such a value.

Theorem 5.29. $NS_{N(B),X(B)}(C_4) = 7$.

Proof. Assume without loss of generality that $N_1 = (1, v_1)$. Then Maximizer sets $X_1 = (2, v_3)$ and the open neighborhood sums of v_1 and v_3 will be $3 + 4 = 7$. $\square$

Corollary 5.30. $NS_{N(B),X(A)}(C_4) = 7$.

Theorem 5.31. $NS_{N(B),X(T)}(C_4) = 6$.

Proof. Assume without loss of generality that $N_1 = (1, v_1)$. If Maximizer sets $X_1 = (4, v_2)$, then the open neighborhood sum of v_1 will be at least 6. If $X_1 = (4, v_2)$ or $X_1 = (4, v_4)$, then Minimizer can set $N_2 = (2, v_4)$ or $N_2 = (2, v_2)$, respectively, to ensure that the maximum open neighborhood sum is no more than 6. If $X_1 = (4, v_3)$, then all open neighborhood sums will equal 5. □

Theorem 5.32. $NS_{N(T),X(B)}(C_4) = 6$.

Proof. Assume without loss of generality that $N_1 = (4, v_1)$. If Maximizer sets $X_1 = (1, v_2)$, then the open neighborhood sum of v_2 will be at least 6. If $X_1 = (1, v_1)$ or $X_1 = (1, v_4)$, then Minimizer can set $N_2 = (3, v_4)$ or $N_2 = (3, v_1)$, respectively, to ensure that the maximum open neighborhood sum is no more than 6. If $X_1 = (1, v_3)$, then all open neighborhood sums will equal 5. □

Theorem 5.33. $NS_{N(T),X(T)}(C_4) = 7$.

Proof. Assume without loss of generality that $N_1 = (4, v_1)$. Then by setting $X_1 = (3, v_3)$, Maximizer will achieve an open neighborhood sum of 7 on vertex v_2. □

Corollary 5.34. $NS_{N(T),X(A)}(C_4) = 7$.

Theorem 5.35. $NS_{N(A),X(B)}(C_4) = 6$.

Proof. From Theorem 5.32 it follows that $NS_{N(A),X(B)}(C_4) \leq 6$. Hence, we need to demonstrate a strategy whereby Maximizer can ensure that the outcome of the game is at least 6. In order to achieve this he simply needs to ensure that no open neighborhood is labeled with the 1 and the 4. Assume without loss of generality that Minimizer's first move is onto

vertex v_1. If $N_1 = (1, v_1)$ or $N_1 = (2, v_1)$, then he can set $X_1 = (2, v_3)$ or $X_1 = (1, v_3)$, respectively, and achieve an outcome of 7. If $N_1 = (3, v_1)$ or $N_1 = (4, v_1)$ he can set $X_1 = (1, v_3)$ or $X_1 = (1, v_2)$. In either case, there will be a vertex with open neighborhood sum of at least 6. □

Theorem 5.36. $NS_{N(A),X(T)}(C_4) = 6$.

Proof. From Theorem 5.31 it follows that $NS_{N(A),X(B)}(C_4) \leq 6$. Hence, we need to demonstrate a strategy whereby Maximizer can ensure that the outcome of the game is at least 6. In order to achieve this he simply needs to ensure that no open neighborhood is labeled with the 1 and the 4. Assume without loss of generality that Minimizer's first move is onto vertex v_1. If $N_1 = (1, v_1)$, then he can set $X_1 = (4, v_2)$. If $N_1 = (2, v_1)$, $N_1 = (3, v_1)$ or $N_1 = (4, v_1)$ he can set $X_1 = (4, v_3)$, $X_1 = (4, v_3)$ or $X_1 = (3, v_3)$ respectively. In each case, there will be a vertex with open neighborhood sum of at least 6. □

Theorem 5.37. $NS_{N(A),X(A)}(C_4) = 7$.

Proof. Assume without loss of generality that Minimizer makes her first move onto vertex v_1. If $N_1 = (1, v_1)$ or $N_1 = (2, v_1)$, then Maximizer can set $X_1 = (2, v_3)$ or $X_1 = (1, v_3)$, respectively, and vertex v_1 will have open neighborhood sum of 7. If $N_1 = (3, v_1)$ or $N_1 = (4, v_1)$, then Maximizer can set $X_1 = (4, v_3)$ or $X_1 = (3, v_3)$, respectively, and vertex v_2 will have open neighborhood sum of 7. □

For any cycle C_n, notice that the maximum possible value for an open neighborhood sum will be $n + (n-1) = 2n - 1$. In many of the possible game scenarios this value can be easily achieved by Maximizer.

Theorem 5.38. *If $n \geq 5$, then $NS_{X(T),N(B)}(C_n) = 2n-1$.*

Proof. Since $2n-1$ is the maximum possible open neighborhood sum, we must demonstrate a strategy whereby Maximizer can always achieve this value. Maximizer can set $X_1 = (n, v_1)$. Since $n \geq 5$, after Minimizer's first move, there is a vertex at distance 2 from $v_1$that is unlabeled, namely v_3 or v_{n-1}. Since Minimizer cannot play the $n-1$ on her first move, Maximizer can play the $n-1$ on his second move on v_3 or v_{n-1} and vertex v_2 or v_n, respectively, will have an open neighborhood sum of $2n-1$. □

Corollary 5.39. *If $n \geq 5$, then $NS_{X(A),N(B)}(C_n) = 2n-1$.*

Theorem 5.40. *If $n \geq 5$, then $NS_{X(A),N(T)}(C_n) = 2n-1$.*

Proof. Since $2n-1$ is the maximum possible open neighborhood sum, we must demonstrate a strategy whereby Maximizer can always achieve this value. Maximizer can set $X_1 = (1, v_1)$. Since $n \geq 5$, Minimizer's first move must place the label n onto a vertex for which another vertex at distance 2 is not yet labeled. On Maximizer's second move he can place the label $n-1$ onto the unlabeled vertex at distance 2, and create an open neighborhood sum of $2n-1$. □

Theorem 5.41. *If $n \geq 5$, then $NS_{X(T),N(T)}(C_n) = 2n-2$.*

Proof. Assume without loss of generality that $X_1 = (n, v_1)$. In order to ensure that no open neighborhood sum exceeds $2n-2$, Minimizer can set $N_1 = (n-1, v_2)$. Since $n \geq 5$, after Minimizer's first move, either v_3 or v_{n-2} is unlabeled. Maximizer can ensure that the outcome of the game is at least $2n-2$ by playing setting $X_2 = (n-2, v_3)$ or $X_2 = (n-2, v_{n-2})$. □

Theorem 5.42. *If $n \geq 5$, then $NS_{X(T),N(A)}(C_n) = 2n-2$.*

Proof. By Theorem 5.41 $NS_{X(T),N(A)}(C_n) \leq 2n-2$, hence, we need to demonstrate a strategy whereby Maximizer can ensure that the outcome of the game is at least $2n-2$. Assume without loss of generality that $X_1 = (n, v_1)$. Since $n \geq 5$, after Minimizer's first move, either v_3 or v_{n-2} is unlabeled, and one of the labels $n-1$ or $n-2$ is available. Maximizer can ensure that the outcome of the game is at least $2n-2$ by playing largest available label either v_3 or v_{n-2}. □

Theorem 5.43. *If $n \geq 5$ and n is odd, then $NS_{X(A),N(A)}(C_n) = 2n-1$.*

Proof. Since $2n-1$ is the maximum possible open neighborhood sum, we must demonstrate a strategy for Maximizer whereby he can achieve this value. Maximizer's strategy will depend on whether $n \equiv 1(\text{Mod } 4)$ or $n \equiv 3(\text{Mod } 4)$.

First consider the case where $n \equiv 1(\text{Mod } 4)$. Maximizer first sets $X_1 = (1, v_1)$. Maximizer then creates the pairs of vertices $S_1 = \{v_2, v_4\}$, $S_2 = \{v_6, v_8\}$,...,$S_{\frac{n-1}{4}} = \{v_{n-3}, v_{n-1}\}$, $S_{\frac{n-1}{4}+1} = \{v_3, v_5\}$, $S_{\frac{n-1}{4}+2} = \{v_7, v_9\}$,..., $S_{\frac{n-1}{2}} = \{v_{n-2}, v_n\}$. After Minimizer makes her move at each turn, if she plays the n or $n-1$ on one of the vertices in set S_k, then Maximizer will play the $n-1$ or n, respectively, on the other vertex in set S_k. This will create an open neighborhood sum of $2n-1$ on the vertex whose open neighborhood consists of the vertices in set S_k. If on her move, Minimizer plays something other than the n or $n-1$ on a vertex in set S_k, then Maximizer will play the smallest available label on the other vertex in set S_k. In this manner, at some turn, Minimizer will be forced to play either the n or the $n-1$, at which time Maximizer can create an open neighborhood sum of $2n-1$.

Next consider the case where $n \equiv 3(\text{Mod } 4)$. Maximizer first sets $X_1 = (1, v_1)$. Maximizer then creates the pairs of vertices $S_1 = \{v_3, v_5\}$, $S_2 = \{v_7, v_9\}, \ldots, S_{\frac{n-1}{4}} = \{v_{n-4}, v_{n-2}\}$, $S_{\frac{n-1}{4}+1} = \{v_n, v_2\}$, $S_{\frac{n-1}{4}+2} = \{v_4, v_6\}, \ldots, S_{\frac{n-1}{2}} = \{v_{n-3}, v_{n-1}\}$. In a similar manner as above, after Minimizer makes her move at each turn, if she plays the n or $n-1$ on one of the vertices in set S_k, then Maximizer will play the $n-1$ or n, respectively, on the other vertex in set S_k. This will create an open neighborhood sum of $2n-1$ on the vertex whose open neighborhood consists of the vertices in set S_k. If on her move, Minimizer plays something other than the n or $n-1$ on a vertex in set S_k, then Maximizer will play the smallest available label on the other vertex in set S_k. In this manner, at some turn, Minimizer will be forced to play either the n or the $n-1$, at which time Maximizer can create an open neighborhood sum of $2n-1$. □

Theorem 5.44. *If $n \geq 5$ and n is even, then $NS_{X(A),N(A)}(C_n) = 2n-2$.*

Proof. From Theorem 5.42 we know that $NS_{X(A),N(A)}(C_n) \geq 2n-2$, hence we need to demonstrate a strategy whereby Minimizer can ensure that the outcome of the game is no more than $2n-2$. Prior to the start of the game she pairs the vertices into the sets $S_1 = \{v_1, v_2\}$, $S_2 = \{v_3, v_4\}, \ldots, S_{\frac{n}{2}} = \{v_{n-1}, v_n\}$. At each turn if Maximizer plays the n or the $n-1$ on a vertex from set S_k, she will play the $n-1$ or n, respectively, on the other vertex in set S_k. If Maximizer plays some other label other than the n and $n-1$ on a vertex from set S_k, Minimizer will play the smallest available label on the other vertex from the set S_k. In this way she will ensure that no vertex has labels n and $n-1$ in its open neighborhood. Therefore the maximum open neighborhood sum will not exceed $2n-2$. □

Theorem 5.45. *If $n \geq 5$ and n is odd, then $NS_{X(B),N(B)}(C_n) = 2n-1$.*

Proof. Maximizer's strategy in this case is similar to that in Theorem 5.43. The only difference is how the game will actually proceed. In this case, Minimizer will play the $n-1$ on her last move and Maximizer will follow it by playing the n and creating an open neighborhood sum of $2n-1$. $\square$

Theorem 5.46. *If $n \geq 6$ and n is even, then $NS_{X(B),N(B)}(C_n) = 2n-2$.*

Proof. First we demonstrate a strategy that Minimizer can use to ensure that the outcome of the game is no more than $2n-2$. Prior to the start of the game she creates the sets $S_1 = \{v_1, v_2\}$, $S_2 = \{v_3, v_4\}, \ldots, S_{\frac{n}{2}} = \{v_{n-1}, v_n\}$. At each turn of the game, if Maximizer plays onto a vertex from a set S_k, then Minimizer will follow by playing onto the other vertex from set S_k. In this manner, there will be a set of vertices whose labels are $n-1$ and n. Hence, there will not be a vertex whose open neighborhood contains both the $n-1$ and n, and so no open neighborhood sum will exceed $2n-2$.

Next we demonstrate a strategy that Maximizer can use to ensure that the outcome of the game is at least $2n-2$. First he sets $X_1 = (1, v_1)$. If $n \equiv 0(\text{Mod } 4)$, he then pairs the vertices into sets $S_1 = \{v_2, v_4\}$, $S_2 = \{v_3, v_5\}$, $S_3 = \{v_6, v_8\}, \ldots, S_{\frac{n-2}{2}} = \{v_{n-2}, v_n\}$, and leaves the vertex v_{n-1} unpaired with any other vertex. If $n \equiv 2(\text{Mod } 4)$, he pairs the vertices into sets $S_1 = \{v_2, v_4\}$, $S_2 = \{v_3, v_5\}$, $S_3 = \{v_6, v_8\}, \ldots, S_{\frac{n-2}{2}} = \{v_{n-3}, v_{n-1}\}$, leaving the vertex v_n unpaired. On all moves until his last, if Minimizer has played onto a vertex that is paired, Maximizer will follow by playing onto that vertex's pair. On all moves until his last, if Minimizer plays onto an unpaired vertex, Maximizer can play on any of the unlabeled vertices, which will be part of a pair. When this occurs, the other vertex that was part of the pair, becomes unpaired. In this manner, except for her final move, Minimizer has the

option to play onto a vertex that is part of a pair, or onto a single unpaired vertex. On her next to last move, Minimizer can then choose to play the label $n-2$ onto one of $\{v_i, v_{i+2}\}$ or onto the unpaired vertex v_k. If she plays onto the unpaired vertex v_k, then the $n-1$ and n will be played on vertices v_i and v_{i+2}, and vertex v_{i+1} will have open neighborhood sum $2n-1$. If she plays onto either v_i or v_{i+2}, then Maximizer will place $n-1$ on vertex v_k on his final move. Minimizer's final move will then be to play the n onto the other vertex in the pair, and vertex v_i will have open neighborhood sum $2n-2$. □

Theorem 5.47. *If $n \geq 5$, then $NS_{X(B),N(T)}(C_n) \geq n + \left\lceil \frac{n}{2} \right\rceil$.*

Proof. Maximizer can set $X_1 = (1, v_1)$. Minimizer will then make her first move as $N_1 = (n, v_i)$. Since $n \geq 5$, there will exist a vertex $u \in V(C_n)$ whose open neighborhood contains v_i and another vertex w that is yet to be labeled. On each move until his last, Minimizer will play on a vertex other than w. If on his last move, vertex w has already been labeled, it was labeled by Minimizer with a label of at least $\left\lceil \frac{n}{2} \right\rceil + 1$, in which case the open neighborhood sum of vertex u will be greater than $n + \left\lceil \frac{n}{2} \right\rceil$. If vertex w has not been labeled on his last move, Maximizer will label vertex w with $\left\lceil \frac{n}{2} \right\rceil$, and the open neighborhood sum of vertex u will be $n + \left\lceil \frac{n}{2} \right\rceil$. □

Next we look at the maximum open neighborhood sum game where Minimizer plays first.

Theorem 5.48. $NS_{N(T),X(T)}(C_n) = 2n - 1$.

Proof. Assume without loss of generality that $N_1 = (n, v_1)$. Then Maximizer sets $X_1 = (n-1, v_3)$ and the open neighborhood sum of vertex v_2 will be $2n-1$. □

Corollary 5.49. $NS_{N(T),X(A)}(C_n) = 2n-1$.

Theorem 5.50. *If $n \geq 5$, then $NS_{N(B),X(T)}(C_n) = 2n-1$.*

Proof. Assume without loss of generality that $N_1 = (1, v_1)$. Maximizer can set $X_1 = (n, v_2)$ and since $n \geq 5$, there will exists 2 vertices at distance 2 from v_2 that have not been labeled, namely v_4 and v_n. On his second move, Maximizer can play the $n-1$ on one of these vertices and create an open neighborhood sum of $2n-1$. $\square$

Corollary 5.51. *If $n \geq 5$, then $NS_{N(B),X(A)}(C_n) = 2n-1$.*

Theorem 5.52. *If $n \geq 5$, then $NS_{N(A),X(T)}(C_n) = 2n-2$.*

Proof. Assume without loss of generality that Minimizer makes her first move onto vertex v_1. First we demonstrate a strategy she can use to ensure that the outcome of the game is no more than $2n-2$. At the start of the game she sets $N_1 = (1, v_1)$. Maximizer must place the label n on his first move. On her second move, Minimizer simply places the label $n-1$ on a vertex adjacent to Maximizer's first move. In this manner, no vertex will have both the n and $n-1$ assigned to vertices in its open neighborhood, and the outcome of the game will be no more than $2n-2$.

