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Convergence analysis of fully discrete finite element approximations for an unsteady doubly diffusive convection model

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CONVERGENCE ANALYSIS OF FULLY DISCRETE FINITE ELEMENT APPROXIMATIONS FOR AN UNSTEADY DOUBLY DIFFUSIVE CONVECTION MODEL

by

TOBIN JACKSON

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We, the undersigned members of the Graduate Faculty of The University of Alabama in Huntsville, certify that we have advised and/or supervised the candidate of the work described in this dissertation. We further certify that we have reviewed the dissertation manuscript and approve it in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Applied Mathematics.

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ABSTRACT

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Title Convergence Analysis of Fully Discrete Finite Element Approximations for an Unsteady Doubly Diffusive Convection Model

Doubly diffusive convection refers to flows driven by density differences caused by the simultaneous occurence of temperature and concentration gradients and has many applications in such diverse areas as oceanography, atmospheric sciences and the study stellar atmospheric instability, to name a few. For Newtonian incompressible flows the Boussinesq model assumes that density varies linearly with temperature and concentration in the bouyancey term of the Navier-Stokes momentum equation. Adding the transport equations for temperature and concentration results in a tightly coupled and highly nonlinear system of partial differential equations.

A fully discrete finite element appoximation for the unsteady model is a discretization in both space and time that uses a finite element approximation for the spatial discretization, also known as the semi-discrete problem. The finite element method is based on the weak form of the governing equations set in the context of function spaces. Error estimates for both space and time discretizations are derived using energy arguments and properties of Sobolev imbeddings of function spaces.

For the semi-discrete problem, a priori error estimates are derived as well as pressure error estimates. However, the main result is an optimal order of convergence achieved by extending the duality argument of Aubin and Nitsche to the doubly diffusive problem.

For time discretization, three different backward Euler schemes are introduced: *fully implicit*, *semi-implicit* and *semi-implicit decoupled.* For each scheme a priori stability estimates are derived and consistency and convergence are demonstrated. Using different versions of the discrete Gronwall lemma it is shown that the *fully implicit* and *semi-implicit* schemes are conditionally convergent whereas the *semi-implicit decoupled* scheme is unconditionally convergent.

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CHAPTER 1

INTRODUCTION

Natural convection refers to flows generated by a buoyancy driving force. Thermosolutal convection, also known as doubly-diffusive flow, refers to flows wherein the buoyancy force is due solely to density differences caused by the simultaneous occurance of temperature and concentration gradients. There are many instances of thermosolutal convection in nature. For example, in oceanography flows can be driven by combined thermal and salinity gradients and the resulting thermosolutal convection has been shown to cause the formation of salt fountains and salt fingers. Other examples of thermosolutal convection in nature include mantle flow in the earth's crust and atmospheric flows driven by temperature and water concentration differences.

Thermosolutal convection for laminar Newtonian flow can be modeled by coupling the Navier-Stokes equations with the transport equations for temperature and concentration. One such model is the Boussinesq model which assumes that fluid properties are constant except for density in the buoyancy term which depends linearly on both temperature and concentration [1]. That is,

$$\rho(\theta, C) = \rho^*[1 - \beta_\theta(\theta - \theta^*) - \beta_C(C - C^*)],$$

where θ is temperature , C is concentration and θ^*, C^* and ρ^* are reference values. β_θ and β_c are buoyancy ratios given by, $\beta_\theta = -1/\rho^*[\partial\rho/\partial\theta]_C$ and $\beta_C = -1/\rho^*[\partial\rho/\partial C]_\theta$.

There are several popular parameter formulations of the governing equations of the Boussinesq model. This disseration will take the same approach as [1] and use the non-dimensional parameter formulation: Prandtl number $Pr = \nu/D$, Lewis number $Le = \kappa/D$, thermal Grashof number G_{r_θ} and solutal Grashof number G_{r_c}, where ν is viscosity, κ is heat conductivity and D is diffusivity. The non-dimensional governing equations using this set of parameters are

Momentum

$$\frac{\partial \mathbf{u}}{\partial t} - \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = (G_{r_\theta}\theta + G_{r_c}C)\mathbf{j} + \mathbf{f} \quad \text{in} \quad \Omega \times [0, T]$$

Continuity

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in} \quad \Omega \times [0, T]$$

Energy

$$\frac{\partial \theta}{\partial t} + (\mathbf{u} \cdot \nabla)\theta = \frac{1}{Pr}\Delta\theta + Q \quad \text{in} \quad \Omega \times [0, T]$$

Concentration

$$\frac{\partial C}{\partial t} + (\mathbf{u} \cdot \nabla)C = \frac{1}{PrLe}\Delta C + \hat{Q} \quad \text{in} \quad \Omega \times [0, T]$$

with boundary conditions:

$$\begin{cases} \mathbf{u}|_{\partial\Omega} = 0 \\ \theta|_{\partial\Omega} = 0 \\ C|_{\partial\Omega} = 0 \end{cases}$$

and initial conditions:

$$\begin{cases} \mathbf{u}(x, 0) = \mathbf{u}_0(x) \\ \theta(x, 0) = \theta_0(x) \\ C(x, 0) = C_0(x), \end{cases}$$

where $\mathbf{u}$ is velocity, p is pressure and f, Q and $\hat{Q}$ are known forcing terms, and $\mathbf{j}$ is the unit vector in the direction of gravitational acceleration. Note that this system of partial differential equations (PDE) are tightly coupled and highly nonlinear thereby making numerical methods an important approach in the study of thermosolutal convection phenomenon. For instance, in 2009 there was a call for contributions to a numerical benchmark problem for the doubly-diffusive flow pertaining to the two-dimensional columnar solidication of binary alloys [2].

Numerical simulations of doubly-diffusive flows for a variety of domain shapes and sets of parameters has been an active area of research, see [1] [3] [4] and [5]. However, theoretical and numerical analysis of the PDEs governing doubly-diffusive flows have not enjoyed the same amount of attention. Previous work has been done on the existence of solutions, see [6], error estimates for the spatial discretization using a spectral method, see [7], and stability analysis, see [8]. To our knowledge, there has been no analysis of the finite element appoximation of the spatial discretization nor analysis of any time discretization schemes. This dissertation contributes a complete theoretical and numerical analysis of the PDEs governing doubly-diffusive flows for both a finite element spatial discretization, also known as the semi-discrete approximation, and three different backward Euler time discretization schemes.

The finite element method (FEM) is a numerical method for finding approximate solutions to PDEs that has become increasingly attractive in computational fluid dynamics. FEM is based on a variational formulation of the governing PDEs in the context of certain function spaces. The spatial domain is discretized and solutions are approximated by piece-wise polynomials in finite dimensional subspaces. This gives the method a rich mathematical foundation for theoretical analysis. From a practical point of view, FEM also has the advantage of coping well with complex domain shapes and unstructured meshes.

The variational formulation of the governing PDEs, also called the weak form, is obtained by multiplying the equations by general test functions from certain function spaces, integrating over the spatial domain and using Green's Theorem on the integrals with diffusive terms to pass a derivative onto the test function.

The semi-discrete error analysis in this dissertation includes a priori stability estimates, convergence analysis and pressure error analysis. Standard techniques for finite element error analysis would lead to sub-optimal order of convergence. However, to determine optimal rate of convergence requires the duality argument of Aubin [9] and Nitsche [10]. The duality argument has been used in the finite element approximation of the Stokes problem [11] and the Navier-Stokes problem [12]. We extend the duality argument for the semi-discrete doubly-diffusive problem to show second-order convergence in the L^2-norm.

With regard to time discretization we introduce three schemes: *fully-implicit*, *semi-implicit* and *semi-implicit decoupled.* The fully-implicit scheme requires that a non-linear system be solved at each time step. Semi-implicit schemes, however, are fast and easier to implement and are thus attractive for large scale unsteady simulations. The semi-implicit scheme in effect linearizes the non-linear terms requiring that only a linear system be solved at each time step. The semi-implicit decoupled scheme *decouples* the momentum equation from the two transport equations allowing two linear systems that may be solved independently. The convergence result of each scheme requires judicious application of one of the following two lemmas: the Discrete Gronwall Lemma I (Lemma 4.5) or the Discrete Gronwall Lemma II (Lemma 4.14). Lemma I is used for the *fully-implicit* and *semi-implicit* scheme and yields *conditional* convergence. That is, convergence is contingent on $\Delta t \leq K$ for some constant K. However, Lemma II, which requires no bound on Δt, is used to show that the *semi-implict decoupled* scheme is *unconditionally* convergent.

CHAPTER 2

PRELIMINARIES

2.1 Function Spaces

In this section many of the function spaces used in this dissertation are introduced. Other function spaces will be introduced as needed. Let Ω be a bounded convex domain in $\mathbb{R}^2$. The set of all square integrable functions on Ω is denoted by $L^2(\Omega)$. The usual inner product and norm are defined on $L^2(\Omega)$ and are denoted $(\cdot,\cdot)$ and $||\cdot||$ respectively. We denote by $L^p(\Omega)$, $1 \leq p < \infty$, the Lebesque space of p-integrable functions. The norm on $L^p(\Omega)$ for $p \geq 1$ is denoted by $||\cdot||_{L^p}$. We also define $L^2_0(\Omega)$ to be a subset of $L^2(\Omega)$ as follows:

$$L^2_0(\Omega) = \left\{ f \in L^2(\Omega) \mid \int_\Omega f \, d\Omega = 0 \right\} .$$

2.1.1 Sobolev Spaces

We denote the Sobolev space $H^k(\Omega)$ to be those functions in L^2 whose weak partial derivatives of order up to and including k are also in L^2. Note that $H^0(\Omega) = L^2(\Omega)$. We associate with the Hilbert spaces $H^1(\Omega)$ and $H^2(\Omega)$ the following Sobolev norms denoted $||\cdot||_1$ and $||\cdot||_2$ respectively. $||u||_1^2 = \int_\Omega \left(u^2 + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 \right) \, d\Omega$

$$||u||_2^2 = \int_\Omega \left(u^2 + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \left(\frac{\partial^2 u}{\partial y^2}\right)^2 + \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 \right) \, d\Omega.$$

We are also interested in those functions in $H^k(\Omega)$ whose derivatives up to order $k-1$ vanish on $\partial\Omega$. We denote this space $H_0^k(\Omega)$. For example, $H_0^1(\Omega) = \left\{ u \in H^1(\Omega) \mid u = 0 \text{ in } \partial\Omega \right\}$.

2.1.2 Vector Function Spaces

In this section several important operators are defined. Let $\Omega \subseteq \mathbb{R}^2$. For scalar fields, $q : \Omega \to \mathbb{R}$, the gradient, denoted ∇q, is defined as a vector that satisfies, $[\nabla q]_i = \frac{\partial q}{\partial x_i}$, $i = 1, 2$. For vector fields, $\mathbf{u} : \Omega \to \mathbb{R}^2$ is the gradient is a matrix defined as: $[\nabla \mathbf{u}]_{ij} = \frac{\partial \mathbf{u}_j}{\partial x_i}$, $1 < i, j \leq 2$. We will follow convention in denoting vector function spaces corresponding to a particular scaler function space. For example, $\mathbf{L}^2(\Omega) := L^2(\Omega) \times L^2(\Omega)$ and $\mathbf{H}_0^1(\Omega) := H_0^1(\Omega) \times H_0^1(\Omega)$. The inner product for functions belonging to $\mathbf{L}^2(\Omega)$ is given by $(\mathbf{u}, \mathbf{v}) = \int_\Omega \mathbf{u} \cdot \mathbf{v} \, d\Omega$. We also define the following inner products: $(\nabla\mathbf{u}, \nabla\mathbf{v}) = \sum_{i,j=1}^{2} \int_\Omega \frac{\partial u_j}{\partial x_i} \frac{\partial v_i}{\partial x_j} \, d\Omega$ and $(\mathbf{u} \cdot \nabla\mathbf{v}, \mathbf{w}) = \sum_{i,j=1}^{2} \int_\Omega u_j \frac{\partial v_i}{\partial x_j} w_i$. We introduce a generic Banach space X and denote by $C([0,T]; X)$ the set of continuous functions that map $[0,T]$ to X with norm $||\mathbf{u}||_{C([0,T];X)} := \max_{0 \leq t \leq T} ||\mathbf{u}(t)||$ and by $L^p([0,T]; X)$ the Banach space of measurable functions with values in X endowed with the norm, $||\mathbf{u}(t)||_{L^p([0,T];X)} = \left(\int_0^T ||\mathbf{u}(t)||^p \, dt \right)^{1/p}$, for $1 \leq p < \infty$, and $||\mathbf{u}(t)||_{L^\infty([0,T];X)} = \operatorname{ess\,sup}_{0 \leq t \leq T} ||\mathbf{u}(t)||$, for $p = \infty$. The Sobolev Space $H^1([0,T]; X)$ consists of those functions $f \in L^2([0,T]; X)$ such that f' exists in the weak sense and $f' \in L^2([0,T]; X)$ and

$$||f||_{H^1([0,T];X)} = \begin{cases} \left(\int_0^T ||f(t)||^2 + ||f'(t)||^2 \, dt \right)^2 & 1 \leq p < \infty, \\ \operatorname{ess\,sup}_{0 \leq t \leq T} ||f(t)|| + ||f'(t)|| & p = \infty. \end{cases}.$$

We denote by $H^{-1}(\Omega)$ the dual space to $H_0^1(\Omega)$. By dual space we mean that $g \in H^{-1}(\Omega)$ if g is a bounded linear functional on $H_0^1(\Omega)$. We also associate with this space the norm $||\cdot||_{-1}$ defined as follows: $||u||_{-1} = \sup_{\phi \in H_0^1(\Omega) \setminus \{0\}} \dfrac{(u, \phi)}{||\phi||_1}$. For details concerning these spaces, see [13] and [14].

Let $\mathcal{V}$ be the set of all real divergence-free C^∞ vector functions having compact support in Ω. Let $\mathbf{H}$ and $\mathbf{V}$ denote the closure of $\mathcal{V}$ in $L^2(\Omega)$ and $H^1(\Omega)$, respectively:

$\mathbf{H} = \overline{\mathcal{V}}^{||\cdot||}$ and $\mathbf{V} = \overline{\mathcal{V}}^{||\cdot||_1}$. So we have $\mathbf{V} = \left\{\mathbf{u} \in \mathbf{H}_0^1(\Omega) \mid \nabla\cdot\mathbf{u} = 0 \text{ in } \Omega\right\}$ and its corresponding dual space $\mathbf{V}^*$ equiped with the norm: $||f||_* := \sup_{z\in\mathbf{V}\setminus\{0\}} \frac{(f,z)}{||z||_1}$.

2.2 Regularity Assumptions

Regarding the regularity of the initial data $\mathbf{u}_0, p_0, \theta_0$ and C_0 and sources $\mathbf{f}, \hat{Q}$ and Q, we will assume, throughout this dissertation, that they satisfy the following regularity assumptions:

Regularity Assumptions

A(i): Domain Ω is a bounded and convex polygon.

A(ii): $\mathbf{u}_0 \in \mathbf{V}\cap\mathbf{H}^2(\Omega)$ and $\theta_0, C_0 \in H_0^1(\Omega)\cap H^2(\Omega)$

A(iii): $f, Q, \hat{Q} \in L^\infty([0,T]; L^2(\Omega)) \cap L^2([0,T]; L^2(\Omega))$

A(iv): $f_t, Q_t, \hat{Q}_t \in L^\infty([0,T]; L^2(\Omega)) \cap L^2([0,T]; L^2(\Omega))$

This implies the following bounds on the solutions, see [7] for details.

$$\sup_{[0,T]}\left(||\mathbf{u}||_2 + ||\mathbf{u}_t|| + \int_0^t ||\nabla\mathbf{u}_t||^2\ dt\right) + \sup_{[0,T]} ||p||_1 \le K \tag{2.1}$$

$$\sup_{[0,T]}\left(||\theta||_2 + ||\theta_t|| + \int_0^t ||\nabla\theta_t||^2\ dt\right) \le K \tag{2.2}$$

$$\sup_{[0,T]}\left(||C||_2 + ||C_t|| + \int_0^t ||\nabla C_t||^2\ dt\right) \le K \tag{2.3}$$

and

$$\int_0^t s\left[||\mathbf{u}_t(s)||_2^2 + ||C_t(s)||_2^2 + ||\theta_t(s)||_2^2 + ||p_t||_1\right]\ ds \le K \tag{2.4}$$

$$t\left[||\mathbf{u}_t||_1^2 + ||\theta_t||_1^2 + ||C_t||_1^2\right] \le K \tag{2.5}$$

The $\mathbf{H}^2$ regularity assumption in A(ii) is made because it is well known, see [12], that even for arbitrarily smooth data the solution of the Navier-Stokes problem may suffer from

$$\lim_{t \to 0} \{||\mathbf{u}(t)||_3 + ||\partial_t \mathbf{u}(t)||_1\} = \infty$$

unless certain non-local (and non-verifiable) compatibility conditions are satisfied for the initial data. For our problem these compatibility conditions are equivelent to the initial data $\mathbf{u}_0, p_0, \theta_0, C_0$ and $\mathbf{f}_0, h_0, Q_0$ satisfying the overdetermined system

$$\Delta p_0 = \nabla \cdot (\mathbf{f}_0 - (\mathbf{u}_0 \cdot \nabla)\mathbf{u}_0 + G_{r_\theta}\theta_0\mathbf{j} + G_{r_c}C_0\mathbf{j}) \quad \text{in } \Omega$$

$$\nabla p_0 = \Delta \mathbf{u}_0 + \mathbf{f}_0 - (\mathbf{u}_0 \cdot \nabla)\mathbf{u}_0 + G_{r_\theta}\theta_0\mathbf{j} + G_{r_c}C_0\mathbf{j} \quad \text{on } \partial\Omega.$$

As these conditions are global and cannot be verified in general for a given data, the maximum degree of regularity is between $\mathbf{H}^2$ and $\mathbf{H}^3$, see [15].

CHAPTER 3

SPATIALLY DISCRETE FINITE ELEMENT APPROXIMATIONS

In this chapter the weak form of the governing equations is introduced and then used in the formulation of a finite element spatial discretization, also referred to as the semi-discrete problem. To simplify calculations, this formulation will assume the spatially discrete velocity is divergence free which causes the pressure term to vanish. There are four major sections in this chapter.

In Section 3.1 *Preliminaries*, the framework for subsequent analysis of the finite element spatial discretization for divergence-free velocity is established including an example and two important approximation properties that will be assumed for most of this disseration. Also, this preliminary section introduces a useful L_2-projection and associated stability and approximation error properties. Section 3.2, *Apriori Stability Estimates*, contains a proof of some important a priori stability estimates (Proposition 3.1) that will be used in other proofs throughout the disseration. The main result of this chapter is the derivation of an optimal order of convergence for the semi-discrete problem (Theorem 3.11) in Section 3.3. In Section 3.4, *Pressure Error Analysis*, pressure, which had previously vanished due to the divergence-free velocity assumption, is reintroduced into the governing equations and previous results are used to determine pressure error estimates.

3.1 Preliminaries

Let $\{\mathcal{T}_h\}_{h>0}$ be a family of finite decompositions of the domain Ω into triangles K with diameter h_K. Let $h = \max\limits_{K\in\mathcal{T}_h} h_K < 1$. So, h is a real positive discretization parameter tending to zero. Let $\mathbf{W}^h$ and W^h be two families of finite dimensional subspaces of $\mathbf{H}_0^1$ and H_0^1, respectively, and let $\mathbf{V^h} = \{\mathbf{u_h} \in \mathbf{W}^h \mid \nabla \cdot \mathbf{u_h} = 0\}$ be a finite dimensional divergence free vector space.

This family, $\{\mathcal{T}_h\}_{h>0}$, of triangulations is assumed to be *shape regular*, that is there exists σ such that $\forall h > 0$, $\forall K \in \mathcal{T}_h$, $\sigma_K = \dfrac{h_K}{\rho_K} \geq \sigma$ where ρ_K is the diameter of the largest ball included in K. Moreover, the family is *quasi-uniform*, that is it is *shape regular* and for every $h > 0$ there exists $\tau_h > 0$ such that $\forall K \in \mathcal{T}_h \quad , \quad h_K \geq \tau_h$. The *quasi-uniformity* condition implies the following inverse inequality for finite diminsions:

$$||\nabla \mathbf{v_h}|| \leq K h^{-1} ||\mathbf{v_h}|| \quad , \quad \forall \mathbf{v_h} \in \mathbf{V^h}. \tag{3.1}$$

Assume that the following approximation property is satisfied for the subspaces $\mathbf{V^h}$ and W^h:

Assumption B: For each $\mathbf{v} \in \mathbf{H}_0^1 \cap \mathbf{H}^2$, $\theta \in H_0^1 \cap H^2$ there exists approximations $\pi_h \mathbf{v} \in \mathbf{V^h}$ and $\tilde{\pi}_h \theta \in W^h$ such that

$$||\mathbf{v} - \pi_h \mathbf{v}|| + h\, ||\nabla(\mathbf{v} - \pi_h \mathbf{v})|| \leq K h^2 ||\mathbf{v}||_2 \tag{3.2}$$

and

$$||\theta - \tilde{\pi}_h \theta|| + h\, ||\nabla(\theta - \tilde{\pi}_h \theta)|| \leq K h^2 ||\theta||_2 \,. \tag{3.3}$$

Also, for each $q \in H^1$, there exists an approximation $\hat{\pi}_h q \in L^h$ such that $||q - \hat{\pi}_h q|| \leq K h\, ||q||_1$.

There are many ways to construct the divergence free finite element subspace $\mathbf{V^h}$, see [16], [17] page 104 and [18] page 72. Most use the fact that the velocity admits a potential function ϕ such that $\mathbf{v} = \text{curl}\phi$, where ϕ is a scalar function in $2D$ and vector function in

$3D$. The construction is more complicated in $3D$ than $2D$. Below we present an example of a finite dimensional subspace $\mathbf{V^h}$ constructed in $2D$ and satisfying the assumption (3.2).

Example 1. *First one constructs an arbitrary finite element space* $\hat{\mathbf{W}}^h$ *in* $\mathbf{H}_0^2(\Omega)$*, where*

$$\mathbf{H}_0^2(\Omega) = \left\{ \mathbf{w} \in \mathbf{H}^2(\Omega) \mid \mathbf{w} = \frac{\partial \mathbf{w}}{\partial \mathbf{n}} = 0 \ on \ \partial\Omega \right\}.$$

Then setting

$$\mathbf{V^h} = curl\hat{\mathbf{W}}^h = \left\{ \mathbf{v_h} \mid \mathbf{v_h} = curl\mathbf{w}_h \ for \ some \ \mathbf{w}_h \in \hat{\mathbf{W}}^h \right\},$$

we get $\mathbf{V^h} \subset \mathbf{V}$*. This kind of element is refered to as a conforming element in the literature. Note this construction requires elements* $\mathbf{w}_h$ *to be continously differentiable. One such element is the Powell-Sabin-Heidl element (PSH), see [19]. PHS basis functions are continuously differentiable, piecewise quadratic polynomials defined on a macro triangle consisting of twelve subtriangles. Each element is uniquely defined by one nodal and two derivative values (in the* x *and* y *directions) at each of the three vertices and three normal derivative values at the midpoint of each side of the triangle, so that each element has twelve degrees of freedom. Alternatively one can use the Powell-Sabin* 6*-split element. For construction of* $\mathbf{V^h}$ *in* $3D$*, readers are referred to [20].*

Let

$$W^h = \left\{ w_h \in C^0(\bar{\Omega}) : w_h|_K \in P_1(K) \forall K \in \mathcal{T}_h \right\},$$

where $\mathcal{T}_h$ *is a triangulation of* Ω *based on* k *elements and is* shape regular *and* quasi-uniform*. Let* $\left\{ \hat{\mathbf{W}}^h \right\}_{h \to 0}$ *be a family of finite element subspaces of* $\mathbf{H}_0^2(\Omega)$ *such that the* $\mathbf{W}^h$*-interpolant* $\pi_h \psi$ *of* $\psi \in \mathbf{H}_0^2(\Omega) \cap \mathbf{H}^{k+2}(\Omega)$ *possesses the approximation property* $||\psi -$

$\pi_h\psi||_1 \le Ch^k||\psi||_{k+2}$ *where* C *is independent of* h, *and* k *is an integer. An interpolent* $\pi_h\mathbf{v} \in \mathbf{V^h}$ *of* $\mathbf{v}$ *may be constructed by setting* $\pi_h\mathbf{v} = curl(\pi_h\psi)$. *The interpolant has the following property: If the above approximation property holds then for all* $\mathbf{v} \in \mathbf{V} \cap \mathbf{H}^{k+1}(\Omega)$, $||\mathbf{v} - \pi_h\mathbf{v}||_1 \le Ch^k\,||\mathbf{v}||_{k+1}$. *Thus for a* k^{th} *degree interpolating polynomial we have*

$$||\mathbf{v} - \pi_h\mathbf{v}|| \le Ch^{k+1}\,||\mathbf{v}||_{k+1}\,. \tag{3.4}$$

The following L^2-projections will be usefull in our error analysis.

Definition 1. *L^2-projections:* $P_h : \mathbf{L}^2(\Omega) \to \mathbf{V^h}$ *and* $\tilde{P}_h : L^2(\Omega) \to W^h$

$$\begin{aligned} (P_h\mathbf{u}, \mathbf{v}) &= (\mathbf{u}, \mathbf{v}) && \forall \mathbf{v} \in \mathbf{V^h} \\ \left(\tilde{P}_h\psi, \phi\right) &= (\psi, \phi) && \forall \phi \in W^h, \end{aligned}$$

where ψ *is either* θ *or* C.

It follows from Definition 1 and Assumption B that,

$$||\mathbf{u} - P_h\mathbf{u}|| + h\,||\nabla(\mathbf{u} - P_h\mathbf{u})|| \le Kh^2\,||\mathbf{u}||_2\,, \qquad \forall \mathbf{u} \in \mathbf{H^2} \cap \mathbf{V} \tag{3.5}$$

$$\left|\left|\psi - \tilde{P}_h\psi\right|\right| + h\left|\left|\nabla\left(\psi - \tilde{P}_h\psi\right)\right|\right| \le Kh^2\,||\psi||_2\,, \qquad \forall \psi \in H^2 \cap H^1_0. \tag{3.6}$$

Also, by Definition 1 and Cauchy Inequality,

$$||P_h\mathbf{u}||^2 = (P_h\mathbf{u}, P_h\mathbf{u}) = (P_h\mathbf{u}, \mathbf{u}) \le ||P_h\mathbf{u}||\,||\mathbf{u}|| \quad , \quad \forall \mathbf{u} \in \mathbf{V}.$$

Thus for $\mathbf{u} \in \mathbf{V}$ and $\phi \in W$, the following L^2**-norm stability** properties hold:

$$||P_h\mathbf{u}|| \le ||\mathbf{u}|| \quad , \quad \forall \mathbf{u} \in \mathbf{V} \tag{3.7}$$

$$\left|\left|\tilde{P}_h\psi\right|\right| \le ||\psi|| \quad , \quad \forall \psi \in \mathbf{W}. \tag{3.8}$$

Moreover, for $\mathbf{u} \in \mathbf{V}$ and $\phi \in H_0^1$, the following $\mathbf{H_0^1}$**-norm stability** properties hold:

$$||\nabla P_h \mathbf{u}|| \leq K \, ||\nabla \mathbf{u}|| \quad , \quad \forall \mathbf{u} \in \mathbf{V} \tag{3.9}$$

$$\left|\left|\nabla \tilde{P}_h \phi\right|\right| \leq K \, ||\nabla \phi|| \quad , \quad \forall \phi \in H_0^1 \tag{3.10}$$

and the following **approximation error** properties hold:

$$||\mathbf{u} - P_h \mathbf{u}|| \leq Kh \, ||\nabla \mathbf{u}|| \quad , \quad \forall \mathbf{u} \in \mathbf{V} \tag{3.11}$$

$$\left|\left|\phi - \tilde{P}_h \phi\right|\right| \leq Kh \, ||\nabla \phi|| \quad , \quad \forall \phi \in H_0^1. \tag{3.12}$$

We are now ready to state the semi-discrete problem which will be the starting point for much of the work done in this chapter. The semi-discrete approximation of the Boussinesq model takes the following form: seek $(\mathbf{u_h}, \theta_h, C_h) \in \mathbf{V^h} \times W^h \times W^h$ such that,

$$\begin{aligned}(\partial_t \mathbf{u_h}, \mathbf{v_h}) + ((\mathbf{u_h} \cdot \nabla)\mathbf{u_h}, \mathbf{v_h}) + (\nabla \mathbf{u_h}, \nabla \mathbf{v_h}) = &((G_{r_\theta} \theta_h + G_{r_c} C_h)\mathbf{j}, \mathbf{v_h}) \\ &+ (\mathbf{f}, \mathbf{v_h}) \qquad \forall \mathbf{v_h} \in \mathbf{V^h}\end{aligned} \tag{3.13}$$

$$(\partial_t \theta_h, \phi_h) + (\mathbf{u_h} \cdot \nabla \theta_h, \phi_h) + \frac{1}{Pr} (\nabla \theta_h, \nabla \phi_h) = (Q, \phi_h) \qquad \forall \phi_h \in W^h \tag{3.14}$$

$$(\partial_t C_h, \psi_h) + (\mathbf{u_h} \cdot \nabla C_h, \psi_h) + \frac{1}{PrLe} (\nabla C_h, \nabla \psi_h) = \left(\hat{Q}, \psi_h\right) \qquad \forall \psi_h \in W^h \tag{3.15}$$

with initial conditions

$$\mathbf{v_h}(0) = \mathbf{v_{h0}} \in \mathbf{V^h}$$
$$\theta_h(0) = \theta_{h0} \in W_h$$
$$C_h(0) = C_{h0} \in W_h,$$

where $(\mathbf{u_{h0}}, \theta_{h0}, C_{h0})$ is an approximation of the initial data $(\mathbf{u}_0, \theta_0, C_0) \in \mathbf{V} \times W \times W$ satisfying uniformly for $h \to 0$:

$$
\begin{aligned}
||\mathbf{u}_0 - \mathbf{u_{h0}}|| &\leq Kh^2 \, ||\mathbf{u}_0||_2 \quad \forall \mathbf{u}_0 \in \mathbf{H}_0^1(\Omega) \cap \mathbf{H}^2(\Omega) \\
||\theta_0 - \theta_{h0}|| &\leq Kh^2 \, ||\theta_0||_2 \quad \forall \theta_0 \in H_0^1(\Omega) \cap H^2(\Omega) \\
||C_0 - C_{h0}|| &\leq Kh^2 \, ||C_0||_2 \quad \forall C_0 \in H_0^1(\Omega) \cap H^2(\Omega).
\end{aligned} \tag{3.16}
$$

3.2 A Priori Stability Estimates

If solutions to the problem (3.13) - (3.15) exist, the following proposition shows that they must satisfy certain a priori estimates.

Proposition 3.1. *The solutions,* $(\mathbf{u_h}, \theta_h, C_h) \in (\mathbf{V^h} \times W^h \times W^h)$*, to the problem* (3.13) - (3.15) *satisfy the following a priori estimates:*

$$
\sup_{[0,T]} (||\mathbf{u_h}||_2 + ||\partial_t \mathbf{u_h}||) + \int_0^t ||\nabla \mathbf{u_h}||^2 \, dt + \int_0^t ||\nabla \partial_t \mathbf{u_h}||^2 \, dt \leq K \tag{3.17}
$$

$$
\sup_{[0,T]} (||\theta_h||_2 + ||\partial_t \theta_h||) + \int_0^t ||\nabla \theta_h||^2 \, dt + \int_0^t ||\nabla \partial_t C_h||^2 \, dt \leq K \tag{3.18}
$$

$$
\sup_{[0,T]} (||C_h||_2 + ||\partial_t C_h||) + \int_0^t ||\nabla C_h||^2 \, dt + \int_0^t ||\nabla \partial_t \theta_h||^2 \, dt \leq K \tag{3.19}
$$

and

$$
\int_{[0,T]} \left(\left|\left|\partial_t^2 \mathbf{u_h}\right|\right|_*^2 + \left|\left|\partial_t^2 C_h\right|\right|_{H^{-1}}^2 + \left|\left|\partial_t^2 \theta_h\right|\right|_{H^{-1}}^2 \right) \, dt \leq K, \tag{3.20}
$$

where K *is a constant.*

Several preliminary results are presented before the proof of Proposition 3.1.

Lemma 3.2 (Uniform Gronwall's Inequality)**.** *Assume that positive locally integrable functions* $y(t)$*,* $g(t)$ *and* $h(t)$ *satisfy* $\frac{dy}{dt} \leq gy + h$ *for* $t \geq 0$ *and there exists positive constants*

ϵ, a_1, a_2 and a_3 such that

$$\int_t^{t+\epsilon} g(s)ds \leq a_1\,, \qquad \int_t^{t+\epsilon} h(s)\ ds \leq a_2 \quad \text{and} \quad \int_t^{t+\epsilon} y(s)\ ds \leq a_3\,,$$

Then

$$y(t+\epsilon) \leq (a_3/\epsilon + a_2)e^{a_1}\,, \qquad t \geq 0\,.$$

Proof. See [21]. □

Lemma 3.3 (Continuous Gronwall lemma)**.** *Suppose $E : [0,T] \to \mathbb{R}$ is C^1 and P, Q and R are continuous and nonnegative functions. If $\frac{dE}{dt} + P(t) \leq R(t)E(t) + Q(t)$ for $t \in [0,T]$ then*

$$E(t) + \int_0^t P(s)\ ds \leq E(0)e^{\Lambda(t)} + e^{\Lambda(t)} \int_0^t Q(s)\ ds$$

where $\Lambda(t) = \int_0^t R(\tau)\ d\tau$.

Proof. See Appendix. □

Property 3.4 (Anti-symmetry Properties)**.** *Suppose $\mathbf{u}$ is a divergence free vector field with zero boundary conditions. Then the following hold:*

1. $(\mathbf{u} \cdot \nabla \mathbf{v}, \mathbf{w}) = -(\mathbf{u} \cdot \nabla \mathbf{w}, \mathbf{v}) \quad \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{H}_0^1(\Omega)$

2. $(\mathbf{u} \cdot \nabla \theta, \phi) = -(\mathbf{u} \cdot \nabla \phi, \theta) \quad \forall \mathbf{u} \in \mathbf{H}_0^1(\Omega)$ *and* $\theta, \phi \in H_0^1(\Omega)$

3. $(\mathbf{u} \cdot \nabla C, \psi) = -(\mathbf{u} \cdot \nabla \psi, C) \quad \forall \mathbf{u} \in \mathbf{H}_0^1(\Omega)$ *and* $C, \psi \in H_0^1(\Omega)$.

Moreover,

$$(\mathbf{u} \cdot \nabla \mathbf{v}, \mathbf{v}) = 0 \quad , \quad (\mathbf{u} \cdot \nabla \theta, \theta) = 0 \quad , \quad (\mathbf{u} \cdot \nabla C, C) = 0.$$

Proof. See Appendix. □

Lemma 3.5. *The inverse Laplacian operator* $(-\Delta)^{-1}$ *and inverse Stokes operator* A^{-1} *satisfy the following relations:*

1. $\exists \quad c_1, c_2, c_3 > 0 \quad$ *such that* $\quad \forall \psi \in \mathbf{H}$

   (a) $c_1 \left|\left|\psi\right|\right|_{-1} \leq \left|\left|(-\Delta)^{-1}\psi\right|\right|_1 \leq c_2 \left|\left|\psi\right|\right|_{-1}$

   (b) $c_3 \left|\left|\psi\right|\right|_{-1}^2 \leq \big((-\Delta)^{-1}\psi, \psi\big)$

2. $\exists \quad c_4, c_5, c_6 > 0 \quad$ *such that* $\quad \forall \mathbf{u} \in \mathbf{H}$

   (a) $c_4 \left|\left|\mathbf{u}\right|\right|_* \leq \left|\left|A^{-1}\mathbf{u}\right|\right|_1 \leq c_5 \left|\left|\mathbf{u}\right|\right|_*$

   (b) $c_6 \left|\left|\mathbf{u}\right|\right|_*^2 \leq \big(A^{-1}\mathbf{u}, \mathbf{u}\big)$.

Proof. See Appendix. □

Proof of Proposition 3.1. The proof will proceed incrementally.

A-priori bounds for $||\theta_h||^2$**,** $||C_h||^2$**,** $||\mathbf{u_h}||^2$ **and** $\int_0^t \Big(||\nabla \mathbf{u_h}||^2 + ||\nabla \theta_h||^2 + ||\nabla C_h||^2\Big)\, dt$.
Setting $\phi_h = \theta_h$ in (3.14), we get

$$(\partial_t \theta_h, \theta_h) + (\mathbf{u_h} \cdot \nabla \theta_h, \theta_h) + \frac{1}{Pr}||\nabla \theta_h||^2 = (Q, \theta_h)\,. \tag{3.21}$$

Since $\mathbf{u_h} \in \mathbf{V^h}$ and $\theta_h \in W^h \subset H_0^1$, Property 3.4 implies $(\mathbf{u_h} \cdot \nabla \theta_h, \theta_h) = 0$ and (3.21) becomes

$$(\partial_t \theta_h, \theta_h) + \frac{1}{Pr}||\nabla \theta_h||^2 = (Q, \theta_h)\,. \tag{3.22}$$

But by the product rule,

$$\frac{d}{dt}\left|\left|\theta_h\right|\right|^2 = \frac{d}{dt}\left(\theta_h, \theta_h\right) = \left(\partial_t \theta_h, \theta_h\right) + \left(\theta_h, \partial_t \theta_h\right) = 2\left(\partial_t \theta_h, \theta_h\right).$$

Applying Cauchy inequality to (3.22), we get

$$\frac{1}{2}\frac{d}{dt}\left|\left|\theta_h\right|\right|^2 + \frac{1}{Pr}\left|\left|\nabla\theta_h\right|\right|^2 \leq \left|\left|Q\right|\right|\left|\left|\theta_h\right|\right|. \tag{3.23}$$

The Poincare inequality states that if $||f|| \in H_0^1(\Omega)$ then $||f|| \leq \lambda\,||\nabla f||$ where λ is a positive constant. Since $\theta_h \in W^h \subset H_0^1$ we can use the Poincare inequality in (3.23) to obtain

$$\frac{1}{2}\frac{d}{dt}\left|\left|\theta_h\right|\right|^2 + \frac{1}{Pr}\left|\left|\nabla\theta_h\right|\right|^2 \leq \lambda\left|\left|Q\right|\right|\left|\left|\nabla\theta_h\right|\right|. \tag{3.24}$$

Next, we multiply by 2 and apply Young's inequality as follows:

$$\begin{aligned}
\frac{d}{dt}\left|\left|\theta_h\right|\right|^2 + \frac{2}{Pr}\left|\left|\nabla\theta_h\right|\right|^2 &\leq 2\lambda\left|\left|Q\right|\right|\left|\left|\nabla\theta_h\right|\right| \\
&= \left(2\lambda\left|\left|Q\right|\right|\sqrt{\frac{Pr}{2}}\right)\left(\frac{\left|\left|\nabla\theta_h\right|\right|}{\sqrt{\frac{Pr}{2}}}\right) \\
&\leq \frac{4Pr\lambda^2\left|\left|Q\right|\right|^2}{4} + \frac{1}{Pr}\left|\left|\nabla\theta_h\right|\right|^2 \\
&= K\left|\left|Q\right|\right|^2 + \frac{1}{Pr}\left|\left|\nabla\theta_h\right|\right|^2. \qquad (3.25)
\end{aligned}$$

This allows us to write $\frac{d}{dt}||\theta_h||^2 + \frac{1}{Pr}||\nabla\theta_h||^2 \leq K||Q||^2$. This technique is sometimes referred to as the kickback argument and is used extensively in this dissertation. Now, integrating with respect to time from 0 to t, we get

$$||\theta_h(t)||^2 - ||\theta_h(0)||^2 + \frac{1}{Pr}\int_0^t ||\nabla\theta_h(s)||^2 \, ds \leq K \int_0^t ||Q(s)||^2 \, ds.$$

Since $||Q||^2$ is bounded by our assumptions on the data and $||\theta_h(0)||^2$ is bounded by the assumptions in (3.16), we have,

$$||\theta_h(t)||^2 + \frac{1}{Pr}\int_0^t ||\nabla\theta_h(s)||^2 \, ds \leq K. \tag{3.26}$$

By setting $\psi_h = C_h$ in (3.15) a similar analysis shows that

$$||C_h(t)||^2 + \frac{1}{Pr}\int_0^t ||\nabla C_h(s)||^2 \, ds \leq K. \tag{3.27}$$

We now turn to velocity. Setting $\mathbf{v_h} = \mathbf{u_h}$ in (3.13) gives us

$$(\partial_t \mathbf{u_h}, \mathbf{u_h}) + ((\mathbf{u_h} \cdot \nabla)\mathbf{u_h}, \mathbf{u_h}) + (\nabla \mathbf{u_h}, \nabla \mathbf{u_h}) = ((G_{r_\theta}\theta + G_{r_c}C)\mathbf{j}, \mathbf{u_h}) + (\mathbf{f}, \mathbf{u_h}).$$

By Property 3.4, $((\mathbf{u_h} \cdot \nabla)\mathbf{u_h}, \mathbf{u_h}) = 0$. Thus, by the Cauchy and Poincare inequalities,

$$\frac{1}{2}\frac{d}{dt}||\mathbf{u}_h||^2 + ||\nabla\mathbf{u_h}||^2 \leq K\,||\theta_h||\,||\nabla\mathbf{u_h}|| + K\,||C_h||\,||\nabla\mathbf{u_h}|| + K\,||\mathbf{f}||\,||\nabla\mathbf{u_h}||\,.$$

Employing the kickback argument yields,

$$\frac{d}{dt}||\mathbf{u_h}||^2 + ||\nabla\mathbf{u_h}||^2 \leq K\left(||\theta_h||^2 + ||C_h||^2 + ||\mathbf{f}||^2\right).$$

Integrating the above with respect to time from 0 to t, we get

$$||\mathbf{u_h}(t)||^2 - ||\mathbf{u_h}(0)||^2 + \int_0^t ||\nabla \mathbf{u_h}(s)||^2 \, ds \leq K \int_0^t \left(||\theta_h(s)||^2 + ||C_h(s)||^2 + ||\mathbf{f}(s)||^2\right) \, ds.$$

Thus by (3.26) and (3.27) and our assumption on the data f and $||\mathbf{u_h}(0)||^2$, we get the necessary bounds:

$$||\mathbf{u_h}(t)||^2 + \int_0^t ||\nabla \mathbf{u_h}(s)||^2 \, ds \leq K. \tag{3.28}$$

A priori bounds for $||\nabla \mathbf{u_h}||$, $||\nabla \theta_h||$ and $||\nabla C_h||$.

