A Mosquito Population Model with Four Positive Equilibrium Points and Sterile Insect Release

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A Mosquito Population Model with Four Positive Equilibrium Points And Sterile Insect Release

by

Maxwell Joseph Fox

An Honors Capstone
submitted in partial fulfillment of the requirements
for the Honors Diploma
to

The Honors College

of

The University of Alabama in Huntsville

04/26/2021

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Dedication

I would like to thank

- Dr. Shangbing Ai, Professor of Mathematical Sciences, for educating me on dynamical systems and guiding me throughout this research project,

- Dr. Mark Pekker, Professor of Mathematical Sciences, for educating me on dynamical systems and numerical bifurcations,

- Dr. Toka Diagana, Professor and Department Chair of the Department of Mathematical Sciences, for guiding me toward research opportunities such as this,

- the UAH Summer 2020 Research and Creative Experiences for Undergraduates (RCEU) Program, for providing me with the opportunity and resources to pursue this topic,

- Dave Cook, Director of Undergraduate Research, Honors College, for guiding me through undergraduate research, including the RCEU Program and the Honors College Capstone,

- all my friends and family, for supporting me through this tumultuous time.
Abstract

A recently proposed ODE system for modeling the interactive dynamics of sterile and wild mosquito populations can have up to four positive equilibria, but only two positive equilibria have been confirmed to exist in the literature. In this thesis, we show that, for a range of parameters, the system does have four positive equilibria. We also obtain the stabilities of these equilibria.
1 Introduction and Main Result

Mosquito-borne diseases, such as malaria and dengue fever, are a global health concern, especially in tropical and subtropical regions. Because of the threat posed by these diseases, population control methods for mosquito populations are a much discussed topic. One such method for controlling mosquito populations is the sterile insect technique (SIT), where sterilized males are released to breed with wild females in order to reduce mosquito birth rates.

To determine the effectiveness and impact of SIT, researchers have dedicated a body of mathematical literature to modeling mosquito populations controlled by various sterile insect release methods. These models involve various methods, such as those that use ordinary differential equations (ODEs) versus those that use delay differential equations (DDEs) to account for a time delay in the mating process. Various sterile insect release rates have also been modeled, such as a constant release rate and one proportional to the wild population, among others (see [1–5] and the references therein).

One such release method involves a proportional release rate with saturation, where the rate of sterile insect release approaches a limit as the wild population becomes very large. Using an ODE system for the wild and sterile populations, it was previously proved [2] that, assuming equal death rates for both populations, there were only up to two positive equilibria for the system. Further discussion has focused on a more general case [3], where it was argued that this result would hold for any death rates greater than zero. However, we prove that for a range of parameters, this system has 4 positive equilibria, the largest possible number of positive equilibria for this system.

The aforementioned ODE system is:

\[
\frac{dw}{dt} = w \left( \frac{w}{1 + w + g} - \mu_1 - \xi_1 (w + g) \right), \quad \frac{dg}{dt} = \frac{bw}{1 + w} - \left( \mu_2 + \xi_2 (w + g) \right) g, \quad (1.1)
\]

where \(w, g\) are the wild and sterile mosquitoes, respectfully, \(\mu_1, \mu_2\) are density independent death rates, and \(\xi_1, \xi_2\) are density dependent death rates. Our main result is the following:

**Theorem 1.1.** (i) The system (1.1) has always an equilibrium point \((0,0)\), which is locally asymptotically stable.

(ii) Let \(\mu_2\) and \(\xi_2\) satisfy \(\mu_2 > \xi_2 > 0\). Then there exists a sufficiently small \(\delta > 0\) such that if the parameters \(\xi_1, b, \mu_2, \xi_2\) satisfy

\[
0 < \xi_1 < \delta, \quad \mu_2 \left( \frac{1}{\xi_1} - 1 \right) - \frac{1}{2} (\mu_2 - \xi_2) < b < \mu_2 \left( \frac{1}{\xi_1} - 1 \right), \quad (1.2)
\]