Next we need to demonstrate a strategy whereby Maximizer can ensure that the outcome of the game is at least $2n-2$. If $N_1 = (n, v_1)$, then he can set $X_1 = (n-1, v_3)$ and vertex v_2 will have open neighborhood sum of $2n-1$. If $N_1 \neq (n, v_1)$, then Maximizer can set $X_1 = (n, v_2)$. Since $n \geq 5$, there will be 2 unlabeled vertices at distance 2 from vertex v_2, namely v_4 and v_n. On his second move, Maximizer can play a label at least as big as $n-2$ on either v_4 or v_n to create an open neighborhood sum of $2n-2$. $\square$

Theorem 5.53. *If $n \geq 5$ and n is odd, then $NS_{N(A),X(A)}(C_n) = 2n-2$.*

Proof. From Theorem 5.52 we can deduce that $NS_{N(A),X(A)}(C_n) \geq 2n-2$, hence, we need to demonstrate a strategy whereby Minimizer can ensure that the outcome of the game is no more than $2n-2$. For her first move she sets $N_1 = (1, v_1)$ and then creates the pairs of vertices $S_1 = \{v_2, v_3\}$, $S_2 = \{v_4, v_5\}$,..., $S_{\frac{n-1}{2}} = \{v_{n-1}, v_n\}$. On each turn if Maximizer plays the n or $n-1$ on a vertex from set S_k, then she will play the $n-1$ or n, respectively, on the other vertex from set S_k. If Maximizer plays some label other than n or $n-1$ on a vertex in set S_k, then Minimizer will play the smallest available label on the other vertex from set S_k. In this way Minimizer ensures that the n and $n-1$ are assigned to adjacent vertices, and hence, no vertex will have open neighborhood sum in excess of $2n-2$. □

Theorem 5.54. *If $n \geq 5$ and n is even, then $NS_{N(A),X(A)}(C_n) = 2n-1$.*

Proof. We must demonstrate a strategy whereby Maximizer can achieve an open neighborhood sum of $2n-1$. His strategy will depend on whether $n \equiv 0(\text{Mod } 4)$ or $n \equiv 2(\text{Mod } 4)$.

First consider the case where $n \equiv 0(\text{Mod } 4)$. Maximizer pairs the vertices into the sets $S_1 = \{v_1, v_3\}$, $S_2 = \{v_5, v_7\}, \ldots, S_{\frac{n}{4}} = \{v_{n-3}, v_{n-1}\}$, $S_{\frac{n}{4}+1} = \{v_2, v_4\}$, $S_{\frac{n}{4}+2} = \{v_6, v_8\}, \ldots,$ $S_{\frac{n}{2}} = \{v_{n-2}, v_n\}$. At each turn, if Minimizer has played the n or $n-1$ on a vertex from set S_k, then Maximizer will play the $n-1$ or n, respectively, on the other vertex from set S_k. If Minimizer plays a label other than n or $n-1$ on a vertex in set S_k, then Maximizer will play the smallest label available. In this manner, at some turn, Minimizer will be forced to play the n or $n-1$, at which time Maximizer will follow by playing the $n-1$ or n, respectively, and he will create an open neighborhood sum of $2n-1$.

Next consider the case where $n \equiv 2(\text{Mod } 4)$. We will assume that Minimizer's first play is onto vertex v_1. Obviously if she plays the n or $n-1$, then Maximizer can play the $n-1$ or n, respectively, and create an open neighborhood sum of $2n-1$. If Minimizer plays some label other than n or $n-1$ on her first move, then Maximizer will set assign the smallest available label to vertex v_2 and then pair the remaining vertices into the sets $S_1 = \{v_3, v_5\}$, $S_2 = \{v_7, v_9\}$,..., $S_{\frac{n-2}{4}} = \{v_{n-3}, v_{n-1}\}$, $S_{\frac{n-2}{4}+1} = \{v_4, v_6\}$, $S_{\frac{n-2}{4}+2} = \{v_8, v_{10}\}$,..., $S_{\frac{n-2}{2}} = \{v_{n-2}, v_n\}$. At each turn, if Minimizer has played the n or $n-1$ on a vertex from set S_k, then Maximizer will play the $n-1$ or n, respectively, on the other vertex from set S_k. If Minimizer plays a label other than n or $n-1$ on a vertex in set S_k, then Maximizer will play the smallest label available on the other vertex from set S_k. In this manner, at some turn, Minimizer will be forced to play the n or $n-1$, at which time Maximizer will follow by playing the $n-1$ or n, respectively, and he will create an open neighborhood sum of $2n-1$. □

Theorem 5.55. *If $n \geq 5$ and n is even, then $NS_{N(B),X(B)}(C_n) = 2n-1$.*

Proof. Maximizer's strategy is similar to that shown in Theorem 5.54. The only difference in this game is that labels will be played from smallest to largest. When the $n-1$ and n are played during the last round of the game, they will create an open neighborhood sum of $2n-1$. □

Theorem 5.56. *If $n \geq 5$ and n is odd, then $NS_{N(B),X(B)}(C_n) = 2n-2$.*

Proof. First we demonstrate a strategy that Minimizer can use to ensure that the outcome of the game is no more than $2n-2$. First she sets $N_1 = (1, v_1)$, and then she creates the sets $S_1 = \{v_2, v_3\}$, $S_2 = \{v_4, v_5\}$,..., $S_{\frac{n-1}{2}} = \{v_{n-1}, v_n\}$. At each turn of the game, if Maximizer

plays onto a vertex from a set S_k, then Minimizer will follow by playing onto the other vertex from set S_k. In this manner, there will be a set of vertices whose labels are $n-1$ and n. Hence, there will not be a vertex whose open neighborhood contains both the $n-1$ and n, and so no open neighborhood sum will exceed $2n-2$.

Next we demonstrate a strategy that Maximizer can use to ensure that the outcome of the game is at least $2n-2$. Assume without loss of generality that $N_1 = (1, v_1)$. Maximizer will set $X_1 = (2, v_3)$. If $n \equiv 1(\text{Mod } 4)$, he then pairs the vertices into sets $S_1 = \{v_2, v_4\}$, $S_2 = \{v_5, v_7\}$, $S_3 = \{v_6, v_8\}, \ldots$, $S_{\frac{n-3}{2}} = \{v_{n-3}, v_{n-1}\}$, and leaves the vertex v_n unpaired with any other vertex. If $n \equiv 3(\text{Mod } 4)$, he pairs the vertices into sets $S_1 = \{v_2, v_4\}$, $S_2 = \{v_5, v_7\}$, $S_3 = \{v_6, v_8\}, \ldots$, $S_{\frac{n-2}{2}} = \{v_{n-2}, v_n\}$, leaving the vertex v_{n-1} unpaired. On all moves until his last, if Minimizer has played onto a vertex that is paired, Maximizer will follow by playing onto that vertex's pair. On all moves until his last, if Minimizer plays onto an unpaired vertex, Maximizer can play on any of the unlabeled vertices, which will be part of a pair. When this occurs, the other vertex that was part of the pair, becomes unpaired. In this manner, except for her final move, Minimizer has the option to play onto a vertex that is part of a pair, or onto a single unpaired vertex. On her next to last move, Minimizer can then choose to play the label $n-2$ onto one of $\{v_i, v_{i+2}\}$ or onto the unpaired vertex v_k. If she plays onto the unpaired vertex v_k, then the $n-1$ and n will be played on vertices v_i and v_{i+2}, and vertex v_{i+1} will have open neighborhood sum $2n-1$. If she plays onto either v_i or v_{i+2}, then Maximizer will place $n-1$ on vertex v_k on his final move. Minimizer's final move will then be to play the n onto the other vertex in the pair, and vertex v_i will have open neighborhood sum $2n-2$. $\square$

Theorem 5.57. *If $n \geq 5$, then $NS_{N(T),X(B)}(C_n) \geq n + \left\lfloor \frac{n}{2} \right\rfloor + 1$.*

Proof. Assume without loss of generality that $N_1 = (n, v_1)$. Notice that Minimizer's last move will be to label a vertex with $n + \left\lfloor \frac{n}{2} \right\rfloor + 1$. On each of his moves prior to his final move, Maximizer will then play on any vertex other than v_3 and v_{n-1}. Since $n \geq 5$, we know that $v_3 \neq v_{n-1}$. If Minimizer has labeled either v_3 or v_{n-1} prior to her final move, then either v_2 or v_n, respectively, will have open neighborhood sum greater than $n + \left\lfloor \frac{n}{2} \right\rfloor + 1$. Otherwise, at least one of v_3 or v_{n-1} will remain unlabeled when Minimizer makes her final move, and either v_2 or v_n will have open neighborhood sum equal $n + \left\lfloor \frac{n}{2} \right\rfloor + 1$. □

By applying Corollary 5.16 to the results of this section, we state the known values for the minimum open neighborhood sum game on C_n in Table 5.2.

5.4 Closed Neighborhood Sums on Cycles

In this section we focus on determining the values for the maximum closed neighborhood sum games on cycle C_n. Once these values are determined, then Corollary 5.16 can be applied to determine the corresponding values for the minimum closed neighborhood sum games. Table 5.3 summarizes the results we will prove in this section. The entries that have not been completed are open problems.

Theorem 5.58. *For cycle C_3 we have*

(i) $NS^-_{X(*),N(*)}[C_3] = NS^-_{N(*),X(*)}[C_3] = NS_{X(*),N(*)}[C_3] = NS_{N(*),X(*)}[C_3] = 6$, and

(ii) $NS^{sp}_{X(*),N(*)}[C_3] = NS^{sp}_{N(*),X(*)}[C_3] = 0$.

Proof. For any bijection $f : V(C_3) \to [3]$ we have that $NS^-[f] = NS[f] = 6$ and $NS^{sp}[f] = 0$. The results of this theorem follow directly from this fact. □

Table 5.2: Value of minimum open neighborhood sum game on C_n

	Order of Graph			
Scenario	3	4	$n \geq 5$, n odd	$n \geq 6$, n even
X(B),N(B)	3	4	3	4
X(B),N(T)	3	5	$\leq n + \lfloor \frac{n}{2} \rfloor + 2$	$\leq n + \lfloor \frac{n}{2} \rfloor + 2$
X(B),N(A)	3	5		
X(T),N(B)	3	5	3	3
X(T),N(T)	3	4	4	4
X(T),N(A)	3	5	4	4
X(A),N(B)	3	4	3	3
X(A),N(T)	3	4	3	3
X(A),N(A)	3	5	3	4
N(B),X(B)	3	3	4	3
N(B),X(T)	3	4	3	3
N(B),X(A)	3	3	3	3
N(T),X(B)	3	4	$\leq n + \lceil \frac{n}{2} \rceil + 1$	$\leq n + \lceil \frac{n}{2} \rceil + 1$
N(T),X(T)	3	3	3	3
N(T),X(A)	3	3	3	3
N(A),X(B)	3	4		
N(A),X(T)	3	4	4	4
N(A),X(A)	3	3	4	3

Theorem 5.59. *For cycle C_4 we have*

(i) $NS^-_{X(*),N(*)}[C_4] = NS^-_{N(*),X(*)}[C_4] = 6$,

(ii) $NS_{X(*),N(*)}[C_4] = NS_{N(*),X(*)}[C_4] = 9$, *and*

(iii) $NS^{sp}_{X(*),N(*)}[C_4] = NS^{sp}_{N(*),X(*)}[C_4] = 3$.

Proof. Let $f : V(C_4) \to [4]$ be an arbitrary bijection. There exists a vertex $u \in V(C_4)$ such that $f(N[u]) = 1+2+3 = 6$. There also exists a vertex $v \in V(C_4)$ such that $f(N[v]) = 2+3+4 = 9$. Thus $NS^-[f] = 6$, $NS[f] = 9$, and $NS^{sp}[f] = 3$. Since f was an arbitrary

Table 5.3: Value of maximum closed neighborhood sum game on C_n

	Order of Graph					
Scenario	3	4	5	6	$n \geq 7$, n odd	$n \geq 8$, n even
X(B),N(B)	6	9	11	12		
X(B),N(T)	6	9	12	13		$\leq 2n+1$
X(B),N(A)	6	9	11	12		
X(T),N(B)	6	9	11	13	$2n+1$	$2n+1$
X(T),N(T)	6	9	12	12	$3n-6$	$3n-6$
X(T),N(A)	6	9	11	12	$2n+1$	$2n+1$
X(A),N(B)	6	9	12	14		$\geq 2n+2$
X(A),N(T)	6	9	12	14	$\geq 3n-6$	$\geq 3n-6$
X(A),N(A)	6	9	12	12	$\geq 2n+2$	$2n+1$
N(B),X(B)	6	9	12	14		
N(B),X(T)	6	9	11	14	$2n+2$	$2n+2$
N(B),X(A)	6	9	12	15	$\geq 2n+2$	$\geq 2n+2$
N(T),X(B)	6	9	12	14		
N(T),X(T)	6	9	11	14	$3n-4$	$3n-4$
N(T),X(A)	6	9	12	15	$3n-4$	$3n-4$
N(A),X(B)	6	9	11	14	$\geq 2n+2$	
N(A),X(T)	6	9	11	13		
N(A),X(A)	6	9	11	15		

bijection, the labeling resulting from a closed neighborhood sum game on C_4 will yield the same values. Therefore, each of the results in the theorem follow. $\square$

We will now look at all maximum closed neighborhood sum games played on C_5. Notice that 12 is the maximum possible closed neighborhood sum across all labelings of the C_5. This value will be achieved if and only if there is a vertex with labels 3, 4, and 5 in its closed neighborhood, or equivalently, if and only if the 1 and the 2 are placed on two adjacent vertices.

Theorem 5.60. $NS_{X(B),N(B)}[C_5] = NS_{X(B),N(A)}[C_5] = 11$.

Proof. Assume without loss of generality that $X_1 = (1, v_1)$. In order to ensure that the outcome of the game is not 12, Minimizer must set $N_1 = (2, v_3)$ or $N_1 = (2, v_4)$. So assume again without loss of generality that $N_1 = (2, v_3)$. Maximizer can then achieve an outcome of 11 by setting $X_2 = (3, v_2)$. In this case the final closed neighborhood sum of v_4 will be 11. $\square$

Theorem 5.61. $NS_{X(B),N(T)}[C_5] = 12$.

Proof. In order to achieve the maximum possible value of 12, Maximizer can set $X_1 = (1, v_1)$ and then regardless of how Minimizer plays, place the 2 adjacent to v_1 on his second move. This ensures that there will be a vertex with labels 3, 4, and 5 in its closed neighborhood at the end of the game. $\square$

Theorem 5.62. $NS_{X(T),N(B)}[C_5] = 11$.

Proof. We first demonstrate a strategy for Minimizer by which she can ensure that the outcome of the game is no more than 11. Suppose, without loss of generality, that $X_1 = (5, v_1)$. Then Minimizer's strategy is as follows:

(i) Set $N_1 = (1, v_2)$.

(ii) If $X_2 = (4, v_5)$, then set $N_2 = (2, v_4)$. If $X_2 = (4, v_3)$ or $X_2 = (4, v_4)$, then set $N_2 = (2, v_5)$.

Next we need to demonstrate a strategy for Maximizer which he can follow to always achieve at least 11. Maximizer's strategy is as follows:

(i) Set $X_1 = (5, v_1)$.

(ii) On the second move, Maximizer plays the 4 adjacent to the 5. Hence, there will be a closed neighborhood sum of at least $2+4+5 = 11$. $\square$

Theorem 5.63. $NS_{X(T),N(T)}[C_5] = 12$.

Proof. Maximizer's strategy in this case is to set $X_1 = (5, v_1)$. Since Minimizer must play the 4 within distance two of v_1, Maximizer simply plays the 3 on his second move in a way to create a closed neighborhood that contains the 3, 4, and 5. $\square$

Theorem 5.64. $NS_{X(T),N(A)}[C_5] = 11$.

Proof. It follow from Theorem 5.62 that $NS_{X(T),N(A)}[C_5] \leq NS_{X(T),N(B)}[C_5] = 11$. Hence, we need to demonstrate a strategy for Maximizer which he can follow to always achieve at least 11. Maximizer's strategy is as follows:

(i) Set $X_1 = (5, v_1)$.

(ii) If $N_1 = (4, *)$ or $N_1 = (3, *)$, then on his second move, Maximizer should play the 3 or the 4, respectively, such that there is a vertex that has labels 3, 4, and 5 in its closed neighborhood. In this case, the value of the game will be 12.

(iii) If $N_1 = (1, *)$ or $N_1 = (2, *)$, then on his second move, Maximizer should play the 4 adjacent to the 5. This will ensure a closed neighborhood sum of at least $2+4+5 = 11$. $\square$

Theorem 5.65. $NS_{X(A),N(*)}[C_5] = 12$.

Proof. We must demonstrate a strategy Maximizer can follow to ensure that the outcome is 12. Maximizer's strategy is as follows:

(i) Set $X_1 = (5, v_1)$.

(ii) If Minimizer plays the 1 or the 2 on her first move, then Maximizer plays the 2 or 1, respectively, adjacent to Minimizer's move. This ensures that there will be a closed neighborhood containing the 3, 4 , and 5.

(iii) If Minimizer plays the 3 or the 4 on her first move, it must be played within distance two of the 5. In this case, Maximizer can play the 4 or the 3, respectively, and creates a closed neighborhood containing the 3, 4, and 5. □

Next we consider the maximum closed neighborhood sum games played on C_5 where Minimizer plays first.

Theorem 5.66. $NS_{N(B),X(B)}[C_5] = NS_{N(B),X(A)}[C_5] = 12.$

Proof. For this game Minimizer must play the 1 on her first move. In order for Maximizer to achieve a game outcome of 12, he simply places the 2 adjacent to the 1. □

Theorem 5.67. $NS_{N(B),X(T)}[C_5] = 11.$

Proof. First we show that Minimizer can ensure a game outcome of no more than 11. In order to accomplish this, Minimizer sets $N_1 = (1, v_1)$. Since Maximizer must play the 5 on his first move, on Minimizer's second move, she can either set $N_2 = (2, v_3)$ or $N_2 = (2, v_4)$. Either of these move will ensure that the 1 and the 2 are not adjacent, and hence, the outcome of the game will be no more than 11.