We now turn to finding estimates for $||\nabla \mathbf{u_h}||$, $||\nabla \theta_h||$ and $||\nabla C_h||$. Performing integration by parts on the terms $(\nabla \theta_h, \nabla \phi_h)$ and $(\nabla C_h, \nabla \psi_h)$ we get $(\nabla \theta_h, \nabla \phi_h) = -(\Delta \theta_h, \phi_h)$ and $(\nabla C_h, \nabla \psi_h) = -(\Delta C_h, \psi_h)$ since $\phi_h, \psi_h \in W^h$. Instead of performing integration by parts on the term $(\nabla \mathbf{u_h}, \nabla \mathbf{v_h})$ we introduce the Stokes operator. The Stokes operator $A : D(A) \to H$ with domain $D(A) \subseteq H$ and range $R(A) = \{Au : u \in D(A)\}$ is defined as follows:

Let $D(A) \subseteq \mathbf{V}$ be the space of all $\mathbf{u} \in \mathbf{V}$ for which there exists some $\mathbf{f} \in \mathbf{H}$ satisfying $(\nabla \mathbf{u}, \nabla \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \mathbf{v} \in \mathbf{V}$. The Riez representation theorem tells us that $D(A)$ is precisely those functions $\mathbf{u} \in \mathbf{V}$ such that the linear functional $\mathbf{v} \longmapsto (\nabla \mathbf{u}, \nabla \mathbf{v}), \mathbf{v} \in \mathbf{V}$ is continuous in the norm $||\mathbf{v}||$. For all $\mathbf{u} \in D(A)$, we define $A\mathbf{u}$ to be the unique element in $\mathbf{H}$ such that $(\nabla \mathbf{u}, \nabla \mathbf{v}) = (A\mathbf{u}, \mathbf{v}) \quad \text{holds} \;\; \forall \mathbf{v} \in \mathbf{H}$. Thus,

$$(\nabla \mathbf{u_h}, \nabla \mathbf{v_h}) = (A\mathbf{u_h}, \mathbf{v_h}). \tag{3.29}$$

Therefore, (3.13)-(3.15) is equivalent to

$$(\partial_t \mathbf{u_h}, \mathbf{v_h}) + ((\mathbf{u_h} \cdot \nabla)\mathbf{u_h}, \mathbf{v_h}) + (A\mathbf{u_h}, \mathbf{v_h}) = ((G_{r_\theta}\theta_h + G_{r_c}C_h)\mathbf{j}, \mathbf{v_h}) + (\mathbf{f}, \mathbf{v_h}) \tag{3.30}$$

$$(\partial_t \theta_h, \phi_h) + (\mathbf{u_h} \cdot \nabla\theta_h, \phi_h) - \frac{1}{Pr}(\Delta\theta_h, \phi_h) = (Q, \phi_h) \tag{3.31}$$

$$(\partial_t C_h, \psi_h) + (\mathbf{u_h} \cdot \nabla C_h, \psi_h) - \frac{1}{PrLe}(\Delta C_h, \psi_h) = \left(\hat{Q}, \psi_h\right). \tag{3.32}$$

In equations (3.30)-(3.32), we set $\mathbf{v_h} = A\mathbf{u_h}$, $\phi_h = \Delta\theta_h$ and $\psi_h = \Delta C_h$:

$$(\partial_t \mathbf{u_h}, A\mathbf{u_h}) + ((\mathbf{u_h} \cdot \nabla)\mathbf{u_h}, A\mathbf{u_h}) + (A\mathbf{u_h}, A\mathbf{u_h}) = ((G_{r_\theta}\theta_h + G_{r_c}C_h)\mathbf{j}, A\mathbf{u_h}) + (\mathbf{f}, A\mathbf{u_h}) \tag{3.33}$$

$$(\partial_t \theta_h, \Delta\theta_h) + (\mathbf{u_h} \cdot \nabla\theta_h, \Delta\theta_h) - \frac{1}{Pr}(\Delta\theta_h, \Delta\theta_h) = (Q, \Delta\theta_h) \tag{3.34}$$

$$(\partial_t C_h, \Delta C_h) + (\mathbf{u_h} \cdot \nabla C_h, \Delta C_h) - \frac{1}{PrLe}(\Delta C_h, \Delta C_h) = \left(\hat{Q}, \Delta C_h\right). \tag{3.35}$$

Note integration by parts and the product rule gives us $(\partial_t \theta_h, \Delta\theta_h) = -(\partial_t \nabla\theta_h, \nabla\theta_h) = -\frac{1}{2}\frac{d}{dt}||\nabla\theta_h||^2$ and $(\partial_t C_h, \Delta C_h) = -(\partial_t \nabla C_h, \nabla C_h) = -\frac{1}{2}\frac{d}{dt}||\nabla C_h||^2$. By (3.29),

$$(\partial_t \mathbf{u_h}, A\mathbf{u_h}) = (\nabla\partial_t \mathbf{u_h}, \nabla\mathbf{u_h}) = (\partial_t \nabla\mathbf{u_h}, \nabla\mathbf{u_h}) = \frac{1}{2}\frac{d}{dt}||\nabla\mathbf{u_h}||^2. \tag{3.36}$$

Also note that Property 3.4 does not apply to the tri-linear form, $|(\mathbf{u_h} \cdot \nabla\mathbf{u_h}, A\mathbf{u_h})|$. Analyzing inequalities that contain terms of this form typically require a combination of Holder's, Ladyzhenskaya and Gagliardo-Nirenberg inequalities to get the left-hand side in a form in which the kickback argument can be used. Holder's inequality states that if $u \in L^p(\Omega)$, $v \in L^q(\Omega)$ and $w \in L^r(\Omega)$ and $\dfrac{1}{p} + \dfrac{1}{q} + \dfrac{1}{r} = 1$, then $|(u \cdot \nabla v, w)| \leq ||u||_{L_p} ||v||_{L_q} ||w||_{L_r}$. Therefore, by Holders inequality with $p = 4$, $q = 4$ and $r = 2$,

$$|(\mathbf{u_h} \cdot \nabla \mathbf{u_h}, A\mathbf{u_h})| \leq ||\mathbf{u_h}||_{L_4} ||\nabla \mathbf{u_h}||_{L_4} ||A\mathbf{u_h}|| . \tag{3.37}$$

The Ladyzhenskaya inequality states that if $u \in H_0^1$ then $||u||_{L^4} \leq 2^{1/4} ||u||^{1/2} ||\nabla u||^{1/2}$. Since $\mathbf{u_h} \in \mathbf{W^h} \subseteq \mathbf{H}_0^1$, $||\mathbf{u_h}||_{L^4} \leq K ||\mathbf{u_h}||^{1/2} ||\nabla \mathbf{u_h}||^{1/2}$. Therefore, $|(\mathbf{u_h} \cdot \nabla \mathbf{u_h}, A\mathbf{u_h})| \leq K ||\mathbf{u_h}||^{1/2} ||\nabla \mathbf{u_h}||^{1/2} ||\nabla \mathbf{u_h}||_{L_4} ||A\mathbf{u_h}||$. By Gagliardo-Nirenberg inequality, Lemma A.2, with $n = s = p = 2$, $q = 4$ and $\alpha = \frac{1}{2}$, $||\nabla \mathbf{u_h}||_{L^4} \leq K ||\nabla \mathbf{u_h}||^{1/2} ||A\mathbf{u_h}||^{1/2}$. Therefore, since $||\mathbf{u_h}||$ is bounded by the previous result,

$$\begin{aligned} |(\mathbf{u_h} \cdot \nabla \mathbf{u_h}, A\mathbf{u_h})| &\leq K ||\mathbf{u_h}||^{1/2} ||\nabla \mathbf{u_h}||^{1/2} ||\nabla \mathbf{u_h}||^{1/2} ||A\mathbf{u_h}||^{1/2} ||A\mathbf{u_h}|| \\ &= K ||\nabla \mathbf{u_h}|| \, ||A\mathbf{u_h}||^{3/2} . \end{aligned}$$

So, using Young's inequality with $p = 4$ and $q = \frac{4}{3}$ to prepare for a kickback argument, we get

$$\begin{aligned} |(\mathbf{u_h} \cdot \nabla \mathbf{u_h}, A\mathbf{u_h})| &\leq K ||\nabla \mathbf{u_h}|| \, ||A\mathbf{u_h}||^{3/2} \\ &\leq K ||\nabla \mathbf{u_h}||^4 + \frac{1}{2} ||A\mathbf{u_h}||^2 . \end{aligned} \tag{3.38}$$

Using (3.36) and (3.38) in (3.33) we get

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left|\left|\nabla \mathbf{u_h}\right|\right|^2 + \left|\left|A\mathbf{u_h}\right|\right|^2 &\leq K\left|\left|\nabla \mathbf{u_h}\right|\right|^4 + \frac{1}{2}\left|\left|A\mathbf{u_h}\right|\right|^2 + K\left|\left|\theta_h\right|\right|\left|\left|A\mathbf{u_h}\right|\right| \\
&+ K\left|\left|C_h\right|\right|\left|\left|A\mathbf{u_h}\right|\right| + \left|\left|\mathbf{f}\right|\right|\left|\left|A\mathbf{u_h}\right|\right| \\
&\leq K\left|\left|\nabla \mathbf{u_h}\right|\right|^4 + \frac{1}{2}\left|\left|A\mathbf{u_h}\right|\right|^2 + \frac{3}{8}\left|\left|A\mathbf{u_h}\right|\right|^2 \\
&+ K\left|\left|\theta_h\right|\right|^2 + K\left|\left|C_h\right|\right|^2 + K\left|\left|\mathbf{f}\right|\right|^2.
\end{aligned}
$$

That is,

$$
\frac{1}{2}\frac{d}{dt}\left|\left|\nabla \mathbf{u_h}\right|\right|^2 + \frac{1}{8}\left|\left|A\mathbf{u_h}\right|\right|^2 \leq K\left|\left|\nabla \mathbf{u_h}\right|\right|^4 + K\left|\left|\theta_h\right|\right|^2 + K\left|\left|C_h\right|\right|^2 + K\left|\left|\mathbf{f}\right|\right|^2. \tag{3.39}
$$

A similar analysis yields,

$$
\begin{aligned}
\left|(\mathbf{u_h}\cdot\nabla\theta_h, \Delta\theta_h)\right| &\leq K\left|\left|\mathbf{u_h}\right|\right|_{L_4}\left|\left|\nabla\theta_h\right|\right|_{L_4}\left|\left|\Delta\theta_h\right|\right| \\
&\leq K\left|\left|\mathbf{u_h}\right|\right|^{1/2}\left|\left|\nabla \mathbf{u_h}\right|\right|^{1/2}\left|\left|\nabla\theta_h\right|\right|^{1/2}\left|\left|\Delta\theta_h\right|\right|^{3/2} \\
&\leq K\left|\left|\nabla \mathbf{u_h}\right|\right|^4 + K\left|\left|\nabla\theta_h\right|\right|^4 + \frac{1}{4Pr}\left|\left|\Delta\theta_h\right|\right|^2.
\end{aligned} \tag{3.40}
$$

Using these estimates in (3.34) we get

$$
\frac{1}{2}\frac{d}{dt}\left|\left|\nabla\theta_h\right|\right|^2 + \frac{1}{2Pr}\left|\left|\nabla\theta_h\right|\right|^2 \leq K\left|\left|\nabla \mathbf{u_h}\right|\right|^4 + K\left|\left|\nabla\theta_h\right|\right|^4 + K\left|\left|Q\right|\right|^2. \tag{3.41}
$$

Similarly, from (3.35) we get

$$
\frac{1}{2}\frac{d}{dt}\left|\left|\nabla C_h\right|\right|^2 + \frac{1}{2PrLe}\left|\left|\nabla C_h\right|\right|^2 \leq K\left|\left|\nabla \mathbf{u_h}\right|\right|^4 + K\left|\left|\nabla C_h\right|\right|^4 + K\left|\left|\hat{Q}\right|\right|^2. \tag{3.42}
$$

Adding (3.39)-(3.42) we get

$$\frac{d}{dt}\left|\left|\nabla(\mathbf{u_h},\theta_h,C_h)\right|\right|^2+\min\left\{\frac{1}{4},\frac{1}{Pr},\frac{1}{PrLe}\right\}\left(\left|\left|A\mathbf{u_h}\right|\right|^2+\left|\left|\Delta\theta_h\right|\right|^2+\left|\left|\Delta C_h\right|\right|^2\right)$$
$$\leq K\left|\left|\nabla(\mathbf{u_h},\theta_h,C_h)\right|\right|^4+K\left(\left|\left|\mathbf{f}\right|\right|^2+\left|\left|Q\right|\right|^2+\left|\left|\hat{Q}\right|\right|^2+K_1\right). \quad (3.43)$$

By the Uniform Gronwall Lemma 3.2,

$$\left|\left|\nabla(\mathbf{u_h},\theta_h,C_h)\right|\right|^2<K. \quad (3.44)$$

A priori bounds for $||\partial_t\mathbf{u_h}||$**,** $||\partial_t\theta_h||$**,** $||\partial_t C_h||$

and $\int_0^T\left(||\nabla\partial_t\mathbf{u_h}||^2+||\nabla\partial_t\theta_h||^2+||\nabla\partial_t C_h||^2\right)\,ds.$

To obtain these bounds we differentiate equations (3.13)-(3.15) with respect to time obtaining,

$$\left(\partial_t^2\mathbf{u_h},\mathbf{v_h}\right)+\left(\partial_t\mathbf{u_h}\cdot\nabla\mathbf{u_h},\mathbf{v_h}\right)+\left(\mathbf{u_h}\cdot\nabla\partial_t\mathbf{u_h},\mathbf{v_h}\right)+\left(\nabla\partial_t\mathbf{u_h},\nabla\mathbf{v_h}\right)=G_{r_\theta}\left(\partial_t\theta_h\mathbf{j},\mathbf{v_h}\right)$$
$$+G_{r_c}\left(\partial_t C_h\mathbf{j},\mathbf{v_h}\right)+\left(\mathbf{f}_t,\mathbf{v_h}\right),\forall\mathbf{v_h}\in\mathbf{V^h} \quad (3.45)$$

$$\left(\partial_t^2\theta_h,\phi_h\right)+\left(\partial_t\mathbf{u_h}\cdot\nabla\theta_h,\phi_h\right)+\left(\mathbf{u_h}\cdot\nabla\partial_t\theta_h,\phi_h\right)+\frac{1}{Pr}\left(\nabla\partial_t\theta_h,\nabla\phi_h\right)=\left(Q_t,\phi_h\right) \quad (3.46)$$
$$\forall\phi_h\in W^h$$

$$\left(\partial_t^2 C_h, \psi_h\right) + \left(\partial_t \mathbf{u_h} \cdot \nabla C_h, \psi_h\right) + \left(\mathbf{u_h} \cdot \nabla \partial_t C_h, \psi_h\right) + \frac{1}{PrLe}\left(\nabla \partial_t C_h, \nabla \psi_h\right) = \left(\hat{Q}_t, \psi_h\right) \tag{3.47}$$

$$\forall \psi_h \in W^h.$$

Setting $\mathbf{v_h} = \partial_t \mathbf{u_h}$ in (3.45) we get by anti-symmetry,

$$\left(\partial_t^2 \mathbf{u_h}, \partial_t \mathbf{u_h}\right) + \left(\partial_t \mathbf{u_h} \cdot \nabla \mathbf{u_h}, \partial_t \mathbf{u_h}\right) + ||\nabla \partial_t \mathbf{u_h}||^2 = G_{r_\theta}\left(\partial_t \theta_h \mathbf{j}, \partial_t \mathbf{u_h}\right) + G_{r_c}\left(\partial_t C_h \mathbf{j}, \partial_t \mathbf{u_h}\right) + \left(\mathbf{f}_t, \partial_t \mathbf{u_h}\right). \tag{3.48}$$

Likewise by setting $\phi_h = \partial_t \theta_h$ and $\psi_h = \partial_t C_h$ in (3.46) and (3.47) we get

$$\left(\partial_t^2 \theta_h, \partial_t \theta_h\right) + \left(\partial_t \mathbf{u_h} \cdot \nabla \theta_h, \partial_t \theta_h\right) + \frac{1}{Pr}||\nabla \partial_t \theta_h||^2 = \left(Q_t, \partial_t \theta_h\right) \tag{3.49}$$

$$\left(\partial_t^2 C_h, \partial_t C_h\right) + \left(\partial_t \mathbf{u_h} \cdot \nabla C_h, \partial_t C_h\right) + \frac{1}{PrLe}||\nabla \partial_t C_h||^2 = \left(\hat{Q}_t, \partial_t C_h\right). \tag{3.50}$$

We now simplify (3.48) using Cauchy and Poincare inequalities,

$$\frac{d}{dt}||\partial_t \mathbf{u_h}||^2 + 2\left(\partial_t \mathbf{u_h} \cdot \nabla \mathbf{u_h}, \partial_t \mathbf{u_h}\right) + 2||\nabla \partial_t \mathbf{u_h}||^2 \leq K||\partial_t \theta_h||\,||\nabla \partial_t \mathbf{u_h}|| + K||\partial_t C_h||\,||\nabla \partial_t \mathbf{u_h}|| + K||\mathbf{f}_t||\,||\nabla \partial_t \mathbf{u_h}||.$$

Employing the kickback argument yields,

$$\frac{d}{dt}||\partial_t \mathbf{u_h}||^2 + \frac{1}{2}||\nabla \partial_t \mathbf{u_h}||^2 \leq K||\partial_t \theta_h||^2 + K||\partial_t C_h||^2 + K||\mathbf{f}_t||^2 + 2\left|\left(\partial_t \mathbf{u_h} \cdot \nabla \mathbf{u_h}, \partial_t \mathbf{u_h}\right)\right|. \tag{3.51}$$

We now estimate $2\left(\partial_t \mathbf{u_h} \cdot \nabla \mathbf{u_h}, \partial_t \mathbf{u_h}\right)$. Using Holder's, Ladyzhenskaya and Young's Inequalities yields,

$$2\left|\left(\partial_t \mathbf{u_h} \cdot \nabla \mathbf{u_h}, \partial_t \mathbf{u_h}\right)\right| \leq K \left|\left|\partial_t \mathbf{u_h}\right|\right|_{L_4}^2 \left|\left|\nabla \mathbf{u_h}\right|\right| \leq \frac{1}{4} \left|\left|\nabla \partial_t \mathbf{u_h}\right|\right|^2 + K \left|\left|\partial_t \mathbf{u_h}\right|\right|^2 .$$

Using this estimate in (3.51), we get

$$\frac{d}{dt} \left|\left|\partial_t \mathbf{u_h}\right|\right|^2 + \frac{1}{4} \left|\left|\nabla \partial_t \mathbf{u_h}\right|\right|^2 \leq K \left|\left|\partial_t \theta_h\right|\right|^2 + K \left|\left|\partial_t C_h\right|\right|^2 + K \left|\left|\mathbf{f}_t\right|\right|^2 + K \left|\left|\partial_t \mathbf{u_h}\right|\right|^2 . \quad (3.52)$$

Turning to temperature, we simplify (3.49). Using Cauchy, Poincare and Young's inequalities and employing the kickback argument yields

$$\frac{d}{dt} \left|\left|\partial_t \theta_h\right|\right|^2 + \frac{1}{2Pr} \left|\left|\nabla \partial_t \theta_h\right|\right|^2 \leq 2\left|\left(\partial_t \mathbf{u_h} \cdot \nabla \theta_h, \partial_t \theta_h\right)\right| + K \left|\left|Q_t\right|\right|^2 . \quad (3.53)$$

We now estimate $2\left|\left(\partial_t \mathbf{u_h} \cdot \nabla \theta_h, \partial_t \theta_h\right)\right|$. Using Holder's, Ladyzhenskaya and Young's inequalities yields,

$$\begin{aligned} 2\left|\left(\partial_t \mathbf{u_h} \cdot \nabla \theta_h, \partial_t \theta_h\right)\right| &\leq K \left|\left|\partial_t \mathbf{u_h}\right|\right|_{L_4} \left|\left|\nabla \theta_h\right|\right| \left|\left|\partial_t \theta_h\right|\right|_{L_4} \\ &\leq \frac{1}{16} \left|\left|\nabla \partial_t \mathbf{u_h}\right|\right|^2 + K \left|\left|\partial_t \mathbf{u_h}\right|\right|^2 + \frac{1}{4Pr} \left|\left|\nabla \partial_t \theta_h\right|\right|^2 + K \left|\left|\partial_t \theta_h\right|\right|^2 . \end{aligned}$$

Using this estimate in (3.53), we get

$$\frac{d}{dt} \left|\left|\partial_t \theta_h\right|\right|^2 + \frac{1}{4Pr} \left|\left|\nabla \partial_t \theta_h\right|\right|^2 \leq \frac{1}{16} \left|\left|\nabla \partial_t \mathbf{u_h}\right|\right|^2 + K \left|\left|\partial_t \mathbf{u_h}\right|\right|^2 + K \left|\left|\partial_t \theta_h\right|\right|^2 + K \left|\left|Q_t\right|\right|^2 . \quad (3.54)$$

Similarly for concentration, we have

$$\frac{d}{dt}\left|\left|\partial_t C_h\right|\right|^2 + \frac{1}{4PrLe}\left|\left|\nabla\partial_t C_h\right|\right|^2 \leq \frac{1}{16}\left|\left|\nabla\partial_t \mathbf{u_h}\right|\right|^2 + K\left|\left|\partial_t \mathbf{u_h}\right|\right|^2 + K\left|\left|\partial_t C_h\right|\right|^2 + K\left|\left|\hat{Q}_t\right|\right|^2. \tag{3.55}$$

Let $Y(t) = ||\partial_t \mathbf{u_h}||^2 + ||\partial_t \theta_h||^2 + ||\partial_t C_h||^2$, $\hat{Y}(t) = ||\nabla\partial_t \mathbf{u_h}||^2 + ||\nabla\partial_t \theta_h||^2 + ||\nabla\partial_t C_h||^2$ and $\alpha = \min\{1/8, 1/4Pr, 1/4PrLe\}$. Adding together the inequalitites (3.52), (3.55) and (3.54) gives us

$$\frac{d}{dt}Y(t) + \alpha\hat{Y}(t) \leq K_1 Y(t) + K_2\left(||\mathbf{f}_t||^2 + ||Q_t||^2 + \left|\left|\hat{Q}_t\right|\right|^2\right).$$

By the Continuous Gronwall Lemma 3.3 and the assumption on the data, we get the necessary bounds.

A priori bounds for $||\mathbf{u_h}||_2$, $||\theta_h||_2$ and $||C_h||_2$

Note that integrating (3.43) with respect to time and using (3.44) yields, $\int_0^t (||A\mathbf{u_h}||^2 + ||\Delta\theta_h||^2 + ||\Delta C_h||^2)\, dt \leq K$. Now, by Holder's inequality, (3.33) implies

$$\begin{aligned} ||A\mathbf{u_h}||^2 \leq\ & ||\partial_t \mathbf{u_h}||\,||A\mathbf{u_h}|| + K\,||\mathbf{u_h}||_{L^4}\,||\nabla\mathbf{u_h}||_{L^4}\,||A\mathbf{u_h}|| \\ & + G_{r_\theta}\,||\theta_h||\,||A\mathbf{u_h}|| + G_{r_c}\,||C_h||\,||A\mathbf{u_h}|| + K\,||f||\,||A\mathbf{u_h}||. \end{aligned} \tag{3.56}$$

Using Gagliardo-Nirenberg and Ladyzhenskaya inequalities and employing the kickback argument,

$$\begin{aligned} ||A\mathbf{u_h}||^2 \leq\ & K\,||\partial_t \mathbf{u_h}||^2 + K\,||\mathbf{u_h}||^2\,||\nabla\mathbf{u_h}||^4 + K\,||\theta_h||^2 \\ & + K\,||C_h||^2 + K\,||f||^2. \end{aligned} \tag{3.57}$$

Since all of the terms on the right-hand side of (3.57) are bounded, we get $||A\mathbf{u_h}||^2 \leq K$ which implies $||\mathbf{u_h}||_2 \leq K$ by the norm equivelence between $||\cdot||_2$ and $||A\cdot||$. To derive a bound for $||\theta_h||_2$, note that using Holder's inequality, (3.34) becomes,

$$\frac{1}{Pr}\,||\Delta\theta_h||^2 \leq ||\partial_t\theta_h||\,||\Delta\theta_h|| + K\,||\mathbf{u_h}||_{L^4}\,||\nabla\theta_h||_{L^4}\,||\Delta\theta_h|| + ||Q||\,||\Delta\theta_h||\,. \tag{3.58}$$

By the Gagliardo-Nirenberg inequality, $||\nabla\theta_h||_{L^4} \leq K\,||\nabla\theta_h||^{1/2}\,||\Delta\theta_h||^{1/2}$. Hence,

$$||\mathbf{u_h}||_{L^4}\,||\nabla\theta_h||_{L^4}\,||\Delta\theta_h|| \leq K\,||\mathbf{u_h}||^{1/2}\,||\nabla\mathbf{u_h}||^{1/2}\,||\nabla\theta_h||^{1/2}\,||\Delta\theta_h||^{3/2}\,.$$

So, by Young's inequality with $p = 4$ and $q = 4/3$,

$$||\mathbf{u_h}||_{L^4}\,||\nabla\theta_h||_{L^4}\,||\Delta\theta_h|| \leq K\,||\mathbf{u_h}||^2\,||\nabla\mathbf{u_h}||^2\,||\nabla\theta_h||^2 + \frac{1}{6Pr}\,||\Delta\theta_h||^2\,. \tag{3.59}$$

Using (3.59) in (3.58) and employing the kickback argument yields,

$$\frac{1}{Pr}\,||\Delta\theta_h||^2 \leq K\,||\partial_t\theta_h||^2 + K\,||\mathbf{u_h}||^2 + ||\nabla\mathbf{u_h}||^2\,||\nabla\theta_h||^2 + ||Q||^2\,.$$

Note that $||Q||^2$ is bounded by our assumptions on the data and all the other terms on the right-hand side are bounded by previous estimates. Therefore by the norm equivalence between $||\cdot||$ and $||\Delta\cdot||$ we get $||\theta_h||_2 \leq K$. Similarly, we can show $||C_h||_2 \leq K$.

A priori bounds for $\left|\left|\partial_t^2 \mathbf{u_h}\right|\right|_*^2$, $\left|\left|\partial_t^2 C_h\right|\right|_{-1}^2$ and $\left|\left|\partial_t^2 \theta_h\right|\right|_{-1}^2$

Setting $\mathbf{v_h} = A^{-1}\partial_t^2\mathbf{u_h}$, $\phi_h = (-\Delta)^{-1}\partial_t^2\theta_h$ and $\psi_h = (-\Delta)^{-1}\partial_t^2 C_h$ in (3.45), (3.46) and (3.47) respectively and by Lemma 3.5 and (B.5), we obtain

$$\left|\left|\partial_t^2 \mathbf{u_h}\right|\right|_*^2 + \frac{1}{2}\frac{d}{dt}\left|\left|\partial_t \mathbf{u_h}\right|\right|^2 \leq \sum_{i=1}^{5} B_i^{\mathbf{u}} \tag{3.60}$$

$$\left|\left|\partial_t^2 \theta_h\right|\right|_{-1}^2 + \frac{1}{2Pr}\frac{d}{dt}\left|\left|\partial_t \theta_h\right|\right|^2 \leq \sum_{i=1}^{3} B_i^{\theta} \tag{3.61}$$

$$\left|\left|\partial_t^2 C_h\right|\right|_{-1}^2 + \frac{1}{2PrLe}\frac{d}{dt}\left|\left|\partial_t C_h\right|\right|^2 \leq \sum_{i=1}^{3} B_i^{c}. \tag{3.62}$$

Estimating the $B_i^{\mathbf{u}}$ terms starting with $B_1^{\mathbf{u}}$, by Holders, Poincare and Ladyzhenskaya inequalities,

$$\begin{aligned} B_1^{\mathbf{u}} := \left|\left((\partial_t \mathbf{u_h}\cdot\nabla)\mathbf{u_h}, A^{-1}\partial_t^2\mathbf{u_h}\right)\right| &\leq \left|\left|\partial_t \mathbf{u_h}\right|\right| \left|\left|\nabla \mathbf{u_h}\right|\right|_{L_4} \left|\left|A^{-1}\partial_t^2\mathbf{u_h}\right|\right|_{L_4} \\ &\leq K\left|\left|\partial_t \mathbf{u_h}\right|\right| \left|\left|\mathbf{u_h}\right|\right|_1^{1/2} \left|\left|\mathbf{u_h}\right|\right|_2^{1/2} \left|\left|A^{-1}\partial_t^2\mathbf{u_h}\right|\right|^{1/2} \left|\left|A^{-1}\partial_t^2\mathbf{u_h}\right|\right|_1^{1/2} \\ &\leq K\left|\left|\partial_t \mathbf{u_h}\right|\right| \left|\left|\mathbf{u_h}\right|\right|_1^{1/2} \left|\left|\mathbf{u_h}\right|\right|_2^{1/2} \left|\left|A^{-1}\partial_t^2\mathbf{u_h}\right|\right|_1. \end{aligned}$$

By previous a priori estimates, $\left|\left|\partial_t \mathbf{u_h}\right|\right|$, $\left|\left|\mathbf{u_h}\right|\right|_1^{1/2}$ and $\left|\left|\mathbf{u_h}\right|\right|_2^{1/2}$ are bounded and by Lemma 3.5, $\left|\left|A^{-1}\partial_t^2\mathbf{u_h}\right|\right|_1 \leq K\left|\left|\partial_t^2\mathbf{u_h}\right|\right|_*$. Thus, by Young's Inequality,

$$B_1^{\mathbf{u}} \leq K\left|\left|\partial_t^2\mathbf{u_h}\right|\right|_* \leq K + \frac{1}{10}\left|\left|\partial_t^2\mathbf{u_h}\right|\right|_*^2.$$

To estimate $B_2^{\mathbf{u}}$, we have

$$B_2^{\mathbf{u}} := \left|\left((\mathbf{u_h}\cdot\nabla)\partial_t\mathbf{u_h}, A^{-1}\partial_t^2\mathbf{u_h}\right)\right| \leq ||\mathbf{u_h}||_{L_4}\,||\nabla\partial_t\mathbf{u_h}||\,\left|\left|A^{-1}\partial_t^2\mathbf{u_h}\right|\right|_{L_4}$$
$$\leq K\,||\nabla\mathbf{u_h}||\,||\nabla\partial_t\mathbf{u_h}||\,\left|\left|A^{-1}\partial_t^2\mathbf{u_h}\right|\right|_1 \leq K\,||\nabla\mathbf{u_h}||\,||\nabla\partial_t\mathbf{u_h}||\,\left|\left|\partial_t^2\mathbf{u_h}\right|\right|_*$$

By previous a priori estimates, $||\nabla\mathbf{u_h}||$ is bounded. Thus, by Young's Inequality,

$$B_2^{\mathbf{u}} \leq K\,||\nabla\partial_t\mathbf{u_h}||\,\left|\left|\partial_t^2\mathbf{u_h}\right|\right|_* \leq \frac{1}{10}\left|\left|\partial_t^2\mathbf{u_h}\right|\right|_*^2 + K\,||\nabla\partial_t\mathbf{u_h}||^2.$$

For $B_3^{\mathbf{u}}$, note $||\partial_t\theta_h||$ is bounded by previous a priori estimates. So by Cauchy, Poincare and Young's Inequalities and Lemma 3.5,

$$B_3^{\mathbf{u}} := \left|G_{r_\theta}\left(\partial_t\theta_h\mathbf{j}, A^{-1}\partial_t^2\mathbf{u_h}\right)\right| \leq K\,||\partial_t\theta_h||\,\left|\left|A^{-1}\partial_t^2\mathbf{u_h}\right|\right| \leq K\left|\left|A^{-1}\partial_t^2\mathbf{u_h}\right|\right|_1$$
$$\leq K\left|\left|\partial_t^2\mathbf{u_h}\right|\right|_* \leq \frac{1}{10}\left|\left|\partial_t^2\mathbf{u_h}\right|\right|_*^2 + K.$$

Similarly, $B_4^{\mathbf{u}} := \left|G_{r_c}\left(\partial_t C_h\mathbf{j}, A^{-1}\partial_t^2\mathbf{u_h}\right)\right| \leq \frac{1}{10}\left|\left|\partial_t^2\mathbf{u_h}\right|\right|_*^2 + K$. Also, by the regularity assumptions, $||\mathbf{f}_t||$ is bounded. So,

$$B_5^{\mathbf{u}} := \left|\left(\mathbf{f}_t, A^{-1}\partial_t^2\mathbf{u_h}\right)\right| \leq ||\mathbf{f}_t||\,\left|\left|A^{-1}\partial_t^2\mathbf{u_h}\right|\right|$$
$$\leq K\left|\left|\partial_t^2\mathbf{u_h}\right|\right|_* \leq \frac{1}{10}\left|\left|\partial_t^2\mathbf{u_h}\right|\right|_*^2 + K.$$

Putting these in (3.60) and multiplying by two, we have

$$\left|\left|\partial_t^2\mathbf{u_h}\right|\right|_*^2 + \frac{d}{dt}\,||\partial_t\mathbf{u_h}||^2 \leq K + K\,||\nabla\partial_t\mathbf{u_h}||^2.$$

Integrating with respect to time from 0 to T, we get

$$\int_0^T \left|\left|\partial_t^2 \mathbf{u_h}\right|\right|_*^2 \, dt + ||\partial_t \mathbf{u_h}(T)||^2 - ||\partial_t \mathbf{u_h}(0)||^2 \leq KT + K \int_0^T ||\nabla \partial_t \mathbf{u_h}||^2 \, dt \leq K$$

since the integral on the right is bounded by previous a priori estimates. Now by Holders, Ladyzhenskaya and Poincare inequalities and Lemma 3.5,

$$\begin{aligned} B_1^\theta &:= \left|\left((\partial_t \mathbf{u_h} \cdot \nabla)\theta_h, (-\Delta)^{-1}\partial_t^2 \theta_h\right)\right| \leq ||\partial_t \mathbf{u_h}||_{L^4} ||\nabla \theta_h|| \left|\left|(-\Delta)^{-1}\partial_t^2 \theta_h\right|\right|_{L^4} \\ &\leq K ||\nabla \partial_t \mathbf{u_h}|| \, ||\nabla \theta_h|| \left|\left|(-\Delta)^{-1}\partial_t^2 \theta_h\right|\right|_1 \leq K ||\nabla \partial_t \mathbf{u_h}|| \, ||\nabla \theta_h|| \left|\left|\partial_t^2 \theta_h\right|\right|_{-1}. \end{aligned}$$

Since $||\nabla \theta_h||$ is bounded by previous a priori estimates, we have by Young's Inequality, $B_1^\theta \leq K ||\nabla \partial_t \mathbf{u_h}||^2 + \frac{1}{6}\left|\left|\partial_t^2 \theta_h\right|\right|_{-1}^2$. Similarly,

$$\begin{aligned} B_2^\theta &:= \left|\left((\mathbf{u_h} \cdot \nabla)\partial_t \theta_h, (-\Delta)^{-1}\partial_t^2 \theta_h\right)\right| \leq ||\mathbf{u_h}||_{L^4} ||\nabla \partial_t \theta_h|| \left|\left|(-\Delta)^{-1}\partial_t^2 \theta_h\right|\right|_{L^4} \\ &\leq K ||\nabla \mathbf{u_h}|| \, ||\nabla \partial_t \theta_h|| \left|\left|(-\Delta)^{-1}\partial_t^2 \theta_h\right|\right|_1 \leq K ||\nabla \mathbf{u_h}|| \, ||\nabla \partial_t \theta_h|| \left|\left|\partial_t^2 \theta_h\right|\right|_{-1}. \end{aligned}$$

Since $||\nabla \mathbf{u_h}||$ is bounded by previous a priori estimates, we have by Young's Inequality, $B_2^\theta \leq \frac{1}{6}\left|\left|\partial_t^2 \theta_h\right|\right|_{-1}^2 + K ||\nabla \partial_t \theta_h||^2$. Finally, by the assumption on the data, $||Q_t||$ is bounded. Thus,

$$\begin{aligned} B_3^\theta &:= \left|\left(Q_t, (-\Delta)^{-1}\partial_t^2 \theta_h\right)\right| \leq ||Q_t|| \left|\left|(-\Delta)^{-1}\partial_t^2 \theta_h\right|\right| \leq K \left|\left|(-\Delta)^{-1}\partial_t^2 \theta_h\right|\right|_1 \\ &\leq K \left|\left|\partial_t^2 \theta_h\right|\right|_{-1} \leq \frac{1}{6}\left|\left|\partial_t^2 \theta_h\right|\right|_{-1}^2 + K. \end{aligned}$$

Putting these in (3.61) and multiplying by two we get

$$\left|\left|\partial_t^2\theta_h\right|\right|_{-1}^2 + \frac{1}{Pr}\frac{d}{dt}\left|\left|\partial_t\theta_h\right|\right|^2 \le K + K\left|\left|\nabla\partial_t\mathbf{u_h}\right|\right|^2 + K\left|\left|\nabla\partial_t\theta_h\right|\right|^2.$$

Integrating from 0 to T with respect to time yields,

$$\begin{aligned}\int_0^T \left|\left|\partial_t^2\theta_h\right|\right|_{-1}^2\,dt + \frac{1}{Pr}\left|\left|\partial_t\theta_h(T)\right|\right|^2 - \frac{1}{Pr}\left|\left|\partial_t\theta_h(0)\right|\right|^2 \le KT \\ + K\int_0^T \left|\left|\nabla\partial_t\mathbf{u_h}\right|\right|^2\,dt + K\int_0^T \left|\left|\nabla\partial_t\theta_h\right|\right|^2\,dt.\end{aligned}$$

Since the two integrals on the right are bounded by previous a priori estimates, we get $\int_0^T \left|\left|\partial_t^2\theta_h\right|\right|_{-1}^2\,dt \le K$. Similarly, we can show $\int_0^T \left|\left|\partial_t^2 C_h\right|\right|_{-1}^2\,dt \le K$. □

The following theorem establishes existence and uniqueness of solutions.