then for sufficiently small \(\mu_1 > 0\), the system (1.1) has additional 4 positive equilibria \((w_i, g_i), i = 1, 2, 3, 4\), with

\[
w_1 < w_2 < w_3 < w_4,
\]
and the property that \((w_1, g_1)\) and \((w_3, g_3)\) are saddle points, and \((w_2, g_2)\) and \((w_4, g_4)\) are stable nodes or foci. In this case, the system (1.1) has also an equilibrium point in the fourth quadrant, which is not biologically interesting.

Theorem 1.1 (ii) is motivated from studying numerical simulations which suggest four positive equilibria for (1.1). These include those given in Fig.1, with the parameters taken as

\[
\mu_1 = 0.01, \quad \mu_2 = 0.5, \quad \xi_1 = \xi_2 = 0.04, \quad b = 5. \tag{1.3}
\]

Using Theorem 1.1, we can better predict the long-term effects of using the specific SIT release method. As the conditions in Theorem 1.1 require a sufficiently small \(\mu_1\), which biologically corresponds to a high wild mosquito lifespan, it is uncertain whether or not such a condition will commonly be found in mosquito populations. We will explore this in future work.

The proof of Theorem 1.1 (i) is straightforward and can be found in [2, 3]. In the following section, we shall prove Theorem 1.1 (ii).

# 2 Proof of Theorem 1.1 (ii)

We divide the proof into two parts. In the first part we show the existence of four positive equilibria, and in the second part we study their stabilities.

## 2.1 Existence of four positive equilibria

The nontrivial equilibrium points \((w, g)\) of (1.1) satisfy the equations

\[
\frac{w}{1 + N} = \mu_1 + \xi_1 N, \quad \frac{bw}{1 + w} = (\mu_2 + \xi_2 N)g,
\]

Figure 1: Graphs of nullclines of system (1.1) in the left panel and some orbits of system (1.1) in the right panel. The parameters in (1.1) are given in (1.3).
where $N := w + g$. Solve for $w$ and $g$ in terms of $N$ to get
\begin{align*}
w &= (1 + N)(\mu_1 + \xi_1 N), \quad g = \frac{b(1 + N)(\mu_1 + \xi_1 N)}{1 + (1 + N)(\mu_1 + \xi_1 N)(\mu_2 + \xi_2 N)}. \quad (2.1)
\end{align*}

Adding these two equations leads to
\begin{align*}
N &= (1 + N)(\mu_1 + \xi_1 N) + \frac{b(1 + N)(\mu_1 + \xi_1 N)}{1 + (1 + N)(\mu_1 + \xi_1 N)(\mu_2 + \xi_2 N)}. \quad (2.2)
\end{align*}

It follows that the positive solutions of (2.2) are the positive roots of the polynomial equation
\begin{align*}
p(N, \mu_1) = 0,
\end{align*}
and vice versa, where
\begin{align*}
p(N, \mu_1) := \left[N - (1 + N)(\mu_1 + \xi_1 N)\right] \left[1 + (1 + N)(\mu_1 + \xi_1 N)\right] (\mu_2 + \xi_2 N) - b(1 + N)(\mu_1 + \xi_1 N).
\end{align*}

Hence, in order to show that the system (1.1) has 4 positive equilibria, using (2.1) it is equivalent to show that the equation (2.3) has 4 positive roots. We note that $P(N, \mu_1)$ is a polynomial of degree 5 and $P(0, \mu_1) = -\mu_1(1 + \mu_1)\mu_2 - b\mu_1 < 0$ for $\mu_1 > 0$ and $p(-\infty, \mu_1) = \infty$, so (2.3) can have at most 4 positive roots.

