Six Degree of Freedom Propagation Tool in MATLAB for a Projectile in the Exo-Atmosphere Using Different Earth Models

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Six Degree of Freedom Propagation Tool in MATLAB for a Projectile in the Exo-Atmosphere Using Different Earth Models

by

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Table of Contents

Abstract ........................................................................................................................................... 4

Project Background .......................................................................................................................... 5
  Purpose of Program ..................................................................................5
  Projectile’s Orbit ...................................................................................... 6
  Different Earth Models ........................................................................... 7

Project Plan of Action .................................................................................................................... 8
  Position and Velocity ............................................................................. 8
  Orientation .............................................................................................. 8

Position and Velocity Solutions ...................................................................................................... 9
  Numerical Solution ............................................................................. 9
  Analytical Solution ............................................................................... 10
  Comparing Numerical to Analytical Solution ........................................... 11

Comparing the Different Earth Models ............................................................................................ 12
  Comparing the Trajectories .................................................................. 12
  Comparing the Velocities ...................................................................... 13

Orientation Solutions ....................................................................................................................... 14
  Analytical Solution ............................................................................. 16
  Numerical Solution ............................................................................. 16
  Plotted Orientation Results ................................................................... 16
  Comparing Numerical to Analytical Solution ........................................... 17

Visual Representations of Actual Results ....................................................................................... 19

Conclusion ................................................................................................................................. 20

Reference List ............................................................................................................................... 21
Abstract

The purpose of this project was to develop a six degree of freedom (X, Y, Z, yaw, pitch and roll) propagation function implemented in MATLAB. The function is capable of determining the position, velocity, yaw, pitch, and roll angles and angular velocity for a projectile on an exo-atmospheric track for a variety of earth models (a spherical or ellipsoidal Earth, including or excluding its rotation). A solution was developed for the only analytically solvable solution, a non-rotating spherical Earth, for the position and velocity and then, for a symmetrically-shaped object, the orientation. The results from the analytical and numerical solutions were then compared to determine the accuracy of the numerical solutions. The different Earth models were also compared to determine how much changing the projectile’s launch point, the shape of the Earth, and including the rotation of the Earth affected the projectile’s flight path and velocity.
Project Background

Purpose of Program:

The program in MATLAB was written to determine the flight path of a generic projectile orbiting the Earth. The specifics of that flight path include: the trajectory (latitude, longitude, altitude), the velocity ($V_x, V_y, V_z$) and the orientation (yaw, pitch and roll angles and angular velocity). The projectile could be either a satellite, an Intermediate to Long Range Ballistic Missile, or some type of debris (from space or a missile) that could interfere with the projectile’s path. The program can only be used once the projectile is exo-atmospheric (100-40,000 km above Earth’s surface). Since there is no air outside the atmosphere and, thus, no external forces (with the exception of gravity) acting on the projectile, there is no need to consider atmospheric effects (drag, etc.).

The purpose of the program is to provide a high-fidelity predictive model of the flight path (trajectory, velocity, and orientation) of an object orbiting the Earth. First, the model predicts the orbital flight path of a satellite. This helps determine if the satellite will fall in or stay out of orbit, if it is orientated in the correct direction to send/receive signals or view a specific part of the Earth. Secondly, it can help determine the flight path of space and missile debris. The purpose of this is to help ascertain the probability that the debris will collide with a satellite, which could then cause the satellite to fall out of orbit and impact the Earth. Thirdly, the program models the exo-atmospheric flight path of an intermediate to long-range ballistic missile, allowing for the ballistic trajectory to be predicted from an initial estimate of position, velocity, and orientation. This prediction can be used to determine future direction of travel, how fast the missile will be traveling, its future orientation, and the position of the likely impact point.
**Projectile’s Orbit:**

When a projectile is launched from Earth, it immediately proceeds into an elliptical orbit. The difference between the orbit of a satellite and a ballistic missile is that while the orbit of the satellite is complete and repeats, a ballistic missile’s orbit is incomplete and does not repeat. This is due to the satellite traveling at a sufficient velocity and/or a high enough altitude needed to achieve a balance between the gravitational pull on the satellite and the inertia of the satellite’s motion (tendency to keep going). Figure 1 depicts a few different types of satellite orbits. For example, a satellite in a sun-synchronous orbit will orbit the Earth at approximately 7.5-8 km/s, while if it were in a geostationary orbit, it would travel at the lower velocity of approximately 2.5-3.5 km/s.