Next we demonstrate a strategy for Maximizer whereby he can achieve an outcome of at least 11. Without loss of generality, assume that $N_1 = (1, v_1)$. Maximizer then sets $X_1 = (5, v_3)$. If Minimizer sets $N_2 = (2, v_2)$, then the outcome of the game will be 12. If Minimizer makes any other move on her second turn, then Maximizer sets $X_2 = (4, v_2)$ and then the closed neighborhood sum of vertex v_3 at the end of the game will be at least 11. □

Theorem 5.68. $NS_{N(T),X(B)}[C_5] = NS_{N(T),X(A)}[C_5] = 12.$

Proof. Assume without loss of generality that $N_1 = (5, v_1)$. Then Maximizer sets $X_1 = (1, v_3)$. If $N_2 = (4, v_2)$, then Maximizer sets $X_2 = (2, v_4)$. If Minimizer makes any other move on his second turn, then Maximizer sets $X_2 = (2, v_2)$. So in both cases, then 1 and the 2 are played on adjacent vertices, and the outcome of the game is 12. $\square$

Theorem 5.69. $NS_{N(T),X(T)}[C_5] = 11$.

Proof. First we show that Minimizer can ensure a game outcome of no more than 11. In order to accomplish this, Minimizer sets $N_1 = (5, v_1)$. Minimizer then makes her second move as follows:

(i) If $X_1 = (4, v_2)$ or $X_1 = (4, v_3)$, then set $N_2 = (3, v_4)$.

(ii) Else if $X_1 = (4, v_4)$ or $X_1 = (4, v_5)$, then set $N_2 = (3, v_3)$.

In each case, there will not be a vertex that has labels 3, 4, and 5 assigned to the vertices in its closed neighborhood, and hence, the outcome of the game will be no more than 11.

Next, we need to demonstrate a strategy for Maximizer whereby he can ensure that the outcome of the game is at least 11. Assume without loss of generality that $N_1 = (5, v_1)$. Then Maximizer can set $X_1 = (4, v_2)$ and then, at the end of the game one of v_1 or v_2 will have closed neighborhood sum of at least 11. $\square$

Theorem 5.70. $NS_{N(A),X(B)}[C_5] = 11$.

Proof. First we show that Minimizer can ensure a game outcome of no more than 11. In order to accomplish this, Minimizer sets $N_1 = (5, v_1)$. Minimizer then makes her second move as follows:

(i) If $X_1 = (1, v_2)$ or $X_1 = (1, v_3)$, then set $N_2 = (2, v_5)$.

(ii) Else if $X_1 = (1, v_4)$ or $X_1 = (1, v_5)$, then set $N_2 = (2, v_2)$.

In this way, Minimizer ensures that the 1 and the 2 are not placed on adjacent vertices, and hence, the outcome of the game will be no more than 11.

Next, we need to demonstrate a strategy whereby Maximizer can ensure a game outcome of at least 11. Without loss of generality, we can assume that Minimizer's first move is on vertex v_1. Maximizer makes his first move as follows:

(i) If $N_1 = (1, v_1)$ or $N_1 = (2, v_1)$, then set $X_1 = (2, v_2)$ or $X_1 = (1, v_2)$, respectively, and the outcome of the game will be 12.

(ii) If $N_1 = (3, v_1)$, then set $X_1 = (1, v_2)$ and then at the end of the game, vertex v_4 will have closed neighborhood sum of 11.

(iii) If $N_1 = (4, v_1)$ or $N_1 = (5, v_1)$, then set $X_1 = (1, v_3)$. If $N_2 = (2, v_5)$, then either vertex v_1 or v_5 will have closed neighborhood sum of at least 11. If $N_2 \neq (2, v_5)$, then the 1 and the 2 will be on adjacent vertices, and the outcome of the game will be 12. □

Theorem 5.71. $NS_{N(A),X(T)}[C_5] = 11$.

Proof. It follows from Theorem 5.69 that $NS_{N(A),X(T)}[C_5] \leq 11$. We now demonstrate that Maximizer can ensure a game outcome of at least 11. Assume without loss of generality that Minimizer's first move is on vertex v_1. Maximizer makes his first move as follows.

(i) If $N_1 = (4, v_1)$ or $N_1 = (5, v_1)$, then set $X_1 = (5, v_2)$ or $X_1 = (4, v_2)$ respectively. At the end of the game either vertex v_1 or v_2 will have closed neighborhood sum of at least 11.

(ii) If $N_1 = (3, v_1)$, then set $X_1 = (5, v_2)$. If $N_2 \neq (4, v_4)$, then the 1 and the 2 will be placed on adjacent vertices, and the outcome of the game will be 12. If $N_2 = (4, v_4)$, then Maximizer sets $X_2 = (2, v_3)$ and vertex v_3 will have closed neighborhood sum of 11.

(iii) If $N_1 = (1, v_1)$ or $N_1 = (2, v_1)$, then set $X_1 = (5, v_4)$. If $N_2 \neq (4, v_2)$, then the 4 and the 5 will be placed on adjacent vertices, and there will be a vertex with closed neighborhood sum of at least 11. If $N_2 = (4, v_2)$, then Maximizer sets $X_2 = (3, v_3)$ and vertex v_3 will have closed neighborhood sum of 12. □

Theorem 5.72. $NS_{N(A),X(A)}[C_5] = 11$.

Proof. From Theorem 5.71 it follows that $NS_{N(A),X(A)}[C_5] \geq 11$. Thus we need to demonstrate a strategy for Minimizer whereby she can ensure that the outcome of the game is no more than 11. Minimizer's strategy is to set $X_1 = (3, v_1)$. Minimizer will then make her second move as follows.

(i) If $X_1 = (1, v_2)$, $X_1 = (1, v_3)$, $X_1 = (1, v_4)$, or $X_1 = (1, v_5)$, then set $N_2 = (2, v_4)$, $N_2 = (2, v_5)$, $N_2 = (2, v_1)$, or $N_2 = (2, v_3)$ respectively. Each of these moves ensures that the 1 and the 2 are not placed on adjacent vertices, and hence, the outcome of the game can be no more than 11.

(ii) If $X_1 = (2, v_2)$, $X_1 = (2, v_3)$, $X_1 = (2, v_4)$, or $X_1 = (2, v_5)$, then set $N_2 = (1, v_4)$, $N_2 = (1, v_5)$, $N_2 = (1, v_1)$, or $N_2 = (1, v_3)$ respectively. Each of these moves ensures that the 1 and the 2 are not placed on adjacent vertices, and hence, the outcome of the game can be no more than 11.

(iii) If $X_1 = (4, v_2)$, $X_1 = (4, v_3)$, $X_1 = (4, v_4)$, or $X_1 = (4, v_5)$, then set $N_2 = (5, v_4)$, $N_2 = (5, v_5)$, $N_2 = (5, v_1)$, or $N_2 = (5, v_3)$ respectively. Each of these moves ensures that

there will not be a vertex with 3, 4, and 5 in its closed neighborhood, and hence, the outcome of the game can be no more than 11.

(iv) If $X_1 = (5, v_2)$, $X_1 = (5, v_3)$, $X_1 = (5, v_4)$, or $X_1 = (5, v_5)$, then set $N_2 = (4, v_4)$, $N_2 = (4, v_5)$, $N_2 = (4, v_1)$, or $N_2 = (4, v_3)$ respectively. Each of these moves ensures that there will not be a vertex with 3, 4, and 5 in its closed neighborhood, and hence, the outcome of the game can be no more than 11. $\square$

We will next look at the maximum closed neighborhood sum games played on the cycle C_6. In some instances, the results will be developed for cycles of order $n \geq 6$ and will carry over to the more general cases for cycles. Notice that maximum possible closed neighborhood sum on C_6 is 15, and this value is achieved if and only if a vertex has labels 4, 5, and 6 in its closed neighborhood. Equivalently, the outcome of the game will be 15 if and only if there is a vertex that has labels 1, 2, and 3 in its closed neighborhood.

Theorem 5.73. $NS_{X(B),N(B)}[C_6] = 12$.

Proof. First we demonstrate a strategy the Minimizer can employ to ensure that the outcome of the game is no more than 12. Notice that all labels will be placed smallest to largest. Minimizer's strategy is to always place her label at distance three from Maximizer's move. This will ensure that no closed neighborhood contains both the 5 and the 6, nor both the 3 and the 4. Hence, no closed neighborhood sum will exceed $2+4+6 = 12$.

Next we need to demonstrate a strategy for Maximizer by which he can ensure that the outcome of the game is at least 12. Maximizer's strategy is as follows:

(i) Set $X_1 = (1, v_1)$.

(ii) If $N_1 = (2, v_2)$, $N_2 = (2, v_3)$, $N_2 = (2, v_5)$, or $N_2 = (2, v_6)$, then set $X_2 = (3, v_3)$, $X_2 = (3, v_2)$, $X_2 = (3, v_6)$, or $X_2 = (3, v_5)$ respectively. Each of these moves will mean there will be a closed neighborhood sum at the end of the game of $4+5+6 = 15$.

(iii) If $N_1 = (2, v_4)$, then set $X_2 = (3, v_3)$. If $N_2 = (4, v_2)$, then v_5 will have closed neighborhood sum of $2+5+6 = 13$. If $N_2 = (4, v_5)$ or $N_2 = (4, v_6)$, then set $X_3 = (5, v_2)$ and in this case v_5 will have closed neighborhood sum of $2+4+6 = 12$. □

Lemma 5.74. *If n is even, then $NS_{X(T),N(B)}[C_n] \leq 2n+1$, $NS_{X(B),N(T)}[C_n] \leq 2n+1$, and $NS_{X(A),N(A)}[C_n] \leq 2n+1$.*

Proof. Prior to the start of the game, Minimizer pairs the labels into the sets $S_1 = \{1, n\}$, $S_2 = \{2, n-1\}, \ldots, S_{\frac{n}{2}} = \{\frac{n}{2}, \frac{n}{2}+1\}$. Notice that the sum of the labels from each set is $n+1$. Minimizer pairs the vertices into the sets $T_1 = \{v_1, v_2\}$, $T_2 = \{v_3, v_4\}, \ldots, T_{\frac{n}{2}} = \{v_{n-1}, v_n\}$. Regardless of which game is being played, at each turn if Maximizer has placed a label from set S_i, then on her turn, Minimizer will place the other label from set S_i. Regardless of which game is being played, at each turn if Maximizer has labeled a vertex from set T_j, then on her turn, Minimizer will label the other vertex from set T_j. Since the maximum value of any label is n, each vertex from set T_j will have a closed neighborhood sum that does not exceed $2n+1$. Therefore at the end of the game, no vertex will have closed neighborhood sum in excess of $2n+1$. □

Theorem 5.75. $NS_{X(B),N(T)}[C_6] = 13$.

Proof. It follows from Lemma 5.74 that $NS_{X(B),N(T)}[C_6] \leq 13$. Hence, we need to demonstrate a strategy by which Maximizer can ensure that the outcome of the game will be at least 13. Maximizer's strategy is as follows:

(i) Set $X_1 = (1, v_1)$.

(ii) If $N_1 = (6, v_2)$, $N_1 = (6, v_3)$, $N_1 = (6, v_4)$, $N_1 = (6, v_5)$, or $N_1 = (6, v_6)$, then Maximizer will set $X_2 = (2, v_6)$, $X_2 = (2, v_2)$, $X_2 = (2, v_2)$, $X_2 = (2, v_6)$, or $X_2 = (2, v_2)$ respectively. Each of these moves will ensure that there will be a vertex whose closed neighborhood sum will be at least $3+4+6 = 13$. □

Theorem 5.76. $NS_{X(B),N(A)}[C_6] = 12$.

Proof. From Theorem 5.73 we know that $NS_{X(B),N(A)}[C_6] \leq 12$. Thus it will suffice to demonstrate a strategy for Maximizer by which he can always ensure that the outcome is at least 12. Maximizer's strategy is as follows:

(i) Set $X_1 = (1, v_1)$.

(ii) On his second move, Maximizer should place the smallest available label adjacent to v_1. Assume without loss of generality that this label is placed on v_6. Then the labels on the set $\{v_2, v_3, v_4, v_5\}$ will have values that are at least as big as $\{2, 4, 5, 6\}$. Hence, one of these vertices will have closed neighborhood sum of at least $2+4+6 = 12$. □

Theorem 5.77. $NS_{X(T),N(B)}[C_6] = 13$.

Proof. From Lemma 5.74 we know that $NS_{X(T),N(B)}[C_6] \leq 13$. Hence we need to demonstrate a strategy for Maximizer by which he can always achieve at least 13. Maximizer's strategy is to set $X_1 = (6, v_1)$. On his second move, Maximizer should play the 5 adjacent to v_1. This will ensure that there will be a vertex with closed neighborhood sum at least $2+5+6 = 13$. □

Lemma 5.78. *If* $n \geq 6$*, then* $NS_{X(A),N(T)}[C_n] \geq NS_{X(T),N(T)}[C_n] \geq 3n-6$.

Proof. It suffices for us to show that $NS_{X(T),N(T)}[C_n] \geq 3n-6$, so consider the game where Maximizer must place the largest remaining label at each turn. We need to demonstrate a strategy for Maximizer such that he can always achieve at least $3n-6$. Maximizer's strategy is as follows:

(i) Set $X_1 = (n, v_1)$.

(ii) If $N_1 = (n-1, v_2)$, then set $X_2 = (n-2, v_3)$. Vertex v_2 will then have closed neighborhood sum of $3n-3$.

(iii) If $N_1 = (n-1, v_n)$, then set $X_2 = (n-2, v_{n-1})$. Vertex v_n will then have closed neighborhood sum of $3n-3$.

(iv) If $N_1 = (n-1, v_3)$, then set $X_2 = (n-2, v_2)$. Vertex v_2 will then have closed neighborhood sum of $3n-3$.

(v) If $N_1 = (n-1, v_{n-1})$, then set $X_2 = (n-2, v_n)$. Vertex v_n will then have closed neighborhood sum of $3n-3$.

(vi) If none of the cases ii-v hold, then Minimizer has played the $n-1$ on a vertex u at distance at least three from v_1. In this case Maximizer sets $X_2 = (n-2, v_2)$. On Maximizer's third move, he will then play the $n-4$ on either v_3 or v_n, one of which must be open. In this case, either vertex v_1 or vertex v_2 will have closed neighborhood sum of $3n-6$. □

Theorem 5.79. $NS_{X(T),N(T)}[C_6] = 12$.

Proof. From Lemma 5.78 we know that $NS_{X(T),N(T)}[C_6] \geq 3n-6 = 12$. To prove that $NS_{X(T),N(T)}[C_6] = 12$, we need to demonstrate a strategy for Minimizer by which she can always achieve an outcome of no more than 12. Assume without loss of generality that

$X_1 = (6, v_1)$. Then Minimizer's strategy is to set $N_1 = (5, v_4)$. Maximizer must then place the 4 on his second move. Minimizer will follow by placing the 3 on her second move at a distance of three from the 4. Thus at the end of the game, no vertex will have both the 5 and 6 in its closed neighborhood, and no vertex will have both the 3 and 4 in its closed neighborhood. Thus the maximum possible value for a closed neighborhood sum will be $2+4+6 = 12$. $\square$

Theorem 5.80. $NS_{X(T),N(A)}[C_6] = 12$.

Proof. It follows from Theorem 5.79 that $NS_{X(T),N(A)}[C_6] \leq NS_{X(T),N(T)}[C_6] = 12$. To show that $NS_{X(T),N(A)}[C_6] = 12$, we need to demonstrate a strategy for Maximizer by which he can always achieve at least this outcome. Maximizer should employ the following strategy:

(i) Set $X_1 = (6, v_1)$.

(ii) On his second move, there will be a label of at least 4 available. Maximizer should play the largest available label adjacent to the 6. This will ensure that there will be a vertex with closed neighborhood sum of at least $2+4+6 = 12$. $\square$

Theorem 5.81. $NS_{X(A),N(B)}[C_6] = 14$.

Proof. We first demonstrate a strategy by which Minimizer can ensure that the outcome is no more than 14. Assume without loss of generality that Maximizer's first move is to place a label on v_1. Minimizer's strategy is as follows:

(i) If $X_1 = (1, v_1)$, $X_1 = (2, v_1)$, or $X_1 = (3, v_1)$, then Minimizer will place the smallest available label on v_4. This will ensure that no vertex will have 1, 2, and 3 assigned to the vertices in its closed neighborhood. In this case, the outcome of the game will be no more than 14.

(ii) If $X_1 = (4, v_1)$, $X_1 = (5, v_1)$, or $X_1 = (6, v_1)$, then Minimizer's first move is to set $N_1 = (1, v_2)$. On Minimizer's second move, either the 2 or the 3 will be available, and either v_5 or $v_6$will be unlabeled. Minimizer's second move will be to place one of these labels on one of these vertices. This will ensure that no vertex will have the 1, 2, and 3 assigned to the vertices in its closed neighborhood, and so again, the outcome of the game will be no more than 14.

Next we need to demonstrate a strategy by which Maximizer can always achieve at least 14. In order to accomplish this, Maximizer should employ the following strategy:

(i) Set $X_1 = (3, v_1)$.

(ii) If $N_1 = (1, v_2)$, $N_1 = (1, v_3)$, $N_1 = (1, v_5)$, or $N_1 = (1, v_6)$, then Maximizer can set $X_2 = (2, v_3)$, $X_2 = (2, v_2)$, $X_2 = (2, v_6)$, or $X_2 = (2, v_5)$ respectively. In each of these cases, there will be a vertex with labels 4, 5, and 6 assigned to its closed neighborhood, and the outcome of the game will be 15.