Below, we use a perturbation argument to show that (2.3) has 4 positive roots. We first consider the case $\mu_1 = 0$ in (2.3), yielding
\begin{align*}
p(N, 0) = Np_0(N) = 0,
\end{align*}
where
\begin{align*}
p_0(N) = \left[1 - \xi_1 (1 + N)\right] \left[1 + \xi_1 N (1 + N)\right] (\mu_2 + \xi_2 N) - b\xi_1 (1 + N).
\end{align*}

We now prove the following result.

Theorem 2.1. Assume (1.2). Then the equation $p_0(N) = 0$ has 3 distinct positive roots $N^0_2 < N^0_3 < N^0_4$ and 1 negative root $N_{-1} < -\frac{\mu_2}{\xi_2}$.

Proof. Since $\mu_2 > \xi_1$ from the assumptions in (1.2), it follows that $N = -\frac{\mu_2}{\xi_2}$ is not a root of $p_0(N) = 0$. Hence, the roots of $p_0(N) = 0$ is equivalent to the solutions of the equation
\begin{align*}
h_1(N) = h_2(N),
\end{align*}
where
\begin{align*}
h_1(N) &= \left[1 - \xi_1 (1 + N)\right] \left[1 + \xi_1 N (1 + N)\right], \quad h_2(N) = \frac{b\xi_1 (1 + N)}{\mu_2 + \xi_2 N}.
\end{align*}
Figure 2: Schematic illustration of the graphs \( y = h_1(N) \) and \( y = h_2(N) \) and their intersections.

Hence, it suffices to study the intersection points of the graphs \( y = h_1(N) \) and \( y = h_2(N) \) in the \((N, y)\) plane. Below we study these graphs qualitatively.

First, we study the graph of \( y = h_1(N) \). It is clear that this graph is a cubic curve, as illustrated by the red curve in Fig. 2. Moreover,

\[
h_1(N) = (1 - \xi_1 - \xi_1N)(1 + \xi_1N + \xi_1N^2) = 1 - \xi_1 - \xi_1^2N + (\xi_1 - 2\xi_1^2)N^2 - \xi_1^2N^3,
\]

and

\[
h_1'(N) = -\xi_1^2 + 2(\xi_1 - 2\xi_1^2)N - 3\xi_1^2N^2 = -\xi_1^2\left[3N^2 - 2\left(\frac{1}{\xi_1} - 2\right)N + 1\right].
\]

Hence \( h_1'(N) = 0 \) has two roots given by the quadratic formula

\[
N_\pm = \frac{2\left(\frac{1}{\xi_1} - 2\right) \pm \sqrt{4\left(\frac{1}{\xi_1} - 2\right)^2 - 12}}{6} = \frac{\left(\frac{1}{\xi_1} - 2\right) \pm \sqrt{\left(\frac{1}{\xi_1} - 2\right)^2 - 3}}{3}.
\]

It follows that for sufficiently small \( \xi_1 > 0 \),

\[
0 < N_- < N_+ , \quad N_- = \frac{\xi_1}{2} + O(\xi_1^2), \quad N_+ = \frac{2}{3\xi_1} + O(1),
\]

\[
h_1(N_-) = h_1\left(\frac{\xi_1}{2} + O(\xi_1^2)\right) = 1 - \xi_1 - \frac{1}{2}\xi_1^3 + \frac{1}{4}\xi_1^3 + O(\xi_1^4) = 1 - \xi_1 - \frac{1}{4}\xi_1^3 + O(\xi_1^4),
\]

\[
h_1(N_+) = h_1\left(\frac{2}{3\xi_1} + O(1)\right) = \xi_1N_+^2 - \xi_1^2N_+^3 + O(1) = \left(\frac{4}{9} - \frac{8}{27}\right)\frac{1}{\xi_1} + O(1) = \frac{4}{27\xi_1} + O(1).
\]

Also, \( h_1(0) = 1 - \xi_1 > 0 \), and \( h_1(N_-) \) is the local minimum and \( h_1(N_+) \) is the local maximum of \( y = h_1(N) \).