![Figure 1: Satellite Orbit Example (4)](image)

However, ballistic missiles travel at lower speeds and/or altitudes such that they cannot balance Earth’s gravity. The two types of missiles that apply to this program (which travel exo-atmospherically) are the Intermediate Range and InterContinental Ballistic Missiles, (IRBM and ICBM, respectively). Table 1 and Figure 2 describe the average maximum altitude, range and velocity expected for the two types of missiles.

<table>
<thead>
<tr>
<th></th>
<th>IRBM</th>
<th>ICBM</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Max Altitude</strong></td>
<td>~400-800 km</td>
<td>~1000-3000 km</td>
</tr>
<tr>
<td><strong>Max Range</strong></td>
<td>~2000-5000 km</td>
<td>~5,500-10,000 km</td>
</tr>
<tr>
<td><strong>Max Velocity</strong></td>
<td>~4-5 km/sec</td>
<td>~6-8 km/sec</td>
</tr>
</tbody>
</table>

Table 1
Different Earth Models

When analyzing a problem such as this, the question of which Earth Model should be used is of prime importance. In this case, the program in MATLAB gives the user the option to choose from 4 different Earth models: 1) a Non-Rotating Spherical Earth, 2) a Non-Rotating Ellipsoidal Earth, 3) a Rotating Spherical Earth, and 4) a Rotating Ellipsoidal Earth (ellipsoidal denoting that the radius from the center of the Earth to the Equator is larger than the radius to the North or South Pole). The next question should be what forces need to be taken into account for the different Earth models. Since models 1 and 2 are non-rotating and there is no drag, the gravitational force is the only external force exerted on the object when using a non-rotating spherical or ellipsoidal earth. Since models 3 and 4 include the rotation of the Earth, the centrifugal and Coriolis forces must be added along with the gravitational force. The centrifugal force is an inertial force directed away from the axis of rotation that appears to act on all objects.
when viewed in a rotating frame of reference. The Coriolis force is also an inertial force that acts perpendicular to the direction of motion and the axis of rotation. On the Earth, the effect tends to cause objects to appear as if they are moving to the right in the northern hemisphere and to the left in the southern hemisphere.

**Project Plan of Action**

**Position and Velocity**

When solving for the position (X, Y, Z in earth-centered earth-fixed, ECEF, coordinates,) and velocity (V_x, V_y, V_z in ECEF coordinates) using the 4 earth models listed previously, the problem is that only Model 1 (Non-Rotating Spherical Earth) has an analytical solution that can be derived by hand. This implies that a numerical solution is needed for Models 2, 3, and 4. The plan of action then became: 1) develop a numerical solution for all 4 Earth Models, 2) develop an analytical solution for the simplest earth model (Non-Rotating Spherical), and 3) (given the same initial conditions and assuming the analytical solution to be exact and true) compare the results from the numerical solution to the analytical solution for that specific case. This is done to check the accuracy of the numerical solution for the position and velocity.

**Orientation**

Changing the Earth model does not affect the orientation (yaw, pitch and roll angles and angular velocity) of the object. However, similar to before, the analytical solution for the orientation can only be solved for a specific case – a symmetrically-shaped object. This implies a numerical solution is needed if the user wishes to use an arbitrarily shaped object. The plan of action for calculating the orientation then became: 1) develop a numerical solution for the arbitrarily shaped object, 2) develop an analytical solution for a symmetrically shaped object, and 3), similar to before, (given the same initial conditions and assuming the analytical solution
to be exact and true) compare the results from the numerical solution to the analytical solution for that specific case. This is done to check the accuracy of the numerical solution for the orientation.

**Position and Velocity Solutions**

**Numerical Solution**

The numerical solution calculates the position and velocity for any of the 4 Earth models using an ODE45 (Ordinary Differential Equation) function solver in MATLAB. The ODE45 function uses the Runge-Kutta method, where two important pieces of information are passed from one step to the next. The accuracy of the solution can be improved by setting a tolerance on the step size, the lower the tolerance, the smaller the step size and the higher the accuracy. For this specific case, the ODE45 function works by taking the initial conditions and the equations for the acceleration and uses them to work backwards in step increments to solve for the position and velocity. The equation of acceleration for the first two non-rotating models is simply the gravitational acceleration,

$$g = -\frac{GM}{r^2} \hat{r}$$

where $G$ (gravitational constant), $M$ (mass of the Earth), and $r$ (position vector) are known. However, for the ellipsoidal (J2 model) earth, the X, Y, and Z components need to be redefined. This is accomplished by choosing different spherical harmonic constants in MATLAB that define the shape of the Earth using the ECEF coordinate system. Since models 3 and 4 use a rotating Earth, they must now include the centrifugal and Coriolis forces as well. This implies that the equation for acceleration for models 3 and 4 becomes