(iii) If $N_1 = (1, v_4)$, then $X_2 = (6, v_2)$. Then on his third move, Maximizer can assign the 5 in such a way to create a closed neighborhood sum of $3+5+6 = 14$. □

Theorem 5.82. $NS_{X(A),N(T)}[C_6] = 14$.

Proof. We first demonstrate a strategy by which Minimizer can ensure that the outcome is no more than 14. Assume without loss of generality that Maximizer's first move is to place a label on v_1. Minimizer's strategy is as follows:

(i) If $X_1 = (4, v_1)$, $X_1 = (5, v_1)$, or $X_1 = (6, v_1)$, then Minimizer will place the largest available label on v_4. This will ensure that no vertex will have 4, 5, and 6 assigned to the

vertices in its closed neighborhood. In this case, the outcome of the game will be no more than 14.

(ii) If $X_1 = (1, v_1)$, $X_1 = (2, v_1)$, or $X_1 = (3, v_1)$, then Minimizer's first move is to set $N_1 = (6, v_2)$. On Minimizer's second move, either the 4 or the 5 will be available, and either v_5 or v_6 will be available. Minimizer's second move will be to place one of these labels on one of these vertices. This will ensure that no vertex will have the 4, 5, and 6 assigned to the vertices in its closed neighborhood, and so again, the outcome of the game will be no more than 14.

Next we need to demonstrate a strategy by which Maximizer can always achieve at least 14. In order to accomplish this, Maximizer should employ the following strategy:

(i) Set $X_1 = (4, v_1)$.

(ii) If $N_1 = (6, v_2)$, $N_1 = (6, v_3)$, $N_1 = (6, v_5)$, or $N_1 = (6, v_6)$, then Maximizer can set $X_2 = (5, v_3)$, $X_2 = (5, v_2)$, $X_2 = (5, v_6)$, or $X_2 = (5, v_5)$ respectively. In each of these cases, there will be a vertex with labels 4, 5, and 6 assigned to its closed neighborhood, and the outcome of the game will be 15.

(iii) If $N_1 = (6, v_4)$, then set $X_2 = (1, v_2)$. Then on his third move, Maximizer can assign the 3 in such a way to create a closed neighborhood sum of $3+5+6 = 14$. □

Theorem 5.83. $NS_{X(A),N(A)}[C_6] = 12$.

Proof. First we demonstrate a strategy the Minimizer can employ to ensure that the outcome of the game is no more than 12. Minimizer's strategy will be to pair the labels up into the following sets: $S_1 = \{1, 2\}$, $S_2 = \{3, 4\}$, and $S_3 = \{5, 6\}$. At each turn Minimizer will choose her label from the same set as Maximizer selected from, and she will place it

at distance three from where Maximizer placed his label. This will ensure that no closed neighborhood contains both the 5 and the 6, nor both the 3 and the 4. Hence, no closed neighborhood sum will exceed $2+4+6=12$.

Next we need to demonstrate a strategy for Maximizer by which he can ensure that the outcome of the game is at least 12. In order to achieve this, Maximizer sets $X_1 = (6, v_1)$ and then on his second move, plays the largest available label adjacent to v_1. This ensures that there will be a closed neighborhood sum of at least $2+4+6=12$. □

Next we consider the maximum closed neighborhood sum games played on C_6 where Minimizer plays first.

Theorem 5.84. $NS_{N(B),X(B)}[C_6] = 14$.

Proof. First we demonstrate a strategy that Minimizer can employ to ensure that the outcome of the game is no more than 14. To accomplish this she sets $N_1 = (1, v_1)$. If $X_1 = (2, v_4)$, then there cannot be a vertex with labels 1, 2, and 3 in its closed neighborhood, and thus the outcome of the game can be no more than 14. If Maximizer places the 2 on a vertex other than v_4, then Minimizer sets $N_2 = (3, v_4)$. In this case, there still cannot be a vertex with labels 1, 2, and 3 in its closed neighborhood, and hence the outcome of the game can be no more than 14.

Next we need to demonstrate a strategy for Maximizer whereby he can ensure that the outcome of the game is at least 14. Assume without loss of generality that $N_1 = (1, v_1)$. Maximizer then sets $X_1 = (2, v_2)$. On his second move, Maximizer uses the following strategy.

(i) If $N_2 = (3, v_3)$ or $N_2 = (3, v_6)$, then regardless of how Maximizer plays, either vertex v_5 or vertex v_4, respectively, will have closed neighborhood sum of 15.

(ii) If $N_2 = (3, v_4)$, then set $X_2 = (4, v_2)$. In this case vertex v_5 will have closed neighborhood sum of 14.

(iii) If $N_2 = (3, v_5)$, then set $X_2 = (4, v_6)$. In this case vertex v_4 will have closed neighborhood sum of 14. $\square$

Theorem 5.85. *If $n \geq 6$, then $NS_{N(B),X(A)}[C_n] \geq NS_{N(B),X(T)}[C_n] \geq 2n+2$.*

Proof. It suffices to show that $NS_{N(B),X(T)}[C_n] \geq 2n+2$. Assume without loss of generality that $N_1 = (1, v_1)$. Then Maximizer sets $X_1 = (n, v_4)$. If $N_2 = (2, v_3)$, then Maximizer sets $X_2 = (n-1, v_5)$ and, since $n \geq 6$, the closed neighborhood sum of v_5 will be at least $n + (n-1) + 3 = 2n+2$. If on her second move Minimizer plays the 2 on some vertex other than v_3, then Maximizer sets $X_2 = (n-1, v_3)$. In this case either vertex v_3 or v_4 will have closed neighborhood sum at least $n + (n-1) + 3 = 2n+2$. $\square$

Theorem 5.86. $NS_{N(B),X(T)}[C_6] = 14$.

Proof. It follows from Theorem 5.85 that $NS_{N(B),X(T)}[C_6] \geq 14$. Hence, we need to describe a strategy that Minimizer can employ to ensure that the outcome is no more than 14. Minimizer sets $N_1 = (1, v_1)$. If $X_1 \neq (6, v_4)$, then Minimizer sets $N_2 = (2, v_4)$. In this case there will not be a vertex with labels 1, 2, and 3 in its closed neighborhood, and hence the outcome of the game will be no more than 14. If $X_1 = (6, v_4)$, then Minimizer sets $N_2 = (2, v_3)$. On her third move, Minimizer can then either set $N_3 = (3, v_5)$ or $N_3 = (3, v_6)$. Again, there will not be a vertex with labels 1, 2, and 3 in its closed neighborhood, and the outcome of the game will be no more than 14. $\square$

Theorem 5.87. $NS_{N(B),X(A)}[C_6] = 15$.

Proof. Notice that 15 is the maximum possible closed neighborhood sum. Hence, we need to demonstrate a strategy by which Maximizer can achieve this value. Assume without loss of generality that $N_1 = (1, v_1)$. Maximizer sets $X_1 = (6, v_4)$. On his second move, Maximizer then employs the following strategy.

(i) If $N_2 = (2, v_2)$ or $N_2 = (2, v_3)$, then set $X_2 = (3, v_3)$ or $X_2 = (3, v_2)$. In either case there will be a vertex with labels 1, 2, and 3 in its closed neighborhood, and hence, the outcome of the game will be 15.

(ii) If $N_2 = (2, v_5)$ or $N_2 = (2, v_6)$, then set $X_2 = (3, v_6)$ or $X_2 = (3, v_5)$. In either case there will be a vertex with labels 1, 2, and 3 in its closed neighborhood, and hence, the outcome of the game will be 15. □

Theorem 5.88. $NS_{N(T),X(B)}[C_6] = 14$.

Proof. First we demonstrate a strategy that Minimizer can employ to ensure that the outcome of the game is no more than 14. To accomplish this she sets $N_1 = (6, v_1)$ and then uses the following strategy.

(i) If $X_1 \neq (1, v_4)$, then set $N_2 = (5, v_4)$. In this case there will not be a vertex with labels 4, 5, and 6 in its closed neighborhood and the outcome of the game will be no more than 14.

(ii) If $X_1 = (1, v_4)$, then set $N_2 = (5, v_3)$. On her third move, Minimizer can either set $N_3 = (4, v_5)$ or $N_3 = (4, v_6)$. In either case, there will not be a vertex that has labels 4, 5, and 6 in its closed neighborhood, and again the outcome of the game will be no more than 14.

Next we need to demonstrate a strategy whereby Maximizer can ensure that the outcome of the game is at least 14. Assume without loss of generality that $N_1 = (6, v_1)$. Maximizer then sets $X_1 = (1, v_4)$ and then uses the following strategy.

(i) If $N_2 = (5, v_2)$ or $N_2 = (5, v_6)$, then set $X_2 = (2, v_5)$ or $X_2 = (2, v_3)$ respectively. In either case, there will be a vertex that has labels 4, 5, and 6 in its closed neighborhood and the outcome of the game will be 15.

(ii) If $N_2 = (5, v_3)$ or $N_2 = (5, v_5)$, then set $X_2 = (2, v_6)$ or $X_2 = (2, v_2)$ respectively. In this case, either vertex v_2 or v_6, respectively, will have closed neighborhood sum of at least 14. $\square$

Theorem 5.89. *For any cycle* $NS_{N(T),X(A)}[C_n] \geq NS_{N(T),X(T)}[C_n] \geq 3n-4$.

Proof. It suffices to show that $NS_{N(T),X(T)}[C_n] \geq 3n-4$. Notice that this result has already been shown for the cases where $n \in \{3,4,5\}$. For the general case, assume without loss of generality that $N_1 = (n, v_1)$. Then Maximizer makes his first move as $X_1 = (n-1, v_2)$. If $N_2 = (n-2, v_3)$, then the closed neighborhood sum of vertex v_2 will be $3n-3$. If Minimizer makes any other move on her second turn, then Maximizer can set $X_2 = (n-3, v_3)$ and vertex v_2 will have closed neighborhood sum of $3n-4$. $\square$

Theorem 5.90. $NS_{N(T),X(T)}[C_6] = 14$.

Proof. From Theorem 5.89 we have that $NS_{N(T),X(T)}[C_6] \geq 14$. So we only need to demonstrate a strategy whereby Minimizer can ensure that the outcome of the game is no more than 14. To accomplish this Minimizer sets $N_1 = (6, v_1)$. If Maximizer sets $X_1 = (5, v_4)$, then there cannot be a vertex with labels 4, 5, and 6 in its closed neighborhood, and hence, the outcome of the game will be no more than 14. If Maximizer makes any other move,

then Minimizer sets $N_2 = (4, v_4)$. Likewise, in this case, there will not be a vertex will labels 4, 5, and 6 in its closed neighborhood, and the outcome of the game will be no more than 14. $\square$

Theorem 5.91. $NS_{N(T),X(A)}[C_6] = 15$.

Proof. Since 15 is the maximum possible closed neighborhood sum, we need to demonstrate a strategy that Maximizer can use to achieve this value. Assume without loss of generality that $N_1 = (6, v_1)$. Maximizer then sets $X_1 = (1, v_4)$ and uses the following strategy on his second move.

(i) If $N_2 = (5, v_2)$ or $N_2 = (5, v_3)$, then set $X_2 = (4, v_3)$ or $X_2 = (4, v_2)$ respectively. Then the closed neighborhood sum of vertex v_2 will be 15.

(ii) If $N_2 = (5, v_5)$ or $N_2 = (5, v_6)$, then set $X_2 = (4, v_6)$ or $X_2 = (4, v_5)$ respectively. Then the closed neighborhood sum of vertex v_6 will be 15. $\square$

Theorem 5.92. $NS_{N(A),X(B)}[C_6] = 14$.

Proof. From Theorem 5.84 it follows that $NS_{N(A),X(B)}[C_6] \leq 14$, thus we need to demonstrate a strategy whereby Maximizer can ensure that the outcome of the game is at least 14. Notice that if the 1 and the 2 are placed on adjacent vertices, then the 5 and the 6 are placed on two of the remaining vertices, say vertices x and y. If x and y are adjacent, then one of the them will have a third label of at least 3. If $d(x, y) = 2$, then the vertex that is adjacent to both x and y will have label at least 3. In either case, there will be a vertex with closed neighborhood sum of at least 14. So, when describing a strategy for Maximizer, it suffices to demonstrate that he can ensure that the 1 and the 2 are placed on adjacent

vertices. Without loss of generality we assume that Minimizer makes her first move onto vertex v_1. To achieve an outcome of at least 14, Maximizer uses the following strategy.

(i) If $N_1 = (1, v_1)$ or $N_1 = (2, v_1)$, then set $X_1 = (2, v_2)$ or $X_1 = (1, v_1)$ respectively. Since 1 and 2 are placed on adjacent vertices, the maximum closed neighborhood sum is at least 14.

(ii) If $N_1 = (4, v_1)$, $N_1 = (5, v_1)$, or $N_1 = (6, v_1)$, then set $X_1 = (1, v_4)$.

(ii.i) If $N_2 = (2, v_3)$ or $N_2 = (2, v_5)$, then the 1 and the 2 will be adjacent, and the maximum closed neighborhood sum will be at least 14.

(ii.ii) If $N_2 = (2, v_2)$ or $N_2 = (2, v_6)$, then set $X_2 = (3, v_3)$ or $X_2 = (3, v_5)$. In either case there will be a vertex with closed neighborhood sum of 15.

(iii) In this case we know that $N_1 = (3, v_1)$. Maximizer will first set $X_1 = (1, v_4)$.

(iii.i) If $N_2 \neq (2, *)$, then Maximizer will place the 2 adjacent to the 1 on his second move to create a closed neighborhood sum of at least 14.

(iii.ii) If $N_2 = (2, v_3)$ or $N_2 = (2, v_5)$, then the 1 and the 2 are adjacent and there will be a closed neighborhood sum of at least 14.

(iii.iii) If $N_2 = (2, v_2)$ or $N_2 = (2, v_6)$, then set $X_2 = (4, v_3)$ or $X_2 = (4, v_5)$ respectively. In this case there will be a closed neighborhood sum of $3 + 5 + 6 = 14$. □

Theorem 5.93. $NS_{N(A),X(T)}[C_6] = 13$.

Proof. First we demonstrate a strategy that Minimizer can use to ensure that the outcome of the game is no more than 13. Notice that if the 5 and the 6 are placed on vertices at distance two from each other, then the maximum possible closed neighborhood sum will

be $3+4+6=13$. Minimizer's strategy is then to set $N_1=(3,v_1)$ and then to make her second move as follows.

(i) If $X_1 \neq (6,v_4)$, then Minimizer places the 5 on a vertex at distance two from where Maximizer made his first move, and then the outcome of the game can be no more than 13.

(ii) If $X_1=(6,v_4)$, then Minimizer sets $N_2=(1,v_3)$.

(ii.i) If $X_2=(5,v_2)$, then the maximum closed neighborhood sum will be $1+5+6=12$.

(ii.ii) If $X_2=(5,v_5)$, then set $N_2=(2,v_6)$ and the maximum closed neighborhood sum will be $2+5+6=13$.

(ii.iii) If $X_2=(5,v_6)$, then set $N_2=(2,v_5)$ and the maximum closed neighborhood sum will be $2+5+6=13$.

Next we need to demonstrate a strategy for Maximizer that he can use to ensure that the outcome of the game is at least 13. We can assume that Minimizer makes her first move onto vertex v_1. Then Maximizer can employ the following strategy.

(i) If $N_1=(5,v_1)$ or $N_1=(6,v_1)$, then set $X_1=(6,v_2)$ or $X_1=(5,v_2)$ respectively. On his second move, Maximizer can place a label at least as big as 3 on either vertex v_3 or v_6 and create a closed neighborhood sum of at least $3+5+6=14$.

(ii) If $N_1=(1,v_1)$, $N_1=(2,v_1)$, $N_1=(3,v_1)$, or $N_1=(4,v_1)$, then set $X_1=(6,v_4)$. If Minimizer does not play the 5 on her second move, then Maximizer can place the 5 adjacent to the 6 and create a closed neighborhood sum of at least $2+5+6=13$. If Minimizer plays the 5 on her second move, then Maximizer can create a closed neighborhood consisting of

the largest remaining label, along with the 5 and 6. That is, Maximizer can create a closed neighborhood sum of at least $3+5+6=14$. □

Theorem 5.94. $NS_{N(A),X(A)}[C_6]=15$.

Proof. Since 15 is the maximum possible closed neighborhood sum, we must demonstrate a strategy whereby Maximizer can achieve this value. Recall that Maximizer can accomplish this by ensuring that there is a vertex with labels 1, 2, and 3 in its closed neighborhood, or equivalently, by ensuring that there is a vertex with labels 4, 5, and 6 in its closed neighborhood. Assume without loss of generality that Minimizer's first move is onto vertex v_1. Maximizer then adopts the following strategy.

(i) If $N_1=(1,v_1)$, $N_1=(2,v_1)$, or $N_1=(3,v_1)$, then set $X_1=(6,v_4)$. The regardless of how Minimizer plays on her second move, Maximizer can create a closed neighborhood sum containing 1, 2, and 3, or containing 4, 5, and 6 on his second move.

(ii) If $N_1=(4,v_1)$, $N_1=(5,v_1)$, or $N_1=(6,v_1)$, then set $X_1=(1,v_4)$. Then regardless of how Minimizer plays on her second move, Maximizer can create a closed neighborhood sum containing 1, 2, and 3, or containing 4, 5, and 6 on his second move. □

We now consider cycles of order at least seven.