Next, we look at the graph of \( h_2(N) \). Since \( h_2(N) = \frac{h_2}{\xi_1} \left(1 - \frac{\mu_2 - \xi_2}{\mu_3 + \xi_2N}\right) \), it follows that the graph of \( y = h_2(N) \) is a hyperbola, as illustrated by blue curves in Fig. 2,
with two branches lying on the left and right sides of the vertical asymptote \( N = \frac{-\mu_2}{\xi_2} \) and horizontal asymptote \( y = \frac{b\xi_1}{\xi_2} \) respectively. When \( \mu_2 > \xi_2 > 0 \), as in (1.2), the left branch is strictly monotone increasing lying above the horizontal asymptote \( y = \frac{b\xi_1}{\xi_2} \), while the right branch is also strictly monotone increasing but lying below this horizontal asymptote. Also, \( h_2(0) = \frac{b\xi_1}{\mu_2} \).

As \( \xi_1 \to 0 \) we have that

\[
h_2(N_-) = \frac{b\xi_1}{\xi_1} \left( 1 - \frac{\mu_2 - \xi_2}{\mu_2 + \xi_2(\xi_1/2 + O(\xi_1^2))} \right)
= \frac{b\xi_1}{\xi_2} \left( 1 - \frac{\mu_2 - \xi_2}{\mu_2} \left( 1 - \frac{\xi_1 \xi_2}{2\mu_2} + O(\xi_1^2) \right) \right)
= \frac{b\xi_1}{\mu_2} \left( 1 + \frac{\mu_2 - \xi_2}{2\mu_2} \xi_1 + O(\xi_1^2) \right).
\]

Applying the assumptions in (1.2), we have the following: For sufficiently small \( \xi_1 > 0 \), the condition \( b < \mu_2(1\xi_1 - 1) \) implies that \( h_1(0) > h_2(0) \), the condition \( b > \mu_2(1\xi_1 - 1) - (\mu_2 - x\xi_2)/2 \) implies that \( h_1(N_j < h_2(N_j) \), and \( \frac{\mu_2}{\xi_2} < \frac{3}{2\mu_1} \) implies that \( h_1(N_+) > h_2(N_+) \). Together with the graphs of \( y = h_1(N) \) and \( y = h_2(N) \) and the intermediate value theorem, we conclude that these two graphs intersect exactly at 3 points \((N_i^0, y_i^0)\), \( i = 2, 3, 4 \), in the right-half plane and one point \((N_{-1}^0, y_{-1}^0)\) to the left of vertical asymptote \( N = \frac{-\mu_2}{\xi_2} \). That is,

\[
N_{-1}^0 < -\frac{\mu_2}{\xi_2} < 0 < N_2^0 < N_3^0 < N_4^0 < N_+^0.
\]

This proves Theorem 2.1.

Next, we use the implicit function theorem to prove existence of four positive roots of (2.3) for sufficiently small \( \mu_1 > 0 \).

**Theorem 2.2. (Implicit Function Theorem)** Suppose \( F(x, y) \) is continuously differentiable in a neighborhood of a point \((x_0, y_0) \in \mathbb{R}^2\) where \( F(x_0, y_0) = 0 \). Then there exists \( \delta > 0, \epsilon > 0 \), and a rectangle defined by \( B = (x_0 - \epsilon, x_0 + \epsilon) \times (y_0 - \delta, y_0 + \delta) \) such that

1. For each \( y \in (y_0 - \delta, y_0 + \delta) \) there exists a unique \( x \in (x_0 - \epsilon, x_0 + \epsilon) \) such that \( F(x, y) = 0 \). This correspondence defines a function \( x = f(y) \) such that

\[
F(x, y) = 0 \text{ for } (x, y) \in B \iff x = f(y).
\]