$$g = -\frac{GM}{r^2} \hat{r} - 2(\Omega \times \mathbf{v}) - \Omega \times (\Omega \times \mathbf{r})$$
where $\Omega$ is the rotation vector of the rotating reference frame with respect to the inertial frame and $v'$ is the velocity relative to the rotating reference frame. Similar to model 2, the X, Y, and Z components of the ellipsoidal earth in model 4 are redefined by choosing different spherical harmonic constants in MATLAB that define the shape of the Earth. Now, an equation for the acceleration for all 4 Earth models is known and can be used to solve for the position and velocity using the ODE45 function in MATLAB for the numerical solution.

### Analytical Solution

The analytical solution for the position and velocity can only be solved using a non-rotating spherical earth (Model 1). If the motion of the planet is not a consideration, the 2-D problem becomes a 1-D problem. This implies that Kepler’s Three Laws of Planetary Motion can be used to solve for the position and velocity. Kepler’s Three Laws are: 1) when a projectile is launched from the surface of the Earth it will go into an elliptical orbit, 2) it takes the same amount of time to travel a large distance near the planet as it does to travel a much smaller distance far away (Equal Areas in Equal Time), and 3) the time to complete the orbit squared is proportional to the semi-major axis cubed. Then, using these three laws and the given initial conditions, the position and velocity can be solved in a straight forward manner. After solving, the position and velocity are:

\[
\|r(t)\| = \frac{h^2}{\mu (1 + e \cos f(t))} \quad (1) \quad \text{and} \quad v = \sqrt{\frac{\mu}{r}} \left( \frac{2}{r} - \frac{1}{a} \right) \quad (2)
\]

where $h$ (angular momentum constant), $\mu$ (gravitational parameter), and $e$ (eccentricity – how much orbit deviates from a perfect circle) can be calculated. However, to solve for $f(t)$, the True Anomaly (angle) or the time-varying angle between the position vector, $r$, and the eccentricity vector, is more involved. However, it can be solved by relating $f(t)$ to the area between the planet
and the projectile that has been covered since the time was equal to zero. This is done by using Kepler’s 2\textsuperscript{nd} Law (Equal Areas in Equal Time), which implies

\[
\frac{t_P}{T} = \frac{\text{Area of FVP}}{\text{Area of ellipse}} = \frac{A_{FVP}}{\pi ab} = \frac{\frac{E(t) - e \sin E(t)}{2\pi}}
\]

Using Figure 3:

where \(t_P\) (time elapsed from periapse), \(T\) (time to complete orbit), and \(E\) (eccentric anomaly) can be calculated using Kepler’s 3\textsuperscript{rd} Law and the mean anomaly. Then \(f(t)\) can finally be solved for,

\[
f(t) = 2 \tan^{-1} \left( \sqrt{\frac{1+e}{1-e}} \tan \left( \frac{E}{2} \right) \right),
\]

which can then be plugged back in to the equation for the position vector above, allowing for the final calculation of the position and velocity.

**Comparing the Numerical and Analytical Solutions**

Now that both a numerical and analytical solution for the position and velocity have been found, for the non-rotating spherical earth, the results can be compared to check the accuracy of the numerical solution (assuming the time-tested analytical solution is true and exact). Using the same initial conditions, Figure 4 and Table 2 display a comparison of the two solutions after propagating for 30 minutes (1800s).
<table>
<thead>
<tr>
<th></th>
<th>Position</th>
<th>Velocity</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Max Percent Error</strong></td>
<td>~0.0000014%</td>
<td>~0.0000029%</td>
</tr>
<tr>
<td><strong>Max Actual Error</strong></td>
<td>~13 m</td>
<td>~0.017 m/s</td>
</tr>
</tbody>
</table>

**Table 2**

As can be seen in Figure 4, the position and velocity results for the numerical and analytical solution overlap extremely well. After calculating the actual numbers, Table 2 shows that (even after propagating for 30 minutes) the percent and actual error is relatively small. This implies that the numerical solution can be considered to be quite accurate and reliable.

**Comparing the Different Earth Models**

**Comparing the Trajectories**

When comparing the trajectories of the 4 different Earth models, the results drastically depend on whether the launch point is near the Equator or the North/South Pole. Figures 5 and 6 and Table 3 compare Earth Model 1 (Non-Rotating Spherical) to Earth Models 2 (Non-Rotating Ellipsoidal), 3 (Rotating Spherical), and 4 (Rotating Ellipsoidal). The figures and table show the impact of changing the shape of Earth, including or excluding the Earth’s rotation, and how the location of the launch point affects the trajectory of the projectile.