Theorem 5.95. *If $n\geq 7$, then $NS_{X(T),N(A)}[C_n]\geq 2n+1$.*

Proof. Maximizer's strategy to ensure the outcome is at least $2n+1$ is as follows:

(i) Set $X_1=(n,v_1)$

(ii) If $N_1\neq(n-1,*)$, then make one of the available moves $X_2=(n-1,v_2)$ or $X_2=(n-1,v_n)$. If $X_2=(n-1,v_2)$, then either v_1 or v_2 will have closed neighborhood

sum of at least $2n+1$. If $X_2 = (n-1, v_n)$, then either v_1 or v_n will have closed neighborhood sum of at least $2n+1$.

(iii) If $N_1 = (n-1, v_2)$, then set $X_2 = (n-2, v_3)$ and v_2 will have closed neighborhood sum of $3n-3 \geq 2n+1$ since $n > 4$.

(iv) If $N_1 = (n-1, v_n)$, then set $X_2 = (n-2, v_{n-1})$ and v_n will have closed neighborhood sum of $3n-3 \geq 2n+1$ since $n > 4$.

(v) If $N_1 = (n-1, v_3)$, then set $X_2 = (n-2, v_2)$ and v_2 will have closed neighborhood sum of $3n-3 \geq 2n+1$ since $n > 4$.

(vi) If $N_1 = (n-1, v_{n-1})$, then set $X_2 = (n-2, v_n)$ and v_n will have closed neighborhood sum of $3n-3 \geq 2n+1$ since $n > 4$.

(vii) If none of the cases ii-vi hold, then Minimizer has played the $n-1$ on a vertex u at distance at least three from v_1. In this case set $X_2 = (n-2, v_2)$. On Maximizer's third move, he will then play the largest label available (either $n-3$ or $n-4$) on either v_3 or v_n, one of which must be unlabeled. Then either v_1 or v_2 will have closed neighborhood sum of at least $3n-6 \geq 2n+1$ since $n \geq 7$. □

Corollary 5.96. *If $n \geq 7$, then $NS_{X(T),N(*)}[C_n] \geq 2n+1$ and $NS_{X(A),N(*)}[C_n] \geq 2n+1$.*

Theorem 5.97. *If $n \geq 7$ and n is odd, then $NS_{X(T),N(B)}[C_n] = 2n+1$.*

Proof. We know from Corollary 5.96 that $NS_{X(T),N(B)}[C_n] \geq 2n+1$. Notice that during each round of the game, Maximizer places the largest remaining label, and Minimizer places the smallest remaining label. Hence, the sum of the labels played during each round is $n+1$. Assume without loss of generality that $X_1 = (n, v_1)$. Minimizer will then use the following strategy to ensure that the outcome does not exceed $2n+1$. Set $N_1 = (1, v_2)$; this

will ensure that the closed neighborhood sums of v_1 and v_2 do not exceed $2n+1$. Minimizer then partitions the unlabeled vertices into the sets $S_1 = \{v_4, v_5\}$, $S_2 = \{v_6, v_7\}$,..., $S_{\frac{n-1}{2}-1} = \{v_{n-1}, v_n\}$. Notice that all vertices belong to one of these sets except for v_3. During each round of the game, if Maximizer places his label one a vertex from set S_k, then Minimizer will place her label on the other vertex from set S_k. Since the total of the labels in set S_k will be $n+1$, neither of these vertices will have closed neighborhood sums in excess of $2n+1$. Furthermore, since the label n was placed on vertex v_1, no vertex, other than v_n, will have a closed neighborhood sum that exceeds $2n$. This strategy describes Minimizer's play until Maximizer places a label on vertex v_3.

When Maximizer places a label on the vertex v_3, we consider the outcomes and Minimizer's strategy on a case by case basis:

(i) If the last play of the game in on v_3, then we know that Maximizer played the $\frac{n+1}{2}$ on v_3. As described above, all other vertices will have closed neighborhood sums that do not exceed $2n$, and possibly $2n+1$ on vertex v_n. Vertex v_3 will have closed neighborhood sum that is less than or equal to $1 + \frac{n+1}{2} + n - 1 < 2n+1$.

(ii) If Maximizer plays on v_3 and vertex v_4 has not been labeled, then Minimizer places her label on v_4. Then the sum of the labels on v_3 and v_4 will be $n+1$ and hence neither will have closed neighborhood sum in excess of $2n$ as the label n was placed on vertex v_1. Minimizer will then remove set S_1 and play the remainder of the game treating v_5 as she has treated v_3 up to this point.

(iii) If Maximizer plays on v_3 and vertex v_4 has been labeled, then we know that the closed neighborhood sum of v_3 will be less than or equal to $1 + (n-2) + (n-1) = 2n-2 < 2n+1$. Minimizer will then place her label on the unlabeled vertex with the smallest index,

say vertex v_j. Notice that since vertex v_{j+1} is unlabeled, and since Maximizer places labels in decreasing order of magnitude, that the sum of the labels on v_j and v_{j+1} will not exceed n. Hence, vertex v_j will have a closed neighborhood less than or equal to $2n-2$. Minimizer will then remove the set containing v_j and will treat v_j as if it were v_3.

The final issue that we need to consider is what the closed neighborhood sum will be when Maximizer plays his last move, which will not be followed by a move from Minimizer. We know that this label will have value of $\frac{n+1}{2}$. Additionally, this vertex will have one neighbor that was labeled by Maximizer (whose value may be as much as n) and one vertex that was labeled by Minimizer (whose value may be as much as $\frac{n-1}{2}$). Hence, the vertex that is last to be labeled will not have a closed neighborhood sum in excess of $\frac{n-1}{2}+\frac{n+1}{2}+n=2n$. □

Theorem 5.98. *If $n \geq 8$ and n is even, then $NS_{X(T),N(B)}[C_n]=2n+1$.*

Proof. From Lemma 5.74 we have that $NS_{X(T),N(B)}[C_n] \leq 2n+1$. It follows from Corollary 5.96 that $NS_{X(T),N(B)}[C_n] \geq 2n+1$. Therefore, $NS_{X(T),N(B)}[C_n]=2n+1$. □

Theorem 5.99. *If $n \geq 7$, then $NS_{X(T),N(T)}[C_n]=3n-6$.*

Proof. From Lemma 5.78 we know that $NS_{X(T),N(T)}[C_n] \geq 3n-6$. Hence, we need to demonstrate a strategy for Minimizer whereby she can ensure that the outcome of the game is no more than $3n-6$. Assume without loss of generality that $X_1=(n,v_1)$. Minimizer then sets $N_1=(n-1,v_4)$ and then adopts the following strategy for her second move.

(i) If $X_2=(n-2,v_2)$ or $X_2=(n-2,v_n)$, then set $N_2=(n-3,v_5)$ or $N_2=(n-3,v_3)$ respectively. In this case the maximum possible closed neighborhood sum that Maximizer

can achieve would occur if he played $X_3 = (n-4, v_n)$ or $X_3 = (n-4, v_2)$ respectively. That is, the maximum possible outcome of the game will be $n + (n-2) + (n-4) = 3n-6$.

(ii) If $X_2 = (n-2, v_3)$, then set $N_2 = (n-3, v_5)$. In this case the maximum closed neighborhood will occur on v_4 and will be $(n-1) + (n-2) + (n-3) = 3n-6$.

(iii) If Maximizer makes any other move on his second turn, then Minimizer can set $N_2 = (n-3, v_3)$ and the maximum possible closed neighborhood sum will be $n + (n-2) + (n-4) = (n-1) + (n-2) + (n-3) = 3n-6$. □

Theorem 5.100. *If $n \geq 7$, then $NS_{X(T),N(A)}[C_n] = 2n+1$.*

Proof. It follows from Corollary 5.96 that $NS_{X(T),N(A)}[C_n] \geq 2n+1$. If n is odd, then it follows from Theorem 5.97 that $NS_{X(T),N(A)}[C_n] \leq 2n+1$. If n is even, then it follows from Theorem 5.98 that $NS_{X(T),N(A)}[C_n] \leq 2n+1$. □

Theorem 5.101. *If $n \geq 5$ and n is odd, then $NS_{X(A),N(A)}[C_n] \geq 2n+2$.*

Proof. In order to achieve an outcome of at least $2n+2$, Maximizer can adopt the following strategy. Before the start of the game, he groups vertices into the pairs $S_1 = \{v_2, v_3\}$, $S_2 = \{v_4, v_5\}$, $S_{\frac{n-1}{2}} = \{v_{n-1}, v_n\}$ and pairs labels into sets $T_1 = \{1, 2\}$, $T_2 = \{n-1, n\}$, $T_3 = \{3, n-2\}$, $T_4 = \{4, n-3\}, \cdots$, $T_{\frac{n-1}{2}} = \{\frac{n-1}{2}, \frac{n+3}{2}\}$. Maximizer then makes his first move as $X_1 = (\frac{n+1}{2}, v_1)$. Notice that all the unlabeled vertices and unused labels are represented in exactly one of the sets. On each of her turns, Minimizer will place one of the unused labels, say from set S_i, on a vertex in one of the sets, say T_j. Maximizer will then follow by playing the other label from set S_i on the unlabeled vertex from set T_j. In this way labels 1 and the 2 will be placed on adjacent vertices, as will the $n-1$ and n. Since $n \geq 5$ there will be a vertex will closed neighborhood sum at least $n + (n-1) + 3 = 2n-2$. □

Corollary 5.102. *If $n \geq 5$ and n is odd, then $NS_{X(A),N(B)}[C_n] \geq 2n+2$.*

Theorem 5.103. *If $n \geq 8$ and n is even, then $NS_{X(A),N(A)}[C_n] = 2n+1$.*

Proof. It follows from Lemma 5.74 that $NS_{X(A),N(A)}[C_n] \leq 2n+1$ and Theorem 5.100 that $NS_{X(A),N(A)}[C_n] \geq 2n+1$. Therefore, $NS_{X(A),N(A)}[C_n] = 2n+1$. □

Next we look at the minimum closed neighborhood sum game played on cycle C_n where Minimizer plays first.

Theorem 5.104. *If $n \geq 7$, then $NS_{N(B),X(T)}[C_n] = 2n+2$.*

Proof. It follows from Theorem 5.85 that $NS_{N(B),X(T)}[C_n] \geq 2n+2$, hence we only need to demonstrate a strategy that Minimizer can use to make sure the outcome is no more than $2n+2$. Without loss of generality we can assume that $N_1 = (1, v_1)$.

If n is odd, then Minimizer pairs the vertices into sets $S_1 = \{v_2, v_3\}$, $S_2 = \{v_4, v_5\}$,..., $S_{\frac{n-1}{2}} = \{v_{n-1}, v_n\}$. If Maximizer plays onto a vertex from set S_k, then Minimizer will follow by playing onto the other vertex from set S_k. In this fashion, the sum of the labels from any set S_k will equal $n+2$. It then follows that the maximum value of any closed neighborhood sum will be $(n+2)+n = 2n+2$.

If n is even, then Minimizer pairs the vertices into sets $S_1 = \{v_2, v_3\}$, $S_2 = \{v_4, v_5\}$,..., $S_{\frac{n-2}{2}} = \{v_{n-2}, v_{n-1}\}$. Notice that all vertices are belong to one of these sets except for v_n. During each round of the game, if Maximizer places his label in set S_k, then Minimizer will also place her label in set S_k. Since the total of the labels in set S_k will be $n+2$, neither vertex from set S_k will have a closed neighborhood sum in excess of $2n+2$. Furthermore, since the label 1 was placed on vertex v_1, the closed neighbor-

hood sum of vertex v_1 will not exceed $2n$. This strategy describes Minimizer's play until Maximizer places a label on vertex v_n.

When Maximizer places a label on the vertex v_n, we consider the outcomes and Minimizer's strategy on a case by case basis:

(i) If the last play of the game in on v_n, then we know that Maximizer played the $\frac{n+1}{2}$. In this case the closed neighborhood sum of v_n will not exceed $n+\frac{n+1}{2}+1<2n+2$.

(ii) If Maximizer plays on v_n and vertex v_{n-1} has not been labeled, then Minimizer places her label on v_{n-1}. Then the sum of the labels on v_n and v_{n-1} will be $n+2$. The maximum possible closed neighborhood sum for v_n will be $(n+2)+1<2n+2$. The maximum possible closed neighborhood sum for v_{n-1} will be $(n+2)+(n-1)=2n+1$. Minimizer will then remove set $S_{\frac{n-2}{2}}$ and play the remainder of the game treating v_{n-2} as she has treated v_n up to this point.

(iii) If Maximizer plays on v_n and vertex v_{n-1} has been labeled, then we know that the closed neighborhood sum of v_n will be less than or equal to $1+n+(n-1)=2n-2<2n+2$. In this case the maximum closed neighborhood sum of v_{n-1} will be $(n+2)+(n-1)=2n+1$. Minimizer will then place her label on the unlabeled vertex with the largest index, say vertex v_k. Notice that since vertex v_{k-1} is unlabeled, and since Maximizer places labels in decreasing order of magnitude, the sum of the labels on v_k and v_{k+1} will not exceed $n+1$. Hence, vertex v_k will have a closed neighborhood less than or equal to $2n+1$. Minimizer will then remove the set containing v_k and will treat v_{k-1} as if it were v_n.

The final issue that we need to consider for Minimizer is what the closed neighborhood sum will be when Maximizer plays his last move, which will not be followed by a

move from Minimizer. We know that this label will have value of $\frac{n+1}{2}$. Additionally, this vertex may have one neighbor that was labeled by Maximizer (whose value may be as much as n) and will have one vertex that was labeled by Minimizer (whose value may be as much as $\frac{n-1}{2}$). Hence, the vertex that is last to be labeled will not have a closed neighborhood sum in excess of $\frac{n-1}{2}+\frac{n+1}{2}+n=2n$. $\square$

Theorem 5.105. *If $n \geq 7$, then $NS_{N(T),X(A)}[C_n] = 3n-4$.*

Proof. In Theorem 5.89 we showed that $NS_{N(T),X(A)}[C_n] \geq 3n-4$, hence we only need to demonstrate a strategy whereby Minimizer can ensure that the maximum closed neighborhood sum does not exceed $3n-4$. On her first move she can set $N_1 = (n, v_1)$. Since $n \geq 7$, vertex v_5 is at least distance two from vertex v_1. On her second move, Minimizer can play a label at least as big as $n-2$ on either vertex either v_4 or v_5. Then the maximum closed neighborhood sum of a vertex whose closed neighborhood includes v_1 will be at most $n+(n-1)+(n-3) = 3n-4$. The closed neighborhood sum of any vertex whose closed neighborhood does not include v_1 will be at most $(n-1)+(n-2)+(n-3) = 3n-6$. Hence, the maximum closed neighborhood sum will be no more than $3n-4$. $\square$

Theorem 5.106. *If $n \geq 7$, then $NS_{N(T),X(T)}[C_n] = 3n-4$.*

Proof. In Theorem 5.89 we showed that $NS_{N(T),X(T)}[C_n] \geq 3n-4$. From Theorem 5.105 it follows that $NS_{N(T),X(T)}[C_n] \leq 3n-4$. Therefore, $NS_{N(T),X(T)}[C_n] = 3n-4$. $\square$

By applying Corollary 5.16 to the results of this section, we state the known values for the minimum closed neighborhood sum game on C_n in Table 5.4.

Table 5.4: Value of minimum closed neighborhood sum game on C_n

	Order of Graph					
Scenario	3	4	5	6	$n \geq 7$, n odd	$n \geq 8$, n even
X(B),N(B)	6	6	7	9		
X(B),N(T)	6	6	6	8		$\geq n+2$
X(B),N(A)	6	6	7	9		
X(T),N(B)	6	6	7	8	$n+1$	$n+2$
X(T),N(T)	6	6	6	9	9	9
X(T),N(A)	6	6	7	9	$n+2$	$n+2$
X(A),N(B)	6	6	6	7		$\leq n+1$
X(A),N(T)	6	6	6	7	≤ 9	≤ 9
X(A),N(A)	6	6	6	9	$\leq n+1$	$n+2$
N(B),X(B)	6	6	6	7		
N(B),X(T)	6	6	7	7	$n+1$	$n+1$
N(B),X(A)	6	6	6	6	$\leq n+1$	$\leq n+1$
N(T),X(B)	6	6	6	7		
N(T),X(T)	6	6	7	7	7	7
N(T),X(A)	6	6	6	6	7	7
N(A),X(B)	6	6	7	7	$\leq n+1$	
N(A),X(T)	6	6	7	8		
N(A),X(A)	6	6	7	6		

5.5 Open Neighborhood Sums on Complete Bipartite Graphs

In this section we focus on determining the values for the maximum open neighborhood sum game on the complete bipartite graph $K_{t,t}$. Table 5.5 summarizes the results we will prove in this section. As in the previous sections, when an entry has not been filled in, it has not been determined. We first consider the games where Maximizer plays first.

Theorem 5.107. $NS_{X(B),N(B)}(K_{t,t}) = t^2 + t$.