2. \( f \) is continuous

3. \( f \) is continuously differentiable and
\[ f'(y) = -\frac{F_y(f(y),y)}{F_x(f(y),y)}, \]

It follows from Theorem 2.1 that when \( \mu_1 = 0 \), the 5th degree polynomial equation (2.4) has 5 solutions

\[ N_{-1} < N_1 := 0 < N_1 < N_2 < N_3 < N_4. \]

Since

\[ \frac{\partial p(N,0)}{\partial N} = p_0(N) + Np_0'(N), \]

it follows that, at the points \( N_i, i = -1,2,3,4 \),

\[ \frac{\partial p(N_i,0)}{\partial N} = N_ip_0'(N_i) \neq 0, \]

and, at the point \( N_1 = 0 \), from the assumption (1.2),

\[ \frac{\partial p(N_1,0)}{\partial N} = p_0(0) = (1 - \xi_1)\mu_2 - b\xi_1 > 0. \]

Applying Theorem (2.2) yields that for sufficiently small \( \mu_1 > 0 \), the equation \( p(N,\mu_1) = 0 \) has five distinct roots \( N_i(\mu_1), i = -1,1,2,3,4 \), with

\[ N_i(\mu_1) = N_i + O(\mu_1), \]

yielding \( N_{-1}(\mu_1) < 0 \) and \( N_i(\mu_1) > 0 \) for \( i = 2,3,4 \). But from \( N_1(\mu_1) = O(\mu_1) \) we cannot determine the sign of \( N_1(\mu_1) \). To this end, we use implicit differentiation to get

\[ N'_1(0) = -\frac{p_{\mu_1}(0,0)}{p_N(0,0)} = \frac{\mu_2 + b}{(1 - \xi_1)\mu_2 - b\xi_1} > 0, \]

where we used \( p_{\mu_1}(0,0) = -(\mu_2 + b) \), and then by Taylor’s theorem, for sufficiently small \( \mu_1 > 0 \),

\[ N_1(\mu_1) = N'_1(0)\mu_1 + O(\mu_1^2) = \frac{\mu_2 + b}{(1 - \xi_1)\mu_2 - b\xi_1}\mu_1 + O(\mu_1^2) > 0. \]

This shows that \( p(N,\mu_1) = 0 \) has 4 distinct positive roots \( N_i(\mu_1), i = 1,2,3,4 \) and 1 negative root \( N_{-1}(\mu_1) < -\frac{\mu_2}{\xi_1} \), and moreover,

\[ p'(N_i(\mu_1),\mu_1) < 0, \quad i = -1,2,4; \quad p'(N_i(\mu_1),\mu_1) > 0, \quad i = 1,3. \quad (2.6) \]

Finally, from (2.1) it follows that, for \( i = -1,1,2,3,4 \), letting

\[ w_i = (1 + N_i)(\mu_1 + \xi_1 N_i), \quad g_i = \frac{b(1 + N_i)(\mu_1 + \xi_1 N_i)}{[1 + (1 + N_i)(\mu_1 + \xi_1 N_i)](\mu_2 + \xi_2 N_i)}, \]

we have that \( (w_i,g_i) \) are all the nontrivial equilibria of (1.1), with \( (w_i,g_i) \), \( i = 1,2,3,4 \) lying in the first quadrant, and \( (w_{-1},g_{-1}) \) lying in the fourth quadrant. This shows that (1.1) has exactly 4 positive equilibria.
2.2 Stability of positive equilibria

First the Jacobian matrix of the vector field of (1.1) at each positive equilibrium \((w_i, g_i), i = 1, 2, 3, 4,\) is

\[
J_i := \begin{pmatrix}
w_i \left( \frac{1+g_i}{(1+N_i)^2} - \xi_1 \right) & -w_i \left( \frac{w_i}{(1+N_i)^2} + \xi_1 \right)
\end{pmatrix},
\]

and the trace of \(J_i\) is

\[
\text{tr} J_i = w_i \left( \frac{1+g_i}{(1+N_i)^2} - \xi_1 \right) - \xi_2 w_i - \xi_2 N_i - \xi_2 g_i.
\]