**Figure 5: Trajectories Near North/South Pole**

**Figure 6: Trajectories Near Equator**
Table 3 shows the actual calculated results for the trajectories, after propagating 33 minutes (2000s), using the same initial conditions except for changing the initial launch point.

<table>
<thead>
<tr>
<th>Non-Rotating Spherical Earth vs Non-Rotating Ellipsoidal Earth</th>
<th>Near North/South Pole</th>
<th>Near Equator</th>
</tr>
</thead>
<tbody>
<tr>
<td>Max Percent Error</td>
<td>Max Actual Error</td>
<td>Max Percent Error</td>
</tr>
<tr>
<td>~0.00038%</td>
<td>~33 km</td>
<td>~0.001%</td>
</tr>
<tr>
<td>~0.015%</td>
<td>~136 km</td>
<td>~0.07%</td>
</tr>
<tr>
<td>~0.018%</td>
<td>~171 km</td>
<td>~0.07%</td>
</tr>
</tbody>
</table>

Table 3

Comparing the Velocities

Now, in the same way as performed above, the velocities of the projectile were compared for the 4 different Earth models. Figures 7 and 8 and Table 4 compare the velocity of Earth Model 1 to the velocities in 2, 3, and 4. Again, the figures and table show the effect of changing the shape of Earth, including or excluding the Earth’s rotation, and how the location of the launch point affects the velocity of the projectile.

Figure 7: Velocities Near North/South Pole  
Figure 8: Velocities Near Equator
Table 4 shows the actual calculated results for the velocities, after propagating 33 minutes (2000s), using the same initial conditions except for changing the initial launch point.

| Orientation Solutions |

Figure 9 shows an example of how the yaw, pitch, and roll axis are orientated for a ballistic missile. A change in the yaw determines how much the object tilts up and down, the pitch determines how much it moves from left to right, and the roll determines how much the object spins. For something such as a
satellite or ballistic missile, the yaw, pitch, and roll angles and angular rates are coupled. This implies that if one axis is given some initial torque in one direction, it will cause the rates at which the other two angles are changing to change at a different rate. Since the projectile is exo-atmospheric (i.e. no drag), there is no external torque applied to the body and the object undergoes a torque-free precession (change in orientation). This implies that a rotation matrix is needed to describe how the angles interact. The user must input a 3x3 inertial matrix and the object’s initial yaw, pitch, and roll angles and angle rates as initial conditions. Since the angular momentum equals the inertia of the object times its angular velocity, the angular momentum can be calculated. The angular momentum and the inertial matrix are defined in the same reference frame, which implies that the yaw, pitch, and roll angles can be defined in the local space frame with respect to the direction of the angular momentum.

Euler’s Angles can then be used to define the following angular velocity equations:

\[
\begin{align*}
\omega_x &= \dot{\phi} \sin \theta \sin \psi + \dot{\theta} \cos \psi \\
\omega_y &= \dot{\phi} \sin \theta \cos \psi - \dot{\theta} \sin \psi \\
\omega_z &= \dot{\psi} \cos \theta + \dot{\phi} 
\end{align*}
\]  

However, the equations only work in the body frame, while our initial conditions are given in the space frame. This implies that the eigenvalues (principle moments of inertia) and eigenvectors of the inertial matrix need to be found to convert the angular velocity in space frame to the body frame. This is accomplished by multiplying the eigenvectors by the angular velocities in the space frame, solving for the angular momentum (a constant), taking the inverse and multiplying the result by the initial angular velocity values. The user now has the angular velocity in the body frame.

Euler’s equations for a free body (no applied torque) are:

\[
\begin{align*}
0 &= I_{xx} \dot{\omega}_x - (I_{yy} - I_{zz}) \omega_y \omega_z \\
0 &= I_{yy} \dot{\omega}_y - (I_{zz} - I_{xx}) \omega_z \omega_x \\
0 &= I_{zz} \dot{\omega}_z - (I_{xx} - I_{yy}) \omega_x \omega_y
\end{align*}
\]
Analytical Solution

The analytical solution for the orientation (yaw, pitch and roll angles and angular velocity) can only be solved using a symmetrically-shaped object. Since the object is symmetric, this implies that the roll angular velocity is constant, the inertial matrix is symmetric, and the moments of inertia about the two axes are equal. These conditions simplify Euler’s equations for a free body such that they can be rewritten as:

\[
\begin{align*}
I_0 \dot{\omega}_x &= (I_0 - I) \omega_y \\
I_0 \dot{\omega}_y &= -(I_0 - I) \omega_x 
\end{align*}
\]  