Theorem 3.6. *Assume the data satisfies assumptions A(i)-A(iv). There exists a solution* $(\mathbf{u_h}, \theta_h, C_h) \in (\mathbf{V^h} \times W^h \times W^h)$ *to* (3.13) - (3.15) *and the solution is unique.*

Proof. Let $\{\phi_i\}_{i=1}^M$ and $\left\{\hat{\phi}_i\right\}_{i=1}^{\hat{M}}$ be the basis for the finite dimensional subspaces $\mathbf{V^h}$ and W^h respectively. The semi-discrete system (3.13) - (3.15) can now be written as a system of ODEs. We begin by writing $\mathbf{v_h}(x,t) = \sum_{i=1}^M c_i(t)\phi_i(x)$, $\theta_h(x,t) = \sum_{i=1}^{\hat{M}} \hat{c}_i(t)\hat{\phi}_i(x)$ and $C_h(x,t) = \sum_{i=1}^{\hat{M}} \tilde{c}_i(t)\tilde{\phi}_i(x)$, for $t \in [0,T]$, where $c_i(t)$, $\hat{c}_i(t)$ and $\tilde{c}_i(t)$ are scaler valued time dependent coefficients. Setting $\mathbf{v} = \phi_j$, $j = 1,\ldots,M$ and $\phi = \hat{\phi}_k$, $\psi = \tilde{\phi}_k$, $k = 1,\ldots,\hat{M}$ in (3.13)-(3.15) results in the following non-linear matrix equation.

$$\frac{d\mathbf{c}}{dt} + A\mathbf{c} + N(\mathbf{c}) = F(t), \tag{3.63}$$

where $\mathbf{c} = ([c]_j, [\hat{c}]_k, [\tilde{c}]_k)$ and $N(\mathbf{c})$ is a quadratic polynomial. Equation (3.63) has a local solution on some interval, see [22], $[0, t]$. Due to the a priori bounds of Proposition 3.1, we can use a standard continuation arguement to show the solution exists on $[0, T]$. □

3.3 Optimal Order Spatial Error Estimate

Turning now to the issue of convergence for space discretization, the weak form for the exact solutions $(\mathbf{u}, \theta, C)$ is

$$
\begin{aligned}
(\partial_t \mathbf{u}, \mathbf{v}) + ((\mathbf{u}\cdot\nabla)\mathbf{u}, \mathbf{v}) + (\nabla\mathbf{u}, \nabla\mathbf{v}) &= ((G_{r_\theta}\theta + G_{r_c}C)\mathbf{j}, \mathbf{v}) \\
&\quad + (\mathbf{f}, \mathbf{v}) \qquad \forall \mathbf{v} \in \mathbf{V}(\Omega) \;\; t \in [0, T]
\end{aligned}
\tag{3.64}
$$

$$
(\partial_t \theta, \phi) + (\mathbf{u}\cdot\nabla\theta, \phi) + \frac{1}{Pr}(\nabla\theta, \nabla\phi) = (Q, \phi) \qquad \forall \phi \in H_0^1(\Omega) \;\; t \in [0, T] \tag{3.65}
$$

$$
(\partial_t C, \psi) + (\mathbf{u}\cdot\nabla C, \psi) + \frac{1}{PrLe}(\nabla C, \nabla\psi) = \left(\hat{Q}, \psi\right) \qquad \forall \psi \in H_0^1(\Omega) \;\; t \in [0, T] \tag{3.66}
$$

with initial conditions $\mathbf{u}(x, 0) = \mathbf{u}_0(x)$, $\theta(x, 0) = \theta_0(x)$ and $C(x, 0) = C_0(x)$ for $x \in \Omega$. Note that (3.64)-(3.66) can be written in the following equivelent form: Seek $\mathbf{u} \in \mathbf{V}$ and $\theta, C \in H_0^1$ such that,

$$
(\partial_t \Phi, \Psi) + \mathtt{a}(\Phi, \Psi) + \mathtt{c}(\Phi, \Phi, \Psi) = \mathtt{d}(\Phi, \Psi) + (F, \Psi), \forall \Psi \in \mathbf{X} \tag{3.67}
$$

with initial conditions $\Phi(0) = \Phi_0 = (\mathbf{u}_0, \theta_0, C_0)$ where $\Phi := (\mathbf{u}, \theta, C)$, $\Psi := (\mathbf{v}, \phi, \psi)$ and $\mathbf{X} = \mathbf{V} \times H_0^1 \times H_0^1$ with norm $||\nabla\Phi|| = \left(||\nabla\mathbf{u}||^2 + ||\nabla\theta||^2 + ||\nabla C||^2\right)^{1/2}$ and

$$\begin{aligned}
\mathtt{a}(\Phi, \Psi) &:= (\nabla\mathbf{u}, \nabla\mathbf{v}) + \frac{1}{Pr}(\nabla\theta, \nabla\phi) + \frac{1}{PrLe}(\nabla C, \nabla\psi) \\
\mathtt{c}(\Phi, \Phi, \Psi) &:= (\mathbf{u}\cdot\nabla\mathbf{u}, \mathbf{v}) + (\mathbf{u}\cdot\nabla\theta, \phi) + (\mathbf{u}\cdot\nabla C, \psi) \\
\mathtt{d}(\Phi, \Psi) &:= ((G_{r_\theta}\theta + G_{r_c}C)\,\mathbf{j}, \mathbf{v}) \\
(F, \Psi) &:= (f, \mathbf{v}) + (Q, \phi) + \left(\hat{Q}, \psi\right).
\end{aligned}$$

Clearly, the bilinear form $\mathtt{a}(\cdot,\cdot)$ is coercive, that is, $\mathtt{a}(\Phi, \Phi) \geq \alpha\,||\nabla\Phi||^2$, where $\alpha = \min\left\{1, \frac{1}{Pr}, \frac{1}{PrLe}\right\}$. To show that $\mathtt{a}(\cdot,\cdot)$ is continuous note that

$$\begin{aligned}
|\mathtt{a}(\Phi, \Psi)| &\leq ||\nabla\mathbf{u}||\,||\nabla\mathbf{v}|| + \frac{1}{Pr}||\nabla\theta||\,||\nabla\phi|| + \frac{1}{PrLe}||\nabla C||\,||\nabla\psi|| \\
&\leq \lambda\,||\nabla\Phi||\,||\nabla\Psi||,
\end{aligned}$$

where $\lambda = 3\max\left\{1, \frac{1}{Pr}, \dfrac{1}{PrLe}\right\}$. Let $\mathbf{X^h} = \mathbf{V^h} \times W^h \times W^h$ be a subspace of $\mathbf{X}$. The finite element Galerkin approximation of the double-diffusive convection model is as follows: seek $\Phi_h(t) : [0, T] \to \mathbf{X^h}$ such that

$$(\partial_t\Phi_h, \Psi_h) + \mathtt{a}(\Phi_h, \Psi_h) + \mathtt{c}(\Phi_h, \Phi_h, \Psi_h) = \mathtt{d}(\Phi_h, \Psi_h) + (F, \Psi_h), \forall\Psi_h \in \mathbf{X^h} \qquad (3.68)$$

and $\Phi_h(0) = \Phi_{0\,h}$, where $\Phi_{0\,h}$ is a suitable approximation of Φ_0, for example $\Phi_{0\,h} = P_h\Phi_0$, where P_h is the L^2 projection from $\mathbf{X} \to \mathbf{X^h}$. Let $\mathbf{e} := \Phi - \Phi_h$ be the error between $\Phi = (\mathbf{u}, \theta, C)$ and $\Phi_h = (\mathbf{u_h}, \theta_h, C_h)$, where Φ and Φ_h are the solutions to (3.67) and (3.68) respectively. The main result of this section is the convergence estimate, $||\mathbf{e}|| + h\,||\nabla\mathbf{e}|| \leq Kh^2$. To prove this however, some auxiliary results are required. To this end, we first introduce an intermediate finite element approximation, Π_h, that satisfies the following auxillary problem.

$$(\partial_t \Pi_h, \Psi_h) + \mathtt{a}(\Pi_h, \Psi_h) + \mathtt{c}(\Phi, \Phi, \Psi_h) = \mathtt{d}(\Phi, \Psi_h) + (F, \Psi_h), \forall \Psi_h \in \mathbf{X^h}, \tag{3.69}$$

with $\Pi_h(0) = P_h\Phi_0$, where $P_h : \mathbf{X} \to \mathbf{X^h}$ is the L^2 orthogonal projection onto $\mathbf{X^h}$. Next, the error $\mathbf{e}$ is decomposed into two parts $\mathbf{e}_L$ and $\mathbf{e}_N$ as follows:

$$\mathbf{e} = \Phi - \Phi_h = (\Phi - \Pi_h) + (\Pi_h - \Phi_h) := \mathbf{e}_L + \mathbf{e}_N.$$

First some estimates for $\mathbf{e}_L$. Note, the solution Φ satisfies (3.67) $\forall \Psi_h \in \mathbf{X^h}$.

$$(\partial_t \Phi, \Psi_h) + \mathtt{a}(\Phi, \Psi_h) + \mathtt{c}(\Phi, \Phi, \Psi_h) = \mathtt{d}(\Phi, \Psi_h) + (F, \Psi_h) \tag{3.70}$$

Subtracting (3.69) from (3.70) gives

$$(\partial_t \mathbf{e}_L, \Psi_h) + \mathtt{a}(\mathbf{e}_L, \Psi_h) = 0, \quad \forall \Psi_h \in \mathbf{X^h}. \tag{3.71}$$

Lemma 3.7. *Let $\Pi_h(t) \in \mathbf{X^h}$ be the solution of (3.69) with initial condition $\Pi_h(0) = P_h\Phi_0$. Then $\mathbf{e}_L := \Phi - \Pi_h$ satisfies,*

$$\int_0^t ||(\Phi - \Pi_h)(\tau)||^2 \ d\tau \leq Kh^4 \int_0^t ||\Phi(s)||_2^2 \ ds, \quad \textit{for } t > 0. \tag{3.72}$$

Proof. Setting $\Psi_h = P_h\mathbf{e}_L$ in (3.71) yields, $(\partial_t \mathbf{e}_L, P_h\mathbf{e}_L) + \mathtt{a}(\mathbf{e}_L, P_h\mathbf{e}_L) = 0$ which can be written,

$$(\partial_t \mathbf{e}_L, \mathbf{e}_L) + \mathtt{a}(\mathbf{e}_L, \mathbf{e}_L) = (\partial_t \mathbf{e}_L, \mathbf{e}_L - P_h\mathbf{e}_L) + \mathtt{a}(\mathbf{e}_L, \mathbf{e}_L - P_h\mathbf{e}_L), \tag{3.73}$$

so that $\mathbf{e}_L$ appears only on the left. By the continuity of $\mathtt{a}(\cdot,\cdot)$ and since $\mathbf{e}_L - P_h\mathbf{e}_L = (\Phi - \Pi_h) - (P_h\Phi - \Pi_h) = \Phi - P_h\Phi$, (3.73) can be written as follows,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\|\mathbf{e}_L\right\|^2 + \alpha\left\|\nabla\mathbf{e}_L\right\|^2 &\le (\partial_t\mathbf{e}_L, \Phi - P_h\Phi) + \mathtt{a}(\mathbf{e}_L, \Phi - P_h\Phi)\\ &\le (\partial_t\mathbf{e}_L, \Phi - P_h\Phi) + K\left\|\nabla\mathbf{e}_L\right\|\left\|\nabla(\Phi - P_h\Phi)\right\|. \qquad (3.74)\end{aligned}$$

Note that since $\mathbf{e}_L = \Phi - \Pi_h$,

$$\begin{aligned}(\partial_t\mathbf{e}_L, \Phi - P_h\Phi) &= (\partial_t(\Phi - P_h\Phi + P_h\Phi - \Pi_h), \Phi - P_h\Phi)\\ &= (\partial_t(\Phi - P_h\Phi), \Phi - P_h\Phi) + (\partial_t(P_h\Phi - \Pi_h), \Phi - P_h\Phi).\end{aligned}$$

But by the orthogonality condition $(\partial_t(P_h\Phi - \Pi_h), \Phi - P_h\Phi) = 0$, thus,

$$(\partial_t\mathbf{e}_L, \Phi - P_h\Phi) = (\partial_t(\Phi - P_h\Phi), \Phi - P_h\Phi) = \frac{1}{2}\frac{d}{dt}\left\|\Phi - P_h\Phi\right\|^2. \qquad (3.75)$$

Using (3.75) in (3.74) yields,

$$\frac{1}{2}\frac{d}{dt}\left\|\mathbf{e}_L\right\|^2 + \alpha\left\|\nabla\mathbf{e}_L\right\|^2 \le \frac{1}{2}\frac{d}{dt}\left\|\Phi - P_h\Phi\right\|^2 + K\left\|\nabla\mathbf{e}_L\right\|\left\|\nabla(\Phi - P_h\Phi)\right\|.$$

Employing the kickback argument,

$$\frac{1}{2}\frac{d}{dt}\left\|\mathbf{e}_L\right\|^2 + \frac{\alpha}{2}\left\|\nabla\mathbf{e}_L\right\|^2 \le \frac{1}{2}\frac{d}{dt}\left\|\Phi - P_h\Phi\right\|^2 + K\left\|\nabla(\Phi - P_h\Phi)\right\|^2. \qquad (3.76)$$

By Proposition 3.1, $\left\|\Phi\right\|_2$ is bounded. So, by the approximation error properties,

$$\left\|\Phi - P_h\Phi\right\| \le Kh^2\left\|\Phi\right\|_2 \le Kh^2 \quad \text{and} \quad \left\|\nabla(\Phi - P_h\Phi)\right\| \le Kh\left\|\Phi\right\|_2 \le Kh.$$

Therefore, integrating yields,

$$||\mathbf{e}_L||^2 + \int_0^t ||\mathbf{e}_L||_1^2 \; ds \le Kh^2. \tag{3.77}$$

Next, we employ the *duality arguement* to derive the optimal order estimate $\int_0^t ||\mathbf{e}_L||^2 \; ds \le Kh^4$. For fixed $h > 0$ and $t > 0$, let $\hat{\Pi}(\tau) \in \mathbf{V}$ be the unique solution of the backward final value problem problem.

$$\left(\partial_\tau \hat{\Pi}, \Psi\right) - \mathsf{a}\left(\hat{\Pi}, \Psi\right) = (\mathbf{e}_L, \Psi)\,,\; 0 \le \tau \le t,\; \forall \Psi \in \mathbf{V} \text{ and } \hat{\Pi}(t) = 0. \tag{3.78}$$

Using the change of variable $\tau \to t - \tau$, we can show

$$\int_0^t \left(\left|\left|\hat{\Pi}_\tau\right|\right|^2 + \left|\left|\hat{\Pi}\right|\right|_2 \right) \; d\tau \le K \int_0^t ||\mathbf{e}_L||^2 \; d\tau. \tag{3.79}$$

Now setting $\Psi = \mathbf{e}_L$ in (3.78), we get

$$\left(\partial_\tau \hat{\Pi}, \mathbf{e}_L\right) - \mathsf{a}\left(\hat{\Pi}, \mathbf{e}_L\right) = ||\mathbf{e}_L||^2 . \tag{3.80}$$

Also, setting $\Psi_h = P_h \hat{\Pi}$ in (3.71) and using $t = \tau$ we get $\left(\partial_\tau \mathbf{e}_L, P_h \hat{\Pi}\right) + \mathsf{a}\left(\mathbf{e}_L, P_h \hat{\Pi}\right) = 0$. Thus,

$$\left(\partial_\tau \mathbf{e}_L, \hat{\Pi}\right) - \left(\partial_\tau \mathbf{e}_L, \hat{\Pi} - P_h \hat{\Pi}\right) + \mathsf{a}\left(\mathbf{e}_L, P_h \hat{\Pi}\right) = 0. \tag{3.81}$$

But $\frac{d}{d\tau}\left(\mathbf{e}_L, \hat{\Pi}\right) = \left(\partial_\tau \mathbf{e}_L, \hat{\Pi}\right) + \left(\mathbf{e}_L, \partial_\tau \hat{\Pi}\right)$. Thus (3.81) becomes

$$\frac{d}{d\tau}\left(\mathbf{e}_L, \hat{\Pi}\right) - \left(\mathbf{e}_L, \partial_\tau \hat{\Pi}\right) - \left(\partial_\tau \mathbf{e}_L, \hat{\Pi} - P_h \hat{\Pi}\right) + \mathsf{a}\left(\mathbf{e}_L, P_h \hat{\Pi}\right) = 0.$$

Therefore,

$$\left(\mathbf{e}_L, \partial_\tau \hat{\Pi}\right) = \frac{d}{d\tau}\left(\mathbf{e}_L, \hat{\Pi}\right) - \left(\partial_\tau \mathbf{e}_L, \hat{\Pi} - P_h\hat{\Pi}\right) + \mathtt{a}\left(\mathbf{e}_L, P_h\hat{\Pi}\right). \tag{3.82}$$

Using (3.82) in (3.80) yields

$$||\mathbf{e}_L||^2 = \frac{d}{d\tau}\left(\mathbf{e}_L, \hat{\Pi}\right) - \left(\partial_\tau \mathbf{e}_L, \hat{\Pi} - P_h\hat{\Pi}\right) + \mathtt{a}\left(\mathbf{e}_L, P_h\hat{\Pi} - \hat{\Pi}\right). \tag{3.83}$$

The second term on the right of (3.83) can be rewritten using the product rule.

$$\left(\partial_\tau \mathbf{e}_L, \hat{\Pi} - P_h\hat{\Pi}\right) = \frac{d}{d\tau}\left(\mathbf{e}_L, \hat{\Pi} - P_h\hat{\Pi}\right) - \left(\mathbf{e}_L, \partial_\tau(\hat{\Pi} - P_h\hat{\Pi})\right). \tag{3.84}$$

But $\mathbf{e}_L := \Phi - \Pi_h$. Therefore,

$$\begin{aligned}\left(\mathbf{e}_L, \partial_\tau(\hat{\Pi} - P_h\hat{\Pi})\right) &= \left(\Phi - P_h\Phi, \partial_\tau(\hat{\Pi} - P_h\hat{\Pi})\right) + \left(P_h\Phi - \Pi_h, \partial_\tau(\hat{\Pi} - P_h\hat{\Pi})\right) \\ &= \left(\Phi - P_h\Phi, \partial_\tau(\hat{\Pi} - P_h\hat{\Pi})\right) = \left(\Phi - P_h\Phi, \partial_\tau\hat{\Pi}\right)\end{aligned}$$

by the orthogonality of P_h. So, (3.84) can now be written as

$$\left(\partial_\tau \mathbf{e}_L, \hat{\Pi} - P_h\hat{\Pi}\right) = \frac{d}{d\tau}\left(\mathbf{e}_L, \hat{\Pi} - P_h\hat{\Pi}\right) - \left(\Phi - P_h\Phi, \partial_\tau\hat{\Pi}\right). \tag{3.85}$$

Using (3.85) in (3.83) we obtain

$$||\mathbf{e}_L||^2 = \frac{d}{d\tau}\left(\mathbf{e}_L, P_h\hat{\Pi}\right) + \left(\Phi - P_h\Phi, \partial_\tau\hat{\Pi}\right) + \mathtt{a}\left(\mathbf{e}_L, P_h\hat{\Pi} - \hat{\Pi}\right). \tag{3.86}$$

Integrating (3.86) with respect to τ yields

$$\int_0^t ||\mathbf{e}_L||^2 \, d\tau = \left(\mathbf{e}_L(t), P_h\hat{\Pi}(t)\right) - \left(\mathbf{e}_L(0), P_h\hat{\Pi}(0)\right) + \int_0^t \left(\Phi - P_h\Phi, \partial_\tau\hat{\Pi}\right) d\tau + \int_0^t \mathtt{a}\left(\mathbf{e}_L, P_h\hat{\Pi} - \hat{\Pi}\right) d\tau. \tag{3.87}$$

But by orthogonality, $\left(\mathbf{e}_L(0), P_h\hat{\Pi}(0)\right) = \left(\Phi_0 - P_h\Phi_0, P_h\hat{\Pi}(0)\right) = 0$ since $P_h\hat{\Pi}(0) \in \mathbf{V^h}$. Moreover, $\left(\mathbf{e}_L(t), P_h\hat{\Pi}(t)\right) = \left(\Phi(t) - \hat{\Pi}_h(t), P_h\hat{\Pi}(t)\right) = 0$ since $\hat{\Pi}(t) = 0$ by (3.78). Therefore, by the continuity of $\mathtt{a}(\cdot,\cdot)$, (3.87) becomes

$$\int_0^t ||\mathbf{e}_L||^2 \, d\tau \le \int_0^t ||\Phi - P_h\Phi|| \left|\left|\partial_\tau\hat{\Pi}\right|\right| d\tau + \alpha \int_0^t ||\nabla\mathbf{e}_L|| \left|\left|\nabla(P_h\hat{\Pi} - \hat{\Pi})\right|\right| d\tau.$$

By Young's inequality, the approximation error properties and (3.77),

$$\begin{aligned}
\int_0^t ||\mathbf{e}_L||^2 \, d\tau &\le \frac{1}{2\epsilon}\int_0^t ||\Phi - P_h\Phi||^2 \, d\tau + \frac{\epsilon}{2}\int_0^t \left|\left|\partial_\tau\hat{\Pi}\right|\right|^2 d\tau \\
&\quad + \alpha \int_0^t ||\nabla\mathbf{e}_L|| \left|\left|\nabla(P_h\hat{\Pi} - \hat{\Pi})\right|\right| d\tau \\
&\le Kh^4 + \frac{\epsilon}{2}\int_0^t \left|\left|\partial_\tau\hat{\Pi}\right|\right|^2 d\tau + K\int_0^t (||\nabla\mathbf{e}_L||\, h) \left|\left|\hat{\Pi}\right|\right|_2 d\tau \\
&\le Kh^4 + \frac{\epsilon}{2}\int_0^t \left|\left|\partial_\tau\hat{\Pi}\right|\right|^2 d\tau + Kh^2\int_0^t ||\nabla\mathbf{e}_L||^2 \, d\tau + \frac{\epsilon}{2}\int_0^t \left|\left|\hat{\Pi}\right|\right|_2^2 d\tau \\
&= Kh^4 + \frac{\epsilon}{2}\int_0^t \left(\left|\left|\partial_\tau\hat{\Pi}\right|\right|^2 + \left|\left|\hat{\Pi}\right|\right|_2^2\right) d\tau.
\end{aligned}$$

By (3.79), $\int_0^t ||\mathbf{e}_L||^2 \, d\tau \le Kh^4 + \frac{\epsilon}{2}K^* \int_0^t ||\mathbf{e}_L||^2 \, d\tau$. So, setting $\epsilon = \frac{1}{K^*}$, we get

$$\int_0^t ||\mathbf{e}_L||^2 \, d\tau \le Kh^4.$$

□

Let P_h^r be the Ritz projection from $\mathbf{X} \to \mathbf{X^h}$ defined by

$$\mathtt{a}\,(P_h^r \Phi, \Psi) = \mathtt{a}\,(\Phi, \Psi) \qquad \forall \Psi \in \mathbf{X^h} \tag{3.88}$$

and decompose $\mathbf{e}_L$ into $\mathbf{e}_{L_1}$ and $\mathbf{e}_{L_2}$ as follows:

$$\mathbf{e}_L = \Phi - \Pi_h = (\Phi - P_h^r \Phi) + (P_h^r \Phi - \Pi_h) := \mathbf{e}_{L_1} + \mathbf{e}_{L_2}.$$

Lemma 3.8. *The error* $\mathbf{e}_{L_1} := (\Phi - P_h^r \Phi)$ *satisfies the following estimate*

$$||\Phi - P_h^r \Phi|| + h\,||\nabla\,(\Phi - P_h^r \Phi)|| \leq K h^2 \tag{3.89}$$

Proof. Setting $\Psi = P_h\,(\Phi - P_h^r \Phi)$ in (3.88) yields

$$\begin{aligned} 0 = \mathtt{a}\,(P_h^r \Phi - \Phi, P_h\,(\Phi - P_h^r \Phi)) &= \mathtt{a}\,(P_h^r \Phi - \Phi, P_h \Phi - P_h^r \Phi) \\ &= \mathtt{a}\,(P_h^r \Phi - \Phi, (P_h \Phi - \Phi) + (\Phi - P_h^r \Phi)). \end{aligned}$$

Thus, $\mathtt{a}\,(P_h^r \Phi - \Phi, P_h^r \Phi - \Phi) = \mathtt{a}\,(P_h^r \Phi - \Phi, P_h \Phi - \Phi)$. By the coercivity and continuity of $\mathtt{a}\,(\cdot,\cdot)$, we get

$$\begin{aligned} \alpha\,||\nabla(P_h^r \Phi - \Phi)||^2 \leq \mathtt{a}\,(P_h^r \Phi - \Phi, P_h^r \Phi - \Phi) &= \mathtt{a}\,(P_h^r \Phi - \Phi, P_h \Phi - \Phi) \\ &\leq K\,||\nabla(P_h^r \Phi - \Phi)||\,||\nabla(P_h \Phi - \Phi)||\,. \end{aligned}$$

So, by the error approximation property and Proposition 3.1, $h\,||\nabla\,(P_h^r \Phi - \Phi)|| \leq K h^2\,||\Phi||_2 \leq K h^2$. Next, to show that $||\Phi - P_h^r \Phi|| \leq K h^2$ we will employ the duality arguement. Let Ξ

be the solution of

$$\mathsf{a}\left(\Xi, \Psi\right) = \left(\Phi - P_h^r\Phi, \Psi\right) \qquad \forall \Psi \in \mathbf{X}. \tag{3.90}$$

Setting $\Psi = \Phi - P_h^r\Phi$ yields $\mathsf{a}\left(\Xi, \Phi - P_h^r\Phi\right) = ||\Phi - P_h^r\Phi||^2$. Moreover, since $P_h\Xi \in \mathbf{X^h}$, we have by definition of the Ritz projection, $\mathsf{a}\left(P_h\Xi, \Phi - P_h^r\Phi\right) = 0$. Combining these last two results yields $||\Phi - P_h^r\Phi||^2 = \mathsf{a}\left(\Xi - P_h\Xi, \Phi - P_h^r\Phi\right)$. By the continuity of $\mathsf{a}\left(\cdot,\cdot\right)$, the error approximation property and Proposition 3.1,

$$||\Phi - P_h^r\Phi||^2 \leq K\,||\nabla\left(\Xi - P_h\Xi\right)||\,||\nabla\left(\Phi - P_h^r\Phi\right)|| \leq Kh\,||\Xi||_2\,Kh\,||\Phi||_2 = Kh^2\,||\Xi||_2\,.$$

But by the stability estimates of (3.90), $||\Xi||_2 \leq ||\Phi - P_h^r\Phi||$. Thus we have the required estimate $||\Phi - P_h^r\Phi|| \leq Kh^2$. □

Lemma 3.9. *The time derivative of the error* $\mathbf{e}_{L_1} := \Phi - P_h^r\Phi$ *satisfies the following estimate*

$$\int_0^t s\left[||\partial_t\left(\Phi - P_h^r\Phi\right)||^2 + h^2\,||\nabla\partial_t\left(\Phi - P_h^r\Phi\right)||^2\right]\,ds \leq Kh^4, \quad \textit{for } t > 0. \tag{3.91}$$

Proof. Differentiating (3.88) with respect to time yields

$$\mathsf{a}\left(\Phi_t - P_h^r\Phi_t, \Psi\right) = 0, \quad \forall\Psi \in \mathbf{X^h}. \tag{3.92}$$

Setting $\Psi = P_h(\Phi_t - P_h^r\Phi_t)$ and adding $\mathsf{a}\left(\Phi_t - P_h^r\Phi_t, \Phi_t\right)$ to both sides of the above gives

$$\mathsf{a}\left(\Phi_t - P_h^r\Phi_t, P_h(\Phi_t - P_h^r\Phi_t)\right) + \mathsf{a}\left(\Phi_t - P_h^r\Phi_t, \Phi_t\right) = \mathsf{a}\left(\Phi_t - P_h^r\Phi_t, \Phi_t\right).$$

This implies

$$\mathtt{a}\,(\Phi_t - P_h^r\Phi_t, P_h\Phi_t - P_h^r\Phi_t) + \mathtt{a}\,(\Phi_t - P_h^r\Phi_t, \Phi_t) = \mathtt{a}\,(\Phi_t - P_h^r\Phi_t, \Phi_t).$$

By the Ritz projection, $\mathtt{a}\,(\Phi_t - P_h^r\Phi_t, P_h\Phi_t) = 0$. Thus,

$$\mathtt{a}\,(\Phi_t - P_h^r\Phi_t, -P_h^r\Phi_t) + \mathtt{a}\,(\Phi_t - P_h^r\Phi_t, \Phi_t) = \mathtt{a}\,(\Phi_t - P_h^r\Phi_t, \Phi_t).$$

This implies, $\mathtt{a}\,(\Phi_t - P_h^r\Phi_t, \Phi_t - P_h^r\Phi_t) = \mathtt{a}\,(\Phi_t - P_h^r\Phi_t, \Phi_t)$, yielding,

$$\alpha\,||\Phi_t - P_h^r\Phi_t||_1^2 = \mathtt{a}\,(\Phi_t - P_h^r\Phi_t, \Phi_t - P_h\Phi_t) \leq K\,||\Phi_t - P_h^r\Phi_t||_1\,||\Phi_t - P_h\Phi_t||_1\,.$$

Therefore, by the approximation error properties,

$$\int_0^t s\,||\Phi_t - P_h^r\Phi_t||_1^2\;ds \leq Kh^2 \int_0^t s\,||\partial_t\Phi||_2^2\;ds \leq Kh^2. \tag{3.93}$$

To derive the L^2-bound, we use the duality argument. Let (Ξ, Φ) be the unique solution of the system

$$\mathtt{a}\,(\Xi, \Psi) = (\partial_t(\Phi - P_h^r\Phi), \Psi) \quad \forall\Psi \in \mathbf{X}. \tag{3.94}$$

It is well known that

$$||\Xi||_2 \leq K\,||\partial_t(\Phi - P_h^r\Phi)||\,. \tag{3.95}$$

So, setting $\Phi = \partial_t(\Phi - P_h^r\Phi)$ in (3.94) yields

$$||\partial_t(\Phi - P_h^r\Phi)||^2 = \mathtt{a}\,(\Xi, \partial_t(\Phi - P_h^r\Phi))$$
$$= \mathtt{a}\,(\Xi - P_h\Xi, \partial_t(\Phi - P_h^r\Phi)) + \mathtt{a}\,(P_h\Xi, \partial_t(\Phi - P_h^r\Phi)). \quad (3.96)$$

Now setting $\Xi = P_h\Xi$ in (3.88) yields $\mathtt{a}\,(\partial_t(\Phi - P_h^r\Phi), P_h\Xi) = 0$. Therefore, (3.96) becomes $||\partial_t(\Phi - P_h^r\Phi)||^2 \le K\,||\Xi - P_h\Xi||_1\,||\partial_t(\Phi - P_h^r\Phi)||_1$. By the approximation error properties, $||\partial_t(\Phi - P_h^r\Phi)||^2 \le Kh\,||\Xi||_2\,||\partial_t(\Phi - P_h^r\Phi)||_1$. Therefore, by (3.95),

$$||\partial_t(\Phi - P_h^r\Phi)||^2 \le Kh\,||\partial_t(\Phi - P_h^r\Phi)||\,||\partial_t(\Phi - P_h^r\Phi)||_1\,.$$

Employing the kickback argument yields $||\partial_t(\Phi - P_h^r\Phi)||^2 \le Kh^2\,||\partial_t(\Phi - P_h^r\Phi)||_1^2$. Integrating after multiplying by s, we get

$$\int_0^t s\,||\partial_t(\Phi - P_h^r\Phi)||^2\ ds\ \le Kh^2 \int_0^t s\,||\partial_t(\Phi - P_h^r\Phi)||_1^2\ ds. \quad (3.97)$$

Using (3.93) and (3.97), we get $\int_0^t s\,||\partial_t(\Phi - P_h^r\Phi)||^2\ ds\ \le Kh^4$. □

Lemma 3.10. *The error* $\mathbf{e}_{L_2} = P_h^r\Phi - \Pi_h$ *satisfies the following estimate*

$$||P_h^r\Phi - \Pi_h|| + h\,||\nabla(P_h^r\Phi - \Pi_h)|| \le Kh^2, \quad \textit{for } t > 0. \quad (3.98)$$

Proof. We begin by recalling Equation (3.71): $(\partial_t\mathbf{e}_L, \Psi_h) + \mathtt{a}\,(\mathbf{e}_L, \Psi_h) = 0$, $\forall\Psi_h \in \mathbf{X^h}$, where $\mathbf{e}_L = \Phi - \Pi_h = (\Phi - P_h^r\Phi) + (P_h^r\Phi - \Pi_h)$. Thus,

$$(\partial_t(P_h^r\Phi - \Pi_h), \Psi_h) + \mathtt{a}\,((\Phi - P_h^r\Phi) + (P_h^r\Phi - \Pi_h), \Psi_h) = (\partial_t(P_h^r\Phi - \Phi), \Psi_h)\ \ \forall\Psi_h \in \mathbf{X^h}.$$

By orthogonality,

$$(\partial_t(P_h^r\Phi - \Pi_h), \Psi_h) + \mathtt{a}\,(P_h^r\Phi - \Pi_h, \Psi_h) = (\partial_t(P_h^r\Phi - \Phi), \Psi_h) \quad \forall \Psi_h \in \mathbf{X^h}.$$

Setting $\Phi_h = P_h^r\Phi - \Pi_h$ yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left|\left|P_h^r\Phi - \Pi_h\right|\right|^2 + \alpha\left|\left|\nabla(P_h^r\Phi - \Pi_h\right|\right|^2 &\leq \left|\left|\partial_t(P_h^r\Phi - \Phi)\right|\right| \left|\left|P_h^r\Phi - \Pi_h\right|\right| \\ &\leq K\left|\left|\partial_t(P_h^r\Phi - \Phi)\right|\right| \left|\left|\nabla(P_h^r\Phi - \Pi_h)\right|\right|.\end{aligned}$$

Employing the kickback arguement and multiplying by t yields

$$\frac{t}{2}\frac{d}{dt}\left|\left|P_h^r\Phi - \Pi_h\right|\right|^2 + \frac{t\alpha}{2}\left|\left|\nabla(P_h^r\Phi - \Pi_h)\right|\right|^2 \leq Kt\left|\left|\partial_t(P_h^r\Phi - \Phi)\right|\right|^2.$$

Note, by the product rule, $t\dfrac{df}{dt} = \dfrac{d}{dt}(tf) - f$ for any $f(t)$. Therefore,

$$\frac{d}{dt}\left[t\left|\left|P_h^r\Phi - \Pi_h\right|\right|^2\right] + \alpha t\left|\left|\nabla(P_h^r\Phi - \Pi_h)\right|\right|^2 \leq Kt\left|\left|\partial_t(P_h^r\Phi - \Phi)\right|\right|^2 + 2\left|\left|P_h^r\Phi - \Pi_h\right|\right|^2.$$

Integrating with respect to t yields

$$\begin{aligned}t\left|\left|P_h^r\Phi - \Pi_h\right|\right|^2 + \alpha\int_0^t s\left|\left|\nabla(P_h^r\Phi - \Pi_h)\right|\right|^2\,ds &\leq K\int_0^t s\left|\left|\partial_t(P_h^r\Phi - \Phi)\right|\right|^2\,ds \\ &\quad + 2\int_0^t \left|\left|P_h^r\Phi - \Pi_h\right|\right|^2\,ds.\end{aligned}$$

Lemma 3.7 and Lemma 3.9 can be used to bound the terms on the right hand side. Thus,

$$t\left|\left|P_h^r\Phi - \Pi_h\right|\right|^2 + \alpha\int_0^t s\left|\left|\nabla(P_h^r\Phi - \Pi_h)\right|\right|^2\,ds \leq Kh^4 + Kh^4 = Kh^4.$$

Therefore, $||P_h^r \Phi - \Pi_h|| \le Kh^2$. By the inverse inequality we get $h\,||\nabla(P_h^r\Phi - \Pi_h)|| \le ||P_h^r \Phi - \Pi_h|| \le Kh^2$ □

By combining Lemma 3.8 and Lemma 3.10, we get

$$||\mathbf{e}_L|| + h\,||\nabla \mathbf{e}_L|| \le Kh^2 \tag{3.99}$$

or $||\Phi - \Pi_h|| + h\,||\nabla(\Phi - \Pi_h)||^2 \le Kh^2$. The following theorem is the main result of this section.

Theorem 3.11. *Let the discrete initial data, $\Phi_{0\,h} := (\mathbf{u}_{0\,h}, \theta_{0\,h}, C_{0\,h}) \in \mathbf{X^h}$, satisfy*

$$||\mathbf{u}_0 - \mathbf{u}_{0\,h}|| + ||\theta_0 - \theta_{0\,h}|| + ||C_0 - C_{0\,h}|| \le Kh^2\,||\Phi_0||_2\,.$$

Then there exists a constant K such that the following estimates hold,

$$||\mathbf{u} - \mathbf{u_h}|| + ||\theta - \theta_h|| + ||C - C_h|| \le Kh^2$$

$$||\nabla(\mathbf{u} - \mathbf{u_h})|| + ||\nabla(\theta - \theta_h)|| + ||\nabla(C - C_h)|| \le Kh.$$

Proof. Recall, $\mathbf{e} = \Phi - \Phi_h = (\mathbf{u} - \mathbf{u_h}, \theta - \theta_h, C - C_h)$ and $\mathbf{e} = \mathbf{e}_L + \mathbf{e}_N$. By (3.99), $||\mathbf{e}_L|| + h\,||\nabla \mathbf{e}_L||^2 \le Kh^2$. To find a similar estimate for $\mathbf{e}_N$, subtract (3.68) from (3.69) to get

$$\begin{aligned}(\partial_t \mathbf{e}_N, \Psi_h) + \mathtt{a}\,(\mathbf{e}_N, \Psi_h) &= \mathtt{c}\,(\Phi_h, \Phi_h, \Psi_h) - \mathtt{c}\,(\Phi, \Phi, \Psi_h) \\ &\quad + \mathtt{d}\,(\Phi - \Phi_h, \Psi_h) \quad, \forall \Psi_h \in \mathbf{X}.\end{aligned} \tag{3.100}$$

Setting $\Psi_h = \mathbf{e}_N$ in (3.100) yields

$$\frac{1}{2}\frac{d}{dt}\left|\left|\mathbf{e}_N\right|\right|^2 + \alpha\left|\left|\nabla\mathbf{e}_N\right|\right|^2 = \mathtt{c}\left(\Phi_h, \Phi_h, \mathbf{e}_N\right) - \mathtt{c}\left(\Phi, \Phi, \mathbf{e}_N\right) + \mathtt{d}\left(\mathbf{e}, \mathbf{e}_N\right). \tag{3.101}$$

But

$$\begin{aligned}\left|\mathtt{d}\left(\mathbf{e}, \mathbf{e}_N\right)\right| = \left|\mathtt{d}\left(\mathbf{e}_L, \mathbf{e}_N\right) + \mathtt{d}\left(\mathbf{e}_N, \mathbf{e}_N\right)\right| &\leq K\left|\left|\mathbf{e}_L\right|\right|\left|\left|\mathbf{e}_N\right|\right| + K\left|\left|\mathbf{e}_N\right|\right|^2 \\ &\leq K\left|\left|\mathbf{e}_L\right|\right|^2 + K\left|\left|\mathbf{e}_N\right|\right|^2. \end{aligned} \tag{3.102}$$

On the other hand, we can estimate the terms involving the trilinear form as follows,

$$\begin{aligned}&\left|\mathtt{c}\left(\Phi_h, \Phi_h, \mathbf{e}_N\right) - \mathtt{c}\left(\Phi, \Phi, \mathbf{e}_N\right)\right| \leq \left|\mathtt{c}\left(\mathbf{e}_L, \Phi_h, \mathbf{e}_N\right) + \mathtt{c}\left(\mathbf{e}_N, \Phi_h, \mathbf{e}_N\right) + \mathtt{c}\left(\Phi, \mathbf{e}_N, \mathbf{e}_L\right)\right| \\ &\leq K\left|\left|\mathbf{e}_L\right|\right|\left|\left|\nabla\Phi_h\right|\right|_{L^4}\left|\left|\mathbf{e}_N\right|\right|_{L^4} + K\left|\left|\mathbf{e}_N\right|\right|_{L^4}\left|\left|\nabla\Phi_h\right|\right|\left|\left|\mathbf{e}_N\right|\right|_{L^4} + K\left|\left|\Phi\right|\right|_\infty\left|\left|\nabla\mathbf{e}_N\right|\right|\left|\left|\mathbf{e}_L\right|\right|,\end{aligned}$$

where the first two estimates are from Holder's inequality and the last estimate is obtained as follows,

$$\begin{aligned}\mathtt{c}\left(\Phi, \mathbf{e}_N, \mathbf{e}_L\right) = \int_\Omega (\Phi\cdot\nabla)\mathbf{e}_N\cdot\mathbf{e}_L\, d\Omega &\leq \int_\Omega \left|\Phi\right|\left|\nabla\mathbf{e}_N\right|\left|\mathbf{e}_L\right|\ d\Omega \\ &\leq \left|\left|\Phi\right|\right|_\infty \int_\Omega \left|\nabla\mathbf{e}_N\right|\left|\mathbf{e}_L\right|\ d\Omega \leq K\left|\left|\Phi\right|\right|_\infty\left|\left|\nabla\mathbf{e}_N\right|\right|\left|\left|\mathbf{e}_L\right|\right|.\end{aligned}$$

But by Agmon inequality, see [23], $\left|\left|\Phi\right|\right|_\infty \leq K\left|\left|\Phi\right|\right|_2^{1/2}\left|\left|\Phi\right|\right|_1^{1/2}$, $\forall\Phi \in \mathbf{H}^2(\Omega)\cap\mathbf{H}_0^1(\Omega)$. However, by Proposition 3.1, $\left|\left|\Phi\right|\right|_2$ is bounded. Also, by Gagliardo inequality and Proposition 3.1, $\left|\left|\Phi_h\right|\right|_{L^4} \leq K\left|\left|\Phi_h\right|\right|^{1/2}\left|\left|\Phi_h\right|\right|_1^{1/2} \leq K$. So, employing Ladyzhenskaya and Gagliardo inequalities, we get

$$\left|\mathsf{c}\left(\Phi_h, \Phi_h, \mathbf{e}_N\right) - \mathsf{c}\left(\Phi, \Phi, \mathbf{e}_N\right)\right| \leq K\left|\left|\mathbf{e}_L\right|\right| \left|\left|\mathbf{e}_N\right|\right|^{1/2} \left|\left|\nabla \mathbf{e}_N\right|\right|^{1/2} + K\left|\left|\mathbf{e}_N\right|\right| \left|\left|\nabla \mathbf{e}_N\right|\right|$$

$$+ K\left|\left|\nabla \mathbf{e}_N\right|\right| \left|\left|\mathbf{e}_L\right|\right| \leq K_1\left|\left|\mathbf{e}_L\right|\right|^2 + \frac{\epsilon}{2}\left|\left|\nabla \mathbf{e}_N\right|\right|^2 + K_2\left|\left|\mathbf{e}_N\right|\right|^2 + \frac{\epsilon}{2}\left|\left|\nabla \mathbf{e}_N\right|\right|^2$$

$$\leq K_1\left|\left|\mathbf{e}_L\right|\right|^2 + K_2\left|\left|\mathbf{e}_N\right|\right|^2 + \epsilon\left|\left|\nabla \mathbf{e}_N\right|\right|^2. \tag{3.103}$$

Taking $\epsilon = \alpha/2$ and inserting estimates (3.102) and (3.103) in (3.101) gives us $\frac{1}{2}\frac{d}{dt}\left|\left|\mathbf{e}_N\right|\right|^2 + \frac{\alpha}{2}\left|\left|\nabla \mathbf{e}_N\right|\right|^2 \leq K_1\left|\left|\mathbf{e}_L\right|\right|^2 + K_2\left|\left|\mathbf{e}_N\right|\right|^2$. Now, by (3.99) we have $\frac{1}{2}\frac{d}{dt}\left|\left|\mathbf{e}_N\right|\right|^2 + \alpha\left|\left|\nabla \mathbf{e}_N\right|\right|^2 \leq K_1 h^4 + K_2\left|\left|\mathbf{e}_N\right|\right|^2$. Therefore, by Continuous Gronwall's Lemma,

$$\left|\left|\mathbf{e}_N\right|\right|^2 \leq K h^4. \tag{3.104}$$

Finally, since $\mathbf{e}_N = \Pi_h - \Phi_h \in \mathbf{X^h}$, by the inverse inequality, $h\left|\left|\nabla \mathbf{e}_N\right|\right| \leq \left|\left|\mathbf{e}_N\right|\right|$. Therefore,

$$h\left|\left|\nabla \mathbf{e}_N\right|\right| \leq K h^2. \tag{3.105}$$

Thus (3.104) and (3.105) imply

$$\left|\left|\mathbf{e}_N\right|\right| + h\left|\left|\nabla \mathbf{e}_N\right|\right| \leq K h^2. \tag{3.106}$$

Using (3.99), (3.106) and the triangle inequality, we get $\left|\left|\mathbf{e}\right|\right| + h\left|\left|\nabla \mathbf{e}\right|\right| \leq K h^2$. □

3.4 Semi-Discrete Spatial Error Estimate for Pressure

In this section we derive an error estimate for the approximation of pressure, $p_h(t)$ of $p(t)$. For pressure error estimates, we make the following assumption on the finite dimensional subspaces L_h and $\mathbf{W}_h$:

Assumption C[Discrete inf-sup condition]: Given two subspaces $L_h \subset L_0^2(\Omega)$ and $\mathbf{W}_h \subset \mathbf{H}_0^1(\Omega)$, there exists a constant $\beta > 0$ such that

$$\inf_{q_h \in L_h} \sup_{\mathbf{v_h} \in \mathbf{W}_h/\{0\}} \frac{(\nabla \cdot \mathbf{v_h}, q_h)}{||\mathbf{v_h}||_1} \geq \beta \, ||q_h|| \, .$$

.