We may assume that \(\mu_1 < \mu_2\) and \(\xi_1 < \xi_2\). Then, using the relation \(\frac{w_i}{1+N_i} = \mu_1 + \xi_1 N_i\), we get

\[
\text{tr} J_i = \frac{1+g_i}{(1+N_i)} (\mu_1 + \xi_1 N_i) - \xi_1 w_i - \xi_2 N_i - \xi_2 g_i
\]

\[
\leq (\mu_1 + \xi_1 N_i) - \xi_1 w_i - \xi_2 N_i - \xi_2 g_i
\]

\[
= (\mu_1 - \mu_2) + (\xi_1 - \xi_2) g_i - \xi_2 N_i < 0.
\]

To determine the sign of \(\det J_i\), two methods are possible:

1. Using a result in [3]. The proof of this result is verified in Appendix A.

2. Using the properties of the nullclines alongside Theorem (2.2). This requires an assumption that the intersections of the nullclines are transversal, so it is an incomplete method.

2.2.1 Method 1

The aforementioned result from [3] is the following: at each \(N = N_i\) with \(i = 1, 2, 3, 4,\) we have that

\[
\det J_i = -\frac{w_i(\mu_1 + \xi_1 N_i) d(G_1(N_i) G_2(N_i))}{1+w_i} dN, \tag{2.7}
\]

where \(G_1(N)\) and \(G_2(N)\) are defined in [3] as

\[
G_1(N) := N - (1+N)(\mu_1 + \xi_1 N), \quad G_2(N) := Q_2 \frac{\mu_2 + \xi_2 N}{\mu_1 + \xi_1 N} \tag{2.8}
\]

where \(Q_2 := \frac{1+(1+N)(\mu_1 + \xi_1 N)}{1+N}\). Note that these satisfies

\[
(1+N)(\mu_1 + \xi_1 N) \left( G_1(N) G_2(N) - b \right) = p(N, \mu_1), \quad \forall N \geq 0,
\]

and hence,

\[
p'(N, \mu_1) = (\mu_1 + \xi_1 + 2\xi_1 N) \left( G_1(N) G_2(N) - b \right) + (1+N)(\mu_1 + \xi_1 N) \frac{d(G_1 G_2)}{dN}.
\]
Since $0 = p(N_i; \mu_1) = (1 + N_i)(\mu_1 + \xi_1 N_i)\left(G_1(N_i)G_2(N_i) - b\right)$, it follows that $G_1(N_i)G_2(N_i) - b = 0$, and hence

$$(\mu_1 + \xi_1 N_i)\frac{d(G_1(N_i)G_2(N_i))}{dN} = \frac{1}{1 + N_i} p'(N_i, \mu_1),$$

which together with (2.7) yields

$$\det J_i = \frac{-w_i}{(1 + w_i)(1 + N_i)} p'(N_i, \mu_1). \quad (2.9)$$

It follows from (2.6) that

$$\det J_1 < 0, \quad \det J_2 > 0, \quad \det J_3 < 0, \quad \det J_4 > 0.$$ 

This, together with $\text{tr} J_i < 0$, yields the stability result claimed in Theorem 1.1 (ii).