Proof. In order to achieve an outcome of at least $t^2 + t$, Maximizer can set $X_1 = (1, v_{1,1})$ and then continue placing his labels on the first partite set until all the vertices in this set are

Table 5.5: Value of maximum open neighborhood sum game on $K_{t,t}$

	Value of t	
Scenario	t odd	t even
X(B),N(B)	t^2+t	t^2+t
X(B),N(T)	$t^2+\lceil \frac{t}{2} \rceil$	$t^2+\lceil \frac{t}{2} \rceil$
X(B),N(A)	$t^2+\lceil \frac{t}{2} \rceil$	$t^2+\lceil \frac{t}{2} \rceil$
X(T),N(B)	$t^2+\lceil \frac{t}{2} \rceil$	$t^2+\lceil \frac{t}{2} \rceil$
X(T),N(T)	t^2+t	t^2+t
X(T),N(A)	$t^2+\lceil \frac{t}{2} \rceil$	$t^2+\lceil \frac{t}{2} \rceil$
X(A),N(B)	$\geq t^2+2t-2$	$\geq t^2+2t-2$
X(A),N(T)	$\geq t^2+2t-2$	$\geq t^2+2t-2$
X(A),N(A)	$t^2+\lceil \frac{t}{2} \rceil$	$t^2+\lceil \frac{t}{2} \rceil$
N(B),X(B)	t^2+2t-1	t^2+2t-1
N(B),X(T)	$t^2+\frac{3t-1}{2}$	t^2+t
N(B),X(A)		
N(T),X(B)	$t^2+\frac{3t-1}{2}$	t^2+t
N(T),X(T)	t^2+2t-1	t^2+2t-1
N(T),X(A)		
N(A),X(B)	$\leq t^2+t$	$\leq t^2+t$
N(A),X(T)		
N(A),X(A)		

labeled. In this way, Maximizer can ensure the labels assigned to the second partite set sum to at least $2+4+\cdots+2t = 2(1+2+\cdots t) = t^2+t$. In order to ensure that the outcome is no greater than t^2+t, Minimizer can, during each round, place her label on the opposite partite set from where Maximizer played. In this way, the absolute difference in the sums of the labels on the partite sets will be no more than t. But since the sum of all labels is $\frac{2t(2t+1)}{2} = 2t^2+t$, the outcome of the game is no more than $\frac{2t^2+t}{2}+\frac{t}{2} = t^2+t$. □

Theorem 5.108. $NS_{X(*),N(A)}(K_{t,t}) = t^2+\lceil \frac{t}{2} \rceil$.

Proof. From Corollary 4.61 we have that $NS(K_{t,t}) = t^2 + \lceil \frac{t}{2} \rceil$. Thus we only need to demonstrate a strategy for Minimizer that she can employ to achieve this value. If t is even, then Minimizer can partition the vertices into the sets $S_1 = \{v_{1,1}, v_{1,2}\}$, $S_2 = \{v_{1,3}, v_{1,4}\}, \ldots, S_{\frac{t}{2}} = \{v_{1,t-1}, v_{1,t}\}$, $S_{\frac{t}{2}+1} = \{v_{2,1}, v_{2,2}\}$, $S_{\frac{t}{2}+2} = \{v_{2,3}, v_{2,4}\}, \ldots,$ $S_t = \{v_{2,t-1}, v_{2,t}\}$. Minimizer will also group the labels into the sets $T_1 = \{1, 2t\}$, $T_2 = \{2, 2t-1\}, \ldots, T_t = \{t, t+1\}$. Then, during each round of the game, if Maximizer plays a label from set T_i on a vertex in set S_k, then Minimizer will play the other label from from T_i on the second vertex from set S_k. In this way, each of the sets of vertices will have labels summing to $2t+1$. As there are $\frac{t}{2}$ such sets in each partite half of the graph $K_{t,t}$, the open neighborhood sum of each vertex in the graph will be $(2t+1)\frac{t}{2} = t^2 + \frac{t}{2}$.

If t is odd, Minimizer partitions the vertices into the sets $S_1 = \{v_{1,1}, v_{1,2}\}$, $S_2 = \{v_{1,3}, v_{1,4}\}, \ldots, S_{\frac{t-1}{2}} = \{v_{1,t-2}, v_{t-1}\}$, $S_{\frac{t-1}{2}+1} = \{v_{2,1}, v_{2,2}\}$, $S_{\frac{t-1}{2}+2} = \{v_{2,3}, v_{2,4}\}, \ldots, S_{t-1} = \{v_{2,t-2}, v_{2,t-1}\}$, $S_t = \{v_{1,t}, v_{2,t}\}$. Minimizer will again group the labels into the sets $T_1 = \{1, 2t\}$, $T_2 = \{2, 2t-1\}, \ldots, T_t = \{t, t+1\}$. Without loss of generality, we can assume that when Maximizer places a label from the set T_t, he places it on a vertex from the set S_t. Then, during each round of the game, if Maximizer plays a label from set T_i on a vertex in set S_k, then Minimizer will play the other label from from T_i on the second vertex from set S_k. In this way, each of the sets of vertices will have labels summing to $2t+1$. At the end of the game vertices in one partite half will have open neighborhood sums of $(2t+1)\frac{t-1}{2} + t = t^2 + \frac{t}{2} - \frac{1}{2} = t^2 + \lfloor \frac{t}{2} \rfloor$ and the vertices in the other partite half will have open neighborhood sums of $(2t+1)\frac{t-1}{2} + t + 1 = t^2 + \frac{t}{2} + \frac{1}{2} = t^2 + \lceil \frac{t}{2} \rceil$. $\square$

Theorem 5.109. $NS_{X(B),N(T)}(K_{t,t}) = t^2 + \lceil \frac{t}{2} \rceil$.

Proof. Notice that Minimizer can follow a strategy very similar to the one in Theorem 5.108. Since Maximizer must pick labels from smallest to largest, he is forced to select the smaller label from set T_i during round i. But the largest remaining label for Minimizer's turn during round i is precisely the larger label from set T_i. By adopting the same strategy for placement of his labels as in Theorem 5.108, Minimizer achieves the outcome of $NS_{X(T),N(B)}(K_{t,t}) = t^2 + \lceil \frac{t}{2} \rceil$. □

Theorem 5.110. $NS_{X(T),N(B)}(K_{t,t}) = t^2 + \lceil \frac{t}{2} \rceil$.

Proof. Notice that Minimizer can follow a similar strategy to the one from Theorem 5.108. Since Maximizer must pick labels from largest to smallest, he is forced to select the larger label from set T_i during round i. Hence the smallest remaining label for Minimizer's turn during round i is precisely the smaller label from set T_i. By adopting the same strategy for placement of her labels as in Theorem 5.108, Minimizer achieves the outcome of $NS_{X(T),N(B)}(K_{t,t}) = t^2 + \lceil \frac{t}{2} \rceil$. □

Theorem 5.111. $NS_{X(T),N(T)}(K_{t,t}) = t^2 + t$.

Proof. In order to achieve an outcome of at least $t^2 + t$, Maximizer can set $X_1 = (2t, v_{1,1})$ and then continue placing his labels on the first partite set until all the vertices in this set are labeled. In this way, Maximizer can ensure an outcome of at least $2t + (2t-2) + (2t-4) + (2t-(2t+2)) = 2t^2 - \frac{2(t-1)t}{2} = t^2 + t$. In order to ensure that the outcome is no greater than $t^2 + t$, Minimizer can, during each round, place her label on the opposite partite set from where Maximizer played. In this way, the absolute difference in the sums of the labels on the partite sets will be no more than t. But since the sum of all labels is $\frac{2t(2t+1)}{2} = 2t^2 + t$, the outcome of the game is no more than $\frac{2t^2+t}{2} + \frac{t}{2} = t^2 + t$. □

Theorem 5.112. $NS_{X(A),N(B)}(K_{t,t}) \geq t^2 + 2t - 2$.

Proof. Without loss of generality we assume that at each turn, once a player decides which partite set they will play on, they label the vertex from that set with the lowest index. In order to achieve a game outcome of at least $t^2 + 2t - 2$, Maximizer can use the following strategy. Set $X_1 = (2t-1, v_{1,1})$. Minimizer will then either play $N_1 = (1, v_{1,2})$ or $N_1 = (1, v_{2,1})$.

If $N_1 = (1, v_{1,2})$, at each turn, Maximizer will play the smallest unused label onto the first partite set until all vertices in that set are labeled, at which time the outcome of the game is determined. In this way Maximizer can ensure that the labels played on the first partite set sum to no more than $(2t-1)+1+(2+4+\cdots+(2t-4)) = (2t-1)+1+\frac{2(t-2)(t-1)}{2} = t^2 - t + 2$. Since the sum of all the labels is $2t^2 + t$, this implies that the vertices in the first partite set have open neighborhood sums of at least $\left(2t^2+t\right) - \left(t^2 - t + 2\right) = t^2 + 2t - 2$.

If $N_1 = (1, v_{2,1})$, at each turn, Maximizer will play the smallest unused label onto the second partite set until all vertices in that set are labeled, at which time the outcome of the game is determined. In this way Maximizer can ensure that the labels played on the second partite set sum to no more than $1+(2+4+\cdots+(2t-2)) = 1+\frac{2(t-1)t}{2} = t^2 - t + 1$. Since the sum of all the labels is $2t^2 + t$, this implies that the vertices in the second partite set have open neighborhood sums of at least $\left(2t^2+t\right) - \left(t^2 - t + 1\right) = t^2 + 2t - 1$. □

Theorem 5.113. $NS_{X(A),N(T)}(K_{t,t}) \geq t^2 + 2t - 2$.

Proof. Without loss of generality we can assume that at each turn, once a player decides which partite set they will play on, they play on the unlabeled vertex from that set with the

lowest index. In order to achieve a game outcome of at least t^2+2t-2, Maximizer can use the following strategy. Set $X_1=(2,v_{1,1})$. Minimizer will then either play $N_1=(2t,v_{1,2})$ or $N_1=(2t,v_{2,1})$.

If $N_1=(2t,v_{1,2})$, at each turn, Maximizer will play the largest unused label onto the first partite set until all vertices in that set are labeled, at which time the outcome of the game is determined. In this way Maximizer can ensure that the labels played on the first partite set sum to at least $2+(2t+(2t-1)+(2t-3)+\cdots+5)=2+2t+3(t-2)+(2+4+\cdots+(2t-4))=5t-4+\frac{2(t-2)(t-1)}{2}=t^2+2t-2$. In this way, the open neighborhood sums of the vertices in the second partite set will be at least t^2+2t-2.

If $N_1=(2t,v_{2,1})$, at each turn, Maximizer will play the largest unused label onto the second partite set until all vertices in that set are labeled, at which time the outcome of the game is determined. In this way Maximizer can ensure that the labels played on the second partite set sum to at least $2t+((2t-1)+(2t-3)+\cdots+3)=2t+(t-1)+(2+4+\cdots+(2t-2))=3t-1+\frac{2(t-1)t}{2}=t^2+2t-1$. In this way, the open neighborhood sums of the vertices in the first partite set will be at least t^2+2t-1. □

Next we will look at the maximum open neighborhood sum game played on $K_{t,t}$ where Minimizer plays first.

Theorem 5.114. $NS_{N(B),X(B)}(K_{t,t})=t^2+2t-1$.

Proof. Without loss of generality we can assume that $N_1=(1,v_{1,1})$. First we need to demonstrate a strategy for Minimizer whereby she can ensure that the outcome of the game is no more than t^2+2t-1. She creates the pairs of vertices $S_1=\{v_{1,2},v_{2,2}\}$, $S_2=\{v_{1,3},v_{2,3}\}$,..., $S_{t-1}=\{v_{1,t},v_{2,t}\}$. Notice that as long as there are at least two un-

labeled vertices in the second partite set, we can assume that a play onto a vertex in the second partite set is made onto a vertex from a set S_k as opposed to onto vertex $v_{2,1}$. At each turn if Maximizer labels a vertex from set S_k with a label x, then Minimizer will follow by labeling the other vertex from set S_k with label $x+1$. On the last play of the game Maximizer will be forced to label vertex $v_{2,1}$ with label $2t$. It then follows that the sum of the labels assigned to the second partite set does not exceed $2t+(3+5+\cdots+(2t-1)) = 2t+(t-1)+(2+4+\cdots+(2t-2)) = 3t-1+\frac{2(t-1)t}{2} = t^2+2t-1$. Hence the vertices in the first partite set will have open neighborhood sums of at most t^2+2t-1.

Next we need to demonstrate a strategy that Maximizer can use to ensure the outcome of the game is at least t^2+2t-1. Since we have assumed $N_1 = (1, v_{1,1})$, Maximizer will set $X_1 = (2, v_{1,2})$. At each subsequent turn Maximizer will place his label on a vertex from the first partite set. Once all vertices from the first partite set have been labeled, the outcome of the game is determined. In this manner, Maximizer can ensure that the sum of the labels in the first partite set is no more than $1+(2+4+(2t-2)) = 1+\frac{2(t-1)t}{2} = t^2-t+1$. Since the sum of all the labels is $2t^2+t$, we know that the sum of the labels assigned to the second partite set will be at least $\left(2t^2+t\right)-\left(t^2-t+1\right) = t^2+2t-1$. Hence the vertices in the first partite set will have open neighborhood sums of at least t^2+2t-1. □

Theorem 5.115. *If t is odd, then $NS_{N(B),X(T)}(K_{t,t}) = t^2+\frac{3t-1}{2}$.*

Proof. Assume without loss of generality that $N_1 = (1, v_{1,1})$. First we demonstrate a strategy that Minimizer can use to ensure that the outcome of the game is no more than $t^2+\frac{3t-1}{2}$. Minimizer pairs the vertices into the sets $S_1 = \{v_{1,2}, v_{1,3}\}$, $S_2 = \{v_{1,4}, v_{1,5}\}, \ldots,$

$S_{\frac{t-1}{2}} = \{v_{1,t-1}, v_{1,t}\}$, $S_{\frac{t-1}{2}+1} = \{v_{2,2}, v_{2,3}\}$, $S_{\frac{t-1}{2}+2} = \{v_{2,4}, v_{2,5}\}$,..., $S_{t-1} = \{v_{2,t-1}, v_{2,t}\}$. We can assume that the last vertex in the second partite set that is labeled is vertex $v_{2,1}$. If Maximizer labels a vertex from set S_k, then Minimizer will follow on her move by labeling the other vertex from set S_k. Notice that as long as vertex $v_{2,1}$ has not been labeled, the sum of the labels assigned to each set will be exactly $2t+2$. After vertex $v_{2,1}$ has been labeled, the outcome of the game is determined. Since vertex $v_{2,1}$ will have label at least $t+1$, we know that the sum of the labels assigned to the second partite set will always exceed those assigned to the first partite set. The maximum possible label for vertex $v_{2,1}$ will be $t+\frac{t+1}{2}$. It follows that the sum of the labels assigned to the second partite set will not exceed $\frac{(2t+2)(t-1)}{2} + \left(t+\frac{t+1}{2}\right) = t^2 + \frac{3t-1}{2}$.

In order to ensure that he can always achieve an outcome of at least $t^2 + \frac{3t-1}{2}$, Maximizer can set $X_1 = (2t, v_{2,1})$. He then creates the pairs of vertices $S_1 = \{v_{1,2}, v_{1,3}\}$, $S_2 = \{v_{1,4}, v_{1,5}\}$,..., $S_{\frac{t-1}{2}} = \{v_{1,t-1}, v_{1,t}\}$, $S_{\frac{t-1}{2}+1} = \{v_{2,2}, v_{2,3}\}$, $S_{\frac{t-1}{2}+2} = \{v_{2,4}, v_{2,5}\}$,..., $S_{t-1} = \{v_{2,t-1}, v_{2,t}\}$. At each turn, if Minimizer labels a vertex from set S_k, Maximizer will label the other vertex from set S_k. In this manner, the sum of the labels assigned to any set S_k will equal $2t+1$. Hence the sum of the labels assigned to the second partite set will equal $2t + \frac{(2t+1)(t-1)}{2} = 2t + \frac{2t^2-t-1}{2} = t^2 + \frac{3t-1}{2}$. □

Theorem 5.116. *If t is even, then $NS_{N(B),X(T)}(K_{t,t}) = t^2 + t$.*

Proof. Assume without loss of generality that $N_1 = (1, v_{1,1})$. First we demonstrate a strategy that Minimizer can use to ensure that the outcome of the game is no more than t^2+t. Minimizer pairs the vertices into the sets $S_1 = \{v_{1,3}, v_{1,4}\}$, $S_2 = \{v_{1,5}, v_{1,6}\}$,..., $S_{\frac{t-2}{2}} = \{v_{1,t-1}, v_{1,t}\}$, $S_{\frac{t-2}{2}+1} = \{v_{2,1}, v_{2,2}\}$, $S_{\frac{t-2}{2}+2} = \{v_{2,3}, v_{2,4}\}$,..., $S_{t-1} = \{v_{2,t-1}, v_{2,t}\}$.

We can assume that the last vertex in the first partite set that is labeled is vertex $v_{1,2}$. If Maximizer labels a vertex from set S_k, then Minimizer will follow on her move by labeling the other vertex from set S_k. Notice that as long as vertex $v_{1,2}$ has not been labeled, the sum of the labels assigned to each set will be exactly $2t+2$. The maximum possible label that can then be assigned to vertex $v_{1,2}$ will be $2t-\frac{t}{2}+1$. Following this approach, the maximum possible sum of the labels assigned to the first partite set will be $1+\left(2t-\frac{t}{2}+1\right)+\left(\frac{t}{2}-1\right)(2t+2)=t^2+\frac{t}{2}$. The sum of the labels assigned to the second partite set will not exceed $\frac{(2t+2)t}{2}=t^2+t$. Hence, the maximum possible open neighborhood sum will be t^2+t.