Lemma 3.12. *The semi-discrete Galerkin approximation p_h of pressure p satisfies*

$$||(p - p_h)(t)|| \leq K \left[h + ||\partial_t \mathbf{e_u}|| + ||\nabla \mathbf{e_u}|| + ||e_\theta|| + ||e_c||\right], \quad \forall t > 0,$$

where $\mathbf{e_u} := \mathbf{u} - \mathbf{u_h}$, $e_c := \theta - \theta_h$ and $e_c := C - C_h$.

Proof. Since $p_h(t)$ satisfies

$$\begin{aligned}(\partial_t \mathbf{u_h}, \mathbf{v_h}) + (\mathbf{u_h} \cdot \nabla \mathbf{u_h}, \mathbf{v_h}) + (\nabla \mathbf{u_h}, \nabla \mathbf{v_h}) - (p_h, \nabla \cdot \mathbf{v_h}) = (\mathbf{f}, \mathbf{v_h}) \\ + G_{r_\theta} \left(\theta_h \mathbf{j}, \mathbf{v_h}\right) + G_{r_c} \left(C_h \mathbf{j}, \mathbf{v_h}\right) \quad \forall \mathbf{v_h} \in W^h \end{aligned} \tag{3.107}$$

and p satisfies

$$\begin{aligned}(\partial_t \mathbf{u}, \mathbf{v}) + (\mathbf{u} \cdot \nabla \mathbf{u}, \mathbf{v}) + (\nabla \mathbf{u}, \nabla \mathbf{v}) - (p, \nabla \cdot \mathbf{v}) = (\mathbf{f}, \mathbf{v}) \\ + G_{r_\theta} \left(\theta_h \mathbf{j}, \mathbf{v}\right) + G_{r_c} \left(C_h \mathbf{j}, \mathbf{v}\right) \quad \forall \mathbf{v} \in H_0^1. \end{aligned} \tag{3.108}$$

Restricting $\mathbf{v}$ in (3.108) to W^h and subtracting (3.107) from (3.108) yields

$$\begin{aligned}&(\partial_t \mathbf{e_u}, \mathbf{v_h}) + (\nabla \mathbf{e_u}, \nabla \mathbf{v_h}) - (p - p_h, \nabla \cdot \mathbf{v_h}) \\ &\quad + (\mathbf{u} \cdot \nabla \mathbf{u}, \mathbf{v_h}) - (\mathbf{u_h} \cdot \nabla \mathbf{u_h}, \mathbf{v_h}) = G_{r_\theta} \left(e_\theta \mathbf{j}, \mathbf{v_h}\right) + G_{r_c} \left(e_c \mathbf{j}, \mathbf{v_h}\right). \end{aligned} \tag{3.109}$$

Therefore,

$$(p - p_h, \nabla \cdot \mathbf{v_h}) = (\partial_t \mathbf{e_u}, \mathbf{v_h}) + (\nabla \mathbf{e_u}, \nabla \mathbf{v_h}) + (\mathbf{u} \cdot \nabla \mathbf{u}, \mathbf{v_h}) - (\mathbf{u_h} \cdot \nabla \mathbf{u_h}, \mathbf{v_h})$$
$$- G_{r_\theta} (e_\theta \mathbf{j}, \mathbf{v_h}) - G_{r_c} (e_c \mathbf{j}, \mathbf{v_h}) . \tag{3.110}$$

But $(\mathbf{u} \cdot \nabla \mathbf{u}, \mathbf{v_h}) - (\mathbf{u_h} \cdot \nabla \mathbf{u_h}, \mathbf{v_h}) = (\mathbf{u} \cdot \nabla \mathbf{e_u}, \mathbf{v_h}) + (\mathbf{e_u} \cdot \nabla \mathbf{u_h}, \mathbf{v_h})$. Therefore,

$$|(p - p_h, \nabla \cdot \mathbf{v_h})| \leq |(\partial_t \mathbf{e_u}, \mathbf{v_h})| + |(\nabla \mathbf{e_u}, \nabla \mathbf{v_h})| + |(\mathbf{u} \cdot \nabla \mathbf{e_u}, \mathbf{v_h})| + |(\mathbf{e_u} \cdot \nabla \mathbf{u_h}, \mathbf{v_h})|$$
$$+ G_{r_\theta} ||e_\theta|| + G_{r_c} ||e_c|| \leq K ||\partial_t \mathbf{e_u}|| \, ||\nabla \mathbf{v_h}|| + ||\nabla \mathbf{e_u}|| \, ||\nabla \mathbf{v_h}|| + K ||\nabla \mathbf{u}|| \, ||\nabla \mathbf{e_u}|| \, ||\nabla \mathbf{v_h}||$$
$$+ K ||\nabla \mathbf{e_u}|| \, ||\nabla \mathbf{u_h}|| \, ||\nabla \mathbf{v_h}|| + K ||e_\theta|| \, ||\mathbf{v_h}|| + K ||e_c|| \, ||\mathbf{v_h}||$$
$$\leq K ||\nabla \mathbf{v_h}|| \left(||\partial_t \mathbf{e_u}|| + ||\nabla \mathbf{e_u}|| + ||e_\theta|| + ||e_c|| \right) . \tag{3.111}$$

Thus we have,

$$\sup_{\mathbf{v_h} \in W^h} \frac{|(p - p_h, \nabla \cdot \mathbf{v_h})|}{||\nabla \mathbf{v_h}||} \leq K \left(||\partial_t \mathbf{e_u}|| + ||\nabla \mathbf{e_u}|| + ||e_\theta|| + ||e_c|| \right) . \tag{3.112}$$

By the triangle inequality,

$$||p - p_h|| \leq ||p - \hat{\pi}_h p|| + ||\hat{\pi}_h p - p_h|| . \tag{3.113}$$

Moreover, by the *inf-sup* condition (Assumption C),

$$
\begin{aligned}
||\hat{\pi}_h p - p_h|| &\leq \sup_{\mathbf{v_h} \in W^h} \frac{(\hat{\pi}_h p - p_h, \nabla \cdot \mathbf{v_h})}{||\nabla \mathbf{v_h}||} \\
&\leq \sup_{\mathbf{v_h} \in W^h} \frac{(\hat{\pi}_h p - p + p - p_h, \nabla \cdot \mathbf{v_h})}{||\nabla \mathbf{v_h}||} \\
&\leq \sup_{\mathbf{v_h} \in W^h} \frac{(\hat{\pi}_h p - p, \nabla \cdot \mathbf{v_h})}{||\nabla \mathbf{v_h}||} + \sup_{\mathbf{v_h} \in W^h} \frac{(p - p_h, \nabla \cdot \mathbf{v_h})}{||\nabla \mathbf{v_h}||} \\
&\leq \sup_{\mathbf{v_h} \in W^h} \frac{||\hat{\pi}_h p - p|| \, ||\nabla \cdot \mathbf{v_h}||}{||\nabla \mathbf{v_h}||} + \sup_{\mathbf{v_h} \in W^h} \frac{(p - p_h, \nabla \cdot \mathbf{v_h})}{||\nabla \mathbf{v_h}||}.
\end{aligned}
$$

Since $||\nabla \cdot \mathbf{v_h}|| \leq K \, ||\nabla \mathbf{v_h}||$,

$$
||\hat{\pi}_h p - p_h|| \leq K \, ||\hat{\pi}_h p - p|| + \sup_{\mathbf{v_h} \in W^h} \frac{(p - p_h, \nabla \cdot \mathbf{v_h})}{||\nabla \mathbf{v_h}||}. \tag{3.114}
$$

Using (3.114) in (3.113) gives

$$
||p - p_h|| \leq K \, ||p - \hat{\pi}_h p|| + \sup_{\mathbf{v_h} \in W^h} \frac{(p - p_h, \nabla \cdot \mathbf{v_h})}{||\nabla \mathbf{v_h}||}. \tag{3.115}
$$

By (3.112) and (3.115) and using the interpolant approximation estimate (3.12) yields,

$$
\begin{aligned}
||p - p_h|| &\leq K \, ||p - \hat{\pi}_h p|| + K \left(||\partial_t \mathbf{e_u}|| + ||\nabla \mathbf{e_u}|| + ||e_\theta|| + ||e_c|| \right) \\
&\leq Kh + K \left(||\partial_t \mathbf{e_u}|| + ||\nabla \mathbf{e_u}|| + ||e_\theta|| + ||e_c|| \right).
\end{aligned}
$$

□

Lemma 3.13. *The error* $\mathbf{e_u} := \mathbf{u} - \mathbf{u_h}$ *satisfies*

$$
||\partial_t \mathbf{e_u}|| \leq \frac{Kh}{\sqrt{t}}
$$

for $t > 0$.

Proof. The error equations governing the errors $\mathbf{e_u}$, e_θ and e_c are derived by subtracting the weak form of the exact solutions (3.64), (3.65) and (3.66) (where $\mathbf{v}$, ϕ and ψ are restricted to $\mathbf{V^h}$, W^h and W^h respectively) from the weak form of the semi-discrete problem (3.13), (3.14) and (3.15) to get

$$(\partial_t \mathbf{e_u}, \mathbf{v_h}) + (\nabla \mathbf{e_u}, \nabla \mathbf{v_h}) + (\mathbf{u} \cdot \nabla \mathbf{u}, \mathbf{v_h}) \\ - (\mathbf{u_h} \cdot \nabla \mathbf{u_h}, \mathbf{v_h}) = G_{r_\theta} (e_\theta \mathbf{j}, \mathbf{v_h}) + G_{r_c} (e_c \mathbf{j}, \mathbf{v_h}) \quad \forall \mathbf{v_h} \in \mathbf{V^h} \tag{3.116}$$

$$(\partial_t e_\theta, \phi_h) + \frac{1}{Pr} (\nabla e_\theta, \nabla \phi_h) + (\mathbf{u} \cdot \nabla \theta, \phi_h) - (\mathbf{u_h} \cdot \nabla \theta_h, \phi_h) = 0 \quad \forall \phi_h \in W^h \tag{3.117}$$

$$(\partial_t e_c, \psi_h) + \frac{1}{PrLe} (\nabla e_c, \nabla \psi_h) + (\mathbf{u} \cdot \nabla C, \psi_h) - (\mathbf{u_h} \cdot \nabla C_h, \psi_h) = 0 \quad \forall \psi_h \in W^h. \tag{3.118}$$

Note that adding and subtracting terms yields the following reductions

$$\begin{aligned}(\mathbf{u} \cdot \nabla \mathbf{u}, \mathbf{v_h}) - (\mathbf{u_h} \cdot \nabla \mathbf{u_h}, \mathbf{v_h}) &= (\mathbf{u} \cdot \nabla \mathbf{u}, \mathbf{v_h}) - (\mathbf{u_h} \cdot \nabla \mathbf{u}, \mathbf{v_h}) \\ &+ (\mathbf{u_h} \cdot \nabla \mathbf{u}, \mathbf{v_h}) - (\mathbf{u_h} \cdot \nabla \mathbf{u_h}, \mathbf{v_h}) \\ &= (\mathbf{e_u} \cdot \nabla \mathbf{u}, \mathbf{v_h}) + (\mathbf{u_h} \cdot \nabla \mathbf{e_u}, \mathbf{v_h}), \end{aligned} \tag{3.119}$$

$$(\mathbf{u} \cdot \nabla \theta, \phi_h) - (\mathbf{u_h} \cdot \nabla \theta_h, \phi_h) = (\mathbf{e_u} \cdot \nabla \theta, \phi_h) + (\mathbf{u_h} \cdot \nabla e_\theta, \phi_h) \tag{3.120}$$

and

$$(\mathbf{u}\cdot\nabla C,\psi_h)-(\mathbf{u_h}\cdot\nabla C_h,\psi_h)=(\mathbf{e_u}\cdot\nabla C,\psi_h)+(\mathbf{u_h}\cdot\nabla e_c,\psi_h)\,. \tag{3.121}$$

Using (3.119)-(3.121) in (3.116)-(3.118) gives

$$\begin{aligned}(\partial_t\mathbf{e_u},\mathbf{v_h})+(\nabla\mathbf{e_u},\nabla\mathbf{v_h})+(\mathbf{e_u}\cdot\nabla\mathbf{u},\mathbf{v_h})+(\mathbf{u_h}\cdot\nabla\mathbf{e_u},\mathbf{v_h})&=G_{r_\theta}\left(e_\theta\mathbf{j},\mathbf{v_h}\right)\\&+G_{r_c}\left(e_c\mathbf{j},\mathbf{v_h}\right)\ \forall\mathbf{v_h}\in\mathbf{V},\end{aligned} \tag{3.122}$$

$$(\partial_t e_\theta,\phi_h)+\frac{1}{Pr}\left(\nabla e_\theta,\nabla\phi_h\right)+(\mathbf{e_u}\cdot\nabla\theta,\phi_h)+(\mathbf{u_h}\cdot\nabla e_\theta,\phi_h)=0,\ \forall\phi_h\in W^h \tag{3.123}$$

and

$$(\partial_t e_c,\psi_h)+\frac{1}{PrLe}\left(\nabla e_c,\nabla\psi_h\right)+(\mathbf{e_u}\cdot\nabla C,\psi_h)+(\mathbf{u_h}\cdot\nabla e_c,\psi_h)=0\ ,\forall\psi_h\in W^h. \tag{3.124}$$

Differentiating the these equations with respect to time and using the product rule on the tri-linear terms yields,

$$\begin{aligned}&\left(\partial_t^2\mathbf{e_u},\mathbf{v_h}\right)+(\nabla\partial_t\mathbf{e_u},\nabla\mathbf{v_h})+(\partial_t\mathbf{e_u}\cdot\nabla\mathbf{u},\mathbf{v_h})+(\mathbf{e_u}\cdot\nabla\partial_t\mathbf{u},\mathbf{v_h})+(\partial_t\mathbf{u_h}\cdot\nabla\mathbf{e_u},\mathbf{v_h})\\&+(\mathbf{u_h}\cdot\nabla\partial_t\mathbf{e_u},\mathbf{v_h})=G_{r_\theta}\left(\partial_t e_\theta\mathbf{j},\mathbf{v_h}\right)+G_{r_c}\left(\partial_t e_c\mathbf{j},\mathbf{v_h}\right)\ \forall\mathbf{v_h}\in\mathbf{V},\end{aligned} \tag{3.125}$$

$$\begin{aligned}\left(\partial_t^2 e_\theta,\phi_h\right)+\frac{1}{Pr}\left(\nabla\partial_t e_\theta,\nabla\partial_t\phi_h\right)+(\partial_t\mathbf{e_u}\cdot\nabla\theta,\phi_h)+(\mathbf{e_u}\cdot\nabla\partial_t\theta,\phi_h)&\\+(\partial_t\mathbf{u_h}\cdot\nabla e_\theta,\phi_h)+(\mathbf{u_h}\cdot\nabla\partial_t e_\theta,\phi_h)=0,\ \forall\phi_h\in W_\theta^h&\end{aligned} \tag{3.126}$$

$$\left(\partial_t^2 e_c, \psi_h\right) + \frac{1}{PrLe}\left(\nabla\partial_t e_c, \nabla\partial_t\psi_h\right) + \left(\partial_t \mathbf{e_u}\cdot\nabla C, \psi_h\right) + \left(\mathbf{e_u}\cdot\nabla\partial_t C, \psi_h\right)$$
$$+ \left(\partial_t \mathbf{u_h}\cdot\nabla e_c, \psi_h\right) + \left(\mathbf{u_h}\cdot\nabla\partial_t e_c, \psi_h\right) = 0,\ \forall\psi_h\in W_c^h. \tag{3.127}$$

Setting $\mathbf{v_h} = P_h\partial_t\mathbf{e_u}$, $\phi_h = \tilde{P}_h\partial_t e_\theta$ and $\psi_h = \tilde{P}_h\partial_t e_c$ in (3.125)-(3.127) respectively yields,

$$\left(\partial_t^2\mathbf{e_u}, P_h\partial_t\mathbf{e_u}\right) + \left(\nabla\partial_t\mathbf{e_u}, \nabla P_h\partial_t\mathbf{e_u}\right) + \left(\partial_t\mathbf{e_u}\cdot\nabla\mathbf{u}, P_h\partial_t\mathbf{e_u}\right) + \left(\mathbf{e_u}\cdot\nabla\partial_t\mathbf{u}, P_h\partial_t\mathbf{e_u}\right)$$
$$+ \left(\partial_t\mathbf{u_h}\cdot\nabla\mathbf{e_u}, P_h\partial_t\mathbf{e_u}\right) + \left(\mathbf{u_h}\cdot\nabla\partial_t\mathbf{e_u}, P_h\partial_t\mathbf{e_u}\right) = G_{r_\theta}\left(\partial_t e_\theta\mathbf{j}, P_h\partial_t\mathbf{e_u}\right) + G_{r_c}\left(\partial_t e_c\mathbf{j}, P_h\partial_t\mathbf{e_u}\right) \tag{3.128}$$

$$\left(\partial_t^2 e_\theta, \tilde{P}_h\partial_t e_\theta\right) + \frac{1}{Pr}\left(\nabla\partial_t e_\theta, \nabla\partial_t\tilde{P}_h e_\theta\right) + \left(\partial_t\mathbf{e_u}\cdot\nabla\theta, \tilde{P}_h\partial_t e_\theta\right) +$$
$$\left(\mathbf{e_u}\cdot\nabla\partial_t\theta, \tilde{P}_h\partial_t e_\theta\right) + \left(\partial_t\mathbf{u_h}\cdot\nabla e_\theta, \tilde{P}_h\partial_t e_\theta\right) + \left(\mathbf{u_h}\cdot\nabla\partial_t e_\theta, \tilde{P}_h\partial_t e_\theta\right) = 0 \tag{3.129}$$

$$\left(\partial_t^2 e_c, \tilde{P}_h\partial_t e_c\right) + \frac{1}{PrLe}\left(\nabla\partial_t e_c, \nabla\partial_t\tilde{P}_h e_c\right) + \left(\partial_t\mathbf{e_u}\cdot\nabla C, \tilde{P}_h\partial_t e_c\right) +$$
$$\left(\mathbf{e_u}\cdot\nabla\partial_t C, \tilde{P}_h\partial_t e_c\right) + \left(\partial_t\mathbf{u_h}\cdot\nabla e_c, \tilde{P}_h\partial_t e_c\right) + \left(\mathbf{u_h}\cdot\nabla\partial_t e_c, \tilde{P}_h\partial_t e_c\right) = 0. \tag{3.130}$$

First simplify the linear terms.

$$\left(\partial_t^2\mathbf{e_u}, P_h\partial_t\mathbf{e_u}\right) = \left(\partial_t^2\mathbf{e_u}, \partial_t\mathbf{e_u}\right) - \left(\partial_t^2\mathbf{e_u}, \partial_t\mathbf{e_u} - P_h\partial_t\mathbf{e_u}\right)$$
$$= \frac{1}{2}\frac{d}{dt}\,||\partial_t\mathbf{e_u}||^2 - \left(\partial_t^2\mathbf{e_u}, \partial_t(\mathbf{e_u} - P_h\mathbf{e_u})\right) \tag{3.131}$$

$$
\begin{aligned}
(\nabla\partial_t\mathbf{e_u}, \nabla P_h\partial_t\mathbf{e_u}) &= (\nabla\partial_t\mathbf{e_u}, \nabla\partial_t\mathbf{e_u}) - (\nabla\partial_t\mathbf{e_u}, \nabla\partial_t(\mathbf{e_u} - P_h\mathbf{e_u})) \\
&= ||\nabla\partial_t\mathbf{e_u}||^2 - (\nabla\partial_t\mathbf{e_u}, \nabla\partial_t(\mathbf{e_u} - P_h\mathbf{e_u}))\,. \qquad (3.132)
\end{aligned}
$$

Similarly,

$$
\left(\partial_t^2 e_\theta, \tilde{P}_h\partial_t e_\theta\right) = \frac{1}{2}\frac{d}{dt}\,||\partial_t e_\theta||^2 - \left(\partial_t^2 e_\theta, \partial_t(e_\theta - \tilde{P}_h e_\theta)\right) \qquad (3.133)
$$

$$
\left(\partial_t^2 e_c, \tilde{P}_h\partial_t e_c\right) = \frac{1}{2}\frac{d}{dt}\,||\partial_t e_c||^2 - \left(\partial_t^2 e_c, \partial_t(e_c - \tilde{P}_h e_c)\right) \qquad (3.134)
$$

and

$$
\left(\nabla\partial_t e_\theta, \nabla(\tilde{P}_h\partial_t e_\theta)\right) = ||\nabla\partial_t e_\theta||^2 - \left(\nabla\partial_t e_\theta, \nabla\partial_t(e_\theta - \tilde{P}_h e_\theta)\right) \qquad (3.135)
$$

$$
\left(\nabla\partial_t e_c, \nabla(\tilde{P}_h\partial_t e_c)\right) = ||\nabla\partial_t e_c||^2 - \left(\nabla\partial_t e_c, \nabla\partial_t(e_c - \tilde{P}_h e_c)\right). \qquad (3.136)
$$

Using (3.131)-(3.136) in (3.128)-(3.130) and moving all the nonlinear terms to the right-hand side yields,

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\,||\partial_t\mathbf{e_u}||^2 + ||\nabla\partial_t\mathbf{e_u}||^2 &\le |(\partial_t\mathbf{e_u}\cdot\nabla\mathbf{u}, P_h\partial_t\mathbf{e_u})| + |(\mathbf{e_u}\cdot\nabla\partial_t\mathbf{u}, P_h\partial_t\mathbf{e_u})| \\
&+ |(\partial_t\mathbf{u_h}\cdot\nabla\mathbf{e_u}, P_h\partial_t\mathbf{e_u})| + |(\mathbf{u_h}\cdot\nabla\partial_t\mathbf{e_u}, P_h\partial_t\mathbf{e_u})| \\
&+ G_{r_\theta}\,(\partial_t e_\theta\mathbf{j}, P_h\partial_t\mathbf{e_u}) + G_{r_c}\,(\partial_t e_c\mathbf{j}, P_h\partial_t\mathbf{e_u}) \\
&+ \left(\partial_t^2\mathbf{e_u}, \partial_t(\mathbf{e_u} - P_h\mathbf{e_u})\right) + (\nabla\partial_t\mathbf{e_u}, \nabla\partial_t(\mathbf{e_u} - P_h\mathbf{e_u}))\,, \qquad (3.137)
\end{aligned}
$$

$$\frac{1}{2}\frac{d}{dt}\left|\left|\partial_t e_\theta\right|\right|^2 + \frac{1}{Pr}\left|\left|\nabla \partial_t e_\theta\right|\right|^2 \leq \left|\left(\partial_t \mathbf{e_u}\cdot\nabla\theta, \tilde{P}_h\partial_t e_\theta\right)\right| + \left|\left(\mathbf{e_u}\cdot\nabla\partial_t\theta, \tilde{P}_h\partial_t e_\theta\right)\right|$$
$$+ \left|\left(\partial_t \mathbf{u_h}\cdot\nabla e_\theta, \tilde{P}_h\partial_t e_\theta\right)\right| + \left|\left(\mathbf{u_h}\cdot\nabla\partial_t e_\theta, \tilde{P}_h\partial_t e_\theta\right)\right|$$
$$+ \left(\partial_t^2 e_\theta, \partial_t(e_\theta - \tilde{P}_h e_\theta)\right) + \frac{1}{Pr}\left(\nabla\partial_t e_\theta, \nabla\partial_t(e_\theta - \tilde{P}_h e_\theta)\right) \tag{3.138}$$

and

$$\frac{1}{2}\frac{d}{dt}\left|\left|\partial_t e_c\right|\right|^2 + \frac{1}{PrLe}\left|\left|\nabla \partial_t e_c\right|\right|^2 \leq \left|\left(\partial_t \mathbf{e_u}\cdot\nabla C, \tilde{P}_h\partial_t e_c\right)\right| + \left|\left(\mathbf{e_u}\cdot\nabla\partial_t C, \tilde{P}_h\partial_t e_c\right)\right|$$
$$+ \left|\left(\partial_t \mathbf{u_h}\cdot\nabla e_c, \tilde{P}_h\partial_t e_c\right)\right| + \left|\left(\mathbf{u_h}\cdot\nabla\partial_t e_c, \tilde{P}_h\partial_t e_c\right)\right|$$
$$+ \left(\partial_t^2 e_c, \partial_t(e_\theta - \tilde{P}_h e_c)\right) + \frac{1}{PrLe}\left(\nabla\partial_t e_c, \nabla\partial_t(e_\theta - \tilde{P}_h e_c)\right). \tag{3.139}$$

Start by estimating the first four nonlinear terms on the right hand side of (3.137) beginning with $|(\partial_t \mathbf{e_u}\cdot\nabla\mathbf{u}, P_h\partial_t\mathbf{e_u})|$. Applying Holders, Ladyzhenskaya and Gagliardo inequalities and using the a priori estimates yields,

$$\begin{aligned}
|(\partial_t \mathbf{e_u}\cdot\nabla\mathbf{u}, P_h\partial_t\mathbf{e_u})| &\leq \left|\left|\partial_t\mathbf{e_u}\right|\right|_{L^4}\left|\left|\nabla\mathbf{u}\right|\right|_{L^4}\left|\left|P_h\partial_t\mathbf{e_u}\right|\right| \\
&\leq \left|\left|\partial_t\mathbf{e_u}\right|\right|_{L^4}\left|\left|\partial_t\mathbf{e_u}\right|\right|\left|\left|\nabla\mathbf{u}\right|\right|_{L^4} \\
&\leq K\left|\left|\partial_t\mathbf{e_u}\right|\right|^{1/2}\left|\left|\nabla\partial_t\mathbf{e_u}\right|\right|^{1/2}\left|\left|\partial_t\mathbf{e_u}\right|\right|\left|\left|\nabla\mathbf{u}\right|\right|^{1/2}\left|\left|\mathbf{u}\right|\right|_2^{1/2} \\
&\leq K\left|\left|\partial_t\mathbf{e_u}\right|\right|^{3/2}\left|\left|\nabla\partial_t\mathbf{e_u}\right|\right|^{1/2}\left|\left|\nabla\mathbf{u}\right|\right|^{1/2}\left|\left|\mathbf{u}\right|\right|_2^{1/2} \\
&\leq K\left|\left|\partial_t\mathbf{e_u}\right|\right|^{3/2}\left|\left|\nabla\partial_t\mathbf{e_u}\right|\right|^{1/2}.
\end{aligned}$$

Using Young's inequality,

$$|(\partial_t \mathbf{e_u} \cdot \nabla \mathbf{u}, P_h \partial_t \mathbf{e_u})| \leq \epsilon \, ||\nabla \partial_t \mathbf{e_u}||^2 + K \, ||\partial_t \mathbf{e_u}||^2 \, . \tag{3.140}$$

Similarly for the term $|(\mathbf{u_h} \cdot \nabla \partial_t \mathbf{e_u}, P_h \partial_t \mathbf{e_u})|$,

$$\begin{aligned} |(\mathbf{u_h} \cdot \nabla \partial_t \mathbf{e_u}, P_h \partial_t \mathbf{e_u})| &\leq ||\mathbf{u_h}||_{L^4} \, ||\nabla \partial_t \mathbf{e_u}|| \, ||P_h \partial_t \mathbf{e_u}||_{L^4} \\ &\leq ||\mathbf{u_h}||^{1/2} \, ||\nabla \mathbf{u_h}||^{1/2} \, ||\nabla \partial_t \mathbf{e_u}|| \, ||P_h \partial_t \mathbf{e_u}||^{1/2} \, ||P_h \nabla \partial_t \mathbf{e_u}||^{1/2} \\ &\leq K \, ||\partial_t \mathbf{e_u}||^{1/2} \, ||\nabla \partial_t \mathbf{e_u}||^{3/2} \, . \end{aligned}$$

Using Young's inequality,

$$|(\mathbf{u_h} \cdot \nabla \mathbf{e_u}, P_h \partial_t \mathbf{e_u})| \leq \epsilon \, ||\nabla \partial_t \mathbf{e_u}||^2 + K \, ||\partial_t \mathbf{e_u}||^2 \, . \tag{3.141}$$

Next estimate the term $|(\mathbf{e_u} \cdot \nabla \partial_t \mathbf{u}, P_h \partial_t \mathbf{e_u})|$ using Holder's and Ladyzhenskaya inequalities.

$$\begin{aligned} |(\mathbf{e_u} \cdot \nabla \partial_t \mathbf{u}, P_h \partial_t \mathbf{e_u})| &\leq ||\mathbf{e_u}||_{L^4} \, ||\nabla \partial_t \mathbf{u}|| \, ||P_h \partial_t \mathbf{e_u}||_{L^4} \\ &\leq K \, ||\mathbf{e_u}||^{1/2} \, ||\nabla \mathbf{e_u}||^{1/2} \, ||\nabla \partial_t \mathbf{u}|| \, ||P_h \partial_t \mathbf{e_u}||^{1/2} \, ||\nabla (P_h \partial_t \mathbf{e_u})||^{1/2} \\ &\leq K \, ||\nabla \mathbf{e_u}|| \, ||\nabla \partial_t \mathbf{u}|| \, ||\partial_t \mathbf{e_u}||^{1/2} \, ||\nabla \partial_t \mathbf{e_u}||^{1/2} \, . \end{aligned}$$

By Theorem 3.11, $||\nabla \mathbf{e_u}|| \leq Kh$. So, by Young's inequality,

$$|(\mathbf{e_u} \cdot \nabla \partial_t \mathbf{u}, P_h \partial_t \mathbf{e_u})| \leq Kh^2 \, ||\nabla \partial_t \mathbf{u}||^2 + K \, ||\partial_t \mathbf{e_u}||^2 + \epsilon \, ||\nabla (\partial_t \mathbf{e_u})||^2 \, . \tag{3.142}$$

Next, to estimate $|(\partial_t \mathbf{u_h} \cdot \nabla \mathbf{e_u}, P_h \partial_t \mathbf{e_u})|$, apply Holder's, Ladyzhenskaya and Poincare inequalities and use Proposition 3.1 to bound $||\partial_t \mathbf{u_h}||^{1/2}$,

$$\begin{aligned}
|(\partial_t \mathbf{u_h} \cdot \nabla \mathbf{e_u}, P_h \partial_t \mathbf{e_u})| &\leq K\, ||\partial_t \mathbf{u_h}||_{L^4}\, ||\nabla \mathbf{e_u}||\, ||P_h \partial_t \mathbf{e_u}||_{L^4} \\
&\leq K\, ||\partial_t \mathbf{u_h}||^{1/2}\, ||\nabla(\partial_t \mathbf{u_h})||^{1/2}\, ||\nabla \mathbf{e_u}||\, ||P_h \partial_t \mathbf{e_u}||^{1/2}\, ||\nabla(P_h \partial_t \mathbf{e_u})||^{1/2} \\
&\leq Kh\, ||\partial_t \mathbf{u_h}||^{1/2}\, ||\nabla(\partial_t \mathbf{u_h})||^{1/2}\, ||\partial_t \mathbf{e_u}||^{1/2}\, ||\nabla \partial_t \mathbf{e_u}||^{1/2} \\
&\leq Kh\, ||\nabla(\partial_t \mathbf{u_h})||\, ||\partial_t \mathbf{e_u}||^{1/2}\, ||\nabla \partial_t \mathbf{e_u}||^{1/2}.
\end{aligned}$$

Applying Young's inequality yields

$$\begin{aligned}
|(\partial_t \mathbf{u_h} \cdot \nabla \mathbf{e_u}, P_h \partial_t \mathbf{e_u})| &\leq Kh^2\, ||\nabla(\partial_t \mathbf{u_h})||^2 + \frac{1}{2}\, ||\partial_t \mathbf{e_u}||\, ||\nabla \partial_t \mathbf{e_u}|| \\
&\leq Kh^2\, ||\nabla(\partial_t \mathbf{u_h})||^2 + K\, ||\partial_t \mathbf{e_u}||^2 + \epsilon\, ||\nabla \partial_t \mathbf{e_u}||^2. \qquad (3.143)
\end{aligned}$$

Now estimate $G_{r_\theta}(\partial_t e_\theta \mathbf{j}, P_h \partial_t \mathbf{e_u})$ using Cauchy, Poincare and Young's inequality,

$$G_{r_\theta}(\partial_t e_\theta \mathbf{j}, P_h \partial_t \mathbf{e_u}) \leq \epsilon\, ||\nabla \partial_t \mathbf{e_u}||^2 + K\, ||\partial_t e_\theta||^2. \qquad (3.144)$$

Similarly,

$$G_{r_c}(\partial_t e_c \mathbf{j}, P_h \partial_t \mathbf{e_u}) \leq \epsilon\, ||\nabla \partial_t \mathbf{e_u}||^2 + K\, ||\partial_t e_c||^2. \qquad (3.145)$$

For the term $(\nabla \partial_t \mathbf{e_u}, \nabla \partial_t (\mathbf{e_u} - P_h \mathbf{e_u}))$ note that $\mathbf{e_u} - P_h \mathbf{e_u} = \mathbf{u} - P_h \mathbf{u}$. So,

$$\begin{aligned}
|(\nabla \partial_t \mathbf{e_u}, \nabla \partial_t (\mathbf{e_u} - P_h \mathbf{e_u}))| &\leq ||\nabla \partial_t \mathbf{e_u}||\, ||\nabla \partial_t (\mathbf{u} - P_h \mathbf{u})|| \\
&\leq \epsilon\, ||\nabla \partial_t \mathbf{e_u}||^2 + K\, ||\nabla \partial_t (\mathbf{u} - P_h \mathbf{u})||^2.
\end{aligned}$$

Therefore by the approximation property in (3.6), we have

$$|(\nabla\partial_t\mathbf{e_u}, \nabla\partial_t(\mathbf{e_u} - P_h\mathbf{e_u}))| \le \epsilon \, ||\nabla\partial_t\mathbf{e_u}||^2 + Kh^2 \, ||\partial_t\mathbf{u}||_2^2 \,. \tag{3.146}$$

For the term $\left(\partial_t^2\mathbf{e_u}, \partial_t(\mathbf{e_u} - P_h\mathbf{e_u})\right)$, using $\mathbf{e_u} - P_h\mathbf{e_u} = \mathbf{u} - P_h\mathbf{u}$ we get,

$$\begin{aligned}\left(\partial_t^2\mathbf{e_u}, \partial_t(\mathbf{e_u} - P_h\mathbf{e_u})\right) &= \left(\partial_t^2(\mathbf{u} - \mathbf{u_h}), \partial_t(\mathbf{u} - P_h\mathbf{u})\right)\\ &= \left(\partial_t^2\mathbf{u} - \partial_t^2\mathbf{u_h}, \partial_t\mathbf{u} - P_h\partial_t\mathbf{u}\right)\\ &= \left(\partial_t^2\mathbf{u}, \partial_t\mathbf{u} - P_h\partial_t\mathbf{u}\right) - \left(\partial_t^2\mathbf{u_h}, \partial_t\mathbf{u} - P_h\partial_t\mathbf{u}\right).\end{aligned}$$

Since $\partial_t^2\mathbf{u_h} \in \mathbf{V^h}$, $\left(\partial_t^2\mathbf{u_h}, \partial_t\mathbf{u} - P_h\partial_t\mathbf{u}\right) = 0$, thus, $\left(\partial_t^2\mathbf{e_u}, \partial_t(\mathbf{e_u} - P_h\mathbf{e_u})\right) = \left(\partial_t^2\mathbf{u}, \partial_t\mathbf{u} - P_h\partial_t\mathbf{u}\right)$. But note that $P_h\partial_t^2\mathbf{u} \in \mathbf{V^h}$ also implies $\left(P_h\partial_t^2\mathbf{u}, \partial_t\mathbf{u} - P_h\partial_t\mathbf{u}\right) = 0$. Thus,

$$\begin{aligned}\left(\partial_t^2\mathbf{e_u}, \partial_t(\mathbf{e_u} - P_h\mathbf{e_u})\right) &= \left(\partial_t^2\mathbf{u}, \partial_t\mathbf{u} - P_h\partial_t\mathbf{u}\right) - \left(P_h\partial_t^2\mathbf{u_h}, \partial_t\mathbf{u} - P_h\partial_t\mathbf{u}\right)\\ &= \left(\partial_t^2(\mathbf{u} - P_h\mathbf{u}), \partial_t(\mathbf{u} - P_h\mathbf{u})\right)\\ &= \frac{1}{2}\frac{d}{dt}\, ||\partial_t(\mathbf{u} - P_h\mathbf{u})||^2 \,.\end{aligned} \tag{3.147}$$

Setting $\epsilon = \frac{1}{14}$ in (3.140)-(3.147) substituting into (3.137) yields

$$\begin{aligned}\frac{d}{dt}\, ||\partial_t\mathbf{e_u}||^2 + ||\nabla\partial_t\mathbf{e_u}||^2 &\le K\, ||\partial_t\mathbf{e_u}||^2 + K\, ||\partial_t e_\theta||^2 + K\, ||\partial_t e_c||^2\\ &+ Kh^2\, ||\nabla\partial_t\mathbf{u}||^2 + \frac{d}{dt}\, ||\partial_t(\mathbf{u} - P_h\mathbf{u})||^2 + Kh^2\, ||\partial_t\mathbf{u}||_2^2 \,.\end{aligned} \tag{3.148}$$

Next estimate the terms on the right hand side of (3.138) . First, let $\mathbf{e} = (\mathbf{e_u}, e_\theta, e_c)$. Beginning with $\left|\left(\partial_t\mathbf{e_u}\cdot\nabla\theta, \tilde{P}_h\partial_t e_\theta\right)\right|$, apply Holder's, Ladyzhenskaya and Gagliardo inequalities and using the a priori estimates.

$$\begin{aligned}\left|\left(\partial_t \mathbf{e_u} \cdot \nabla\theta, \tilde{P}_h \partial_t e_\theta\right)\right| &\leq ||\partial_t \mathbf{e_u}||_{L^4} ||\nabla\theta||_{L^4} \left|\left|\tilde{P}_h \partial_t e_\theta\right|\right| \\ &\leq K ||\partial_t \mathbf{e_u}||^{1/2} ||\nabla\partial_t \mathbf{e_u}||^{1/2} ||\nabla\theta||^{1/2} ||\theta||_2^{1/2} ||\partial_t e_\theta|| \\ &\leq K ||\partial_t \mathbf{e}||^{3/2} ||\nabla\partial_t \mathbf{e_u}||^{1/2}.\end{aligned}$$

Using Young's inequality,

$$\left|\left(\partial_t \mathbf{e_u} \cdot \nabla\theta, \tilde{P}_h \partial_t e_\theta\right)\right| \leq \epsilon ||\nabla\partial_t \mathbf{e_u}||^2 + K ||\partial_t \mathbf{e}||^2. \tag{3.149}$$

To estimate $\left|\left(\mathbf{u_h} \cdot \nabla\partial_t e_\theta, \tilde{P}_h \partial_t e_\theta\right)\right|$ apply Holder's, Poincare, Ladyzhenskaya and Gagliardo inequalities and use the a priori estimates.

$$\begin{aligned}\left|\left(\mathbf{u_h} \cdot \nabla\partial_t e_\theta, \tilde{P}_h \partial_t e_\theta\right)\right| &\leq ||\mathbf{u_h}||_{L^4} ||\nabla\partial_t e_\theta|| \left|\left|\tilde{P}_h \partial_t e_\theta\right|\right|_{L^4} \\ &\leq K ||\mathbf{u_h}||^{1/2} ||\nabla\mathbf{u_h}||^{1/2} ||\nabla\partial_t e_\theta|| \left|\left|\nabla\tilde{P}_h \partial_t e_\theta\right|\right|^{1/2} \left|\left|\tilde{P}_h \partial_t e_\theta\right|\right|^{1/2} \\ &\leq K ||\nabla\partial_t e_\theta|| \, ||\nabla\partial_t e_\theta||^{1/2} ||\partial_t e_\theta||^{1/2} \\ &\leq K ||\nabla\partial_t e_\theta||^{3/2} ||\partial_t e_\theta||^{1/2} \\ &\leq \frac{\delta}{Pr} ||\nabla\partial_t e_\theta||^2 + K ||\partial_t e_\theta||^2.\end{aligned} \tag{3.150}$$

Next we estimate $\left|\left(\mathbf{e_u}\cdot\nabla\partial_t\theta,\tilde{P}_h\partial_t e_\theta\right)\right|$ using Holder's and Ladyzhenskaya inequalities.

$$\begin{aligned}\left|\left(\mathbf{e_u}\cdot\nabla\partial_t\theta,\tilde{P}_h\partial_t e_\theta\right)\right| &\le ||\mathbf{e_u}||_{L^4}\,||\nabla\partial_t\theta||\,\left|\left|\tilde{P}_h\partial_t e_\theta\right|\right|_{L^4}\\ &\le K\,||\mathbf{e_u}||^{1/2}\,||\nabla\mathbf{e_u}||^{2}\,||\nabla\partial_t\theta||\,\left|\left|\nabla(\tilde{P}_h\partial_t e_\theta)\right|\right|^{1/2}\left|\left|\tilde{P}_h\partial_t e_\theta\right|\right|^{1/2}\\ &\le K\,||\mathbf{e_u}||^{1/2}\,||\nabla\mathbf{e_u}||^{2}\,||\nabla\partial_t\theta||\,||\nabla\partial_t e_\theta||^{1/2}\,||\partial_t e_\theta||^{1/2}.\end{aligned}$$

By Theorem 3.11, $||\nabla\mathbf{e_u}||^{1/2}\le Kh^{1/2}$. So, by Young's inequality,

$$\left|\left(\mathbf{e_u}\cdot\nabla\partial_t\theta,\tilde{P}_h\partial_t e_\theta\right)\right|\le Kh^2\,||\nabla\partial_t\theta||^2+\frac{1}{4}\,||\partial_t e_\theta||^2+\frac{\delta}{Pr}\,||\nabla\partial_t e_\theta||^2. \tag{3.151}$$