The yaw, pitch, and roll velocities can then be solved for in the following form:

\[
\begin{align*}
\omega_{\text{yaw}} &= A \cos(\Omega t) \\
\omega_{\text{pitch}} &= A \sin(\Omega t) \\
\omega_{\text{roll}} &= \text{initial constant}
\end{align*}
\]

where \( \Omega = \omega_{\text{roll}}(I - I_0)/I_0 \) and \( A \) are constants that can be determined by initial conditions. \( \omega_{\text{yaw}} \) and \( \omega_{\text{pitch}} \) can then be calculated as a function of time and then converted back to the space frame.

Numerical Solution

The numerical solution follows the same process of solving for the angular velocity in the body frame as the analytical solution. However, now Euler’s equations for a free body cannot be simplified using symmetric conditions. One needs to solve for the angular acceleration in all three equations and then input them into the ODE45 function in MATLAB. The ODE function then works backwards in step increments to solve for the yaw, pitch, and roll angles and angular velocities.
Plotted Orientation Results

The results for the orientation (yaw, pitch, and roll angles and angular velocities) were then plotted to determine if they were reasonable, assuming stabilizing initial conditions are given. Stabilized indicates that the projectile is not tumbling because the object is spinning significantly faster than it is tilting up and down or left and right (i.e. the roll angular velocity is considerably higher than the yaw and pitch). Figure 10 displays the yaw, pitch, and roll angles and Figure 11 displays the yaw, pitch, and roll angular velocities. For the results in Figure 10, one can tell the yaw and pitch angles oscillate back and forth, while the roll keeps spinning and steadily increasing its angle over time (the angle, in radians, does not restart its count after one period but accumulates). For the results in Figure 11, it can be seen that the yaw and pitch have a similar angular speed, which is lower than the angular speed of the roll. The combined results are what would be expected and considered reasonable for a stabilized object in the exo-atmosphere.
Comparing Numerical to Analytical Solution

Now that both a numerical and an analytical solution for the orientation have been found for a symmetricaly shaped object, the results can be compared to check the accuracy of the numerical solution (assuming the time-tested analytical solution is true and exact). Using the same initial conditions, Figures 12 and 13 display a comparison of the two solutions.

Figure 12 displays both the numerical and analytical angular velocity after 30 seconds. At this point in time, the two solutions overlap so well the difference is indistinguishable on this scale. Figure 13 compares the same two solutions, except now after 2000 seconds, at which point a small difference appears between the two solutions. After propagating for 33 minutes (2000s) the max percent error was ~0.011% and the max actual error was ~0.0034 rad/s. This implies that the numerical solution can be consider to be relatively accurate and reliable.
Visual Representations of Actual Results

For Orientation:

Example of a Stable Missile:

Example of a “Wobbling” Missile

For Trajectory:

Combine MATLAB with Google Earth for a better visualization:

Example for a Satellite:

Example for a Long-range Missile:
Conclusion

After completing the project, the developed MATLAB propagation tool can now be used to predict what trajectory an object in the exo-atmosphere will follow, how fast the object is moving at any location along that path, how that object is orientated, and how fast that orientation is changing at any given time. This can be done using any of the 4 Earth models - Non-Rotating Spherical, Non-Rotating Ellipsoidal, Rotating Spherical, or Rotating Ellipsoidal, while using an arbitrarily shaped object. A solution for the only analytically solvable case was derived for the position and velocity (for a Non-Rotating Spherical Earth) and then for the orientation (for a symmetrically shaped object). While assuming the time-tested analytical solutions to be exact and true, they were compared to the numerical solution for those specific cases. After comparing, the numerical solution for the position and velocity was determined to be accurate to within $10^{-6}$ percent and $10^{-2}$ percent for the orientation. It was also shown how changing the Earth’s shape and including or excluding its rotation, along with changing its launch point, all significantly affect both the projectile’s flight path and velocity. Overall, the rotation of the Earth affects the trajectory of the projectile significantly more than the shape of the Earth, as expected. The results show that the shape of the Earth makes a larger difference near the North/South Pole and less of a difference at the Equator, while the rotation of the Earth makes less of a difference at the North/South Pole and a significantly larger difference at the Equator. All of these factors play a key role in predicting the exo-atmospheric flight path of 1) a satellite orbiting the Earth, 2) space or missile debris that could collide with the missile or satellite along their path, and 3) the flight path of an Intermediate to Long-range Ballistic Missile.
Reference List