In order to ensure that he can always achieve an outcome of at least t^2+t, Maximizer can set $X_1=(2t,v_{2,1})$, and then at each turn he will place the largest available label onto the vertex from the second partite with the smallest index. Under this strategy, the sum of the labels assigned to any pair of vertices $\{v_{2,2i-1},v_{2,2i}\}$ will be at least $2t+2$. Hence, the sum of the labels assigned to the second partite set will be at least $\frac{t(2t+2)}{2}=t^2+t$. $\square$

Theorem 5.117. *If t is odd, then $NS_{N(T),X(B)}(K_{t,t})=t^2+\frac{3t-1}{2}$.*

Proof. Assume without loss of generality that $N_1=(2t,v_{1,1})$. First we demonstrate a strategy that Minimizer can use to ensure that the outcome of the game is no more than $t^2+\frac{3t-1}{2}$. Minimizer pairs the vertices into the sets $S_1=\{v_{1,2},v_{1,3}\}$, $S_2=\{v_{1,4},v_{1,5}\}$,..., $S_{\frac{t-1}{2}}=\{v_{1,t-1},v_{1,t}\}$, $S_{\frac{t-1}{2}+1}=\{v_{2,2},v_{2,3}\}$, $S_{\frac{t-1}{2}+2}=\{v_{2,4},v_{2,5}\}$,..., $S_{t-1}=\{v_{2,t-1},v_{2,t}\}$. We can assume that the last vertex in the second partite set that is labeled is vertex $v_{2,1}$. If Maximizer labels a vertex from set S_k, then Minimizer will follow on her move by labeling the other vertex from set S_k. Notice that as long as vertex $v_{2,1}$ has not been labeled,

the sum of the labels assigned to each set will be exactly $2t$. After vertex $v_{2,1}$ has been labeled, the outcome of the game is determined. Also, vertex $v_{2,1}$ will have label of at least $\frac{t+1}{2}$ and no more than t. Thus, the sum of the labels in the second partite set will be at least $\frac{t+1}{2}+\frac{(t-1)2t}{2}=t^2-\frac{(t-1)}{2}$ and no more than $t+\frac{(t-1)2t}{2}=t^2+t$. The maximum possible sum of the labels assigned to the second partite set will be $t+\frac{(t-1)2t}{2}=t^2$. Since the sum of all labels is $2t^2+t$, the sum of the labels on the first partite set will be at least $\left(2t^2+t\right)-\left(t^2+t\right)=t^2$, and will be no more than $\left(2t^2+t\right)-\left(t^2-\frac{(t-1)}{2}\right)=t^2+\frac{3t-1}{2}$. Therefore, the maximum possible open neighborhood sum will be $t^2+\frac{3t-1}{2}$.

In order to ensure that he can always achieve an outcome of at least $t^2+\frac{3t-1}{2}$, Maximizer can set $X_1=(1,v_{2,1})$. He then creates the pairs of vertices $S_1=\{v_{1,2},v_{1,3}\}$, $S_2=\{v_{1,4},v_{1,5}\},\ldots$, $S_{\frac{t-1}{2}}=\{v_{1,t-1},v_{1,t}\}$, $S_{\frac{t-1}{2}+1}=\{v_{2,2},v_{2,3}\}$, $S_{\frac{t-1}{2}+2}=\{v_{2,4},v_{2,5}\},\ldots$, $S_{t-1}=\{v_{2,t-1},v_{2,t}\}$. At each turn, if Minimizer labels a vertex from set S_k, Maximizer will label the other vertex from set S_k. In this manner, the sum of the labels assigned to any set S_k will equal $2t+1$. Hence the sum of the labels assigned to the first partite set will equal $2t+\frac{(2t+1)(t-1)}{2}=2t+\frac{2t^2-t-1}{2}=t^2+\frac{3t-1}{2}$. Therefore, the maximum open neighborhood sum will be $t^2+\frac{3t-1}{2}$. □

Theorem 5.118. *If t is even, then $NS_{N(T),X(B)}(K_{t,t})=t^2+t$.*

Proof. Assume without loss of generality that $N_1=(2t,v_{1,1})$. First we demonstrate a strategy that Minimizer can use to ensure that the outcome of the game is no more than t^2+t. Minimizer pairs the vertices into the sets $S_1=\{v_{1,3},v_{1,4}\}$, $S_2=\{v_{1,5},v_{1,6}\},\ldots$, $S_{\frac{t-2}{2}}=\{v_{1,t-1},v_{1,t}\}$, $S_{\frac{t-2}{2}+1}=\{v_{2,1},v_{2,2}\}$, $S_{\frac{t-2}{2}+2}=\{v_{2,3},v_{2,4}\},\ldots$, $S_{t-1}=\{v_{2,t-1},v_{2,t}\}$. We can assume that the last vertex in the first partite set that is labeled is vertex $v_{1,2}$. If

Maximizer labels a vertex from set S_k, then Minimizer will follow on her move by labeling the other vertex from set S_k. Notice that as long as vertex $v_{1,2}$ has not been labeled, the sum of the labels assigned to each set will be exactly $2t$. The maximum possible label that can then be assigned to vertex $v_{1,2}$ will be t. Following this approach, the maximum possible sum of the labels assigned to the first partite set will be $2t+t+\frac{2t(t-2)}{2}=t^2+t$. The sum of the labels assigned to the second partite set will not exceed $\frac{(2t+1)t}{2}=t^2+\frac{t}{2}$. Hence, the maximum possible open neighborhood sum will be t^2+t.

In order to ensure that he can always achieve an outcome of at least t^2+t, Maximizer can set $X_1=(1,v_{2,1})$, and then at each turn he will place the smallest available label onto the vertex from the second partite with the smallest index. Under this strategy, the sum of the labels assigned to any pair of vertices $\{v_{2,2i-1},v_{2,2i}\}$ will be no more than $2t$. Hence, the sum of the labels assigned to the second partite set will be no more than $\frac{t(2t)}{2}=t^2$. This implies that the labels assigned to the first partite set will sum to at least $\left(2t^2+t\right)-t^2=t^2+t$. □

Theorem 5.119. $NS_{N(T),X(T)}(K_{t,t})=t^2+2t-1$.

Proof. Without loss of generality we can assume that $N_1=(2t,v_{1,1})$. First we need to demonstrate a strategy for Minimizer whereby she can ensure that the outcome of the game is no more than t^2+2t-1. She creates the pairs of vertices $S_1=\{v_{1,2},v_{2,2}\}$, $S_2=\{v_{1,3},v_{2,3}\}$,..., $S_{t-1}=\{v_{1,t},v_{2,t}\}$. Notice that as long as there are at least two unlabeled vertices in the second partite set, we can assume that a play onto a vertex in the second partite set is made onto a vertex from a set S_k as opposed to onto vertex $v_{2,1}$. At each turn if Maximizer labels a vertex from set S_k with a label x, then Minimizer will fol-

low by labeling the other vertex from set S_k with label $x-1$. On the last play of the game, Maximizer will be forced to label vertex $v_{2,1}$ with label 1. We know that the sum of the labels assigned to the first partite set is at least as much as the sum of the labels assigned to the second partite set. It then follows that the sum of the labels assigned to the first partite set does not exceed $2t+(3+5+\cdots+(2t-1)) = 2t+(t-1)+(2+4+\cdots+(2t-2)) = 3t-1+\frac{2(t-1)t}{2} = t^2+2t-1$. Hence the vertices in the second partite set will have open neighborhood sums of at most t^2+2t-1.

Next we need to demonstrate a strategy that Maximizer can use to ensure the outcome of the game is at least t^2+2t-1. Since we have assumed $N_1 = (2t, v_{1,1})$, Maximizer will set $X_1 = (2t-1, v_{1,2})$. At each subsequent turn Maximizer will place his label on a vertex from the first partite set. Once all vertices from the first partite set have been labeled, the outcome of the game is determined. In this manner, Maximizer can ensure that the sum of the labels in the first partite set is at least $2t+((2t-1)+(2t-3)+\cdots+3) = 2t+(t-1)+\frac{2(t-1)t}{2} = t^2+2t-1$. Hence the vertices in the second partite set will have open neighborhood sums of at least t^2+2t-1. $\square$

Theorem 5.120. $NS_{N(A),X(B)}(K_{t,t}) \leq t^2+t$.

Proof. We can assume without loss of generality that Minimizer makes her first move onto vertex $v_{1,1}$. We need to demonstrate a that she can use to ensure that the outcome of the game is no more than t^2+t. Her first move will be to set $N_1 = (t+1, v_{1,1})$.

If t is even, then it follows from Theorem 5.118 that $NS_{N(A),X(B)}(K_{t,t}) \leq t^2+t$. If t is odd, then Maximizer pairs the vertices into sets $S_1 = \{v_{1,2}, v_{1,3}\}$, $S_2 = \{v_{1,4}, v_{1,5}\}$,..., $S_{\frac{t-1}{2}} = \{v_{1,t-1}, v_{1,t}\}$, $S_{\frac{t-1}{2}+1} = \{v_{2,2}, v_{2,3}\}$, $S_{\frac{t-1}{2}+2} = \{v_{2,4}, v_{2,5}\}$,..., $S_{t-1} = \{v_{2,t-1}, v_{2,t}\}$. If

Maximizer places a label on a vertex from the second partite set, as long as there are at least three unlabeled vertices in that set, we can assume that it is placed on a vertex with index at least two. As long as there are at least three unlabeled vertices in the second partite set when Maximizer plays, if he labels a vertex from set S_k, Minimizer will label the other vertex from set S_k such that the sum of the labels assigned to the vertices from set S_k is $2t+1$. If on any turn vertex $v_{2,1}$ is the only unlabeled vertex from the second partite set, Minimizer will place the label t on vertex $v_{2,1}$. In this way, if Minimizer is the one to label vertex $v_{2,1}$, then the sum of the labels from the first partite set will be $t^2 + \left\lfloor \frac{t}{2} \right\rfloor$ and the sum of the labels from the second partite set will be $t^2 + \left\lceil \frac{t}{2} \right\rceil$. If Maximizer labels vertex $v_{2,1}$, then it will have label no smaller than $\frac{t+1}{2}$ and no bigger than t. Hence, the sum of the labels assigned to the second partite set will be at least $\frac{(2t+1)(t-1)}{2} + \frac{t+1}{2} = t^2$ and no more than $\frac{(2t+1)(t-1)}{2} + t = t^2 + \left\lfloor \frac{t}{2} \right\rfloor$. Since the sum of all the labels is $2t^2+t$, the sum of the labels on the first partite set will be least $t^2 + \left\lceil \frac{t}{2} \right\rceil$ and no more than t^2+t. Hence the maximum possible open neighborhood sum will be t^2+t. □

By applying Corollary 5.17 to the results of this section, we state the known values for the minimum open neighborhood sum game on $K_{t,t}$ in Table 5.6. Also notice that for any bijection $f : V\left(K_{p,q}\right) \to [p+q]$ that there are at most two distinct open neighborhood sums. Specifically, any vertices that belong to the same partite set will have equal neighborhood sums. This implies that if $NS\left[K_{p,q}\right] = NS[f]$, then we must also have $NS^-\left[K_{p,q}\right] = NS^-[f]$ and $NS^{sp}\left[K_{p,q}\right] = NS^{sp}[f]$. For our competitive game scenarios on $K_{p,q}$, this implies that a strategy to minimize (maximize) the maximum open neighborhood sum will also minimize (maximize) the open neighborhood spread. For any given game

Table 5.6: Value of minimum open neighborhood sum game on $K_{t,t}$

	Value of t	
Scenario	t odd	t even
X(B),N(B)	t^2	t^2
X(B),N(T)	$t^2+\lfloor\frac{t}{2}\rfloor$	$t^2+\lfloor\frac{t}{2}\rfloor$
X(B),N(A)	$t^2+\lfloor\frac{t}{2}\rfloor$	$t^2+\lfloor\frac{t}{2}\rfloor$
X(T),N(B)	$t^2+\lfloor\frac{t}{2}\rfloor$	$t^2+\lfloor\frac{t}{2}\rfloor$
X(T),N(T)	t^2	t^2
X(T),N(A)	$t^2+\lfloor\frac{t}{2}\rfloor$	$t^2+\lfloor\frac{t}{2}\rfloor$
X(A),N(B)	$\leq t^2-t+2$	$\leq t^2-t+2$
X(A),N(T)	$\leq t^2-t+2$	$\leq t^2-t+2$
X(A),N(A)	$t^2+\lfloor\frac{t}{2}\rfloor$	$t^2+\lfloor\frac{t}{2}\rfloor$
N(B),X(B)	t^2-t+1	t^2-t+1
N(B),X(T)	$t^2-\frac{t-1}{2}$	t^2
N(B),X(A)		
N(T),X(B)	$t^2-\frac{t-1}{2}$	t^2
N(T),X(T)	t^2-t+1	t^2-t+1
N(T),X(A)		
N(A),X(B)	$\geq t^2$	$\geq t^2$
N(A),X(T)		
N(A),X(A)		

scenario, in order to achieve a minimum open neighborhood spread value, Minimizer can employ her strategy to minimize the maximum value of an open neighborhood sum. Likewise, for any given game scenario, in order to achieve a maximum open neighborhood spread value, Maximizer can employ his strategy to maximize the (maximum) value of an open neighborhood sum. For example, since $NS_{X(B),N(B)}\left(K_{p,q}\right)=t^2+t$, we can infer that $NS^{sp}_{X(B),N(B)}\left(K_{p,q}\right)=2\left[\left(t^2+t\right)-\left(\frac{2t^2+t}{2}\right)\right]=t$. Applying this result, we summarize all the know minimum open neighborhood spread values for the competitive games on $K_{t,t}$ in Table 5.7.

Table 5.7: Value of minimum open neighborhood spread game on $K_{t,t}$

	Value of t	
Scenario	t odd	t even
X(B),N(B)	t	t
X(B),N(T)	1	0
X(B),N(A)	1	0
X(T),N(B)	1	0
X(T),N(T)	t	t
X(T),N(A)	1	0
X(A),N(B)	$\geq 3t-4$	$\geq 3t-4$
X(A),N(T)	$\geq 3t-4$	$\geq 3t-4$
X(A),N(A)	1	0
N(B),X(B)	$3t-2$	$3t-2$
N(B),X(T)	$2t-1$	t
N(B),X(A)		
N(T),X(B)	$2t-1$	t
N(T),X(T)	$3t-2$	$3t-2$
N(T),X(A)		
N(A),X(B)	$\leq t$	$\leq t$
N(A),X(T)		
N(A),X(A)		

CHAPTER 6

CONCLUSION AND OPEN PROBLEMS

In this chapter will summarize the key contributions of this dissertation and recap some of the more significant problems that remain open for further research.

6.1 Conclusion

The primary purpose of this dissertation was to develop a more generalized framework for studying graph labeling problems than what traditional YES/NO classification approaches have provided. We developed such a framework by defining the parameters $NS_W^-(G;D)$, $NS_W(G;D)$, and $NS_W^{sp}(G;D)$ for the graph labeling problem where the vertices of the graph G are labeled with the elements from the weight set W, and then the minimax, maximin, and spread values of the D-neighborhood vertex sums are determined. This approach is a generalization to the problem of classifying graphs that are Σ or Σ'-labeled, or more generally, D-vertex magic. Specifically we showed that graph G is D-vertex magic if and only if $NS^{sp}(G;D) = 0$.

We demonstrated two significant advantages of this approach over traditional approaches. First of all, each of these parameters is a measure of how equitably one can distribute the weights from set W onto the D-neighborhoods of the graph G. For any graph G,

weight set W, and distance set D, each of the parameters provide a measure of how close to a perfectly equitable distribution of weights one can achieve. We demonstrated how these measures could be calculated for several families of graphs. For example, while the unions for C_4's are the only Σ-labeled graphs, we determined the exact values of $NS^-(kC_t)$, $NS(kC_t)$, and $NS^{sp}(kC_t)$ for the union of all cycles of equal order. Having a measure of how close to Σ-labeled an arbitrary graph can be is useful as there are apparently very few Σ-labeled graphs.

The second significant advantage of our approach is that it allowed us to relate our graph labeling problem to other well developed concepts in graph theory. Specifically we were able to use the fractional packing number and fractional domination number of a graph to provide bounds for $NS^-_W(G;D)$ and $NS_W(G;D)$ for all graphs G, all label sets W, and all neighborhood sets D. These bounds allowed us to show that D-neighborhood vertex magic constants are unique. More generally we showed that anytime $NS^{sp}_W(G;D) = 0$, there will exist a unique constant $c = NS^-_W(G;D) = NS_W(G;D)$. This result is significant, not only because it answers a previously open question in the graph labeling research, but also because of the connection it makes between the graph labeling problem and other areas of research within graph theory.

Our framework also suggests how other graph labeling problems could be generalized. In Appendix B we briefly describe one such generalization. However, many of the types of graph labeling problems described by Gallian [5] can be generalized in a similar sort of way. It is hoped that by doing so, similar types of connections to know graph theory results can be found.

6.2 Open Problems

Of course this research exposes as many new questions as it answers. We close with a few of the more significant open problems that will be the subject of future research.

Problem 6.1. Theorem 3.48 establishes the important result that for any graph G, label set W, and distance set D, we have $NS^-_W(G;D) \le \frac{\sigma_W}{\rho_f(G;D)} = \frac{\sigma_W}{\gamma_f(G;D)} \le NS_W(G;D)$. Though as shown in Appendix A, at least when $n \le 10$, $W = [n]$, and $D = \{0,1\}$, there are very few graphs that achieve this bound. Can the bounds for $NS^-_W(G;D)$ and $NS_W(G;D)$ be improved, specifically by finding a stronger relationship with the packing and domination parameters of the graph?