Applying Holder's and Ladyzhenskaya inequalities to the term $\left|\left(\partial_t\mathbf{u_h}\cdot\nabla e_\theta,\tilde{P}_h\partial_t e_\theta\right)\right|$ yields,

$$\begin{aligned}\left|\left(\partial_t\mathbf{u_h}\cdot\nabla e_\theta,\tilde{P}_h\partial_t e_\theta\right)\right| &\le ||\partial_t\mathbf{u_h}||_{L^4}\,||\nabla e_\theta||\,\left|\left|\tilde{P}_h\partial_t e_\theta\right|\right|_{L^4}\\ &\le K\,||\partial_t\mathbf{u_h}||^{1/2}\,||\nabla(\partial_t\mathbf{u_h})||^{1/2}\,||\nabla e_\theta||\,\left|\left|\tilde{P}_h\partial_t e_\theta\right|\right|^{1/2}\left|\left|\nabla(\tilde{P}_h\partial_t e_\theta)\right|\right|^{1/2}\\ &\le Kh\,||\nabla(\partial_t\mathbf{u_h})||\,||\partial_t e_\theta||^{1/2}\,||\nabla\partial_t e_\theta||^{1/2}.\end{aligned}$$

Applying Young's inequality gives,

$$\left|\left(\partial_t\mathbf{u_h}\cdot\nabla e_\theta,\tilde{P}_h\partial_t e_\theta\right)\right|\le Kh^2\,||\nabla\partial_t\mathbf{u_h}||^2+\frac{1}{4}\,||\partial_t e_\theta||^2+\frac{\delta}{Pr}\,||\nabla\partial_t e_\theta||^2. \tag{3.152}$$

Using $e_\theta - \tilde{P}_h e_\theta = \theta - \tilde{P}_h\theta$, we have,

$$\begin{aligned}\frac{1}{Pr}\left(\nabla\partial_t e_\theta, \nabla\partial_t(e_\theta - \tilde{P}_h e_\theta)\right) &\leq \frac{1}{Pr}\,||\nabla\partial_t e_\theta||\,\left|\left|\nabla\partial_t(\theta - \tilde{P}_h\theta)\right|\right| \\ &\leq \frac{\delta}{Pr}\,||\nabla\partial_t e_\theta||^2 + K\left|\left|\nabla\partial_t(\theta - \tilde{P}_h\theta)\right|\right|^2 \\ &\leq \frac{\delta}{Pr}\,||\nabla\partial_t e_\theta||^2 + Kh^2\,||\partial_t\theta||_2^2\,. \end{aligned} \tag{3.153}$$

For the term $\left(\partial_t^2 e_\theta, \partial_t(e_\theta - \tilde{P}_h e_\theta)\right)$ first note that by definition $\left(\tilde{P}_h\theta, \phi_h\right) = (\theta, \phi_h)$ for all $\phi_h \in W_h$. Thus, $\left(\tilde{P}_h\theta - \theta, \phi_h\right) = 0$, $\forall\phi_h \in W_h$. Using $e_\theta - \tilde{P}_h e_\theta = \theta - \tilde{P}_h\theta$ we get,

$$\begin{aligned}\left(\partial_t^2 e_\theta, \partial_t(e_\theta - \tilde{P}_h e_\theta)\right) &= \left(\partial_t^2(\theta - \theta_h), \partial_t(\theta - \tilde{P}_h\theta)\right) \\ &= \left(\partial_t^2\theta - \partial_t^2\theta_h, \partial_t\theta - \tilde{P}_h\partial_t\theta\right) \\ &= \left(\partial_t^2\theta, \partial_t\theta - \tilde{P}_h\partial_t\theta\right) - \left(\partial_t^2\theta_h, \partial_t\theta - \tilde{P}_h\partial_t\theta\right). \end{aligned}$$

Also, note $\partial_t^2\theta_h \in W_h$, $\left(\partial_t^2\theta_h, \partial_t\theta - P_h\partial_t\theta\right) = 0$. This yields,

$$\left(\partial_t^2 e_\theta, \partial_t(e_\theta - \tilde{P}_h e_\theta)\right) = \left(\partial_t^2\theta, \partial_t\theta - \tilde{P}_h\partial_t\theta\right).$$

But $\tilde{P}_h\partial_t^2\theta \in W_h$ also implies $\left(\tilde{P}_h\partial_t^2\theta, \partial_t\theta - \tilde{P}_h\partial_t\theta\right) = 0$. Thus,

$$\begin{aligned}\left(\partial_t^2 e_\theta, \partial_t(e_\theta - \tilde{P}_h e_\theta)\right) &= \left(\partial_t^2\theta, \partial_t\theta - \tilde{P}_h\partial_t\theta\right) - \left(\tilde{P}_h\partial_t^2\theta, \partial_t\theta - \tilde{P}_h\partial_t\theta\right) \\ &= \left(\partial_t^2(\theta - \tilde{P}_h\theta), \partial_t(\theta - \tilde{P}_h\theta)\right) = \frac{1}{2}\frac{d}{dt}\left|\left|\partial_t(\theta - \tilde{P}_h\theta)\right|\right|^2. \end{aligned} \tag{3.154}$$

Setting $\delta = \frac{1}{8}$ in (3.149)-(3.154) and using these estimates in (3.138), we get

$$\frac{d}{dt} \|\partial_t e_\theta\|^2 + \frac{1}{Pr} \|\nabla \partial_t e_\theta\|^2 \leq \epsilon \|\nabla \partial_t \mathbf{e_u}\|^2 + K \|\partial_t \mathbf{e}\|^2 + Kh^2 \|\nabla \partial_t \theta\|^2 + Kh^2 \|\nabla \partial_t \mathbf{u_h}\|^2 + \frac{d}{dt} \left\|\partial_t(\theta - \tilde{P}_h \theta)\right\|^2 + Kh^2 \|\partial_t \theta\|_2^2 . \tag{3.155}$$

Using similar estimates in (3.139) yields

$$\frac{d}{dt} \|\partial_t e_c\|^2 + \frac{1}{PrLe} \|\nabla \partial_t e_c\|^2 \leq \epsilon \|\nabla \partial_t \mathbf{e_u}\|^2 + K \|\partial_t \mathbf{e}\|^2 + Kh^2 \|\nabla \partial_t C\|^2 + Kh^2 \|\nabla \partial_t \mathbf{u_h}\|^2 + \frac{d}{dt} \left\|\partial_t(C - \tilde{P}_h C)\right\|^2 + Kh^2 \|\partial_t C\|_2^2 . \tag{3.156}$$

Setting $\epsilon = \frac{1}{4}$ in (3.155) and (3.156), summing (3.148), (3.155) and (3.156) and multiplying both sides of the resulting inequality by t yields,

$$t\frac{d}{dt} \|\partial_t \mathbf{e}\|^2 + \alpha t \|\nabla \partial_t \mathbf{e}\|^2 \leq Kt \|\partial_t \mathbf{e}\|^2 + Kh^2 t \left(\|\partial_t \mathbf{u}\|_2^2 + \|\partial_t \theta\|_2^2 + \|\partial_t C\|_2^2\right) + Kh^2 t \left(\|\nabla \partial_t \theta\|^2 + \|\nabla \partial_t C\|^2 + \|\nabla \partial_t \mathbf{u}\|^2 + \|\nabla \partial_t \mathbf{u_h}\|^2\right) + t\frac{d}{dt} \|\partial_t(\mathbf{u} - P_h \mathbf{u})\|^2 + t\frac{d}{dt} \left\|\partial_t(\theta - \tilde{P}_h \theta)\right\|^2 + t\frac{d}{dt} \left\|\partial_t(C - \tilde{P}_h C)\right\|^2 , \tag{3.157}$$

where $\alpha = \min\left\{\frac{1}{2}, \frac{1}{Pr}, \frac{1}{PrLe}\right\}$. Note that by the product rule,

$$\frac{d}{dt}\left(t \|\partial_t \mathbf{e}\|^2\right) = \|\partial_t \mathbf{e}\|^2 + t\frac{d}{dt} \|\partial_t \mathbf{e}\|^2 \Rightarrow t\frac{d}{dt} \|\partial_t \mathbf{e}\|^2 = \frac{d}{dt}\left(t \|\partial_t \mathbf{e}\|^2\right) - \|\partial_t \mathbf{e}\|^2 .$$

Likewise,

$$t\frac{d}{dt} \|\partial_t(\mathbf{u} - P_h \mathbf{u})\|^2 = \frac{d}{dt}\left(t \|\partial_t(\mathbf{u} - P_h \mathbf{u})\|^2\right) - \|\partial_t(\mathbf{u} - P_h \mathbf{u})\|^2$$

$$t\frac{d}{dt}\left|\left|\partial_t(\theta - \tilde{P}_h\theta)\right|\right|^2 = \frac{d}{dt}\left(t\left|\left|\partial_t(\theta - \tilde{P}_h\theta)\right|\right|^2\right) - \left|\left|\partial_t(\theta - \tilde{P}_h\theta)\right|\right|^2$$

and

$$t\frac{d}{dt}\left|\left|\partial_t(C - \tilde{P}_hC)\right|\right|^2 = \frac{d}{dt}\left(t\left|\left|\partial_t(C - \tilde{P}_hC)\right|\right|^2\right) - \left|\left|\partial_t(C - \tilde{P}_hC)\right|\right|^2.$$

Thus (3.157) can be written

$$\begin{aligned}
&\frac{d}{dt}\left(t\,||\partial_t\mathbf{e}||^2\right) + \alpha t\,||\nabla\partial_t\mathbf{e}||^2 \le Kt\,||\partial_t\mathbf{e}||^2 + Kh^2t\left(||\partial_t\mathbf{u}||_2^2 + ||\partial_t\theta||_2^2 + ||\partial_tC||_2^2\right) \\
&+ Kh^2t\left(||\nabla\partial_t\theta||^2 + ||\nabla\partial_tC||^2 + ||\nabla\partial_t\mathbf{u}||^2 + ||\nabla\partial_t\mathbf{u_h}||^2\right) \\
&+ \frac{d}{dt}\left(t\,||\partial_t(\mathbf{u} - P_h\mathbf{u})||^2\right) + \frac{d}{dt}\left(t\left|\left|\partial_t(\theta - \tilde{P}_h\theta)\right|\right|^2\right) + \frac{d}{dt}\left(t\left|\left|\partial_t(C - \tilde{P}_hC)\right|\right|^2\right) \\
&- ||\partial_t(\mathbf{u} - P_h\mathbf{u})||^2 - \left|\left|\partial_t(\theta - \tilde{P}_h\theta)\right|\right|^2 - \left|\left|\partial_t(C - \tilde{P}_hC)\right|\right|^2. \qquad (3.158)
\end{aligned}$$

Applying the continous gronwall Lemma 3.3 to (3.158) yields,

$$\begin{aligned}
t\,||\partial_t\mathbf{e}||^2 + \alpha\int_0^t s\,||\nabla\partial_t\mathbf{e}||^2\;ds &\le Kh^2\int_0^t s\left(||\partial_t\mathbf{u}||_2^2 + ||\partial_t\theta||_2^2 + ||\partial_tC||_2^2\right)\;ds \\
&+ Kh^2\int_0^t s\left(||\nabla\partial_t\theta||^2 + ||\nabla\partial_tC||^2 + ||\nabla\partial_t\mathbf{u}||^2 + ||\nabla\partial_t\mathbf{u_h}||^2\right)\;ds \\
&+ t\,||\partial_t(\mathbf{u} - P_h\mathbf{u})||^2 + t\left|\left|\partial_t(\theta - \tilde{P}_h\theta)\right|\right|^2 + t\left|\left|\partial_t(C - \tilde{P}_hC)\right|\right|^2. \qquad (3.159)
\end{aligned}$$

By a priori bounds of the exact solution (2.1), (2.2), (2.3) and (2.4) and a priori bounds of the semi-discrete solution (3.17)

$$\int_0^t s\left(||\partial_t\mathbf{u}||_2^2 + ||\partial_t\theta||_2^2 + ||\partial_tC||_2^2\right)\;ds \le K$$

and

$$\int_0^t s\left(||\nabla\partial_t\theta||^2 + ||\nabla\partial_t C||^2 + ||\nabla\partial_t \mathbf{u}||^2 + ||\nabla\partial_t \mathbf{u_h}||^2\right)\, ds \leq K.$$

By Assumption B (3.2),

$$t\,||\partial_t(\mathbf{u} - P_h\mathbf{u})||^2 + t\left|\left|\partial_t(\theta - \tilde{P}_h\theta)\right|\right|^2 + t\left|\left|\partial_t(C - \tilde{P}_h C)\right|\right|^2 \leq tKh^2\left(||\partial_t\mathbf{u}||_1^2 + ||\partial_t\theta||_1^2 + ||\partial_t C||_1^2\right).$$

But by (2.5),

$$t\left(||\partial_t\mathbf{u}||_1^2 + ||\partial_t\theta||_1^2 + ||\partial_t C||_1^2\right) \leq K.$$

Using these results in (3.159), we get

$$t\,||\partial_t\mathbf{e}||^2 + \alpha\int_0^t s\,||\nabla\partial_t\mathbf{e}||^2\, ds \leq Kh^2, \tag{3.160}$$

which implies

$$||\partial_t\mathbf{e}|| \leq \frac{Kh}{\sqrt{t}}.$$

□

Theorem 3.14. *The pressure approximation satisfies the following error estimate:*

$$||p - p_h|| \leq \frac{Kh}{\sqrt{t}},$$

for $t > 0$

Proof. The result follows by combining Lemma 3.12-Lemma 3.13 and the results $||e_\theta|| \leq Kh^2$ and $||e_c|| \leq Kh^2$ from Theorem 3.11. □

CHAPTER 4

TIME DISCRETIZATION APPROXIMATIONS

4.1 Overview

In Chapter 3, error estimates were derived for the semi-discrete spatial discretization. This chapter considers three backward Euler time discretization schemes: *fully implicit*, *semi-implicit* and *semi-implicit decoupled*. When these schemes are applied to the semi-discrete equations (3.13), (3.14) and (3.15), we get a fully discrete approximation.

The *fully implicit* scheme requires that a non-linear system be solved, using for example Newton's method, at each time step. The *semi-implicit* scheme, however, approximates this non-linear system by linearizing it thereby requiring only a linear system be solved at each time step. The *semi-implicit decoupled* scheme is a variation of the *semi-implicit* scheme that decouples the momentum equation from the two transport equations giving the linear system a block structure that can be exploited by a parallel programming implementation.

In this chapter, stability, consistency, convergence and pressure error results are derived for the *fully implicit*, *semi-implicit* and *semi-implicit decoupled* time discretization schemes. The convergence results will require discrete versions of Gronwall's Lemma. The *fully implicit* and *semi-implicit* schemes use a version that establishes conditional convergence, whereas the *semi-implicit decoupled* scheme uses a version of the discrete Gronwall Lemma yielding unconditional convergence.

All three schemes will assume a uniform partition, $t_0 < t_1 < \cdots < t_N$, of $[0, T]$ with $\Delta t = T/N$.

4.2 Fully Implicit Backward Euler Scheme

We introduce the fully implicit backward euler scheme as follows:
seek $(\mathbf{u_h}^{n+1}, \theta_h^{n+1}, C_h^{n+1}) \in \mathbf{V^h} \times W^h \times W^h$ such that

$$\left(\frac{\mathbf{u_h}^{n+1} - \mathbf{u_h}^n}{\Delta t}, \mathbf{v}\right) + \left(\mathbf{u_h}^{n+1} \cdot \nabla \mathbf{u_h}^{n+1}, \mathbf{v}\right) + \left(\nabla \mathbf{u_h}^{n+1}, \nabla \mathbf{v}\right) = \left(\mathbf{f}^{n+1}, \mathbf{v}\right) + G_{r_\theta}\left(\theta_h^{n+1}\mathbf{j}, \mathbf{v}\right) + G_{r_c}\left(C_h^{n+1}\mathbf{j}, \mathbf{v}\right) \quad (4.1)$$

$$\left(\frac{\theta_h^{n+1} - \theta_h^n}{\Delta t}, \phi\right) + \left(\mathbf{u_h}^{n+1} \cdot \nabla \theta_h^{n+1}, \phi\right) + \frac{1}{Pr}\left(\nabla \theta_h^{n+1}, \nabla \phi\right) = \left(Q^{n+1}, \phi\right) \quad (4.2)$$

$$\left(\frac{C_h^{n+1} - C_h^n}{\Delta t}, \psi\right) + \left(\mathbf{u_h}^{n+1} \cdot \nabla C_h^{n+1}, \psi\right) + \frac{1}{PrLe}\left(\nabla C_h^{n+1}, \nabla \psi\right) = \left(\hat{Q}^{n+1}, \psi\right) \quad (4.3)$$

$$\forall (\mathbf{v}, \phi, \psi) \in \mathbf{V^h} \times W^h \times W^h \quad , \quad n = 0, 1, \ldots, N-1,$$

and

$$\mathbf{u_h}^0 = \mathbf{u_h}(0),\ \theta_h^0 = \theta_h(0) \text{ and } C_h^0 = C_h(0),$$

where

$$f^{n+1} = f(t_{n+1}) \quad , \quad Q^{n+1} = Q(t_{n+1}) \quad \text{and} \quad \hat{Q}^{n+1} = \hat{Q}(t_{n+1}).$$

4.2.1 Fully Implicit: Stability Bounds

We now present a priori stability bounds in the following lemma.

Lemma 4.1. (Fully Implicit Apriori Stability Bounds) *The solutions* $\mathbf{u_h}^n$*,* θ_h^n *and* C_h^n *of equations* (4.1)*,* (4.2) *and* (4.3) *satisfy the following apriori bounds:*

$$\max_{1\leq m\leq N} ||\theta_h^m||^2 + \frac{\Delta t}{Pr}\sum_{n=0}^{N-1}||\nabla\theta_h^{n+1}||^2 \leq M_\theta$$
$$\max_{1\leq m\leq N} ||C_h^m||^2 + \frac{\Delta t}{PrLe}\sum_{n=0}^{N-1}||\nabla C_h^{n+1}||^2 \leq M_C$$
$$\max_{1\leq m\leq N} ||\mathbf{u}_h^m||^2 + \frac{\Delta t}{2}\sum_{n=0}^{N-1}||\nabla\mathbf{u_h}^{n+1}||^2 \leq M.$$

Before proving this lemma, we state a property of inner products that will be used extensively in this chapter.

Property 4.2.

$$(a-b,a) = \frac{1}{2}||a||^2 - \frac{1}{2}||b||^2 + \frac{1}{2}||a-b||^2$$

Proof. See Appendix. □

Proof. (Fully Implicit A priori Stability Bounds)

We start with temperature. Setting $\phi = \theta_h^{n+1}$ in (4.2) yields

$$\left(\frac{\theta_h^{n+1}-\theta_h^n}{\Delta t}, \theta_h^{n+1}\right) + \left(\mathbf{u_h}^{n+1}\cdot\nabla\theta_h^{n+1}, \theta_h^{n+1}\right) + \frac{1}{Pr}\left(\nabla\theta_h^{n+1}, \nabla\theta_h^{n+1}\right) = \left(Q^{n+1}, \theta_h^{n+1}\right).$$

The second term on the left vanishes due to anti-symmetry. By Cauchy's Inequality and Property 4.2 we get,

$$\frac{1}{2\Delta t}(||\theta_h^{n+1}||^2 - ||\theta_h^n||^2 + ||\theta_h^{n+1}-\theta_h^n||^2) + \frac{1}{Pr}||\nabla\theta_h^{n+1}||^2 \leq ||Q^{n+1}||\ ||\theta_h^{n+1}||. \tag{4.4}$$

Employing the kickback argument yields,

$$\frac{1}{\Delta t}\left(||\theta_h^{n+1}||^2 - ||\theta_h^n||^2 + ||\theta_h^{n+1} - \theta_h^n||^2\right) + \frac{1}{Pr}||\nabla\theta_h^{n+1}||^2 \leq K||Q^{n+1}||^2. \tag{4.5}$$

The first two terms on the left-hand side telescope when summing the inequality (4.5) from $n = 0$ to $N - 1$ yielding

$$||\theta_h^N||^2 + \sum_{n=0}^{N-1}||\theta_h^{n+1} - \theta_h^n||^2 + \frac{\Delta t}{Pr}\sum_{n=0}^{N-1}||\nabla\theta_h^{n+1}||^2 \leq K\Delta t\sum_{n=0}^{N-1}||Q^{n+1}||^2 + ||\theta_h^0||^2. \tag{4.6}$$

By the assumption on the data, the summation on the right-hand side and $||\theta_h^0||^2$ is bounded. Therefore, summing (4.5) from $n = 0$ to m we get,

$$||\theta_h^m||^2 + \sum_{n=0}^{m}||\theta_h^{n+1} - \theta_h^n||^2 + \frac{\Delta t}{Pr}\sum_{n=0}^{m}||\nabla\theta_h^{n+1}||^2 \leq M_\theta.$$

Therefore we have the following stability bounds for θ_h^m:

$$\max_{1\leq m\leq N}||\theta_h^m||^2 < M_\theta \quad \texttt{and} \quad \frac{\Delta t}{Pr}\sum_{n=0}^{N-1}||\nabla\theta_h^{n+1}||^2 \leq M_\theta. \tag{4.7}$$

Similarly, we have the following stability bounds for C_h^m:

$$\max_{1\leq m\leq N}||C_h^m||^2 < M_c \quad \texttt{and} \quad \frac{\Delta t}{PrLe}\sum_{n=0}^{N-1}||\nabla C_h^{n+1}||^2 \leq M_c. \tag{4.8}$$

Setting $\mathbf{v} = \mathbf{u_h}^{n+1}$ in (4.1) yields

$$\left(\frac{\mathbf{u_h}^{n+1} - \mathbf{u_h}^n}{\Delta t}, \mathbf{u_h}^{n+1}\right) + \left(\mathbf{u_h}^{n+1}\cdot\nabla\mathbf{u_h}^{n+1}, \mathbf{u_h}^{n+1}\right) + \left(\nabla\mathbf{u_h}^{n+1}, \nabla\mathbf{u_h}^{n+1}\right) = \left(\mathbf{f}^{n+1}, \mathbf{u_h}^{n+1}\right)$$
$$+ G_{r_\theta}\left(\theta_h^{n+1}\mathbf{j}, \mathbf{u_h}^{n+1}\right) + G_{r_c}\left(C_h^{n+1}\mathbf{j}, \mathbf{u_h}^{n+1}\right).$$

Using Property 3.4, Property 4.2 and the Cauchy and Poincare inequalities yields,

$$\frac{1}{2\Delta t}\left(||\mathbf{u_h}^{n+1}||^2 - ||\mathbf{u_h}^{n}||^2\right) + \frac{1}{2\Delta t}||\mathbf{u_h}^{n+1} - \mathbf{u_h}^{n}||^2 + ||\nabla \mathbf{u_h}^{n+1}||^2 \leq K||\mathbf{f}^{n+1}||\ ||\nabla \mathbf{u_h}^{n+1}|| + K||\theta^{n+1}||\ ||\nabla \mathbf{u_h}^{n+1}|| + K||C^{n+1}||\ ||\nabla \mathbf{u_h}^{n+1}||. \quad (4.9)$$

Employing the kickback argument yields,

$$||\mathbf{u_h}^{n+1}||^2 - ||\mathbf{u_h}^{n}||^2 + ||\mathbf{u_h}^{n+1} - \mathbf{u_h}^{n}||^2 + \frac{\Delta t}{2}||\nabla \mathbf{u_h}^{n+1}||^2 \leq K\Delta t||\mathbf{f}^{n+1}||^2 + K\Delta t||\theta^{n+1}||^2 + K\Delta t||C^{n+1}||^2. \quad (4.10)$$

Next, we sum (4.10) from $n = 0$ to $N-1$. However, note that $\Delta t\sum_{n=0}^{N-1}\left|\left|\mathbf{f}^{n+1}\right|\right|^2$ is bounded by the assumption on the data and $\Delta t\sum_{n=0}^{N-1}\left|\left|\theta^{n+1}\right|\right|^2$ and $\Delta t\sum_{n=0}^{N-1}\left|\left|C^{n+1}\right|\right|^2$ are bounded by the previous result. Also, $\left|\left|\mathbf{u_h}^0\right|\right|^2$ is bounded. Therefore we get,

$$\left|\left|\mathbf{u_h}^N\right|\right|^2 + \sum_{n=0}^{N-1}\left|\left|\mathbf{u_h}^{n+1} - \mathbf{u_h}^{n}\right|\right|^2 + \frac{\Delta t}{2}\sum_{n=0}^{N-1}\left|\left|\nabla \mathbf{u_h}^{n+1}\right|\right|^2 \leq K. \quad (4.11)$$

Summing (4.10) from $n = 0$ to m, we get

$$||\mathbf{u_h}^m||^2 + \sum_{n=0}^{m}||\mathbf{u_h}^{n+1} - \mathbf{u_h}^{n}||^2 + \frac{\Delta t}{2}\sum_{n=0}^{m}||\nabla \mathbf{u_h}^{n+1}||^2 \leq M. \quad (4.12)$$

Therefore we have the following stability bounds for $\mathbf{u_h}^m$

$$\max_{1\leq m\leq N}||\mathbf{u}_h^m||^2 < M \ \texttt{ and } \ \frac{\Delta t}{2}\sum_{n=0}^{N-1}||\nabla \mathbf{u_h}^{n+1}||^2 \leq M. \quad (4.13)$$

□

4.2.2 Fully Implicit: Consistency

We now present the following consistency result.

Theorem 4.3. (Fully-Implicit Consistency)

The Fully-Implicit Backward Euler Scheme is consistent. Moreover, the residuals in a weak sense vanish linearly with Δt*.*

Proof. We obtain residuals by substituting the exact solutions ($u_h(t_n)$, $\theta_h(t_n)$, $C_h(t_n)$) of (3.13)-(3.15) into (4.1)-(4.3).

$$\left(\frac{\mathbf{u_h}(t_{n+1}) - \mathbf{u_h}(t_n)}{\Delta t}, \mathbf{v}\right) + (\mathbf{u_h}(t_{n+1}) \cdot \nabla \mathbf{u_h}(t_{n+1}), \mathbf{v}) +$$

$$(\nabla \mathbf{u_h}(t_{n+1}), \nabla \mathbf{v}) = (\mathbf{f}(t_{n+1}), \mathbf{v}) + G_{r_\theta}\,(\theta_h(t_{n+1})\mathbf{j}, \mathbf{v}) + G_{r_c}\,(C_h(t_{n+1})\mathbf{j}, \mathbf{v})$$

$$+ \left(R_{\mathbf{u}}^{n+1}, \mathbf{v}\right) \tag{4.14}$$

$$\left(\frac{\theta_h(t_{n+1}) - \theta_h(t_n)}{\Delta t}, \phi\right) + (\mathbf{u_h}(t_{n+1}) \cdot \nabla \theta_h(t_{n+1}), \phi) +$$

$$\frac{1}{Pr}\,(\nabla \theta_h(t_{n+1}), \nabla \phi) = (Q(t_{n+1}), \phi) + \left(R_{\theta}^{n+1}, \phi\right) \tag{4.15}$$

$$\left(\frac{C_h(t_{n+1}) - C_h(t_n)}{\Delta t}, \psi\right) + (\mathbf{u_h}(t_{n+1}) \cdot \nabla C_h(t_{n+1}), \psi) +$$

$$\frac{1}{PrLe}\,(\nabla C_h(t_{n+1}), \nabla \psi) = \left(\hat{Q}(t_{n+1}), \psi\right) + \left(R_{c}^{n+1}, \psi\right). \tag{4.16}$$

Subtracting (4.14)-(4.16) from the weak form (3.13)-(3.15) at t_{n+1} yields,

$$\left(\partial_t \mathbf{u_h}(t_{n+1}) - \frac{\mathbf{u_h}(t_{n+1}) - \mathbf{u_h}(t_n)}{\Delta t}, \mathbf{v}\right) = \left(R_{\mathbf{u}}^{n+1}, \mathbf{v}\right) \qquad , \forall \mathbf{v} \in \mathbf{V^h}$$
$$\left(\partial_t \theta_h(t_{n+1}) - \frac{\theta_h(t_{n+1}) - \theta_h(t_n)}{\Delta t}, \phi\right) = \left(R_{\theta}^{n+1}, \phi\right) \qquad , \forall \phi \in \mathbf{W}^h$$
$$\left(\partial_t C_h(t_{n+1}) - \frac{C_h(t_{n+1}) - C_h(t_n)}{\Delta t}, \psi\right) = \left(R_{c}^{n+1}, \psi\right) \qquad , \forall \psi \in \mathbf{W}^h.$$

Using Taylor expansions we get the following expressions for the residuals.

$$R_{\mathbf{u}}^{n+1} = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} (t-t_n)\mathbf{u}_{h\,tt}(t)\ dt$$
$$R_{\theta}^{n+1} = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} (t-t_n)\theta_{h\,tt}(t)\ dt$$
$$R_{c}^{n+1} = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} (t-t_n)C_{h\,tt}(t)\ dt.$$

Note that

$$\left|\left|R_c^{n+1}\right|\right|_{-1} = \sup_w \frac{\left(w, R_c^{n+1}\right)}{||w||_1} \implies \left(\psi, R_c^{n+1}\right) \le \left|\left|R_c^{n+1}\right|\right|_{-1} ||\psi||_1 .$$

So, to prove consistency it is sufficient to show that $\left|\left|R_c^{n+1}\right|\right|_{-1} = \mathcal{O}(\sqrt{\Delta t})$ since this implies that $\left|(R_c^{n+1}, \psi)\right| \to 0$ as $\Delta t \to 0$. To this end observe,

$$\begin{aligned}
\left|\left|R_c^{n+1}\right|\right|_{-1} &= \\
\left|\left|\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} (t-t_n)C_{h\,tt}(t)\ dt\right|\right|_{-1} &= \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} (t-t_n)\,||C_{h\,tt}(t)||_{-1}\ dt \\
&\le \frac{1}{\Delta t}\left(\int_{t_n}^{t_{n+1}} (t-t_n)^2\ dt\right)^{1/2}\left(\int_{t_n}^{t_{n+1}} ||C_{h\,tt}(t)||_{-1}^2\ dt\right)^{1/2} \\
&= \mathcal{O}(\sqrt{\Delta t}). \qquad (4.17)
\end{aligned}$$

Similarly, $||R_{\mathbf{u}}||_{-1} = \mathcal{O}(\sqrt{\Delta t})$ and $||R_\theta||_{-1} = \mathcal{O}(\sqrt{\Delta t})$. □

4.2.3 Fully Implicit: Convergence

In this section we show that the total error between the fully implicit scheme (4.1)-(4.3) and the exact solutions of the semi-discrete equations $(\mathbf{u_h}(t_n), \theta_h(t_n), C_h(t_n))$ is order Δt. By total error at the n^{th} time-step we mean the 3-tuple $\mathbf{e}^n = (\mathbf{e}_\mathbf{u}^n, e_\theta^n, e_c^n)$, where $\mathbf{e}_\mathbf{u}^n := \mathbf{u_h}(t_n) - \mathbf{u_h}^n$, $e_\theta^n := \theta_h(t_n) - \theta_h^n$ and $e_c^n := C_h(t_n) - C_h^n$. We summarize the result in the following theorem. Note that for the fully implicit case we must assume Δt is sufficiently small for the result to hold. As we shall see, this restriction is not present for the *semi-implicit decoupled* case.

Theorem 4.4 (Fully-Implicit Convergence)**.** *Assume that* $||\mathbf{e}^0||^2 = \mathcal{O}((\Delta t)^2)$. *Provided* Δt *is sufficiently small, the error between the solutions* $(\mathbf{u_h}^n, \theta_h^n, C_h^n)$ *of* (4.1)-(4.3) *and the exact solutions* $(\mathbf{u_h}(t_n), \theta_h(t_n), C_h(t_n))$ *of the semi-discrete equations* (3.13)-(3.15) *satisfy the following:*

$$||\mathbf{u_h}(t_n) - \mathbf{u_h}^n||^2 + ||\theta_h(t_n) - \theta_h^n||^2 + ||C_h(t_n) - C_h^n||^2 = \mathcal{O}(\Delta t^2). \tag{4.18}$$

Moreover,

$$\sum_{n=0}^{N-1} \left(||\mathbf{e}^{n+1} - \mathbf{e}^n||^2 \right) + \Delta t \sum_{n=0}^{N-1} ||\nabla \mathbf{e}^{n+1}||^2 = \mathcal{O}(\Delta t^2),$$

where

$$\begin{aligned} \mathbf{e}^n &= (\mathbf{e}_\mathbf{u}^n, e_\theta^n, e_c^n) \\ &= (\mathbf{u_h}(t_n) - \mathbf{u_h}^n, \theta_h(t_n) - \theta_h^n, C_h(t_n) - C_h^n). \end{aligned}$$

The convergence proof requires the following lemma.

Lemma 4.5 (Discrete Gronwall Lemma I). *Let τ, C and a_n, b_n, c_n, d_n be nonnegative numbers such that*

$$a_n + \tau \sum_{i=0}^{n} b_i \leq \tau \sum_{i=0}^{n} d_i a_i + \tau \sum_{i=0}^{n} c_i + C,$$

for $n \geq 0$. Suppose that $\tau d_i < 1$ for all i. Then

$$a_n + \tau \sum_{i=0}^{n} b_i \leq \exp\left(\tau \sum_{i=0}^{n} \frac{d_i}{1-\tau d_i}\right)\left(\tau \sum_{i=0}^{n} c_i + C\right),$$

for all $n \geq 0$.

Proof. See, [24] □

Proof. (Fully-Implicit Convergence) First we work with concentration. Subtracting (4.3) from (4.16), we get

$$\left(\frac{e_c^{n+1}-e_c^n}{\Delta t}, \psi\right) + (\mathbf{u_h}(t_{n+1}) \cdot \nabla C_h(t_{n+1}), \psi) - \left(\mathbf{u_h}^{n+1} \cdot \nabla C_h^{n+1}, \psi\right) + \frac{1}{PrLe}\left(\nabla e_c^{n+1}, \nabla \psi\right) = \left(R_c^{n+1}, \psi\right).$$

Adding and subtracting the term $(\mathbf{u_h}^{n+1} \cdot \nabla C_h(t_{n+1}), \psi)$ introduces the concentration error, e_c^{n+1}, yielding,

$$\left(\frac{e_c^{n+1}-e_c^n}{\Delta t}, \psi\right) + \left(\mathbf{u_h}^{n+1} \cdot \nabla e_c^{n+1}, \psi\right) + \left(\mathbf{e_u}^{n+1} \cdot \nabla C_h(t_{n+1}), \psi\right) + \frac{1}{PrLe}\left(\nabla e_c^{n+1}, \nabla \psi\right) = \left(R_c^{n+1}, \psi\right), \ \forall \psi \in W^h. \tag{4.19}$$

Since $e_c^{n+1} = C_h(t_{n+1}) - C_n^{n+1} \in W^h$, ψ can be set equal to e_c^{n+1} in (4.19). Also, $\left(e_c^{n+1}, R_c^{n+1}\right) \leq \left|\left|R_c^{n+1}\right|\right|_{-1} \ \left|\left|e_c^{n+1}\right|\right|_1$ and $||\nabla \cdot||$ and $||\cdot||_1$ are equivalent norms in H_0^1. Therefore,

$$\frac{1}{2\Delta t}\left(\left|\left|e_c^{n+1}\right|\right|^2 - ||e_c^n||^2 + \left|\left|e_c^{n+1} - e_c^n\right|\right|^2\right) +$$
$$\left(\mathbf{e}_{\mathbf{u}}^{n+1} \cdot \nabla C_h(t_{n+1}), e_c^{n+1}\right) + \frac{1}{PrLe}\left|\left|\nabla e_c^{n+1}\right|\right|^2 \leq K\left|\left|R_c^{n+1}\right|\right|_{-1}\left|\left|\nabla e_c^{n+1}\right|\right|.$$

Employing the kickback argument and multiplying both sides by $2\Delta t$ yields,

$$\left|\left|e_c^{n+1}\right|\right|^2 - ||e_c^n||^2 + \left|\left|e_c^{n+1} - e_c^n\right|\right|^2 +$$
$$\frac{\Delta t}{PrLe}\left|\left|\nabla e_c^{n+1}\right|\right|^2 \leq 2\Delta t\left|\left(\mathbf{e}_{\mathbf{u}}^{n+1} \cdot \nabla C_h(t_{n+1}), e_c^{n+1}\right)\right| + \Delta t K\left|\left|R_c^{n+1}\right|\right|_{-1}^2. \tag{4.20}$$

Estimating the term $2\Delta t\left|\left(\mathbf{e}_{\mathbf{u}}^{n+1} \cdot \nabla C_h(t_{n+1}), e_c^{n+1}\right)\right|$ using Holder's, Ladyzhenskaya and Young's Inequalities and Proposition 3.1, for $||\nabla C_h||$ yields,

$$\begin{aligned}
2\Delta t\left|\left(\mathbf{e}_{\mathbf{u}}^{n+1} \cdot \nabla C_h(t_{n+1}), e_c^{n+1}\right)\right| &\leq K\Delta t\left|\left|\mathbf{e}_{\mathbf{u}}^{n+1}\right|\right|_{L^4}||\nabla C_h(t_{n+1})||\left|\left|e_c^{n+1}\right|\right|_{L^4} \\
&\leq K\Delta t\left|\left|\mathbf{e}_{\mathbf{u}}^{n+1}\right|\right|^{1/2}\left|\left|\nabla \mathbf{e}_{\mathbf{u}}^{n+1}\right|\right|^{1/2}\left|\left|\nabla e_c^{n+1}\right|\right|^{1/2}\left|\left|e_c^{n+1}\right|\right|^{1/2} \\
&\leq K\Delta t\left(\left|\left|\mathbf{e}_{\mathbf{u}}^{n+1}\right|\right|\left|\left|\nabla \mathbf{e}_{\mathbf{u}}^{n+1}\right|\right| + \left|\left|e_c^{n+1}\right|\right|\left|\left|\nabla e_c^{n+1}\right|\right|\right) \\
&\leq \frac{\Delta t}{4}\left|\left|\nabla \mathbf{e}_{\mathbf{u}}^{n+1}\right|\right|^2 + K\Delta t\left|\left|\mathbf{e}_{\mathbf{u}}^{n+1}\right|\right|^2 \\
&+ \frac{\Delta t}{2PrLe}\left|\left|\nabla e_c^{n+1}\right|\right|^2 + K\Delta t\left|\left|e_c^{n+1}\right|\right|^2.
\end{aligned}$$

Applying this estimate to (4.20), we get

$$\left|\left|e_c^{n+1}\right|\right|^2 - ||e_c^n||^2 + \left|\left|e_c^{n+1} - e_c^n\right|\right|^2 + \frac{\Delta t}{2PrLe}\left|\left|\nabla e_c^{n+1}\right|\right|^2 \leq \frac{\Delta t}{4}\left|\left|\nabla \mathbf{e}_{\mathbf{u}}^{n+1}\right|\right|^2$$
$$+ K\Delta t\left|\left|\mathbf{e}_{\mathbf{u}}^{n+1}\right|\right|^2 + K\Delta t\left|\left|e_c^{n+1}\right|\right|^2 + K\Delta t\left|\left|R_c^{n+1}\right|\right|_{-1}^2. \tag{4.21}$$

Similarly, for temperature,

$$\left|\left|e_{\theta}^{n+1}\right|\right|^2 - ||e_{\theta}^{n}||^2 + \left|\left|e_{\theta}^{n+1} - e_{\theta}^{n}\right|\right|^2 + \frac{\Delta t}{2Pr}\left|\left|\nabla e_{\theta}^{n+1}\right|\right|^2 \leq \frac{\Delta t}{4}\left|\left|\nabla \mathbf{e}_{\mathbf{u}}^{n+1}\right|\right|^2$$
$$+ K\Delta t\left|\left|\mathbf{e}_{\mathbf{u}}^{n+1}\right|\right|^2 + K\Delta t\left|\left|e_{\theta}^{n+1}\right|\right|^2 + K\Delta t\left|\left|R_{\theta}^{n+1}\right|\right|_{-1}^2. \quad (4.22)$$

Turning to velocity. Subtracting (4.1) from (4.14) yields

$$\left(\frac{\mathbf{e}_{\mathbf{u}}^{n+1} - \mathbf{e}_{\mathbf{u}}^{n}}{\Delta t}, \mathbf{v}\right) + (\mathbf{u_h}(t_{n+1}) \cdot \nabla \mathbf{u_h}(t_{n+1}), \mathbf{v}) - \left(\mathbf{u_h}^{n+1} \cdot \nabla \mathbf{u_h}^{n+1}, \mathbf{v}\right) +$$
$$\left(\nabla \mathbf{e}_{\mathbf{u}}^{n+1}, \nabla \mathbf{v}\right) = G_{r_\theta}\left(e_{\theta}^{n+1}\mathbf{j}, \mathbf{v}\right) + G_{r_c}\left(e_{c}^{n+1}\mathbf{j}, \mathbf{v}\right) + \left(R_{\mathbf{u}}^{n+1}, \mathbf{v}\right).$$

Adding and subtracting $(\mathbf{u_h}^{n+1} \cdot \nabla \mathbf{u_h}(t_{n+1}), \mathbf{v})$, we get

$$\left(\frac{\mathbf{e}_{\mathbf{u}}^{n+1} - \mathbf{e}_{\mathbf{u}}^{n}}{\Delta t}, \mathbf{v}\right) + \left(\mathbf{u_h}^{n+1} \cdot \nabla \mathbf{e}_{\mathbf{u}}^{n+1}, \mathbf{v}\right) + \left(\mathbf{e}_{\mathbf{u}}^{n+1} \cdot \nabla \mathbf{u_h}(t_{n+1}), \mathbf{v}\right) +$$
$$\left(\nabla \mathbf{e}_{\mathbf{u}}^{n+1}, \nabla \mathbf{v}\right) = G_{r_\theta}\left(e_{\theta}^{n+1}\mathbf{j}, \mathbf{v}\right) + G_{r_c}\left(e_{c}^{n+1}\mathbf{j}, \mathbf{v}\right) + \left(R_{\mathbf{u}}^{n+1}, \mathbf{v}\right).$$

Since $\mathbf{e}_{\mathbf{u}}^{n+1} = \mathbf{u_h}(t_{n+1}) - \mathbf{u_h}^{n+1} \in \mathbf{V^h}$, we set $\mathbf{v} = \mathbf{e}_{\mathbf{u}}^{n+1}$. By the anti-symmetric property, $\left(\mathbf{u_h}^{n+1} \cdot \nabla \mathbf{e}_{\mathbf{u}}^{n+1}, \mathbf{e}_{\mathbf{u}}^{n+1}\right) = 0$. Thus,

$$\left|\left|\mathbf{e}_{\mathbf{u}}^{n+1}\right|\right|^2 - ||\mathbf{e}_{\mathbf{u}}^{n}||^2 + \left|\left|\mathbf{e}_{\mathbf{u}}^{n+1} - \mathbf{e}_{\mathbf{u}}^{n}\right|\right|^2 + 2\Delta t\left|\left|\nabla \mathbf{e}_{\mathbf{u}}^{n+1}\right|\right|^2 \leq 2\Delta t\left|\left(\mathbf{e}_{\mathbf{u}}^{n+1} \cdot \nabla \mathbf{u_h}(t_{n+1}), \mathbf{e}_{\mathbf{u}}^{n+1}\right)\right|$$
$$+ 2\Delta t G_{r_\theta}\left(e_{\theta}^{n+1}\mathbf{j}, \mathbf{e}_{\mathbf{u}}^{n+1}\right) + 2\Delta t G_{r_c}\left(e_{c}^{n+1}\mathbf{j}, \mathbf{e}_{\mathbf{u}}^{n+1}\right) + 2\Delta t\left(R_{\mathbf{u}}^{n+1}, \mathbf{e}_{\mathbf{u}}^{n+1}\right). \quad (4.23)$$

Since $||\nabla \mathbf{u_h}(t_{n+1})||$ is bounded by Proposition 3.1, Holder's, Ladyzhenskaya and Young's Inequalities yields,

$$2\Delta t \left| \left(\mathbf{e}_{\mathbf{u}}^{n+1} \cdot \nabla \mathbf{u_h}(t_{n+1}), \mathbf{e}_{\mathbf{u}}^{n+1} \right) \right| \leq K \Delta t \left| \left| \mathbf{e}_{\mathbf{u}}^{n+1} \right| \right|_{L^4}^2 ||\nabla \mathbf{u_h}(t_{n+1})||$$
$$\leq K \Delta t \left| \left| \nabla \mathbf{e}_{\mathbf{u}}^{n+1} \right| \right| \left| \left| \mathbf{e}_{\mathbf{u}}^{n+1} \right| \right| \leq \frac{\Delta t}{8} \left| \left| \nabla \mathbf{e}_{\mathbf{u}}^{n+1} \right| \right|^2 + K \Delta t \left| \left| \mathbf{e}_{\mathbf{u}}^{n+1} \right| \right|^2 .$$

Also, by Cauchy, Poincare and Young's Inequalities

$$2G_{r_\theta} \Delta t \left| \left(e_\theta^{n+1} \, \mathbf{j}, \mathbf{e}_{\mathbf{u}}^{n+1} \right) \right| \leq K \Delta t \left| \left| e_\theta^{n+1} \right| \right|^2 + \frac{\Delta t}{8} \left| \left| \nabla \mathbf{e}_{\mathbf{u}}^{n+1} \right| \right|^2 .$$

Similarly,

$$2G_{r_c} \Delta t \left| \left(e_c^{n+1} \, \mathbf{j}, \mathbf{e}_{\mathbf{u}}^{n+1} \right) \right| \leq K \Delta t \left| \left| e_c^{n+1} \right| \right|^2 + \frac{\Delta t}{8} \left| \left| \nabla \mathbf{e}_{\mathbf{u}}^{n+1} \right| \right|^2 .$$

To estimate $\left| \left(R_{\mathbf{u}}^{n+1} \, \mathbf{j}, \mathbf{e}_{\mathbf{u}}^{n+1} \right) \right|$ observe

$$\left| \left| R_{\mathbf{u}}^{n+1} \right| \right|_* = \sup_{\mathbf{v}} \frac{\left| \left(R_{\mathbf{u}}^{n+1}, \mathbf{v} \right) \right|}{||\nabla \mathbf{v}||} \implies \left| \left| \nabla \mathbf{e}_{\mathbf{u}}^{n+1} \right| \right| \left| \left| R_{\mathbf{u}}^{n+1} \right| \right|_* \geq \left| \left(R_{\mathbf{u}}^{n+1}, \mathbf{e}_{\mathbf{u}}^{n+1} \right) \right| .$$

By Cauchy and Young's Inequality,

$$2\Delta t \left| \left(R_{\mathbf{u}}^{n+1} \, \mathbf{j}, \mathbf{e}_{\mathbf{u}}^{n+1} \right) \right| \leq 2\Delta t \left| \left| R_{\mathbf{u}}^{n+1} \right| \right|_* \left| \left| \nabla \mathbf{e}_{\mathbf{u}}^{n+1} \right| \right|$$
$$\leq K \Delta t \left| \left| R_{\mathbf{u}}^{n+1} \right| \right|_*^2 + \frac{\Delta t}{8} \left| \left| \nabla \mathbf{e}_{\mathbf{u}}^{n+1} \right| \right|^2 .$$

Using the above estimates in (4.23) yields,

$$\left|\left|\mathbf{e}_{\mathbf{u}}^{n+1}\right|\right|^2 - ||\mathbf{e}_{\mathbf{u}}^{n}||^2 + \left|\left|\mathbf{e}_{\mathbf{u}}^{n+1} - \mathbf{e}_{\mathbf{u}}^{n}\right|\right|^2 + \frac{3}{2}\Delta t \left|\left|\nabla \mathbf{e}_{\mathbf{u}}^{n+1}\right|\right|^2 \leq K\Delta t \left|\left|\mathbf{e}_{\mathbf{u}}^{n+1}\right|\right|^2 + K\Delta t \left(\left|\left|e_{\theta}^{n+1}\right|\right|^2 + \left|\left|e_{c}^{n+1}\right|\right|^2 + \left|\left|R_{\mathbf{u}}^{n+1}\right|\right|_*^2\right). \tag{4.24}$$

Since the inequalities (4.22), (4.21) and (4.24) remain coupled, we add them together and introduce the following *total* error notation: $\mathbf{e} = (\mathbf{e}_{\mathbf{u}}, e_c, e_\theta)$. Note that $||\mathbf{e}||^2 = ||\mathbf{e}_{\mathbf{u}}||^2 + ||e_\theta||^2 + ||e_c||^2$ and $||\nabla\mathbf{e}||^2 = ||\nabla\mathbf{e}_{\mathbf{u}}||^2 + ||\nabla e_\theta||^2 + ||\nabla e_c||^2$. Let

$$\alpha = \min\left\{\frac{1}{2Pr\cdot Le}, \frac{1}{2Pr}, 1\right\}.$$

Adding inequalities (4.21), (4.22) and (4.24),

$$\left|\left|\mathbf{e}^{n+1}\right|\right|^2 - ||\mathbf{e}^{n}||^2 + \left|\left|\mathbf{e}^{n+1} - \mathbf{e}^{n}\right|\right|^2 + \alpha\Delta t \left|\left|\nabla \mathbf{e}^{n+1}\right|\right|^2 \leq K\Delta t \left|\left|\mathbf{e}^{n+1}\right|\right|^2 + K\Delta t \left(||R_{\mathbf{u}}||_*^2 + ||R_\theta||_{-1}^2 + ||R_c||_{-1}^2\right) \tag{4.25}$$

and summing (4.25) from $n = 0$ to $n = N - 1$ yields

$$\left|\left|\mathbf{e}^{N}\right|\right|^2 - \left|\left|\mathbf{e}^{0}\right|\right|^2 + \sum_{n=0}^{N-1}\left(\left|\left|\mathbf{e}^{n+1} - \mathbf{e}^{n}\right|\right|^2\right) + \alpha\Delta t \sum_{n=0}^{N-1}\left|\left|\nabla \mathbf{e}^{n+1}\right|\right|^2 \quad \leq K\Delta t \sum_{n=0}^{N-1}\left|\left|\mathbf{e}^{n+1}\right|\right|^2 + K\Delta t \sum_{n=0}^{N-1}\left(||R_{\mathbf{u}}||_*^2 + ||R_\theta||_{-1}^2 + ||R_c||_{-1}^2\right). \tag{4.26}$$

Next, to see that $\sum_{n=0}^{N-1} \left|\left|R_c^{n+1}\right|\right|_{-1}^2 = \mathcal{O}(\Delta t)$, observe,

$$\begin{aligned}
\sum_{n=0}^{N-1} \left|\left|R_c^{n+1}\right|\right|_{-1}^2 &= \sum_{n=0}^{N-1} \left|\left| \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} (t - t_n) C_{h\,tt}\ dt \right|\right|_{-1}^2 \\
&= \sum_{n=0}^{N-1} \left(\frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} (t - t_n) \left|\left|C_{h\,tt}\right|\right|_{-1}\ dt \right) \left(\frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} (t - t_n) \left|\left|C_{h\,tt}\right|\right|_{-1}\ dt \right) \\
&\le \sum_{n=0}^{N-1} \frac{1}{(\Delta t)^2} \left(\int_{t_n}^{t_{n+1}} (t - t_n)^2\ dt \right) \left(\int_{t_n}^{t_{n+1}} \left|\left|C_{h\,tt}\right|\right|_{-1}^2\ dt \right) \\
&= \frac{1}{(\Delta t)^2} \sum_{n=0}^{N-1} \frac{(\Delta t)^3}{3} \int_{t_n}^{t_{n+1}} \left|\left|C_{h\,tt}\right|\right|_{-1}^2\ dt \\
&= \frac{\Delta t}{3} \int_0^T \left|\left|C_{h\,tt}\right|\right|_{-1}^2\ dt \\
&= \mathcal{O}(\Delta t),
\end{aligned}$$

where the last integral is bounded by Proposition 3.1. Similarly, $\sum_{n=0}^{N-1} \left|\left|R_\theta^{n+1}\right|\right|_{-1}^2 = \mathcal{O}(\Delta t)$ and $\sum_{n=0}^{N-1} \left|\left|R_{\mathbf{u}}^{n+1}\right|\right|_{*}^2 = \mathcal{O}(\Delta t)$. Therefore, the last term in (4.26) has order $\mathcal{O}((\Delta t)^2)$. Since we are assuming $\left|\left|\mathbf{e}^0\right|\right| = \mathcal{O}(\Delta t)$,

$$\left|\left|\mathbf{e}^N\right|\right|^2 + \sum_{n=0}^{N-1} \left(\left|\left|\mathbf{e}^{n+1} - \mathbf{e}^n\right|\right|^2 \right) + \alpha \Delta t \sum_{n=0}^{N-1} \left|\left|\nabla \mathbf{e}^{n+1}\right|\right|^2 \le K \Delta t \sum_{n=0}^{N-1} \left|\left|\mathbf{e}^{n+1}\right|\right|^2 + \mathcal{O}((\Delta t)^2). \tag{4.27}$$

Assuming that Δt is sufficiently small, the Discrete Gronwall Lemma 4.5 yields,

$$\left|\left|\mathbf{e}^N\right|\right|^2 + \alpha \Delta t \sum_{n=0}^{N-1} \left|\left|\nabla \mathbf{e}^{n+1}\right|\right|^2 \le K (\Delta t)^2, \quad \text{where} \quad K < \infty \quad \text{as} \quad \Delta t \to 0.$$

Thus, $||\mathbf{e}^n||^2 \le K(\Delta t)^2$, for $n \ge 0$ and by (4.27), $\sum_{n=0}^{N-1} \left(\left|\left|\mathbf{e}^{n+1} - \mathbf{e}^n\right|\right|^2 \right) \le K(\Delta t)^2$. $\square$

Remark 1. *Since* $||\mathbf{e}^n|| \le K\Delta t$, *we can write* $|\ ||\mathbf{u_h}(t_n)|| - ||\mathbf{u_h}^n||\ | \le ||\mathbf{u_h}(t_n) - \mathbf{u_h}^n|| \le K\Delta t$. *Thus,* $||\mathbf{u_h}^n|| \le ||\mathbf{u_h}(t_n)|| + K\Delta t \le K$. *Moreover, since* $\sum_{n=1}^{N} ||\nabla(\mathbf{u_h}(t_n) - \mathbf{u_h}^n)||^2 \le$

$K\Delta t$, we have, $||\nabla(\mathbf{u_h}^n - \mathbf{u_h}(t_n))|| \leq K \implies ||\nabla \mathbf{u_h}^n|| \leq K$. *That is,* $||\mathbf{u_h}^n||$ *and* $||\nabla \mathbf{u_h}^n||$ *are bounded.*

4.2.4 Fully Implicit: Pressure Error

We have been dealing with pressure by assuming divergence free velocity which causes the pressure terms to vanish. Let $\epsilon(t) := p(t) - p_h(t)$ be the spatial discretization error of pressure and $\epsilon^n := p_h(t_n) - p_h^n$ be the time discretization error of pressure at time $t = t_n$. So, the total error in pressure is $p(t_n) - p_h^n = \epsilon(t_n) + \epsilon^n$. The following theorem provides an estimate of ϵ^n for the fully implicit scheme.

Theorem 4.6. *Assuming that the* inf-sup *condition holds (Assumption C), the error of the pressure term between the semi-discrete approximation, $p_h(t_n)$, and the fully implicit approximation, p_h^n, satisfies the following:*

$$\left(\sum_{n=1}^{N-1} \Delta t\, ||\epsilon^n||^2\right)^{1/2} = \mathcal{O}(\sqrt{\Delta t}),$$

provided Δt is sufficiently small.

Proof. Once the velocity, temperature and concentration $\{(\mathbf{u_h}^n, \theta_h^n, C_h^n)\}$ are determined, we can compute the approximations p_h^n for the pressure $p_h(t_n)$ from the weak formulations of the doubly diffusive model in non-divergence-free function spaces. We seek $(\mathbf{u_h}, \theta_h, C_h) \in \mathbf{W}^h \times W^h \times W^h$ and $p_h^n \in L^h$ such that

$$\begin{aligned}\left(p_h^{n+1}, \nabla \cdot \mathbf{v_h}\right) = \left(\frac{\mathbf{u_h}^{n+1} - \mathbf{u_h}^n}{\Delta t}, \mathbf{v_h}\right) + \left(\nabla \mathbf{u_h}^{n+1}, \nabla \mathbf{v_h}\right) + \left(\mathbf{u_h}^{n+1} \cdot \nabla \mathbf{u_h}^{n+1}, \mathbf{v_h}\right) \\ - \left(\left(G_{r_\theta} \theta_h^{n+1} + G_{r_c} C_h^{n+1}\right) \mathbf{j}, \mathbf{v_h}\right) - \left(\mathbf{f}^{n+1}, \mathbf{v_h}\right) \quad \forall \mathbf{v_h} \in \mathbf{W}^h, \qquad (4.28)\end{aligned}$$

where $\mathbf{W}^h$, W^h and L^h are subspaces of $\mathbf{H}_0^1$, H_0^1 and L_0^2 satisfying the *inf-sup* condition. Let $\epsilon^n := p_h(t_n) - p_h^n, n = 1, 2, \ldots, N$ be the error in pressure. Then the error ϵ^n satisfies the equation,

$$\begin{aligned}(\epsilon^n, \nabla \cdot \mathbf{v_h}) = &\left(\frac{\mathbf{e_u}^{n+1} - \mathbf{e_u}^n}{\Delta t}, \mathbf{v_h}\right) + \left(\nabla \mathbf{e_u}^{n+1}, \nabla \mathbf{v_h}\right) + \left(\mathbf{u_h}(t_{n+1}) \cdot \nabla \mathbf{e_u}^{n+1}, \mathbf{v_h}\right) \\ &+ \left(\mathbf{e_u}^{n+1} \cdot \nabla \mathbf{u_h}^{n+1}, \mathbf{v_h}\right) - \left(\left(G_{r_\theta} e_\theta^{n+1} + G_{r_c} e_c^{n+1}\right) \mathbf{j}, \mathbf{v_h}\right) + \left(R_\mathbf{u}^{n+1}, \mathbf{v_h}\right). \quad (4.29)\end{aligned}$$

Note that since $\epsilon^n \in L_h$, the *inf-sup* condition implies $K \left|\left|\epsilon^n\right|\right| \leq \sup_{\mathbf{v_h} \in \mathbf{W^h}} \dfrac{(\epsilon^n, \nabla \cdot \mathbf{v_h})}{\left|\left|\mathbf{v_h}\right|\right|_1}$. Using this and the Poincare inequality in (4.29) we have,

$$\begin{aligned}\left|\left|\epsilon^n\right|\right| \leq &\frac{K}{\Delta t} \left|\left|\mathbf{e_u}^{n+1} - \mathbf{e_u}^n\right|\right| + K \left|\left|\nabla \mathbf{e_u}^{n+1}\right|\right| + K \left|\left|\nabla \mathbf{e_u}^{n+1}\right|\right| \left|\left|\mathbf{u_h}(t_{n+1})\right|\right| \\ &+ K \left|\left|\mathbf{u_h}^{n+1}\right|\right|_1 \left|\left|\nabla \mathbf{e_u}^{n+1}\right|\right| + K \left|\left|\nabla e_\theta^{n+1}\right|\right| + K \left|\left|\nabla e_c^{n+1}\right|\right| + K \left|\left|R_\mathbf{u}^{n+1}\right|\right|_* .\end{aligned}$$

By Proposition 3.1, $\left|\left|\mathbf{u_h}(t_{n+1})\right|\right|$ is bounded and by Remark 1, $\left|\left|\mathbf{u_h}^{n+1}\right|\right|_1$ is bounded. Also, by (4.17), $\left|\left|R_\mathbf{u}^{n+1}\right|\right|_* \leq K\sqrt{\Delta t}$. Therefore,

$$\Delta t \left|\left|\epsilon^n\right|\right| \leq K \left|\left|\mathbf{e}^{n+1} - \mathbf{e}^n\right|\right| + K \Delta t \left|\left|\nabla \mathbf{e}^{n+1}\right|\right| + K \left(\Delta t\right)^{3/2}.$$

Squaring both sides and applying Young's inequality to the right hand side yields,

$$\begin{aligned}(\Delta t)^2 \left|\left|\epsilon^n\right|\right|^2 &\leq \left(K \left|\left|\mathbf{e}^{n+1} - \mathbf{e}^n\right|\right| + K \Delta t \left|\left|\nabla \mathbf{e}^{n+1}\right|\right| + K \left(\Delta t\right)^{3/2}\right)^2 \\ &\leq K \left|\left|\mathbf{e}^{n+1} - \mathbf{e}^n\right|\right|^2 + K(\Delta t)^2 \left|\left|\nabla \mathbf{e}^{n+1}\right|\right|^2 + K(\Delta t)^3.\end{aligned}$$

Summing from $n = 0$ to $n = N - 1$ and using the estimates from Theorem 4.4 gives,

$$(\Delta t)^2 \sum_{n=0}^{N-1} ||\epsilon^n||^2 \leq K \sum_{n=0}^{N-1} \left|\left|\mathbf{e}^{n+1} - \mathbf{e}^n\right|\right|^2 + K(\Delta t)^2 \sum_{n=0}^{N-1} \left|\left|\nabla \mathbf{e}^{n+1}\right|\right|^2 + K(\Delta t)^3 \sum_{n=0}^{N-1} 1$$

$$\leq K(\Delta t)^2 + K(\Delta t)^3 + K(\Delta t)^2 \leq K(\Delta t^2),$$

which implies $\left(\sum_{n+1}^{N-1} \Delta t \, ||\epsilon^n||^2\right)^{1/2} = \mathcal{O}(\sqrt{\Delta t})$. □

4.3 Semi-Implicit Backward Euler Scheme

In this section we study the stability, consistency and convergence of a semi-implicit backward euler scheme to discretize time. The particular semi-implicit scheme we consider is the following: seek $(\mathbf{u_h}^{n+1}, \theta_h^{n+1}, C_h^{n+1}) \in \mathbf{V^h} \times W^h \times W^h$ such that

$$\left(\frac{\mathbf{u_h}^{n+1} - \mathbf{u_h}^n}{\Delta t}, \mathbf{v}\right) + \left(\mathbf{u_h}^n \cdot \nabla \mathbf{u_h}^{n+1}, \mathbf{v}\right) + \left(\nabla \mathbf{u_h}^{n+1}, \nabla \mathbf{v}\right) = \left(\mathbf{f}^{n+1}, \mathbf{v}\right) + G_{r_\theta}\left(\theta_h^{n+1}\mathbf{j}, \mathbf{v}\right) + G_{r_c}\left(C_h^{n+1}\mathbf{j}, \mathbf{v}\right) \quad (4.30)$$

$$\left(\frac{\theta_h^{n+1} - \theta_h^n}{\Delta t}, \phi\right) + \left(\mathbf{u_h}^n \cdot \nabla \theta_h^{n+1}, \phi\right) + \frac{1}{Pr}\left(\nabla \theta_h^{n+1}, \nabla \phi\right) = \left(Q^{n+1}, \phi\right) \quad (4.31)$$

$$\left(\frac{C_h^{n+1} - C_h^n}{\Delta t}, \psi\right) + \left(\mathbf{u_h}^n \cdot \nabla C_h^{n+1}, \psi\right) + \frac{1}{PrLe}\left(\nabla C_h^{n+1}, \nabla \psi\right) = \left(\hat{Q}^{n+1}, \psi\right) \quad (4.32)$$

$$\forall (\mathbf{v}, \phi, \psi) \in \mathbf{V^h} \times W^h \times W^h \quad , \quad n = 0, 1, \ldots, N-1,$$

and

$$\mathbf{u_h}^0 = \mathbf{u_h}(0),\ \theta_h^0 = \theta_h(0) \text{ and } C_h^0 = C_h(0).$$

4.3.1 Semi-Implicit: Stability Bounds

We now present a priori stability bounds in the following lemma.

Lemma 4.7. (Semi-implicit A priori Stability Bounds) *The solutions* $\mathbf{u_h}^n$*,* θ_h^n *and* C_h^n *of equations* (4.30)*,* (4.31) *and* (4.32) *satisfy the following a priori bounds:*

$$\max_{1\le m\le N} ||\theta_h^m||^2 + \frac{\Delta t}{Pr}\sum_{n=0}^{N-1} ||\nabla\theta_h^{n+1}||^2 \le M_\theta$$
$$\max_{1\le m\le N} ||C_h^m||^2 + \frac{\Delta t}{PrLe}\sum_{n=0}^{N-1} ||\nabla C_h^{n+1}||^2 \le M_C$$
$$\max_{1\le m\le N} ||\mathbf{u}_h^m||^2 + \frac{\Delta t}{2}\sum_{n=0}^{N-1} ||\nabla \mathbf{u_h}^{n+1}||^2 \le M.$$

Proof. We start with temperature. Setting $\phi = \theta_h^{n+1}$ in (4.31) yields

$$\left(\frac{\theta_h^{n+1}-\theta_h^n}{\Delta t}, \theta_h^{n+1}\right) + \left(\mathbf{u_h}^n \cdot \nabla\theta_h^{n+1}, \theta_h^{n+1}\right) + \frac{1}{Pr}\left(\nabla\theta_h^{n+1}, \nabla\theta_h^{n+1}\right) = \left(Q^{n+1}, \theta_h^{n+1}\right).$$

By Property 3.4, Property 4.2 and Cauchy inequality,

$$\frac{1}{2\Delta t}(||\theta_h^{n+1}||^2 - ||\theta_h^n||^2 + ||\theta_h^{n+1}-\theta_h^n||^2) + \frac{1}{Pr}||\nabla\theta_h^{n+1}||^2 \le ||Q^{n+1}||\ ||\theta_h^{n+1}||. \tag{4.33}$$

Note that the expression (4.4) in the a priori stability proof of previous fully implicit scheme is the same as (4.33). Since the assumptions have not changed the previous results for temperature, concentration and velocity hold for the semi-implicit case. Therefore we have the following stability bounds for θ_h^m, C_h^m and $\mathbf{u}_h^m$:

$$\max_{1\le m\le N} ||\theta_h^m||^2 + \frac{\Delta t}{Pr}\sum_{n=0}^{N-1} ||\nabla\theta_h^{n+1}||^2 \le M_\theta \tag{4.34}$$

$$\max_{1\le m\le N}||C_h^m||^2 + \frac{\Delta t}{PrLe}\sum_{n=0}^{N-1}||\nabla C_h^{n+1}||^2 \le M_C \tag{4.35}$$

$$\max_{1\le m\le N}||\mathbf{u}_h^m||^2 + \frac{\Delta t}{2}\sum_{n=0}^{N-1}||\nabla \mathbf{u_h}^{n+1}||^2 \le M. \tag{4.36}$$

□

4.3.2 Semi-Implicit: Consistency

We now present the following consistency result.

Theorem 4.8. (Semi-Implicit Consistency)

The Semi-Implicit Backward Euler Scheme is consistent. Moreover, the residuals in a weak sense vanish linearly with Δt*.*

Proof. We obtain residuals by substituting the exact solutions $(u_h(t_n),\, \theta_h(t_n),\, C_h(t_n))$ of (3.13)-(3.15) into (4.30)-(4.32)

$$\left(\frac{\mathbf{u_h}(t_{n+1}) - \mathbf{u_h}(t_n)}{\Delta t}, \mathbf{v}\right) + (\mathbf{u_h}(t_n)\cdot\nabla\mathbf{u_h}(t_{n+1}), \mathbf{v}) + (\nabla\mathbf{u_h}(t_{n+1}), \nabla\mathbf{v}) =$$
$$(\mathbf{f}(t_{n+1}), \mathbf{v}) + G_{r_\theta}\,(\theta_h(t_{n+1})\mathbf{j}, \mathbf{v}) + G_{r_c}\,(C_h(t_{n+1})\mathbf{j}, \mathbf{v}) + \left(R_{\mathbf{u}}^{n+1}, \mathbf{v}\right) \tag{4.37}$$

$$\left(\frac{\theta_h(t_{n+1}) - \theta_h(t_n)}{\Delta t}, \phi\right) + (\mathbf{u_h}(t_n)\cdot\nabla\theta_h(t_{n+1}), \phi) +$$
$$\frac{1}{Pr}\,(\nabla\theta_h(t_{n+1}), \nabla\phi) = (Q(t_{n+1}), \phi) + \left(R_\theta^{n+1}, \phi\right) \tag{4.38}$$

$$\left(\frac{C_h(t_{n+1}) - C_h(t_n)}{\Delta t}, \psi\right) + (\mathbf{u_h}(t_n) \cdot \nabla C_h(t_{n+1}), \psi) +$$

$$\frac{1}{PrLe} (\nabla C_h(t_{n+1}), \nabla \psi) = \left(\hat{Q}(t_{n+1}), \psi\right) + \left(R_c^{n+1}, \psi\right). \qquad (4.39)$$

Subtracting the exact solutions ($u_h(t_n)$, $\theta_h(t_n)$, $C_h(t_n)$) of (4.30)-(4.32) from the weak form equations (3.13)-(3.15) at t_{n+1} yields,

$$\left(\partial_t \mathbf{u_h}(t_{n+1}) - \frac{\mathbf{u_h}(t_{n+1}) - \mathbf{u_h}(t_n)}{\Delta t}, \mathbf{v}\right) +$$

$$(\mathbf{u_h}(t_{n+1}) - \mathbf{u_h}(t_n) \cdot \nabla \mathbf{u_h}(t_{n+1}), \mathbf{v}) = \left(R_{\mathbf{u}}^{n+1}, \mathbf{v}\right) \qquad , \forall \mathbf{v} \in \mathbf{V^h} \qquad (4.40)$$

$$\left(\partial_t \theta_h(t_{n+1}) - \frac{\theta_h(t_{n+1}) - \theta_h(t_n)}{\Delta t}, \phi\right) +$$

$$(\mathbf{u_h}(t_{n+1}) - \mathbf{u_h}(t_n) \cdot \nabla \theta_h(t_{n+1}), \phi) = \left(R_\theta^{n+1}, \phi\right) \qquad , \forall \phi \in \mathbf{W}^h \qquad (4.41)$$

$$\left(\partial_t C_h(t_{n+1}) - \frac{C_h(t_{n+1}) - C_h(t_n)}{\Delta t}, \psi\right) +$$

$$(\mathbf{u_h}(t_{n+1}) - \mathbf{u_h}(t_n) \cdot \nabla C_h(t_{n+1}), \psi) = \left(R_c^{n+1}, \psi\right) \qquad , \forall \psi \in \mathbf{W}^h. \qquad (4.42)$$

To estimate the trilinear term in the temperature equation, (4.41), we first define, using the fundamental theorem of calculus,

$$\hat{\mathbf{u}} = \int_{t_n}^{t_{n+1}} \partial_t \mathbf{u_h}(s)\, ds = \mathbf{u_h}(t_{n+1}) - \mathbf{u_h}(t_n). \qquad (4.43)$$

Substituting the above and using Holder's and Ladyzhenskaya Inequalities, we have

$$
\begin{aligned}
&|(\mathbf{u_h}(t_{n+1}) - \mathbf{u_h}(t_n) \cdot \nabla\theta_h(t_{n+1}), \phi)| = |(\hat{\mathbf{u}} \cdot \nabla\theta_h(t_{n+1}), \phi)| \\
&\leq K\,||\hat{\mathbf{u}}||_{L^4} \;\; ||\nabla\theta_h(t_{n+1})|| \;\; ||\phi||_{L^4} \leq K\,||\hat{\mathbf{u}}||^{1/2} \;\; ||\nabla\hat{\mathbf{u}}||^{1/2} \;\; ||\nabla\phi||\,. \qquad (4.44)
\end{aligned}
$$

But

$$
||\hat{\mathbf{u}}||^{1/2} = \left|\left|\int_{t_n}^{t_{n+1}} \partial_t \mathbf{u_h}(s)\,ds\right|\right|^{1/2} = \left[\int_\Omega \left(\int_{t_n}^{t_{n+1}} \partial_t \mathbf{u_h}(s)\,ds\right)^2 d\Omega\right]^{1/4}, \qquad (4.45)
$$

and by Cauchy-Schwarz

$$
\begin{aligned}
\int_{t_n}^{t_{n+1}} \partial_t \mathbf{u_h}(s)\,ds &\leq \left(\int_{t_n}^{t_{n+1}} (\partial_t \mathbf{u_h}(s))^2\,ds\right)^{1/2} \left(\int_{t_n}^{t_{n+1}} 1^2\right)^{1/2} \\
&= (\Delta t)^{1/2} \left(\int_{t_n}^{t_{n+1}} (\partial_t \mathbf{u_h}(s))^2\,ds\right)^{1/2}. \qquad (4.46)
\end{aligned}
$$

Using (4.46) in (4.45) and applying Fubini's Theorem, we get

$$
\begin{aligned}
||\hat{\mathbf{u}}||^{1/2} &\leq \left[\int_\Omega \Delta t \int_{t_n}^{t_{n+1}} (\partial_t \mathbf{u_h}(s))^2\,ds\;d\Omega\right]^{1/4} \\
&= (\Delta t)^{1/4} \left[\int_{t_n}^{t_{n+1}} \int_\Omega (\partial_t \mathbf{u_h}(s))^2\,d\Omega\;ds\right]^{1/4} = (\Delta t)^{1/4} \left[\int_{t_n}^{t_{n+1}} ||\partial_t \mathbf{u_h}(s)||^2 \;\; ds\right]^{1/4}.
\end{aligned}
$$

Since $||\partial_t \mathbf{u_h}(s)||$ is bounded by Proposition 3.1, we have

$$
||\hat{\mathbf{u}}||^{1/2} \leq K(\Delta t)^{1/4}. \qquad (4.47)
$$

Similarly, $||\nabla\hat{\mathbf{u}}||^{1/2} = \left[\int_\Omega \Delta t \int_{t_n}^{t_{n+1}} (\nabla\partial_t \mathbf{u_h}(s))^2 \, ds \; d\Omega\right]^{1/4}$. By Proposition 3.1, the inner integral is bounded giving us

$$||\nabla\hat{\mathbf{u}}||^{1/2} \leq K(\Delta t)^{1/4}. \tag{4.48}$$

Using (4.47) and (4.48) in (4.44) we get

$$|(\mathbf{u}((t_{n+1})) - \mathbf{u}(t_n) \cdot \nabla\theta(t_{n+1}), \phi)| \leq K\sqrt{\Delta t}\, ||\nabla\phi|| \,. \tag{4.49}$$

The terms on the left in (4.40), (4.41) and (4.42) can be Taylor expanded as follows:

$$\partial_t \mathbf{u_h}(t_{n+1}) - \frac{\mathbf{u_h}(t_{n+1}) - \mathbf{u_h}(t_n)}{\Delta t} = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} (t - t_n)\partial_t^2 \mathbf{u_h}(t) \; dt \tag{4.50}$$

$$\partial_t \theta_h(t_{n+1}) - \frac{\theta_h(t_{n+1}) - \theta_h(t_n)}{\Delta t} = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} (t - t_n)\partial_t^2 \theta_h(t) \; dt \tag{4.51}$$

$$\partial_t C_h(t_{n+1}) - \frac{C_h(t_{n+1}) - C_h(t_n)}{\Delta t} = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} (t - t_n)\partial_t^2 C_h(t) \; dt. \tag{4.52}$$

So using (4.49) and (4.51), we get

$$\left|(R_\theta^{n+1}, \phi)\right| \leq K\sqrt{\Delta t}\, ||\nabla\phi|| + \left|\left(\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} (t - t_n)\partial_t^2 \theta_h(t) \; dt, \phi\right)\right|. \tag{4.53}$$

But

$$\left|\left(\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} (t - t_n)\partial_t^2 \theta_h(t) \; dt, \phi\right)\right| = \left|\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} (t - t_n)\left(\partial_t^2 \theta_h(t), \phi\right) \; dt\right| \tag{4.54}$$

$$\leq \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} (t - t_n)\left|\left(\partial_t^2 \theta_(t), \phi\right)\right| \; dt. \tag{4.55}$$

Note that by the definition of $||\cdot||_{-1}$, $\dfrac{\left|(\partial_t^2\theta_h(t),\phi)\right|}{||\nabla\phi||} \leq \left|\left|\partial_t^2\theta_h\right|\right|_{-1}$. So, (4.55) becomes

$$\begin{aligned}
&\left|\left(\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} (t-t_n)\partial_t^2\theta_h(t)\ dt, \phi\right)\right| \leq \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} (t-t_n)\left|\left|\partial_t^2\theta_h(t)\right|\right|_{-1} ||\nabla\phi||\ dt \\
&= \frac{||\nabla\phi||}{\Delta t}\int_{t_n}^{t_{n+1}} (t-t_n)\left|\left|\partial_t^2\theta_h(t)\right|\right|_{-1}\ dt \\
&\leq \frac{||\nabla\phi||}{\Delta t}\left[\int_{t_n}^{t_{n+1}} (t-t_n)^2\ dt\right]^{1/2}\left[\int_{t_n}^{t_{n+1}} \left|\left|\partial_t^2\theta_h(t)\right|\right|_{-1}^2\ dt\right]^{1/2} \leq K\sqrt{\Delta t}\,||\nabla\phi||. \qquad (4.56)
\end{aligned}$$

Using (4.56) in (4.53), we get $\left|(R_\theta^{n+1},\phi)\right| \leq K\sqrt{\Delta t}\,||\nabla\phi||$ or $\dfrac{\left|(R_\theta^{n+1},\phi)\right|}{||\nabla\phi||} \leq K\sqrt{\Delta t} = \mathcal{O}(\sqrt{\Delta t})$. Therefore, $\left|\left|R_\theta^{n+1}\right|\right|_{-1} = \mathcal{O}(\sqrt{\Delta t})$. Similarly, $\left|\left|R_C^{n+1}\right|\right|_{-1} = \mathcal{O}(\sqrt{\Delta t})$ and $\left|\left|R_{\mathbf{u}}^{n+1}\right|\right|_{*} = \mathcal{O}(\sqrt{\Delta t})$. Therefore the scheme is consistent. □

4.3.3 Semi-Implicit: Convergence

In this section we show that the total error between the semi-implicit scheme (4.30)-(4.32) and the exact solutions of the semi-discrete equations $(\mathbf{u_h}(t_n), \theta_h(t_n), C_h(t_n))$ is order (Δt) in L^2-norm. We summarize the result in the following theorem.

Theorem 4.9. (Semi-Implicit Convergence)

Assume that $\left|\left|\mathbf{e}^0\right|\right|^2 = \mathcal{O}((\Delta t)^2)$ *and* $\left|\left|\nabla\mathbf{e}^0\right|\right|^2 = \mathcal{O}(\Delta t)$. *Assuming* Δt *is sufficiently small, the error between the solutions* $(\mathbf{u_h}^n, \theta_h^n, C_h^n)$ *of* (4.30)-(4.32) *and the exact solutions* $(\mathbf{u_h}(t_n), \theta_h(t_n), C_h(t_n))$ *of the semi-discrete equations* (3.13)-(3.15) *satisfy the following:*

$$||\mathbf{u_h}(t_n) - \mathbf{u_h}^n||^2 + ||\theta_h(t_n) - \theta_h^n||^2 + ||C_h(t_n) - C_h^n||^2 = \mathcal{O}(\Delta t^2).$$

Moreover, $\sum_{n=0}^{N-1}\left(\left|\left|\mathbf{e}^{n+1} - \mathbf{e}^n\right|\right|^2\right) = \mathcal{O}(\Delta t^2)$.