Problem 6.2. Theorem 4.29 provides a complete characterization of the parameters $NS(kC_t)$, $NS^-(kC_t)$, and $NS^{sp}(kC_t)$. What are the values of $NS(G)$, $NS^-(G)$, and $NS^{sp}(G)$ when G is the union of cycles not all of the same order?

Problem 6.3. What are the values of $NS^-[C_n]$, $NS[C_n]$, and $NS^{sp}[C_n]$ for $n \equiv 3(Mod 6)$ and $n \ge 21$?

Problem 6.4. What are the values of $NS^-[kC_t]$, $NS[kC_t]$, and $NS^{sp}[kC_t]$ when $k \ge 2$ and $t \notin \{3,6\}$?

Problem 6.5. When graph G is the union of $k > 2$ complete graphs, not all of the same order, what are the values of $NS^-[G]$, $NS[G]$, and $NS^{sp}[G]$?

Problem 6.6. When graph G is the union of $k \ge 2$ complete graphs, what are the values of $NS^-(G)$, $NS(G)$, and $NS^{sp}(G)$?

Problem 6.7. For q odd, what are the values of $NS^{-}(T_{2\times q})$, $NS(T_{2\times q})$, and $NS^{sp}(T_{2\times q})$?

Problem 6.8. For $i \not\equiv 0(Mod4)$, what is $NS^{sp}[T_{2\times i}]$? We have bijections that show that this value is no more than three and a proof that it is at least two. However, no case has been found where the spread is two.

Problem 6.9. What are the open and closed neighborhood sum values for paths, grids, cylinders, tori, and rooks graphs?

Problem 6.10. What are the remaining game values on cycles and complete bipartite graphs from Tables 5.1, 5.3 and 5.5.

APPENDICES

APPENDIX A

Σ'-LABELED GRAPHS

In this appendix we provide a complete list of the graphs of order $n \leq 10$ that are Σ'-labeled. By Corollary 3.19 it follows that the only Σ-labeled graphs of order $n \leq 10$ are the complements of these graphs. For many of the graphs shown, there will typically be many Σ'-labelings; only a single such labeling will be shown. These graphs and their labelings were found by an exhaustive computer search. The purpose of listing these graphs in this appendix is to demonstrate the apparently small number of Σ'-labeled graphs, and thus to establish the usefulness of having the more general parameters $NS_W^-(G;D)$, $NS_W(G;D)$, and $NS_W^{sp}(G;D)$.

We established in Corollary 4.2 that K_n is Σ'-labeled for all n. More generally, Theorem 4.60 established that kK_t is Σ'-labeled if and only if k is odd, or t is even. We established in Theorems 4.7 and 4.12 that pK_2 and $K_1 \bigcup pK_2$ are Σ'-labeled for all p. Figures for these graphs will not be shown in this appendix.

Many of the remaining graphs are the union of complete graphs of different order. However, some of these Σ'-labeled graphs are connected, non-regular graphs. Except for the graphs mentioned in the previous paragraph, the graphs shown in Figures A.1, A.2, A.3 and A.4 are all the Σ'-labeled graphs of order 10 or less.

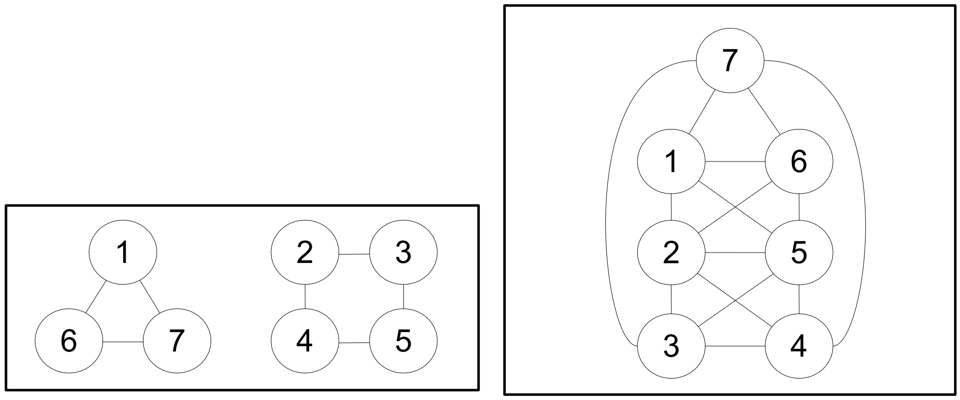

Figure A.1: Σ'-labeled graphs of order 7

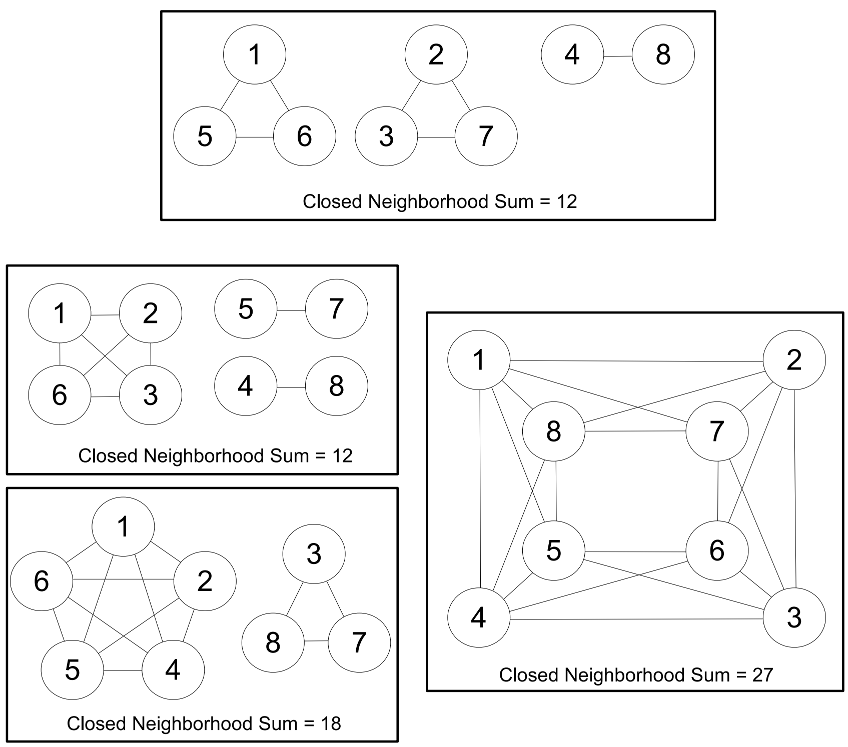

Figure A.2: Σ'-labeled graphs of order 8

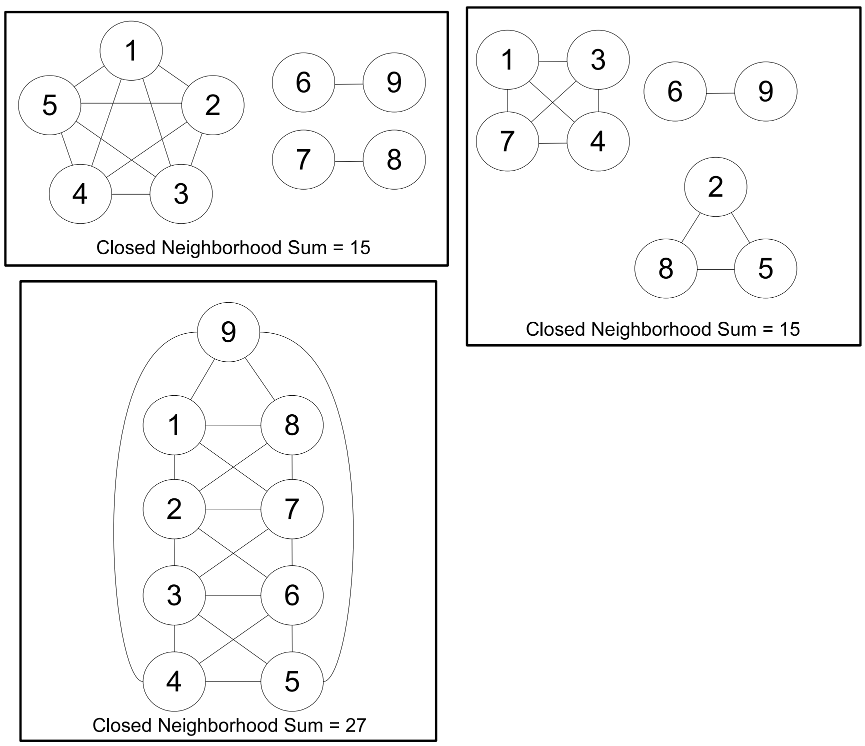

Figure A.3: Σ'-labeled graphs of order 9

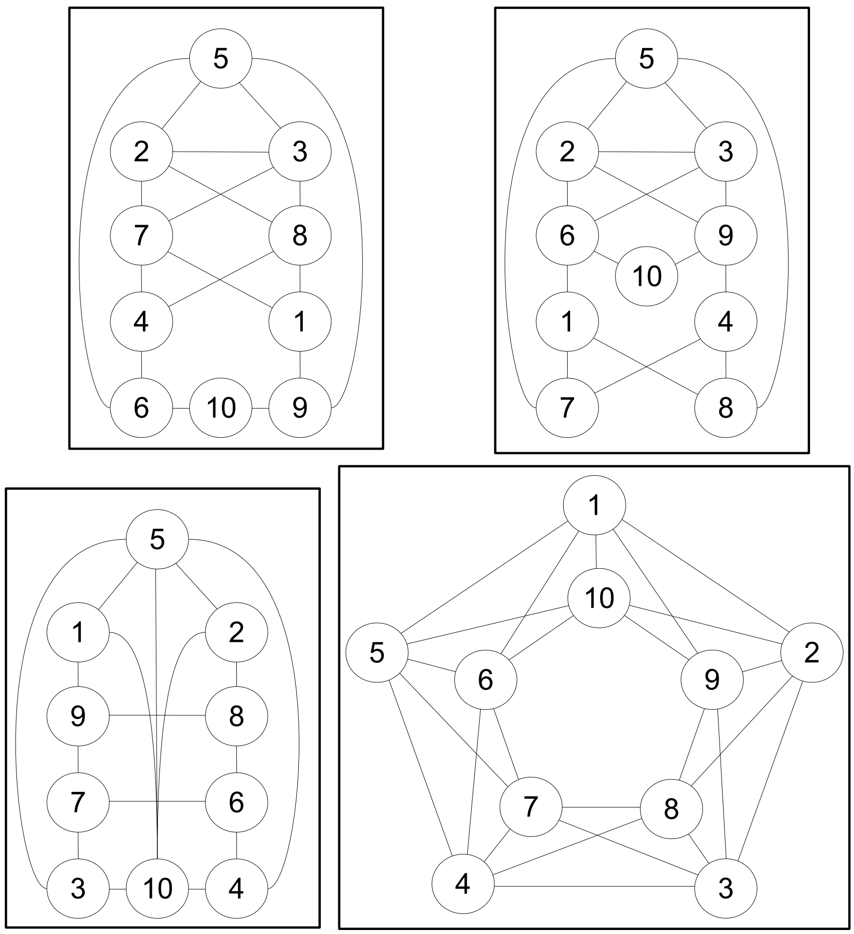

Figure A.4: Σ'-labeled graphs of order 10

APPENDIX B

EXTENSION TO EDGE-MAGIC LABELINGS

The purpose of this appendix is to demonstrate how the ideas behind the development of the parameters $NS_W(G;D)$, $NS_W^-(G;D)$, and $NS_W^{sp}(G;D)$ can be applied to other graph labeling problems. A graph G is said to be *edge-magic* (or *edge-magic total*) if there exists a bijection $f : V(G) \bigcup E(G) \to [m+n]$ and a constant c such that for all $uv \in E(G)$, $c = f(uv) + f(u) + f(v)$. In Example 2.19 we demonstrated edge-magic labelings for C_5 that produced distinct edge-magic total constants. Most research in this area focuses on determining whether specific classes of graphs are edge-magic or not. We suggest the problem could be generalized utilizing the following definitions.

Definition B.1. For a graph G and a bijection $f : V(G) \bigcup E(G) \to [m+n]$ define $EM(f) = max\{f(uv) + f(u) + f(v) | uv \in E(G)\}$.

Definition B.2. For a graph G define $EM(G) = min\{EM(f) | f : V(G) \bigcup E(G) \to [m+n]$ is a bijection$\}$.

Definition B.3. For a graph G and a bijection $f : V(G) \bigcup E(G) \to [m+n]$ define $EM^-(f) = min\{f(uv) + f(u) + f(v) | uv \in E(G)\}$.

Definition B.4. For a graph G define $EM^{-}(G) = max\{EM^{-}(f) | f : V(G) \bigcup E(G) \to [m+n]$ is a bijection$\}$.

Definition B.5. For a graph G and a bijection $f : V(G) \bigcup E(G) \to [m+n]$ define $EM^{sp}(f) = EM(f) - EM^{-}(f)$.

Definition B.6. For a graph G define $EM^{sp}(G) = min\{EM^{sp}(f) | f : V(G) \bigcup E(G) \to [m+n]$ is a bijection$\}$.

The labelings in Example 2.19 demonstrate that $EM(C_5) \leq 14$, $EM^{-}(C_5) \geq 19$, and $EM^{sp}(C_5) = 0$. Clearly $EM^{sp}(G) = 0$ if and only if G is edge-magic.

As with the neighborhood sums problem, this type of generalization provides more information about a graph than simply classifying it as edge-magic or not. For example, Chapter 2 of Wallis [22] is devoted to edge-magic labelings of graphs. One result that is stated is that for $n > 6$, K_6 is not edge-magic. This is equivalent to stating the for $n > 6$, $EM^{sp}(K_n) > 0$. With the above generalization, we can ask the question, for $n > 6$, what are the values of $EM(K_n)$, $EM^{-}(K_n)$ and $EM^{sp}(K_n)$? There are numerous other results and open research problems provided in [22] that could be generalized in a similar fashion.

Many of the other types of graph labeling problems discussed by Gallian [5] could be generalized in a similar fashion.

REFERENCES

[1] R. Anstee, R. Ferguson, and J. Griggs. Permutations with low discrepancy consecutive k-sums. *J. Combinatorial Theory (series A)*, 100:302–321, 2002.

[2] S. Arumugam, D. Froncek, and N. Kamatchi. Distance magic graphs - a survey. 2010 IWOGL conference presentation, 2010.

[3] M.S. Bazaraa, J.J. Jarvis, and H.D. Sherali. *Linear Programming and Network Flows*. Marcel Dekker, Inc., 1990.

[4] S. Beena. On sigma and sigma prime labelled graphs. *Discrete Mathematics*, 309:1783–1787, 2009.

[5] J. Gallian. A dynamic survey of graph labeling. *The Electronic Journal of Combinatorics*, 16, 2010.

[6] R.D. Godbold and P.J. Slater. All cycles are edge-magic. *Bulletin of the ICA*, 22:93–97, 1998.

[7] D.L. Grinstead and P.J. Slater. Fractional domination and fractional packing in graphs. *Congressus Numerantium*, 71:153–172, 1990.

[8] T.W. Haynes, Hedetniemi S.T., and P.J. Slater. *Domination in Graphs Advanced Topics*. Marcel Dekker, Inc., 1998.

[9] T.W. Haynes, Hedetniemi S.T., and P.J. Slater. *Fundamentals of Domination in Graphs*. Marcel Dekker, Inc., 1998.

[10] C. Huang, A. Kotzig, and A. Rosa. Further results on tree labellings. *Util. Math.*, 21c:31–48, 1982.

[11] M. Miller, C. Rodger, and R. Simanjuntak. Distance magic labelings of graphs. *Australasian Journal of Combinatorics*, 28:305–315, 2003.

[12] F.A. O'Neal and P.J. Slater. An introduction to closed/open neighborhood sums: Minimax, maximin, and spread. *Mathematics in Computer Science*, to appear.

[13] F.A. O'Neal and P.J. Slater. An introduction to distance d magic graphs. *J. Indones. Math. Soc.*, to appear.

[14] F.A. O'Neal and P.J. Slater. The minimax, maximin, and spread values for open neighborhood sums for 2-regular graphs. *Cong. Num.*, to appear.

[15] F.A. O'Neal and P.J. Slater. Uniqueness of vertex magic constants. *SIAM J. Discrete Math.*, submitted.

[16] A. Schneider and P.J. Slater. Minimax neighborhood sums. *Cong. Num.*, 188:75–83, 2007.

[17] A. Schneider and P.J. Slater. Minimax open and closed neighborhood sums. *AKCE J. Graphs. Combin.*, 6(1):183–190, 2009.

[18] J.L. Sewell and P.J. Slater. Distance independence in graphs. *Discussiones Mathematicae Graph Theory*, 31:397–409, 2011.

[19] R. Simanjuntak. Distance magic labelings and antimagic coverings of graphs. 2010 IWOGL conference presentation, 2010.

[20] K.A. Sugeng, D. Froncek, M. Miller, J. Ryan, and J. Walker. On distance magic labelings of graphs. *J. Combin. Math. Combin. Comput.*, 7:39–48, 2009.

[21] V. Vilfred. *Sigma labelled graphs and circulant graphs*. Ph. d. thesis, University of Kerala, India, 1994.

[22] W.D. Wallis. *Magic Graphs*. Birkhauser, 2001.