Proof. First we work with concentration. Subtracting (4.32) from (4.39), we get

$$\left(\frac{e_c^{n+1}-e_c^n}{\Delta t},\psi\right)+(\mathbf{u_h}(t_n)\cdot\nabla C_h(t_{n+1}),\psi)-\left(\mathbf{u_h}^n\cdot\nabla C_h^{n+1},\psi\right)+$$

$$\frac{1}{PrLe}\left(\nabla e_c^{n+1},\nabla\psi\right)=\left(R_c^{n+1},\psi\right).$$

Adding and subtracting the term $(\mathbf{u_h}^n\cdot\nabla C_h(t_{n+1}),\psi)$ yields,

$$\left(\frac{e_c^{n+1}-e_c^n}{\Delta t},\psi\right)+\left(\mathbf{u_h}^n\cdot\nabla e_c^{n+1},\psi\right)+(\mathbf{e_u}^n\cdot\nabla C_h(t_{n+1}),\psi)+$$

$$\frac{1}{PrLe}\left(\nabla e_c^{n+1},\nabla\psi\right)=\left(R_c^{n+1},\psi\right) \tag{4.57}$$

$$\forall\psi\in W^h.$$

Since $e_c^{n+1}=C_h(t_{n+1})-C_n^{n+1}\in W^h$, we set $\psi=e_c^{n+1}$ in (4.57).

$$\frac{1}{2\Delta t}\left(||e_c^{n+1}||^2-||e_c^n||^2+||e_c^{n+1}-e_c^n||^2\right)+\left(\mathbf{e_u}^n\cdot\nabla C_h(t_{n+1}),e_c^{n+1}\right)+$$

$$\frac{1}{PrLe}||\nabla e_c^{n+1}||^2=\left(R_c^{n+1},e_c^{n+1}\right).$$

As noted earlier $\left(e_c^{n+1},R_c^{n+1}\right)\leq||R_c^{n+1}||_{-1}\ ||e_c^{n+1}||_1$. Since $||\nabla w||$ and $||w||_1$ are equivalent norms for $w\in H_0^1$, we have

$$\frac{1}{2\Delta t}\left(||e_c^{n+1}||^2-||e_c^n||^2+||e_c^{n+1}-e_c^n||^2\right)+\left(\mathbf{e_u}^n\cdot\nabla C_h(t_{n+1}),e_c^{n+1}\right)+$$

$$\frac{1}{PrLe}||\nabla e_c^{n+1}||^2\leq K||R_c^{n+1}||_{-1}\ ||\nabla e_c^{n+1}||.$$

Applying Young's Inequality and the kickback argument, we get

$$||e_c^{n+1}||^2 - ||e_c^n||^2 + ||e_c^{n+1} - e_c^n||^2 + \frac{\Delta t}{PrLe}||\nabla e_c^{n+1}||^2 \leq 2\Delta t \left| \left(\mathbf{e_u}^n \cdot \nabla C_h(t_{n+1}), e_c^{n+1} \right) \right| + \Delta t K ||R_c^{n+1}||_{-1}^2. \tag{4.58}$$

We now estimate the term $2\Delta t \left| \left(\mathbf{e}_\mathbf{u}^n \cdot \nabla C_h(t_{n+1}), e_c^{n+1} \right) \right|$. First, we define the constant $\alpha = \min\left(\frac{1}{2Pr}, \frac{1}{2PrLe}, 1 \right)$. By Proposition 3.1, $||\nabla C_h||$ is bounded. So, by Holder's, Ladyzhenskaya and Young's Inequalities

$$\begin{aligned}
2\Delta t \left| \left(\mathbf{e}_\mathbf{u}^n \cdot \nabla C_h(t_{n+1}), e_c^{n+1} \right) \right| &\leq K\Delta t \, ||\mathbf{e}_\mathbf{u}^n||_{L_4} \, ||\nabla C_h(t_{n+1})|| \, \left|\left| e_c^{n+1} \right|\right|_{L_4} \\
&\leq K\Delta t \, ||\mathbf{e}_\mathbf{u}^n||^{1/2} \, ||\nabla \mathbf{e}_\mathbf{u}^n||^{1/2} \, \left|\left| e_c^{n+1} \right|\right|^{1/2} \left|\left| \nabla e_c^{n+1} \right|\right|^{1/2} \\
&\leq K\Delta t \left|\left| e_c^{n+1} \right|\right|^2 + \frac{\Delta t}{2PrLe} \left|\left| \nabla e_c^{n+1} \right|\right|^2 \\
&+ K\Delta t \, ||\mathbf{e}_\mathbf{u}^n||^2 + \frac{\alpha \Delta t}{3} \, ||\nabla \mathbf{e}_\mathbf{u}^n||^2 .
\end{aligned}$$

Applying this estimate to (4.58), we get

$$\left|\left| e_c^{n+1} \right|\right|^2 - ||e_c^n||^2 + \left|\left| e_c^{n+1} - e_c^n \right|\right|^2 + \frac{\Delta t}{2PrLe} \left|\left| \nabla e_c^{n+1} \right|\right|^2 \leq \frac{\alpha \Delta t}{3} \, ||\nabla \mathbf{e}_\mathbf{u}^n||^2 + K\Delta t \, ||\mathbf{e}_\mathbf{u}^n||^2 + K\Delta t \left|\left| e_c^{n+1} \right|\right|^2 + K\Delta t \left|\left| R_c^{n+1} \right|\right|_{-1}^2 . \tag{4.59}$$

Similarly by subtracting (4.31) from (4.38) and estimating the terms as above, we get

$$\left|\left|e_\theta^{n+1}\right|\right|^2 - ||e_\theta^n||^2 + \left|\left|e_\theta^{n+1} - e_\theta^n\right|\right|^2 + \frac{\Delta t}{2Pr}\left|\left|\nabla e_\theta^{n+1}\right|\right|^2 \leq \frac{\alpha \Delta t}{3}||\nabla \mathbf{e}_\mathbf{u}^n||^2 + K\Delta t\,||\mathbf{e}_\mathbf{u}^n||^2 + K\Delta t\left|\left|e_\theta^{n+1}\right|\right|^2 + K\Delta t\left|\left|R_\theta^{n+1}\right|\right|_{-1}^2 . \tag{4.60}$$

We now turn to velocity. Subtracting (4.30) from (4.37), we get

$$\left(\frac{\mathbf{e}_\mathbf{u}^{n+1} - \mathbf{e}_\mathbf{u}^n}{\Delta t}, \mathbf{v}\right) + (\mathbf{u_h}(t_n) \cdot \nabla \mathbf{u_h}(t_{n+1}), \mathbf{v}) - \left(\mathbf{u_h}^n \cdot \nabla \mathbf{u_h}^{n+1}, \mathbf{v}\right) + \left(\nabla \mathbf{e}_\mathbf{u}^{n+1}, \nabla \mathbf{v}\right) = G_{r_\theta}\left(e_\theta^{n+1}\mathbf{j}, \mathbf{v}\right) + G_{r_c}\left(e_c^{n+1}\mathbf{j}, \mathbf{v}\right) + \left(R_\mathbf{u}^{n+1}, \mathbf{v}\right).$$

Adding and subtracting $(\mathbf{u_h}^n \cdot \nabla \mathbf{u_h}(t_{n+1}), \mathbf{v})$, we get

$$\left(\frac{\mathbf{e}_\mathbf{u}^{n+1} - \mathbf{e}_\mathbf{u}^n}{\Delta t}, \mathbf{v}\right) + \left(\mathbf{u_h}^n \cdot \nabla \mathbf{e}_\mathbf{u}^{n+1}, \mathbf{v}\right) + (\mathbf{e_u}^n \cdot \nabla \mathbf{u_h}(t_{n+1}), \mathbf{v}) + \left(\nabla \mathbf{e}_\mathbf{u}^{n+1}, \nabla \mathbf{v}\right) = G_{r_\theta}\left(e_\theta^{n+1}\mathbf{j}, \mathbf{v}\right) + G_{r_c}\left(e_c^{n+1}\mathbf{j}, \mathbf{v}\right) + \left(R_\mathbf{u}^{n+1}, \mathbf{v}\right).$$

Since $\mathbf{e}_\mathbf{u}^{n+1} = \mathbf{u_h}(t_{n+1}) - \mathbf{u_h}^{n+1} \in \mathbf{V^h}$, we can set $\mathbf{v} = \mathbf{e}_\mathbf{u}^{n+1}$ in the above equation. By the anti-symmetric property, $\left(\mathbf{u_h}^n \cdot \nabla \mathbf{e}_\mathbf{u}^{n+1}, \mathbf{e}_\mathbf{u}^{n+1}\right) = 0$ yielding,

$$\left|\left|\mathbf{e}_\mathbf{u}^{n+1}\right|\right|^2 - ||\mathbf{e}_\mathbf{u}^n||^2 + \left|\left|\mathbf{e}_\mathbf{u}^{n+1} - \mathbf{e}_\mathbf{u}^n\right|\right|^2 + 2\Delta t\left|\left|\nabla \mathbf{e}_\mathbf{u}^{n+1}\right|\right|^2 \leq 2\Delta t\left|\left(\mathbf{e_u}^n \cdot \nabla \mathbf{u_h}(t_{n+1}), \mathbf{e}_\mathbf{u}^{n+1}\right)\right| + 2\Delta t G_{r_\theta}\left(e_\theta^{n+1}\mathbf{j}, \mathbf{e}_\mathbf{u}^{n+1}\right) + 2\Delta t G_{r_c}\left(e_c^{n+1}\mathbf{j}, \mathbf{e}_\mathbf{u}^{n+1}\right) + 2\Delta t\left(R_\mathbf{u}^{n+1}, \mathbf{e}_\mathbf{u}^{n+1}\right). \tag{4.61}$$

We now estimate the tri-linear term $2\Delta t \left|\left(\mathbf{e_u}^n \cdot \nabla \mathbf{u_h}(t_{n+1}), \mathbf{e_u}^{n+1}\right)\right|$ as follows. By Holder's Inequality

$$2\Delta t \left|\left(\mathbf{e_u}^n \cdot \nabla \mathbf{u_h}(t_{n+1}), \mathbf{e_u}^{n+1}\right)\right| \leq K \left|\left|\mathbf{e_u}^n\right|\right|_{L_4} \left|\left|\nabla \mathbf{u_h}(t_{n+1})\right|\right| \left|\left|\mathbf{e_u}^{n+1}\right|\right|_{L_4}.$$

By Proposition 3.1, $||\nabla \mathbf{u_h}||$ is bounded. So by Ladyzhenskaya, Poincare and Young's Inequalities,

$$\begin{aligned}
2\Delta t \left|\left(\mathbf{e_u}^n \cdot \nabla \mathbf{u_h}(t_{n+1}), \mathbf{e_u}^{n+1}\right)\right| &\leq K\Delta t \left|\left|\mathbf{e_u}^n\right|\right|^{1/2} \left|\left|\nabla \mathbf{e_u}^n\right|\right|^{1/2} \left|\left|\mathbf{e_u}^{n+1}\right|\right|^{1/2} \left|\left|\nabla \mathbf{e_u}^{n+1}\right|\right|^{1/2} \\
&\leq K\Delta t \left|\left|\mathbf{e_u}^n\right|\right|^{1/2} \left|\left|\nabla \mathbf{e_u}^n\right|\right|^{1/2} \left|\left|\nabla \mathbf{e_u}^{n+1}\right|\right| \\
&\leq K\Delta t \left|\left|\mathbf{e_u}^n\right|\right| \left|\left|\nabla \mathbf{e_u}^n\right|\right| + \frac{\Delta t}{4} \left|\left|\nabla \mathbf{e_u}^{n+1}\right|\right|^2 \\
&\leq K\Delta t \left|\left|\mathbf{e_u}^n\right|\right|^2 + \frac{\alpha \Delta t}{3} \left|\left|\nabla \mathbf{e_u}^n\right|\right|^2 + \frac{\Delta t}{4} \left|\left|\nabla \mathbf{e_u}^{n+1}\right|\right|^2. \qquad (4.62)
\end{aligned}$$

Next we estimate the terms on the right-hand side of (4.61) using a combination of Cauchy, Poincare and Young's Inequalities,

$$\begin{aligned}
2\Delta t G_{r_\theta} \left|\left(e_\theta^{n+1}\,\mathbf{j}, \mathbf{e_u}^{n+1}\right)\right| &\leq K\Delta t \left|\left|e_\theta^{n+1}\right|\right| \left|\left|\mathbf{e_u}^{n+1}\right|\right| \leq K\Delta t \left|\left|e_\theta^{n+1}\right|\right| \left|\left|\nabla \mathbf{e_u}^{n+1}\right|\right| \\
&\leq K\Delta t \left|\left|e_\theta^{n+1}\right|\right|^2 + \frac{\Delta t}{4} ||\nabla \mathbf{e_u}^{n+1}||^2.
\end{aligned}$$

Similarly,

$$2\Delta t G_{r_c} \left|\left(e_c^{n+1}\,\mathbf{j}, \mathbf{e_u}^{n+1}\right)\right| \leq K\Delta t \left|\left|e_c^{n+1}\right|\right|^2 + \frac{\Delta t}{4} ||\nabla \mathbf{e_u}^{n+1}||^2$$

and

$$2\Delta t \left|\left(R_{\mathbf{u}}^{n+1}\,\mathbf{j}, \mathbf{e}_{\mathbf{u}}^{n+1}\right)\right| \leq K\Delta t \left|\left|R_{\mathbf{u}}^{n+1}\right|\right|_*^2 + \frac{\Delta t}{4}\left|\left|\nabla \mathbf{e}_{\mathbf{u}}^{n+1}\right|\right|^2.$$

Using the above estimates (4.61) becomes

$$\begin{aligned}\left|\left|\mathbf{e}_{\mathbf{u}}^{n+1}\right|\right|^2 - ||\mathbf{e}_{\mathbf{u}}^{n}||^2 + \left|\left|\mathbf{e}_{\mathbf{u}}^{n+1} - \mathbf{e}_{\mathbf{u}}^{n}\right|\right|^2 + \Delta t||\nabla \mathbf{e}_{\mathbf{u}}^{n+1}||^2 \quad &\leq \frac{\alpha \Delta t}{3}||\nabla \mathbf{e}_{\mathbf{u}}^{n}||^2 + K\Delta t\,||\mathbf{e}_{\mathbf{u}}^{n}||^2 \\ + K\Delta t\left(\left|\left|e_{\theta}^{n+1}\right|\right|^2 + \left|\left|e_{c}^{n+1}\right|\right|^2\right) + K\Delta t\left|\left|R_{\mathbf{u}}^{n+1}\right|\right|_*^2. \qquad (4.63)\end{aligned}$$

Let $\mathbf{e} = (\mathbf{e}_{\mathbf{u}}, e_c, e_\theta)$. Adding inequalities (4.59), (4.60) and (4.63) gives us

$$\begin{aligned}\left|\left|\mathbf{e}^{n+1}\right|\right|^2 - ||\mathbf{e}^{n}||^2 + \left|\left|\mathbf{e}^{n+1} - \mathbf{e}^{n}\right|\right|^2 + \alpha\Delta t\left|\left|\nabla \mathbf{e}^{n+1}\right|\right|^2 \leq \alpha\Delta t\,||\nabla \mathbf{e}^{n}||^2 \\ + K\Delta t\left(\left|\left|\mathbf{e}^{n+1}\right|\right|^2 + ||\mathbf{e}^{n}||^2 + \left|\left|R_{\mathbf{u}}^{n+1}\right|\right|_*^2 + \left|\left|R_{\theta}^{n+1}\right|\right|_{-1}^2 + \left|\left|R_{c}^{n+1}\right|\right|_{-1}^2\right). \qquad (4.64)\end{aligned}$$

Summing (4.64) from $n = 0$ to $n = N-1$, we get

$$\begin{aligned}\left|\left|\mathbf{e}^{N}\right|\right|^2 + \sum_{n=0}^{N-1}\left|\left|\mathbf{e}^{n+1} - \mathbf{e}^{n}\right|\right|^2 + \alpha\Delta t\left|\left|\nabla \mathbf{e}^{N}\right|\right|^2 \leq \alpha\Delta t\left|\left|\nabla \mathbf{e}^{0}\right|\right|^2 + K\Delta t\sum_{n=0}^{N-1}\left|\left|\mathbf{e}^{n+1}\right|\right|^2 \\ + K\Delta t\sum_{n=0}^{N-1}\left(\left|\left|R_{\mathbf{u}}^{n+1}\right|\right|_*^2 + \left|\left|R_{\theta}^{n+1}\right|\right|_{-1}^2 + \left|\left|R_{c}^{n+1}\right|\right|_{-1}^2\right). \qquad (4.65)\end{aligned}$$

To show that $\sum_{n=0}^{N-1}||R_\theta||_{-1}^2 = \mathcal{O}(\Delta t)$, we begin by estimating $\left|\left(R_{\theta}^{n+1}, \phi\right)\right|$. By (4.41), we have

$$\begin{aligned}\left|\left(R_{\theta}^{n+1}, \phi\right)\right| \leq \left|\left(\partial_t \theta_h(t_{n+1}) - \frac{\theta_h(t_{n+1}) - \theta_h(t_n)}{\Delta t}, \phi\right)\right| \\ + \left|\left(\left(\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n)\right)\cdot \nabla \theta_h(t_{n+1}), \phi\right)\right|. \qquad (4.66)\end{aligned}$$

Let

$$X = \left| \left(\partial_t \theta_h(t_{n+1}) - \frac{\theta_h(t_{n+1}) - \theta_h(t_n)}{\Delta t}, \phi \right) \right|$$

and

$$Y = |((\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n)) \cdot \nabla \theta_h(t_{n+1}), \phi)|$$

and write

$$\sum_{n=0}^{N-1} \left| \left(R_\theta^{n+1}, \phi \right) \right|^2 = \sum_{n=0}^{N-1} X^2 + \sum_{n=0}^{N-1} 2XY + \sum_{n=0}^{N-1} Y^2.$$

To estimate $\sum_{n=0}^{N-1} X^2$, we start with (4.51) and (4.54) and employ the Cauchy Inequality.

$$\begin{aligned}
\sum_{n=0}^{N-1} X^2 &= \sum_{n=0}^{N-1} \left| \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} (t - t_n) \left(\partial_t^2 \theta_h(t), \phi \right) \, dt \right|^2 \\
&= \frac{1}{(\Delta t)^2} \sum_{n=0}^{N-1} \left(\int_{t_n}^{t_{n+1}} (t - t_n) \left(\partial_t^2 \theta_h(t), \phi \right) \, dt \right) \left(\int_{t_n}^{t_{n+1}} (t - t_n) \left(\partial_t^2 \theta_h(t), \phi \right) \, dt \right) \\
&\leq \frac{1}{(\Delta t)^2} \sum_{n=0}^{N-1} \left(\int_{t_n}^{t_{n+1}} (t - t_n)^2 \, dt \right) \left(\int_{t_n}^{t_{n+1}} \left| \left(\partial_t^2 \theta_h(t), \phi \right) \right|^2 \, dt \right) \\
&= \frac{T \Delta t}{3} \int_0^T \left| \left(\partial_t^2 \theta_h(t), \phi \right) \right|^2 \, dt \leq \frac{T \Delta t}{3} \int_0^T \left| \left| \partial_t^2 \theta_h(t) \right| \right|_{-1}^2 ||\nabla \phi||^2 \, dt \\
&= \frac{T \Delta t}{3} ||\nabla \phi||^2 \int_0^T \left| \left| \partial_t^2 \theta_h(t) \right| \right|_{-1}^2 \, dt \leq K \Delta t \, ||\nabla \phi||^2 ,
\end{aligned} \tag{4.67}$$

where the last integral is bounded by Proposition 3.1. To estimate $\sum_{n=0}^{N-1} Y^2$, we use the definition of $\hat{\mathbf{u}}$ in (4.43), square the result obtained in (4.44), employ Poincare's inequality and the inequality from (4.48).

$$\sum_{n=0}^{N-1} Y^2 \leq \sum_{n=0}^{N-1} ||\hat{\mathbf{u}}|| \, ||\nabla \hat{\mathbf{u}}|| \, ||\nabla \phi||^2 \leq ||\nabla \phi||^2 \sum_{n=0}^{N-1} ||\nabla \hat{\mathbf{u}}||^2 \leq K \Delta t \, ||\nabla \phi||^2 . \tag{4.68}$$

Similarly by Young's inequality and (4.67)-(4.68), we have $\sum_{n=0}^{N-1} 2XY \leq K\Delta t \, ||\nabla\phi||^2$. Therefore, we have $\sum_{n=0}^{N-1} \frac{\left|\left(R_\theta^{n+1}, \phi\right)\right|^2}{||\nabla\phi||^2} \leq K\Delta t$. That is,

$$\sum_{n=0}^{N-1} \left|\left|R_\theta^{n+1}\right|\right|_{-1}^2 = \mathcal{O}(\Delta t). \tag{4.69}$$

Similarly

$$\sum_{n=0}^{N-1} \left|\left|R_{\mathbf{u}}^{n+1}\right|\right|_{-1}^2 = \mathcal{O}(\Delta t) \tag{4.70}$$

and

$$\sum_{n=0}^{N-1} \left|\left|R_c^{n+1}\right|\right|_{-1}^2 = \mathcal{O}(\Delta t). \tag{4.71}$$

So we write (4.65) as

$$\begin{aligned} \left|\left|\mathbf{e}^N\right|\right|^2 + \sum_{n=0}^{N-1} \left|\left|\mathbf{e}^{n+1} - \mathbf{e}^n\right|\right|^2 + \alpha\Delta t \left|\left|\nabla\mathbf{e}^N\right|\right|^2 &\leq \left|\left|\mathbf{e}^0\right|\right|^2 + \alpha\Delta t \left|\left|\nabla\mathbf{e}^0\right|\right|^2 \\ &+ K\Delta t \sum_{n=0}^{N-1} \left|\left|\mathbf{e}^{n+1}\right|\right|^2 + K(\Delta t)^2. \end{aligned} \tag{4.72}$$

Thus by the Discrete Gronwall Lemma 4.5, we get the desired estimate. □

4.3.4 Semi-Implicit: Pressure Error

The following theorem provides an estimate for ϵ^n for the semi-implicit scheme.

Theorem 4.10. *Assuming that the* inf-sup *condition holds (Assumption C), the error of the pressure term between the semi-discrete approximation, $p_h(t_n)$, and the fully implicit* *approximation, p_h^n, satisfies the following:*

$$\left(\sum_{n=1}^{N-1} \Delta t\,||\epsilon^n||^2\right)^{1/2} = \mathcal{O}(\sqrt{\Delta t}),$$

provided Δt is sufficiently small.

Proof. Once the velocity, temperature and concentration $\{(\mathbf{u_h^n}, \theta_h^n, C_h^n)\}$ are determined, we can compute the approximations p_h^n for the pressure $p_h(t_n)$ from the weak formulations of the doubly diffusive model in non-divergence-free function spaces. We seek $(\mathbf{u_h}, \theta_h, C_h) \in \mathbf{W}^h \times W^h \times W^h$ and $p_h^n \in L^h$ such that

$$\begin{aligned}\left(p_h^{n+1}, \nabla\cdot\mathbf{v_h}\right) = \left(\frac{\mathbf{u_h^{n+1}} - \mathbf{u_h^n}}{\Delta t}, \mathbf{v_h}\right) + \left(\nabla\mathbf{u_h^{n+1}}, \nabla\mathbf{v_h}\right) + \left(\mathbf{u_h^n}\cdot\nabla\mathbf{u_h^{n+1}}, \mathbf{v_h}\right) \\ - \left(\left(G_{r_\theta}\theta_h^{n+1} + G_{r_c}C_h^{n+1}\right)\mathbf{j}, \mathbf{v_h}\right) - \left(\mathbf{f}^{n+1}, \mathbf{v_h}\right) \qquad \forall \mathbf{v_h}\in\mathbf{W}^h,\end{aligned} \tag{4.73}$$

where $\mathbf{W}^h$, W^h and L^h are subspaces of $\mathbf{H}_0^1$, H_0^1 and L_0^2 satisfying the *inf-sup* condition. Let $\epsilon^n := p_h(t_n) - p_h^n, n = 1, 2, \ldots, N$ be the error in pressure. Then the error ϵ^n satisfies the equation,

$$\begin{aligned}(\epsilon^n, \nabla\cdot\mathbf{v_h}) = \left(\frac{\mathbf{e_u^{n+1}} - \mathbf{e_u^n}}{\Delta t}, \mathbf{v_h}\right) + \left(\nabla\mathbf{e_u^{n+1}}, \nabla\mathbf{v_h}\right) + \left(\mathbf{u_h}(t_n)\cdot\nabla\mathbf{e_u^{n+1}}, \mathbf{v_h}\right) \\ + \left(\mathbf{e_u^n}\cdot\nabla\mathbf{u_h^{n+1}}, \mathbf{v_h}\right) - \left(\left(G_{r_\theta}e_\theta^{n+1} + G_{r_c}e_c^{n+1}\right)\mathbf{j}, \mathbf{v_h}\right) + \left(R_{\mathbf{u}}^{n+1}, \mathbf{v_h}\right).\end{aligned} \tag{4.74}$$

Note that the *inf-sup* condition implies,

$$K\,||\epsilon^n|| \leq \sup_{\mathbf{v_h}\in\mathbf{W^h}} \frac{(\epsilon^n, \nabla\cdot\mathbf{v_h})}{||\mathbf{v_h}||_1}, \quad \text{since} \;\; \epsilon^n \in L_h.$$

Using this and the Poincare inequality in (4.74) we have,

$$\begin{aligned}||\epsilon^n|| \leq \frac{K}{\Delta t}\left|\left|\mathbf{e}_{\mathbf{u}}^{n+1} - \mathbf{e}_{\mathbf{u}}^{n}\right|\right| + K\left|\left|\nabla\mathbf{e}_{\mathbf{u}}^{n+1}\right|\right| + K\left|\left|\nabla\mathbf{e}_{\mathbf{u}}^{n+1}\right|\right| ||\mathbf{u_h}(t_n)|| \\ + K\left|\left|\mathbf{u_h}^{n+1}\right|\right|_1 ||\nabla\mathbf{e}_{\mathbf{u}}^{n}|| + K\left|\left|\nabla e_{\theta}^{n+1}\right|\right| + K\left|\left|\nabla e_{c}^{n+1}\right|\right| + K\left|\left|R_{\mathbf{u}}^{n+1}\right|\right|_* .\end{aligned}$$

By Proposition 3.1, $||\mathbf{u_h}(t_n)||$ is bounded and by Remark 1, $\left|\left|\mathbf{u_h}^{n+1}\right|\right|_1$ is bounded. Therefore,

$$\Delta t\, ||\epsilon^n|| \leq K\left|\left|\mathbf{e}^{n+1} - \mathbf{e}^{n}\right|\right| + K\Delta t\left|\left|\nabla\mathbf{e}^{n+1}\right|\right| + K\Delta t\left|\left|R_{\mathbf{u}}^{n+1}\right|\right|_* .$$

Squaring both sides and applying Young's inequality to the right hand side yields,

$$\begin{aligned}(\Delta t)^2\, ||\epsilon^n||^2 &\leq \left(K\left|\left|\mathbf{e}^{n+1} - \mathbf{e}^{n}\right|\right| + K\Delta t\left|\left|\nabla\mathbf{e}^{n+1}\right|\right| + K\Delta t\left|\left|R_{\mathbf{u}}^{n+1}\right|\right|_{-1}\right)^2 \\ &\leq K\left|\left|\mathbf{e}^{n+1} - \mathbf{e}^{n}\right|\right|^2 + K(\Delta t)^2\left|\left|\nabla\mathbf{e}^{n+1}\right|\right|^2 + K(\Delta t)^2\left|\left|R_{\mathbf{u}}^{n+1}\right|\right|_*^2 .\end{aligned}$$

Summing from $n = 0$ to $n = N - 1$ and using (4.70) and the estimates from Theorem 4.9 yields,

$$(\Delta t)^2 \sum_{n=0}^{N-1} ||\epsilon^n||^2 \leq K \sum_{n=0}^{N-1} \left|\left|\mathbf{e}^{n+1} - \mathbf{e}^n\right|\right|^2 + K(\Delta t)^2 \sum_{n=0}^{N-1} \left|\left|\nabla \mathbf{e}^{n+1}\right|\right|^2 + K(\Delta t)^2 \sum_{n=0}^{N-1} \left|\left|R_{\mathbf{u}}^{n+1}\right|\right|_*^2$$
$$\leq K(\Delta t)^2 + K(\Delta t)^3 + K(\Delta t)^3 \leq K(\Delta t^2),$$

which implies $\left(\sum_{n+1}^{N-1} \Delta t \, ||\epsilon^n||^2\right)^{1/2} = \mathcal{O}(\sqrt{\Delta t})$. □

4.4 Semi-Implicit Decoupled Scheme

The semi-implicit decoupled scheme is stated below:
seek $(\mathbf{u_h}^{n+1}, \theta_h^{n+1}, C_h^{n+1}) \in \mathbf{V^h} \times W^h \times W^h$ such that

$$\left(\frac{\mathbf{u_h}^{n+1} - \mathbf{u_h}^n}{\Delta t}, \mathbf{v}\right) + \left(\mathbf{u_h}^n \cdot \nabla \mathbf{u_h}^{n+1}, \mathbf{v}\right) + \left(\nabla \mathbf{u_h}^{n+1}, \nabla \mathbf{v}\right) = \left(\mathbf{f}^{n+1}, \mathbf{v}\right) + G_{r_\theta}\left(\theta_h^n \mathbf{j}, \mathbf{v}\right) + G_{r_c}\left(C_h^n \mathbf{j}, \mathbf{v}\right) \tag{4.75}$$

$$\left(\frac{\theta_h^{n+1} - \theta_h^n}{\Delta t}, \phi\right) + \left(\mathbf{u_h}^n \cdot \nabla \theta_h^{n+1}, \phi\right) + \frac{1}{Pr}\left(\nabla \theta_h^{n+1}, \nabla \phi\right) = \left(Q^{n+1}, \phi\right) \tag{4.76}$$

$$\left(\frac{C_h^{n+1} - C_h^n}{\Delta t}, \psi\right) + \left(\mathbf{u_h}^n \cdot \nabla C_h^{n+1}, \psi\right) + \frac{1}{PrLe}\left(\nabla C_h^{n+1}, \nabla \psi\right) = \left(\hat{Q}^{n+1}, \psi\right) \tag{4.77}$$

$$\forall (\mathbf{v}, \phi, \psi) \in \mathbf{V^h} \times W^h \times W^h \quad , \quad n = 0, 1, \ldots, N-1,$$

and

$$\mathbf{u_h}^0 = \mathbf{u_h}(0), \; \theta_h^0 = \theta_h(0) \text{ and } C_h^0 = C_h(0).$$

4.4.1 Semi-Implicit Decoupled: Stability Bounds

We now present a priori stability bounds in the following lemma.

Lemma 4.11. (Semi-Implicit Decoupled: A priori Stability Bounds) *The solutions* $\mathbf{u_h}^n$*,* θ_h^n *and* C_h^n *of equations* (4.75)*,* (4.76) *and* (4.77) *satisfy the following a priori bounds:*

$$\max_{1\le m\le N} ||\theta_h^m||^2 + \frac{\Delta t}{Pr}\sum_{n=0}^{N-1} ||\nabla\theta_h^{n+1}||^2 \le M_\theta$$
$$\max_{1\le m\le N} ||C_h^m||^2 + \frac{\Delta t}{PrLe}\sum_{n=0}^{N-1} ||\nabla C_h^{n+1}||^2 \le M_C$$
$$\max_{1\le m\le N} ||\mathbf{u}_h^m||^2 + \frac{\Delta t}{2}\sum_{n=0}^{N-1} ||\nabla\mathbf{u_h}^{n+1}||^2 \le M.$$

Proof. Note, the only difference between this scheme and the previous *semi-implicit* scheme is in the right-hand side of the momentum equation. Therefore, from the previous proof we have the following stability bounds for θ_h^m and C_h^m.

$$\max_{1\le m\le N} ||\theta_h^m||^2 + \frac{\Delta t}{Pr}\sum_{n=0}^{N-1} ||\nabla\theta_h^{n+1}||^2 \le M_\theta$$

$$\max_{1\le m\le N} ||C_h^m||^2 + \frac{\Delta t}{PrLe}\sum_{n=0}^{N-1} ||\nabla C_h^{n+1}||^2 \le M_C.$$

We proceed with velocity. Setting $\mathbf{v} = \mathbf{u_h}^{n+1}$ in (4.75) yields

$$\left(\frac{\mathbf{u_h}^{n+1} - \mathbf{u_h}^n}{\Delta t}, \mathbf{u_h}^{n+1}\right) + \left(\mathbf{u_h}^n \cdot \nabla\mathbf{u_h}^{n+1}, \mathbf{u_h}^{n+1}\right) + \left(\nabla\mathbf{u_h}^{n+1}, \nabla\mathbf{u_h}^{n+1}\right) = \left(\mathbf{f}^{n+1}, \mathbf{u_h}^{n+1}\right)$$
$$+ G_{r_\theta}\left(\theta_h^n\mathbf{j}, \mathbf{u_h}^{n+1}\right) + G_{r_c}\left(C_h^n\mathbf{j}, \mathbf{u_h}^{n+1}\right).$$

By Property 3.4, $\left(\mathbf{u_h}^n \cdot \nabla \mathbf{u_h}^{n+1}, \mathbf{u_h}^{n+1}\right) = 0$. So by Cauchy and Poincare inequalities and Property 4.2,

$$\frac{1}{2\Delta t}(||\mathbf{u_h}^{n+1}||^2 - ||\mathbf{u_h}^n||^2 + ||\mathbf{u_h}^{n+1} - \mathbf{u_h}^n||^2) + ||\nabla \mathbf{u_h}^{n+1}||^2 \leq K||\mathbf{f}^{n+1}|| \; ||\nabla \mathbf{u_h}^{n+1}|| + K\,||\theta_h^n||\,\left|\left|\nabla \mathbf{u_h}^{n+1}\right|\right| + K\left|\left|C_h^h\right|\right|\left|\left|\nabla \mathbf{u_h}^{n+1}\right|\right|. \tag{4.78}$$

Since $||\theta_h^n||^2$, $||C_h^n||^2$ and $\left|\left|\mathbf{f}^{n+1}\right|\right|^2$ are bounded, employing the kickback argument yields,

$$||\mathbf{u_h}^{n+1}||^2 - ||\mathbf{u_h}^n||^2 + \Delta t||\nabla \mathbf{u_h}^{n+1}||^2 \leq K\Delta t. \tag{4.79}$$

Since $\left|\left|\mathbf{u_h}^0\right|\right|^2$ is bounded, summing (4.79) from $n = 0$ to $N - 1$ yields

$$||\mathbf{u_h}^N||^2 + \Delta t \sum_{n=0}^{N-1} ||\nabla \mathbf{u_h}^{n+1}||^2 \leq M.$$

□

4.4.2 Semi-Implicit Decoupled: Consistency

We now present the following consistency result.

Theorem 4.12. (Semi-Implicit Decoupled: Consistency)

The Semi-Implicit Decoupled Scheme is consistent. Moreover, the residuals in a weak sense vanish linearly with Δt.

Proof. We obtain residuals by substituting the exact solutions ($u_h(t_n)$, $\theta_h(t_n)$, $C_h(t_n)$) of (3.13)-(3.15) into (4.75)-(4.77)

$$\left(\frac{\mathbf{u_h}(t_{n+1}) - \mathbf{u_h}(t_n)}{\Delta t}, \mathbf{v}\right) + (\mathbf{u_h}(t_n) \cdot \nabla \mathbf{u_h}(t_{n+1}), \mathbf{v}) +$$

$$(\nabla \mathbf{u_h}(t_{n+1}), \nabla \mathbf{v}) = (\mathbf{f}(t_{n+1}), \mathbf{v}) + G_{r_\theta}(\theta_h(t_n)\mathbf{j}, \mathbf{v}) + G_{r_c}(C_h(t_n)\mathbf{j}, \mathbf{v}) + \left(R_{\mathbf{u}}^{n+1}, \mathbf{v}\right) \tag{4.80}$$

$$\left(\frac{\theta_h(t_{n+1}) - \theta_h(t_n)}{\Delta t}, \phi\right) + (\mathbf{u_h}(t_n) \cdot \nabla \theta_h(t_{n+1}), \phi) +$$

$$\frac{1}{Pr}(\nabla \theta_h(t_{n+1}), \nabla \phi) = (Q(t_{n+1}), \phi) + \left(R_\theta^{n+1}, \phi\right) \tag{4.81}$$

$$\left(\frac{C_h(t_{n+1}) - C_h(t_n)}{\Delta t}, \psi\right) + (\mathbf{u_h}(t_n) \cdot \nabla C_h(t_{n+1}), \psi) +$$

$$\frac{1}{PrLe}(\nabla C_h(t_{n+1}), \nabla \psi) = \left(\hat{Q}(t_{n+1}), \psi\right) + \left(R_c^{n+1}, \psi\right). \tag{4.82}$$

Subtracting (4.80)-(4.82) from the weak form (3.13)-(3.15) at t_{n+1} yields,

$$\left(\partial_t \mathbf{u_h}(t_{n+1}) - \frac{\mathbf{u_h}(t_{n+1}) - \mathbf{u_h}(t_n)}{\Delta t}, \mathbf{v}\right) +$$

$$G_{r_\theta}(\theta_h(t_{n+1}) - \theta_h(t_n), \mathbf{v}) + G_{r_c}(C_h(t_{n+1}) - C_h(t_n), \mathbf{v}) +$$

$$((\mathbf{u_h}(t_{n+1}) - \mathbf{u_h}(t_n)) \cdot \nabla \mathbf{u_h}(t_{n+1}), \mathbf{v}) = \left(R_{\mathbf{u}}^{n+1}, \mathbf{v}\right) \quad , \forall \mathbf{v} \in \mathbf{V^h} \tag{4.83}$$

$$\left(\partial_t \theta_h(t_{n+1}) - \frac{\theta_h(t_{n+1}) - \theta_h(t_n)}{\Delta t}, \phi\right) +$$

$$((\mathbf{u_h}(t_{n+1}) - \mathbf{u_h}(t_n)) \cdot \nabla \theta_h(t_{n+1}), \phi) = \left(R_\theta^{n+1}, \phi\right) \quad , \forall \phi \in \mathbf{W}^h \tag{4.84}$$

$$\left(\partial_t C_h(t_{n+1}) - \frac{C_h(t_{n+1}) - C_h(t_n)}{\Delta t}, \psi\right) +$$
$$((\mathbf{u_h}(t_{n+1}) - \mathbf{u_h}(t_n)) \cdot \nabla C_h(t_{n+1}), \psi) = \left(R_c^{n+1}, \psi\right) \qquad , \forall \psi \in \mathbf{W}^h. \tag{4.85}$$

Since (4.84) and (4.85) are the same as (4.41) and (4.42) from the previous *semi-implicit* proof, we have $\left|\left|R_\theta^{n+1}\right|\right|_{-1} = \mathcal{O}(\sqrt{\Delta t})$ and $\left|\left|R_c^{n+1}\right|\right|_{-1} = \mathcal{O}(\sqrt{\Delta t})$. Note that (4.83) is the same as (4.40) in the *semi-implicit* proof except for the terms

$$G_{r_\theta}\left(\theta_h(t_{n+1}) - \theta_h(t_n), \mathbf{v}\right) \quad \text{and} \quad G_{r_c}\left(C_h(t_{n+1}) - C_h(t_n), \mathbf{v}\right)$$

on the left-hand side. If these two terms are bounded by $K\sqrt{\Delta t}\,||\nabla \mathbf{v}||$, then the rest of the proof proceeds as in the *semi-implicit* case. To this end, define

$$\hat{\theta} = \int_{t_n}^{t_{n+1}} \partial_t \theta_h(s)\, ds = \theta_h(t_{n+1}) - \theta_h(t_n). \tag{4.86}$$

By Cauchy Inequality

$$|(\theta_h(t_{n+1}) - \theta_h(t_n), \mathbf{v})| = \left|\left(\hat{\theta}, \mathbf{v}\right)\right| \leq K \left|\left|\hat{\theta}\right|\right| \;\; ||\mathbf{v}||\,. \tag{4.87}$$

But

$$\left|\left|\hat{\theta}\right|\right| = \left|\left|\int_{t_n}^{t_{n+1}} \partial_t \theta_h(s)\, ds\right|\right| = \left[\int_\Omega \left(\int_{t_n}^{t_{n+1}} \partial_t \theta_h(s)\, ds\right)^2 d\Omega\right]^{1/2} \tag{4.88}$$

and by Cauchy-Schwarz

$$\int_{t_n}^{t_{n+1}} \partial_t \theta_h(s)\, ds \leq \left(\int_{t_n}^{t_{n+1}} (\partial_t \theta_h(s))^2\, ds \right)^{1/2} \left(\int_{t_n}^{t_{n+1}} 1^2 \right)^{1/2}$$
$$= (\Delta t)^{1/2} \left(\int_{t_n}^{t_{n+1}} (\partial_t \theta_h(s))^2\, ds \right)^{1/2}. \tag{4.89}$$

Using (4.89) in (4.88) and applying Fubini's Theorem, we get

$$\left|\left|\hat{\theta}\right|\right| \leq \left[\int_\Omega \Delta t \int_{t_n}^{t_{n+1}} (\partial_t \theta_h(s))^2\, ds\ d\Omega \right]^{1/2} = (\Delta t)^{1/2} \left[\int_{t_n}^{t_{n+1}} \int_\Omega (\partial_t \theta_h(s))^2\, d\Omega\ ds \right]^{1/2}$$
$$= (\Delta t)^{1/2} \left[\int_{t_n}^{t_{n+1}} ||\partial_t \theta_h(s)||^2\ \ ds \right]^{1/2}.$$

Since $||\partial_t \theta_h(s)||$ is bounded [see Proposition 3.1], we have

$$\left|\left|\hat{\theta}\right|\right| \leq K(\Delta t)^{1/2}. \tag{4.90}$$

Using this in (4.87), we get $(\theta_h(t_{n+1}) - \theta_h(t_n), \mathbf{v}) \leq K\sqrt{\Delta t}\, ||\nabla \mathbf{v}||$. Similarly, $(C_h(t_{n+1}) - C_h(t_n), \mathbf{v}) \leq K\sqrt{\Delta t}\, ||\nabla \mathbf{v}||$. So, in a manner similar to the *semi-implicit* case, we get $\left|\left|R_{\mathbf{u}}^{n+1}\right|\right|_* = \mathcal{O}(\sqrt{\Delta t})$. Therefore, the scheme is consistent. □

4.4.3 Semi-Implicit Decoupled: Convergence

Theorem 4.13. (Semi-Implicit Decoupled Convergence)

Assume that $\left|\left|\mathbf{e}^0\right|\right|^2 = \mathcal{O}((\Delta t)^2)$ *and* $\left|\left|\nabla \mathbf{e}^0\right|\right|^2 = \mathcal{O}(\Delta t)$. *The error between the solutions* $(\mathbf{u_h}^n, \theta_h^n, C_h^n)$ *of* (4.30)-(4.32) *and the exact solutions* $(\mathbf{u_h}(t_n), \theta_h(t_n), C_h(t_n))$ *of the semi-*

discrete equations (3.13)-(3.15) *satisfy the following:*

$$||\mathbf{u_h}(t_n) - \mathbf{u_h}^n||^2 + ||\theta_h(t_n) - \theta_h^n||^2 + ||C_h(t_n) - C_h^n||^2 = \mathcal{O}(\Delta t^2).$$

Moreover,

$$\sum_{n=0}^{N-1} \left(\left|\left|\mathbf{e}^{n+1} - \mathbf{e}^n\right|\right|^2 \right) = \mathcal{O}(\Delta t^2),$$

where

$$\begin{aligned}\mathbf{e}^n &= (\mathbf{e}_\mathbf{u}^n, e_\theta^n, e_c^n) \\ &= (\mathbf{u_h}(t_n) - \mathbf{u_h}^n, \theta_h(t_n) - \theta_h^n, C_h(t_n) - C_h^n)\,.\end{aligned}$$

The convergence proof requires the following lemma.

Lemma 4.14 (Discrete Gronwall Lemma II)**.** *Let C, τ, a_n, b_n, c_n, and d_n be nonnegative numbers such that*

$$a_m + \tau \sum_{n=1}^{m} b_n \leq \tau \sum_{n=0}^{m-1} a_n d_n + \tau \sum_{n=0}^{m-1} c_n + C\,, \quad m \geq 1\,.$$

Then

$$a_m + \tau \sum_{n=1}^{m} b_n \leq \exp\left(\tau \sum_{n=0}^{m-1} d_n \right) \left(\tau \sum_{n=0}^{m-1} c_n + C \right), \quad m \geq 1\,.$$

Proof. See, [24]. □

Proof. (Semi-Implicit Decoupled Convergence) The temperature and concentration equations are the same as in the previous *semi-implicit* case. However, we must proceed in a slightly different manner. Starting with the concentration inequality, (4.58), from the *semi-implicit* proof,

$$||e_c^{n+1}||^2 - ||e_c^n||^2 + ||e_c^{n+1} - e_c^n||^2 + \frac{\Delta t}{PrLe}||\nabla e_c^{n+1}||^2 \leq 2\Delta t \left|\left(\mathbf{e_u}^n \cdot \nabla C_h(t_{n+1}), e_c^{n+1}\right)\right| + \Delta t K||R_c^{n+1}||_{-1}^2. \tag{4.91}$$

We now estimate the term $2\Delta t \left|\left(\mathbf{e}_\mathbf{u}^n \cdot \nabla C_h(t_{n+1}), e_c^{n+1}\right)\right|$. By Proposition 3.1, $||C_h||_2$ is bounded. So, by Holder's, Ladyzhenskaya and Young's Inequalities

$$\begin{aligned}
2\Delta t \left|\left(\mathbf{e}_\mathbf{u}^n \cdot \nabla C_h(t_{n+1}), e_c^{n+1}\right)\right| &\leq K\Delta t \,||\mathbf{e}_\mathbf{u}^n||\, ||\nabla C_h(t_{n+1})||_{L^4} \left|\left|e_c^{n+1}\right|\right|_{L^4} \\
&\leq K\Delta t \,||\mathbf{e}_\mathbf{u}^n||\, ||\nabla C_h||^{1/2} ||C_h||_2^{1/2} \left|\left|e_c^{n+1}\right|\right|^{1/2} \left|\left|\nabla e_c^{n+1}\right|\right|^{1/2} \\
&\leq K\Delta t \,||\mathbf{e}_\mathbf{u}^n|| \left|\left|\nabla e_c^{n+1}\right|\right| \\
&\leq \frac{\Delta t}{2PrLe} \left|\left|\nabla e_c^{n+1}\right|\right|^2 + K\Delta t \,||\mathbf{e}_\mathbf{u}^n||^2 .
\end{aligned}$$

Applying this estimate to (4.91), we get

$$\left|\left|e_c^{n+1}\right|\right|^2 - ||e_c^n||^2 + \left|\left|e_c^{n+1} - e_c^n\right|\right|^2 + \frac{\Delta t}{2PrLe} \left|\left|\nabla e_c^{n+1}\right|\right|^2 \leq K\Delta t \,||\mathbf{e}_\mathbf{u}^n||^2 + K\Delta t \left|\left|R_c^{n+1}\right|\right|_{-1}^2 . \tag{4.92}$$

Similarly,

$$\left|\left|e_\theta^{n+1}\right|\right|^2 - ||e_\theta^n||^2 + \left|\left|e_\theta^{n+1} - e_\theta^n\right|\right|^2 + \frac{\Delta t}{2Pr} \left|\left|\nabla e_\theta^{n+1}\right|\right|^2 \leq K\Delta t \,||\mathbf{e}_\mathbf{u}^n||^2 + K\Delta t \left|\left|R_\theta^{n+1}\right|\right|_{-1}^2 . \tag{4.93}$$

Turning now to velocity, subtracting (4.83) from the exact solution $u_h(t_n)$ of (3.15) yields,

$$\left(\frac{\mathbf{e_u}^{n+1} - \mathbf{e_u}^n}{\Delta t}, \mathbf{v}\right) + (\mathbf{u_h}(t_n) \cdot \nabla \mathbf{u_h}(t_{n+1}), \mathbf{v}) - \left(\mathbf{u_h}^n \cdot \nabla \mathbf{u_h}^{n+1}, \mathbf{v}\right)$$
$$+ \left(\nabla \mathbf{e_u}^{n+1}, \nabla \mathbf{v}\right) = G_{r_\theta} \left(e_\theta^n \mathbf{j}, \mathbf{v}\right) + G_{r_c} \left(e_c^n \mathbf{j}, \mathbf{v}\right) + \left(R_{\mathbf{u}}^{n+1}, \mathbf{v}\right).$$

Adding and subtracting $(\mathbf{u_h}^n \cdot \nabla \mathbf{u_h}(t_{n+1}), \mathbf{v})$, we get

$$\left(\frac{\mathbf{e_u}^{n+1} - \mathbf{e_u}^n}{\Delta t}, \mathbf{v}\right) + \left(\mathbf{u_h}^n \cdot \nabla \mathbf{e_u}^{n+1}, \mathbf{v}\right) + (\mathbf{e_u}^n \cdot \nabla \mathbf{u_h}(t_{n+1}), \mathbf{v}) + \left(\nabla \mathbf{e_u}^{n+1}, \nabla \mathbf{v}\right)$$
$$= G_{r_\theta} \left(e_\theta^n \mathbf{j}, \mathbf{v}\right) + G_{r_c} \left(e_c^n \mathbf{j}, \mathbf{v}\right) + \left(R_{\mathbf{u}}^{n+1}, \mathbf{v}\right).$$

Since $\mathbf{e_u}^{n+1} = \mathbf{u_h}(t_{n+1}) - \mathbf{u_h}^{n+1} \in \mathbf{V^h}$, we can set $\mathbf{v} = \mathbf{e_u}^{n+1}$ in the above equation. By the anti-symmetric property, $\left(\mathbf{u_h}^n \cdot \nabla \mathbf{e_u}^{n+1}, \mathbf{e_u}^{n+1}\right) = 0$. Thus,

$$\left|\left|\mathbf{e_u}^{n+1}\right|\right|^2 - \left|\left|\mathbf{e_u}^n\right|\right|^2 + \left|\left|\mathbf{e_u}^{n+1} - \mathbf{e_u}^n\right|\right|^2 + 2\Delta t \left|\left|\nabla \mathbf{e_u}^{n+1}\right|\right|^2$$
$$\leq 2\Delta t \left|\left(\mathbf{e_u}^n \cdot \nabla \mathbf{u_h}(t_{n+1}), \mathbf{e_u}^{n+1}\right)\right| + 2\Delta t G_{r_\theta} \left(e_\theta^n \mathbf{j}, \mathbf{e_u}^{n+1}\right)$$
$$+ 2\Delta t G_{r_c} \left(e_c^n \mathbf{j}, \mathbf{e_u}^{n+1}\right) + 2\Delta t \left(R_{\mathbf{u}}^{n+1}, \mathbf{e_u}^{n+1}\right). \tag{4.94}$$

The tri-linear term $2\Delta t \left|\left(\mathbf{e_u}^n \cdot \nabla \mathbf{u_h}(t_{n+1}), \mathbf{e_u}^{n+1}\right)\right|$ can be estimated in a similar manner as the *semi-implicit* proof, (4.62), yielding,

$$2\Delta t \left|\left(\mathbf{e_u}^n \cdot \nabla \mathbf{u_h}(t_{n+1}), \mathbf{e_u}^{n+1}\right)\right| \leq K\Delta t \left|\left|\mathbf{e_u}^n\right|\right|^2 + \alpha \Delta t \left|\left|\nabla \mathbf{e_u}^n\right|\right|^2 + \frac{\Delta t}{4} \left|\left|\nabla \mathbf{e_u}^{n+1}\right|\right|^2, \tag{4.95}$$

where $\alpha = \min\left(\frac{1}{2Pr}, \frac{1}{2PrLe}, 1\right)$. Next we estimate the terms on the right-hand side of (4.94) using a combination of Cauchy, Poincare and Young's Inequalities.

$$\begin{aligned}
2\Delta t G_{r_\theta} \left|\left(e_\theta^n\, \mathbf{j}, \mathbf{e_u}^{n+1}\right)\right| &\le K\Delta t\, ||e_\theta^n||^2 + \frac{\Delta t}{4}||\nabla \mathbf{e_u}^{n+1}||^2 \\
2\Delta t G_{r_c} \left|\left(e_c^n\, \mathbf{j}, \mathbf{e_u}^{n+1}\right)\right| &\le K\Delta t\, ||e_c^n||^2 + \frac{\Delta t}{4}||\nabla \mathbf{e_u}^{n+1}||^2 \\
2\Delta t \left|\left(R_{\mathbf{u}}^{n+1}\, \mathbf{j}, \mathbf{e_u}^{n+1}\right)\right| &\le K\Delta t \left|\left|R_{\mathbf{u}}^{n+1}\right|\right|_*^2 + \frac{\Delta t}{4}\left|\left|\nabla \mathbf{e_u}^{n+1}\right|\right|^2.
\end{aligned}$$

Using these estimates (4.94) becomes,

$$\begin{aligned}
\left|\left|\mathbf{e_u}^{n+1}\right|\right|^2 - ||\mathbf{e_u}^n||^2 + \left|\left|\mathbf{e_u}^{n+1} - \mathbf{e_u}^n\right|\right|^2 + \Delta t||\nabla \mathbf{e_u}^{n+1}||^2 &\le \alpha\Delta t\, ||\nabla \mathbf{e_u}^n||^2 + K\Delta t\, ||\mathbf{e_u}^n||^2 \\
&+ K\Delta t \left(||e_\theta^n||^2 + ||e_c^n||^2\right) + K\Delta t \left|\left|R_{\mathbf{u}}^{n+1}\right|\right|_*^2. \qquad (4.96)
\end{aligned}$$

Adding inequalities (4.92), (4.93) and (4.96) gives us

$$\begin{aligned}
\left|\left|\mathbf{e}^{n+1}\right|\right|^2 - ||\mathbf{e}^n||^2 + \left|\left|\mathbf{e}^{n+1} - \mathbf{e}^n\right|\right|^2 + \alpha\Delta t \left|\left|\nabla \mathbf{e}^{n+1}\right|\right|^2 &\le \alpha\Delta t\, ||\nabla \mathbf{e}^n||^2 \\
+ K\Delta t \left(||\mathbf{e}^n||^2\right) + K\Delta t \left(\left|\left|R_{\mathbf{u}}^{n+1}\right|\right|_*^2 + \left|\left|R_\theta^{n+1}\right|\right|_{-1}^2 + \left|\left|R_c^{n+1}\right|\right|_{-1}^2\right)&. \qquad (4.97)
\end{aligned}$$

Summing (4.97) from $n = 0$ to $n = N-1$, we get

$$\begin{aligned}
\left|\left|\mathbf{e}^N\right|\right|^2 + \sum_{n=0}^{N-1} \left|\left|\mathbf{e}^{n+1} - \mathbf{e}^n\right|\right|^2 + \alpha\Delta t \left|\left|\nabla \mathbf{e}^N\right|\right|^2 &\le \left|\left|\mathbf{e}^0\right|\right|^2 + \alpha\Delta t \left|\left|\nabla \mathbf{e}^0\right|\right|^2 \\
+ K\Delta t \sum_{n=0}^{N-1} ||\mathbf{e}^n||^2 + K\Delta t \sum_{n=0}^{N-1} \left(\left|\left|R_{\mathbf{u}}^{n+1}\right|\right|_*^2 + \left|\left|R_\theta^{n+1}\right|\right|_{-1}^2 + \left|\left|R_c^{n+1}\right|\right|_{-1}^2\right)&. \qquad (4.98)
\end{aligned}$$

From the *semi-implicit* proof $\sum_{n=0}^{N-1} \left|\left|R_\theta^{n+1}\right|\right|_{-1}^2 = \mathcal{O}(\Delta t)$ and $\sum_{n=0}^{N-1} \left|\left|R_c^{n+1}\right|\right|_{-1}^2 = \mathcal{O}(\Delta t)$. To show that $\sum_{n=0}^{N-1} \left|\left|R_{\mathbf{u}}^{n+1}\right|\right|_*^2 = \mathcal{O}(\Delta t)$, we begin by estimating $\left|\left(R_{\mathbf{u}}^{n+1}, \mathbf{v}\right)\right|$. By (4.83) we have

$$\left|\left(R_{\mathbf{u}}^{n+1},\mathbf{v}\right)\right| \leq \left|\left(\partial_t \mathbf{u_h}(t_{n+1}) - \frac{\mathbf{u_h}(t_{n+1}) - \mathbf{u_h}(t_n)}{\Delta t}, \mathbf{v}\right)\right|$$
$$+ \left|\left((\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n)) \cdot \nabla \mathbf{u_h}(t_{n+1}), \mathbf{v}\right)\right|$$
$$+ G_{r_\theta}\left|(\theta_h(t_{n+1}) - \theta_h(t_n), \mathbf{v})\right| + G_{r_c}\left|(C_h(t_{n+1}) - C_h(t_n), \mathbf{v})\right|$$

Let

$$X = \left|\left(\partial_t \mathbf{u_h}(t_{n+1}) - \frac{\mathbf{u_h}(t_{n+1}) - \mathbf{u_h}(t_n)}{\Delta t}, \mathbf{v}\right)\right|,$$

$$Y = \left|\left((\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n)) \cdot \nabla \mathbf{u_h}(t_{n+1}), \mathbf{v}\right)\right|$$

and

$$Z = G_{r_\theta}\left|(\theta_h(t_{n+1}) - \theta_h(t_n), \mathbf{v})\right| + G_{r_c}\left|(C_h(t_{n+1}) - C_h(t_n), \mathbf{v})\right|$$

and write

$$\sum_{n=0}^{N-1}\left|\left(R_{\mathbf{u}}^{n+1},\mathbf{v}\right)\right|^2 \leq \sum_{n=0}^{N-1} X^2 + \sum_{n=0}^{N-1} Y^2 + \sum_{n=0}^{N-1} Z^2 + \sum_{n=0}^{N-1} 2XY + \sum_{n=0}^{N-1} 2XZ + \sum_{n=0}^{N-1} 2YZ.$$

From the *semi-implicit* proof, $\sum_{n=0}^{N-1} X^2 \leq K\Delta t\,||\nabla \mathbf{v}||$, $\sum_{n=0}^{N-1} Y^2 \leq K\Delta t\,||\nabla \mathbf{v}||$ and $\sum_{n=0}^{N-1} 2XY \leq K\Delta t\,||\nabla \mathbf{v}||$. To show $\sum_{n=0}^{N-1} Z^2 \leq K\Delta t\,||\nabla \mathbf{v}||$ start with the definition of $\hat{\theta}$ in (4.86), square both sides of (4.87), use (4.90) and sum both sides from $n = 0$ to $N-1$. This yields, $\sum_{n=0}^{N-1}\left|(\theta_h(t_{n+1}) - \theta_h(t_n), \mathbf{v})\right|^2 \leq K\Delta t\,||\nabla \mathbf{v}||$. Defining $\hat{C}$ in a similar way we get the analogous result, $\sum_{n=0}^{N-1}\left|(C_h(t_{n+1}) - C_h(t_n), \mathbf{v})\right|^2 \leq K\Delta t\,||\nabla \mathbf{v}||$. It is also clear from (4.87) and (4.90) that

$$\sum_{n=0}^{N-1}\left|(C_h(t_{n+1}) - C_h(t_n), \mathbf{v})\right|\left|(\theta_h(t_{n+1}) - \theta_h(t_n), \mathbf{v})\right| \leq K\Delta t\,||\nabla \mathbf{v}||\,.$$

Finally, by Young's inequality, $\sum_{n=0}^{N-1} 2XZ \leq K\Delta t\,||\nabla \mathbf{v}||$ and $\sum_{n=0}^{N-1} 2YZ \leq K\Delta t\,||\nabla \mathbf{v}||$. Using these results and the definition of the dual space norm, (4.98) can be written as

$$\left|\left|\mathbf{e}^N\right|\right|^2 + \sum_{n=0}^{N-1} \left|\left|\mathbf{e}^{n+1} - \mathbf{e}^n\right|\right|^2 + \alpha\Delta t\left|\left|\nabla\mathbf{e}^N\right|\right|^2 \leq \left|\left|\mathbf{e}^0\right|\right|^2 + \alpha\Delta t\left|\left|\nabla\mathbf{e}^0\right|\right|^2 + K\Delta t\sum_{n=0}^{N-1} ||\mathbf{e}^n||^2 + K(\Delta t)^2.$$

Thus by the Discrete Gronwall II Lemma 4.14, we get the desired estimate. □

4.4.4 Semi-Implicit Decoupled: Pressure Error

Theorem 4.15. *Assuming that the* inf-sup *condition holds (Assumption C), the error of the pressure term between the semi-discrete approximation,* $p_h(t_n)$*, and the semi-implicit decoupled approximation,* p_h^n*, satisfies the following:*

$$\left(\sum_{n=1}^{N-1} \Delta t\,||\epsilon^n||^2\right)^{1/2} = \mathcal{O}(\sqrt{\Delta t}).$$

Proof. Once the velocity, temperature and concentration $\{(\mathbf{u_h}^n, \theta_h^n, C_h^n)\}$ are determined, we can compute the approximations p_h^n for the pressure $p_h(t_n)$ from the weak formulations of the doubly diffusive model in non-divergence-free function spaces. We seek $(\mathbf{u_h}, \theta_h, C_h) \in \mathbf{W}^h \times W^h \times W^h$ and $p_h^n \in L^h$ such that

$$\left(p_h^{n+1}, \nabla\cdot\mathbf{v_h}\right) = \left(\frac{\mathbf{u_h}^{n+1} - \mathbf{u_h}^n}{\Delta t}, \mathbf{v_h}\right) + \left(\nabla\mathbf{u_h}^{n+1}, \nabla\mathbf{v_h}\right) + \left(\mathbf{u_h}^n\cdot\nabla\mathbf{u_h}^{n+1}, \mathbf{v_h}\right) - \left(\left(G_{r_\theta}\theta_h^n + G_{r_c}C_h^n\right)\mathbf{j}, \mathbf{v_h}\right) - \left(\mathbf{f}^{n+1}, \mathbf{v_h}\right) \quad \forall \mathbf{v_h} \in \mathbf{W}^h, \tag{4.99}$$

where $\mathbf{W}^h$, W^h and L^h are subspaces of $\mathbf{H}_0^1$, H_0^1 and L_0^2 satisfying the *inf-sup* condition $(B3)$. Let

$$\epsilon^n := p_h(t_n) - p_h^n, \qquad n = 1, 2, \ldots, N$$

be the error in pressure. Then the error ϵ^n satisfies the equation,

$$\begin{aligned}(\epsilon^n, \nabla \cdot \mathbf{v_h}) = &\left(\frac{\mathbf{e_u}^{n+1} - \mathbf{e_u}^n}{\Delta t}, \mathbf{v_h}\right) + \left(\nabla \mathbf{e_u}^{n+1}, \nabla \mathbf{v_h}\right) + \left(\mathbf{u_h}(t_n) \cdot \nabla \mathbf{e_u}^{n+1}, \mathbf{v_h}\right) \\ &+ \left(\mathbf{e_u}^n \cdot \nabla \mathbf{u_h}^{n+1}, \mathbf{v_h}\right) - \left((G_{r_\theta} e_\theta^n + G_{r_c} e_c^n)\mathbf{j}, \mathbf{v_h}\right) + \left(R_{\mathbf{u}}^{n+1}, \mathbf{v_h}\right). \end{aligned} \tag{4.100}$$

Note that the *inf-sup* condition (Assumption Bii) implies,

$$K \left|\left|\epsilon^n\right|\right| \leq \sup_{\mathbf{v_h} \in \mathbf{W^h}} \frac{(\epsilon^n, \nabla \cdot \mathbf{v_h})}{\left|\left|\mathbf{v_h}\right|\right|_1}, \quad \text{since} \ \ \epsilon^n \in L_h.$$

Using this and the Poincare inequality in (4.100) we have,

$$\begin{aligned}\left|\left|\epsilon^n\right|\right| \leq &\frac{K}{\Delta t} \left|\left|\mathbf{e_u}^{n+1} - \mathbf{e_u}^n\right|\right| + K \left|\left|\nabla \mathbf{e_u}^{n+1}\right|\right| + K \left|\left|\nabla \mathbf{e_u}^{n+1}\right|\right| \left|\left|\mathbf{u_h}(t_n)\right|\right| \\ &+ K \left|\left|\mathbf{u_h}^{n+1}\right|\right|_1 \left|\left|\nabla \mathbf{e_u}^n\right|\right| + K \left|\left|\nabla e_\theta^n\right|\right| + K \left|\left|\nabla e_c^n\right|\right| + K \left|\left|R_{\mathbf{u}}^{n+1}\right|\right|_*. \end{aligned}$$

By Proposition 3.1, $\left|\left|\mathbf{u_h}(t_n)\right|\right|$ is bounded and by Remark 1, $\left|\left|\mathbf{u_h}^{n+1}\right|\right|_1$ is bounded. Therefore,

$$\begin{aligned}\Delta t \left|\left|\epsilon^n\right|\right| \leq &K \left|\left|\mathbf{e}^{n+1} - \mathbf{e}^n\right|\right| + K \Delta t \left|\left|\nabla \mathbf{e}^{n+1}\right|\right| \\ &+ K \Delta t \left|\left|\nabla \mathbf{e}^n\right|\right| + K \Delta t \left|\left|R_{\mathbf{u}}^{n+1}\right|\right|_*. \end{aligned}$$

Squaring both sides and applying Young's inequality to the right hand side yields,

$$(\Delta t)^2 \left|\left|\epsilon^n\right|\right|^2 \leq K \left|\left|\mathbf{e}^{n+1} - \mathbf{e}^n\right|\right|^2 + K(\Delta t)^2 \left|\left|\nabla \mathbf{e}^{n+1}\right|\right|^2$$
$$+ K(\Delta t)^2 \left|\left|\nabla \mathbf{e}^n\right|\right|^2 + K(\Delta t)^2 \left|\left|R_{\mathbf{u}}^{n+1}\right|\right|_*^2.$$

Summing from $n = 0$ to $n = N - 1$ and using (4.70) and the estimates from Theorem 4.13 gives,

$$(\Delta t)^2 \sum_{n=0}^{N-1} \left|\left|\epsilon^n\right|\right|^2 \leq K \sum_{n=0}^{N-1} \left|\left|\mathbf{e}^{n+1} - \mathbf{e}^n\right|\right|^2 + K(\Delta t)^2 \sum_{n=0}^{N-1} \left|\left|\nabla \mathbf{e}^{n+1}\right|\right|^2$$
$$+ K(\Delta t)^2 \left|\left|\nabla \mathbf{e}^0\right|\right|^2 + K(\Delta t)^2 \sum_{n=0}^{N-1} \left|\left|R_{\mathbf{u}}^{n+1}\right|\right|_*^2$$
$$\leq K(\Delta t)^2 + K(\Delta t)^3 + K(\Delta t)^3 \leq K(\Delta t^2),$$

which implies

$$\left(\sum_{n+1}^{N-1} \Delta t \left|\left|\epsilon^n\right|\right|^2 \right)^{1/2} = \mathcal{O}(\sqrt{\Delta t}).$$

□

CHAPTER 5

CONCLUSIONS

In this dissertation we derived error estimates for the spatially discrete finite element approximation problem for doubly-diffusive flows in Chapter 3 and error estimates for three different time discretization schemes in Chapter 4. All the results assume the *regularity* restrictions on the data given in (2.1)-(2.5).

In Section 3.2, a priori stability estimates, Proposition 3.1, were derived for the semi-discrete problem. These estimates are used throughout the dissertation, in particular they are used to establish the main result of Chapter 3, the optimal order of convergence of the semi-discrete problem, Theorem 3.11. With additional assumptions on the discrete initial data, the optimal order of convergence in the L^2 and H^1 norm were found to be $\mathcal{O}(h^2)$ and $\mathcal{O}(h)$, respectively. Finally, the semi-discrete error analysis is made complete with Theorem 3.14 where the order of convergence of pressure in the L^2 norm is shown to be $\mathcal{O}(h)$.

Chapter 4 introduces three backward Euler time discretization schemes: *fully implicit*, *semi-implicit* and *semi-implicit decoupled*. These three schemes have important differences from an implementation standpoint. The *fully implicit* scheme requires that a non-linear system be solved at each time step, whereas the *semi-implicit* scheme yields a linear system to be solved at each time step. The *semi-implicit decoupled* scheme yields two smaller linear systems that can be solved independant of one another. A priori stability estimates and consistency results were determined for all three schemes. The main result of Chapter 4 is the $\mathcal{O}(\Delta t)$ order of convergence for all three schemes. However, the *fully implicit* and *semi-*

implicit schemes were found to be conditionally convergent and the *semi-implicit decoupled* scheme was found to be unconditionally convergent.

APPENDICES

APPENDIX A

MATHEMATICAL PRELIMINARIES

These fundamental estimates are employed throughout the dissertation.

Cauchy Inequality If $f, g \in L^2(\Omega)$ then $\int_\Omega |fg| \; d\Omega \leq ||f|| \; ||g||$.

Poincare Inequality If $u \in H_0^1(\Omega)$ then $||u|| \leq \lambda ||\nabla u||$, where λ is a positive constant.

Ladyzhenskaya Inequality (see [17] or [25]) If $u \in H_0^1(\Omega)$, then $||u||_{L^4} \leq 2^{\frac{1}{4}} ||u||^{1/2} ||\nabla u||^{1/2}$

Young Inequality If $a, b > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$, then $ab \leq \dfrac{a^p}{p} + \dfrac{b^q}{q}$. Also,

$$ab \leq \frac{a^p \epsilon^p}{p} + \frac{b^q}{q\epsilon^q} \quad \text{for } \epsilon > 0.$$

Holder Inequality if $u \in L^p(\Omega)$, $v \in L^q(\Omega)$, $w \in L^r(\Omega)$ and $\dfrac{1}{p} + \dfrac{1}{q} + \dfrac{1}{r} = 1$, then $uvw \in L^1(\Omega)$ and

$$\int_\Omega |uvw| \; d\Omega \leq ||u||_p \, ||v||_q \, ||w||_r \, .$$

Application of the *Holder Inequality* with $p = 4$, $q = 2$ and $r = 4$.

$$\begin{aligned} |(\mathbf{u} \cdot \nabla \mathbf{v}, \mathbf{w})| = \left| \sum_{i,j=1}^{2} \int_\Omega u_j \frac{\partial v_i}{\partial x_j} w_i \; d\Omega \right| &\leq \sum_{i,j=1}^{2} ||u_i||_{L_4} \left|\left| \frac{\partial v_j}{\partial x_i} \right|\right| ||w_j||_{L_4} \\ &\leq K \, ||\mathbf{u}||_{L_4} \, ||\nabla \mathbf{v}|| \, ||\mathbf{w}||_{L_4} \, . \end{aligned}$$

The following continuous Gronwall inequality is used in the semi-discrete error analysis.

Lemma A.1 (Gronwall lemma)**.** *Suppose* $E : [0,T] \to \mathbb{R}$ *is* C^1 *and* P*,* Q *and* R *are continuous and nonnegative functions. Assume that that*

$$\frac{dE}{dt} + P(t) \leq R(t)E(t) + Q(t) \qquad \text{for} \quad t \in [0,T], \tag{A.1}$$

then

$$E(t) + \int_0^t P(s)\ ds \leq E(0)e^{\Lambda(t)} + e^{\Lambda(t)} \int_0^t Q(s)\ ds,$$
$$\text{where} \ \ \Lambda(t) = \int_0^t R(\tau)\ d\tau.$$

Proof. Begin by writing

$$E'(t) - R(r)E(t) \leq Q(t) - P(t).$$

Multiplying the above by the integrating factor $e^{-\Lambda(t)}$, where $\Lambda(t) = \int_0^t R(\tau)\ d\tau$, we get

$$e^{-\Lambda(t)}E'(t) - e^{-\Lambda(t)}R(r)E(t) \leq (Q(t) - P(t))e^{-\Lambda(t)}.$$

Since $\frac{d}{dt}\left(e^{-\Lambda(t)}E(t)\right) = e^{-\Lambda(t)}E'(t) - R(t)e^{-\Lambda(t)}E(t)$, we have the following inequality

$$\frac{d}{dt}\left(e^{-\Lambda(t)}E(t)\right) \leq (Q(t) - P(t))e^{-\Lambda(t)}.$$

Integrating we get $e^{-\Lambda(t)}E(t) - E(0) \leq \int_0^t (Q(s) - P(s))\, e^{-\Lambda(s)}\ ds$. Therefore,

$$\begin{aligned} E(t) &\leq E(0)e^{\Lambda(t)} + e^{\Lambda(t)} \int_0^t (Q(s) - P(s))\, e^{-\Lambda(s)}\ ds \\ \Rightarrow E(t) + e^{\Lambda(t)} \int_0^t P(s) e^{-\Lambda(s)}\ ds &\leq E(0)e^{\Lambda(t)} + e^{\Lambda(t)} \int_0^t Q(s) e^{-\Lambda(s)}\ ds \\ \Rightarrow E(t) + \int_0^t e^{\int_s^t R(\tau)\ d\tau} P(s)\ ds &\leq E(0)e^{\Lambda(t)} + e^{\Lambda(t)} \int_0^t Q(s) e^{-\Lambda(s)}\ ds. \end{aligned}$$

Since $R \geq 0$, $e^{\int_s^t R(\tau)\ d\tau} \geq 1$ and $e^{-\Lambda(t)} = e^{-\int_0^t R(\tau)\ d\tau} \leq 1$. Therefore we have

$$E(t) + \int_0^t P(s)\ ds \leq E(0)e^{\Lambda(t)} + e^{\Lambda(t)} \int_0^t Q(s)\ ds. \tag{A.2}$$

□

Lemma A.2 (Gagliardo-Nirenberg Inequality). *Let* $\mathbf{v} \in W^{1,p}(\mathbb{R}^n)$*, where* n *is the dimension of the space. For every fixed number* $p, s \geq 1$*, there exists a constant* C *depending only on* n*,*p *and* s *such that*

$$||\mathbf{v}||_{L^q} \leq C\, ||\nabla \mathbf{v}||_{L^p}^{\alpha}\, ||\mathbf{v}||_{L^s}^{\alpha},$$

where $\alpha \in [0,1]$*,* p*,*$q \geq 1$ *and* s *satisfies the following relation*

$$\alpha = \left(\frac{1}{s} - \frac{1}{q}\right)\left(\frac{1}{n} - \frac{1}{p} + \frac{1}{s}\right)^{-1}.$$

Proof. see [26] □

Property A.3.

$$(a - b, a) = \frac{1}{2}||a||^2 - \frac{1}{2}||b||^2 + \frac{1}{2}||a - b||^2$$

Proof. First expand $||a-b||^2 = (a-b, a-b) = ||a||^2 - 2(a,b) + ||b||^2$ to get

$$(a,b) = \frac{1}{2}||a||^2 + \frac{1}{2}||b||^2 - \frac{1}{2}||a-b||^2.$$

Next, write $||a-b||^2 = (a-b, a-b) = (a, a-b) - (b, a-b)$. Combine these two results to get,

$$\begin{aligned}
(a, a-b) &= ||a-b||^2 + (b, a-b) \\
&= ||a-b||^2 + (b,a) - ||b||^2 \\
&= ||a-b||^2 + \left(\frac{1}{2}||a||^2 + \frac{1}{2}||b||^2 - \frac{1}{2}||a-b||^2\right) - ||b||^2 \\
&= \frac{1}{2}||a||^2 - \frac{1}{2}||b||^2 + \frac{1}{2}||a-b||^2.
\end{aligned}$$

□

The tri-linear forms $(\mathbf{u} \cdot \nabla \mathbf{v}, \mathbf{w})$, $(\mathbf{u} \cdot \nabla \theta, \phi)$ and $(\mathbf{u} \cdot \nabla C, \psi)$ satisfy the following Anti-symmetry properties.

Lemma A.4. (Anti-symmetry Properties) *Suppose* $\mathbf{u}$ *is a divergence free vector field, that is* $\nabla \cdot \mathbf{u} = 0$*, with zero boundary conditions. Then the following hold:*

(i) $(\mathbf{u} \cdot \nabla \mathbf{v}, \mathbf{w}) = -(\mathbf{u} \cdot \nabla \mathbf{w}, \mathbf{v}) \quad \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{H}_0^1(\Omega)$

(ii) $(\mathbf{u} \cdot \nabla \theta, \phi) = -(\mathbf{u} \cdot \nabla \phi, \theta) \quad \forall \mathbf{u} \in \mathbf{H}_0^1(\Omega)$ *and* $\theta, \phi \in H_0^1(\Omega)$

(iii) $(\mathbf{u} \cdot \nabla C, \psi) = -(\mathbf{u} \cdot \nabla \psi, C) \quad \forall \mathbf{u} \in \mathbf{H}_0^1(\Omega)$ *and* $C, \psi \in H_0^1(\Omega)$

$$(\mathbf{u} \cdot \nabla \mathbf{v}, \mathbf{v}) = 0 \quad , \quad (\mathbf{u} \cdot \nabla \theta, \theta) = 0 \quad , \quad (\mathbf{u} \cdot \nabla C, C) = 0.$$

The following is a proof of *(ii)* whereby *(iii)* follows directly and *(i)* by extension.

Proof. Since $\mathbf{u}|_{\partial\Omega} = 0$, using integration by parts yields,

$$
\begin{aligned}
(\mathbf{u} \cdot \nabla\theta, \phi) &= \int_\Omega \left(u_1 \frac{\partial \theta}{\partial x} + u_2 \frac{\partial \theta}{\partial y} \right) \phi \, d\Omega \\
&= \int_{\partial\Omega} (u_1 \theta n_x + u_2 \theta n_y) \, \phi \, ds - \int_\Omega \theta \left(\frac{\partial}{\partial x}(u_1 \phi) + \frac{\partial}{\partial y}(u_2 \phi) \right) \, d\Omega \\
&= - \int_\Omega \theta \left(\frac{\partial}{\partial x}(u_1 \phi) + \frac{\partial}{\partial y}(u_2 \phi) \right) \, d\Omega.
\end{aligned}
$$

Using the product rule yields,

$$
\begin{aligned}
(\mathbf{u} \cdot \nabla\theta, \phi) &= - \int_\Omega \theta \left(\phi \frac{\partial u_1}{\partial x} + \phi \frac{\partial u_2}{\partial y} + u_1 \frac{\partial \phi}{\partial x} + u_2 \frac{\partial \phi}{\partial y} \right) \, d\Omega \\
&= - \int_\Omega \theta \phi \nabla \cdot \mathbf{u} \, d\Omega - \int_\Omega \theta \mathbf{u} \cdot \nabla \phi \, d\Omega.
\end{aligned}
$$

But $\nabla \cdot \mathbf{u} = 0$, therefore

$$
(\mathbf{u} \cdot \nabla\theta, \phi) = -(\mathbf{u} \cdot \nabla\phi, \theta).
$$

□

APPENDIX B

INVERSE LAPLACIAN AND INVERSE STOKES OPERATORS

Some of the error analysis in this dissertation will make use of the inverse Laplacian operator $(-\Delta)^{-1}$ and the inverse Stoke's operator A^{-1}. The operator $(-\Delta)^{-1}$ is defined as the inverse of the Laplace operator $\Delta : L^2(\Omega) \to L^2(\Omega)$ in the following sense. Given $\psi \in L^2(\Omega)$, we have by definition of Δ that $\phi = (-\Delta)^{-1}\psi$ is the solution of the following Poisson problem:

$$\begin{aligned} -\Delta\phi &= \psi \text{ in } \Omega \\ \phi &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{B.1}$$

The regularity results for (B.1) gives

$$\left|\left|(-\Delta)^{-1}\psi\right|\right|_s = ||\phi||_s \le K\,||\psi||_{s-2} \quad \text{for} \quad s = 1, 2 \tag{B.2}$$

and

$$\big((-\Delta)^{-1}\psi, \psi\big) = (\phi, \psi) = (-\Delta\phi, \phi) = ||\nabla\phi||^2 \le K\,||\psi||_{-1}^2 . \tag{B.3}$$

We define the Stokes operator $A : D(A) \to H$ with domain $D(A) \subseteq H$ and range $R(A) = \{Au : u \in D(A)\}$ as follows:

Let $D(A) \subseteq \mathbf{V}$ be the space of all $\mathbf{u} \in \mathbf{V}$ for which there exists some $\mathbf{f} \in \mathbf{H}$ satisfying

$$(\nabla \mathbf{u}, \nabla \mathbf{v}) = (\mathbf{f}, \mathbf{v}) \quad , \quad \mathbf{v} \in \mathbf{V}. \tag{B.4}$$

The Riez representation theorem tells us that $D(A)$ is precisely those functions $\mathbf{u} \in \mathbf{V}$ such that the linear functional

$$\mathbf{v} \longmapsto (\nabla \mathbf{u}, \nabla \mathbf{v}) \quad , \quad \mathbf{v} \in \mathbf{V}$$

is continuous in the norm $||\mathbf{v}||$. For all $\mathbf{u} \in D(A)$, we define $A\mathbf{u}$ to be the unique element in $\mathbf{H}$ such that

$$(\nabla \mathbf{u}, \nabla \mathbf{v}) = (A\mathbf{u}, \mathbf{v}) \quad \text{holds} \;\; \forall \mathbf{v} \in \mathbf{H}. \tag{B.5}$$

Thus by (B.4) and (B.5), we have $A\mathbf{u} = \mathbf{f}$ with $\mathbf{f}$ in (B.4).

Theorem B.1 (Stokes operator properties)**.** *Let $A : D(A) \to H$ be the Stokes operator. Then we have*

(i) A is positive self-adjoint with dense domain $D(A) \subseteq \mathbf{H}$. It holds that $N(A) = \{0\}$ and the inverse $A^{-1} : D(A^{-1}) \to \mathbf{H}$ with domain $D(A^{-1}) = R(A)$ is also positive self-adjoint.

(ii) Let $\mathbf{u} \in \mathbf{V}$, $\mathbf{f} \in \mathbf{H}$. The following three statements are equivelent:

(a) $\mathbf{u}$ is a weak solution of the Stokes system (B.4) *with force $\mathbf{f}$.*

(b) $\mathbf{u} \in D(A)$ and $A\mathbf{u} = \mathbf{f}$

(c) There exists $p \in L_0^2(\Omega)$ satisfying, $-\Delta \mathbf{u} + \nabla p = \mathbf{f}$, in the sense of distributions.

(iii) If Ω is bounded, then $D(A^{-1}) = R(A) = H$ and A^{-1} is a bounded operator with operator norm, $||A^{-1}|| \leq K$.

(iv) If Ω is bounded and $\partial\Omega$ is C^2, then $D(A) = \mathbf{V} \cap \mathbf{H}^2$, $A\mathbf{u} = -P_H \Delta \mathbf{u}$ and $||\mathbf{u}||_2 + ||\nabla p|| \leq K ||\mathbf{f}|| \quad \forall \mathbf{u} \in D(A)$, where $P_H : L^2(\Omega) \to \mathbf{H}$ is the Helmholtz operator.

Proof. see [27] □

The operator A^{-1} is defined as the inverse of the Stokes operator $A = -P_H \Delta : \mathbf{V} \to \mathbf{H}$ where $P_H : \mathbf{L}^2 \to \mathbf{H}$ is the projection onto

$$\mathbf{H} = \left\{ \mathbf{v} \in \mathbf{L}^2(\Omega) \mid \nabla \cdot \mathbf{v} = 0 \text{ in } \Omega \text{ and } \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega \right\}.$$

The Stokes operator A is defined for $\mathbf{v} \in D(A)$, where

$$D(A) = \{\mathbf{v} \in \mathbf{V} \mid A\mathbf{v} \in \mathbf{H}\} = \left\{ \mathbf{v} \in \mathbf{H}^2(\Omega) \cap \mathbf{V} \right\} = \mathbf{H}^2 \cap \mathbf{V}.$$

A is an unbounded, positive, self-adjoint closed operator onto $\mathbf{H}$. Given $\mathbf{u} \in \mathbf{H}$, by the definition of A, $\mathbf{v} = A^{-1}\mathbf{u}$ is a solution of the following Stokes problem:

$$\begin{aligned} -\Delta \mathbf{v} + \nabla p &= \mathbf{u} \text{ in } \Omega \\ \nabla \cdot \mathbf{v} &= 0 \text{ in } \Omega \\ \mathbf{v} &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{B.6}$$

The regularity results for (B.6), see [17], give us

$$||\mathbf{v}||_s + ||p||_{s-1} \leq K \, ||\mathbf{u}||_{s-2} \text{ for } s = 1, 2.$$

That is,

$$\left|\left|A^{-1}\mathbf{u}\right|\right|_s \leq K \left|\left|\mathbf{u}\right|\right|_{s-2} \text{ for } s = 1, 2. \tag{B.7}$$

This implies $A^{-1} : \mathbf{H} \to D(A) = \mathbf{H}^2 \cap \mathbf{V}$ is bounded.

Lemma B.2. *The inverse Laplacian operator* $(-\Delta)^{-1}$ *and inverse Stokes operator* A^{-1} *satisfy the following relations:*

(i) $\exists \quad c_1, c_2, c_3 > 0$ *such that* $\forall \psi \in \mathbf{H}$

(a) $c_1 \left|\left|\psi\right|\right|_{-1} \leq \left|\left|(-\Delta)^{-1}\psi\right|\right|_1 \leq c_2 \left|\left|\psi\right|\right|_{-1}$

(b) $c_3 \left|\left|\psi\right|\right|_{-1}^2 \leq \left((-\Delta)^{-1}\psi, \psi\right)$

(ii) $\exists \quad c_4, c_5, c_6 > 0$ *such that* $\forall \mathbf{u} \in \mathbf{H}$

(a) $c_4 \left|\left|\mathbf{u}\right|\right|_* \leq \left|\left|A^{-1}\mathbf{u}\right|\right|_1 \leq c_5 \left|\left|\mathbf{u}\right|\right|_*$

(b) $c_6 \left|\left|\mathbf{u}\right|\right|_*^2 \leq \left(A^{-1}\mathbf{u}, \mathbf{u}\right)$.

Proof. Let ψ and ϕ be as in the Stokes problem (B.1).

(ia):

$$\begin{aligned}\left|\left|\psi\right|\right|_{-1} &= \sup_{\hat{\psi} \in H_0^1} \frac{\left(\psi, \hat{\psi}\right)}{\left|\left|\hat{\psi}\right|\right|_1} = \sup_{\hat{\psi} \in H_0^1} \frac{\left(-\Delta\phi, \hat{\psi}\right)}{\left|\left|\hat{\psi}\right|\right|_1} = \sup_{\hat{\psi} \in H_0^1} \frac{\left(\nabla\phi, \nabla\hat{\psi}\right)}{\left|\left|\hat{\psi}\right|\right|_1} \\ &\leq K_1 \left|\left|\nabla\phi\right|\right|_1 = \left|\left|\phi\right|\right|_1 = \left|\left|(-\Delta)^{-1}\psi\right|\right|_1\end{aligned}$$

Thus, $\left|\left|\psi\right|\right|_{-1} \leq K_1 \left|\left|(-\Delta)^{-1}\psi\right|\right|_1$. On the other hand from (B.2) we have $\left|\left|(-\Delta)^{-1}\psi\right|\right|_1 \leq K_2 \left|\left|\psi\right|\right|_{-1}$. Therefore, $\left|\left|\psi\right|\right|_{-1} \leq K_1 \left|\left|(-\Delta)^{-1}\psi\right|\right|_1 \leq K_2 \left|\left|\psi\right|\right|_{-1}$.

(ib):

$$\left((-\Delta)^{-1}\psi, \psi\right) = (\phi, \psi) = (\phi, (-\Delta\phi)) = (\nabla\phi, \nabla\phi)$$
$$= ||\nabla\phi||^2 \geq K\,||\phi||_1^2 = \left|\left|(-\Delta)^{-1}\psi\right|\right|_1^2 \geq K\,||\psi||_{-1}^2\,.$$

(iia): First,

$$||\mathbf{u}||_* = \sup_{\mathbf{w}\in\mathbf{V}} \frac{(\mathbf{u}, \mathbf{w})}{||\mathbf{w}||_1} = \sup_{\mathbf{w}\in\mathbf{V}} \frac{(A\mathbf{v}, \mathbf{w})}{||\mathbf{w}||_1} = \sup_{\mathbf{w}\in\mathbf{V}} \frac{(\nabla\mathbf{v}, \nabla\mathbf{w})}{||\mathbf{w}||_1}$$
$$\leq K\,||\mathbf{v}||_1 = K\left|\left|A^{-1}\mathbf{u}\right|\right|_1\,. \tag{B.8}$$

Next, $\left|\left|A^{-1}\mathbf{u}\right|\right|_1^2 \leq \left|\left|\nabla(A^{-1}\mathbf{u})\right|\right|^2 = \left(\mathbf{u}, A^{-1}\mathbf{u}\right) \leq ||\mathbf{u}||_*\left|\left|A^{-1}\mathbf{u}\right|\right|_1$. This yields $\left|\left|A^{-1}\mathbf{u}\right|\right|_1 \leq ||\mathbf{u}||_*$.

(iib):

$$\left(A^{-1}\mathbf{u}, \mathbf{u}\right) = (\mathbf{v}, \mathbf{u}) = (\mathbf{v}, A\mathbf{v}) = (\nabla\mathbf{v}, \nabla\mathbf{v}) = ||\nabla\mathbf{v}||^2$$
$$\geq K\,||\mathbf{v}||_1^2 = K\left|\left|A^{-1}\mathbf{u}\right|\right|_1^2 \geq K\,||\mathbf{u}||_*^2$$

Therefore,

$$\left(A^{-1}\mathbf{u}, \mathbf{u}\right) \geq K\,||\mathbf{u}||_*^2$$

□

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