### 2.2.2 Method 2

We consider the nullclines of system (1.1) defined by

$$F(w, g) := \frac{w}{1 + w + g} - \mu_1 - \xi_1(w + g), \quad G(w, g) := \frac{bw}{1 + w} - g\left(\mu_2 + \xi_2(w + g)\right). \quad (2.10)$$

Notice that for all $w \geq 0, g \geq 0$,

$$F_g = -\frac{w}{(1 + w + g)^2} - \xi_1 < 0, \quad G_g = -\mu_2 - \xi_2(w + g) - \xi_2g < 0.$$

Thus, we can use implicit differentiation to determine functions

$$g = \varphi_1(w), \quad g = \varphi_2(w).$$

where

$$F(w, \varphi_1(w)) = 0, \quad G(w, \varphi_2(w)) = 0.$$

Let $\varphi(w) = \varphi_1(w) - \varphi_2(w)$ for $w \geq 0$. Assuming the intersections of (2.10) are transversal, then $\varphi(w)$ should appear qualitatively like the graph shown above, and

$$\varphi'(w_1) > 0, \quad \varphi'(w_2) < 0, \quad \varphi'(w_3) > 0, \quad \varphi'(w_4) < 0.$$ 

Observe that

$$\det J(w_i, g_i) = w_i F_g G_g \left(\frac{F_w}{F_g} - \frac{G_w}{G_g}\right) = -w_i F_g G_g \varphi'(w).$$

Thus

$$\det J(w_1, g_1) < 0, \quad \det J(w_2, g_2) > 0, \quad \det J(w_3, g_3) < 0, \quad \det J(w_4, g_4) > 0.$$ 

Which, together with $\text{tr} J < 0$, implies the stability results claimed in Theorem 1.1 (ii).
3 Conclusion

By establishing a range of parameters under which the system can have four positive equilibria, we can better predict the long-term effects of using a specific SIT release method. As the conditions for Theorem 1.1 require a sufficiently small $\mu_1$, corresponding to a high wild mosquito lifespan, it is uncertain whether or not such a condition will commonly be found in mosquito populations. We shall further explore the effect that other ranges of parameters have on the equilibria and their stabilities, including if there exist periodic solutions to system 1.1.

Another area of interest will be to show when the intersections of the nullclines featured in (2.10) will be transversal. In effect, we would need to show that $\varphi(w_i) \neq 0$ for $i = 1, 2, 3, 4$ and for the parameters listed in 2.1. It is important to note that, based on the above, $\varphi(w_i) = 0$ is equivalent to $\det J_i = 0$. Since the proof using Method 1 suggests that $\det J_i \neq 0$, then it suggests that each equilibrium will be transversal for the parameters in 2.1. If Method 2 is to be complete, however, a separate method of proof is needed to avoid circular reasoning.

We hope that future research can further explore the effect that the parameters have on the global dynamics of the model (1.1), including the existence of limit cycles. Many other works have investigated periodic solutions for various models (including [3, 5]), so we are confident that future work can yield a result for this model.

Appendix A: Verifying Equation (2.7)

For the reader’s convenience, we have printed each step of the original proof of (2.7), alongside justification in order to verify its accuracy.
Thus, it is evident that

\[
\frac{1}{w_i} \det J_i = \left[ \xi_1 - \frac{1 + g_i}{(1 + N_i)^2} \right] (\mu_2 + \xi_2 N_i) + \left[ \frac{w_i}{(1 + N_i)^2} + \xi_1 \right] \left[ \frac{b}{(1 + w_i)^2} - \xi_2 g_i \right]
\]

\[
= \left[ \xi_1 - \frac{1 + g_i}{(1 + N_i)^2} \right] (\mu_2 + \xi_2 N_i) - \frac{\xi_2 g_i (1 + g_i)}{(1 + N_i)^2} + \frac{w_i}{(1 + N_i)^2 + \xi_1} \frac{b}{(1 + w_i)^2} - \frac{\xi_2 g_i}{1 + N_i}.
\]

Using identities derived from \( G(w, g) = 0 \) in (2.10),

\[
\frac{1}{w_i} \det J_i = \left[ \xi_1 - \frac{1 + g_i}{(1 + N_i)^2} \right] (\mu_2 + \xi_2 N_i) + \left[ \frac{w_i}{(1 + N_i)^2 (1 + w_i)^2} + \xi_1 \frac{b}{1 + N_i} \right] - \frac{\xi_2 bw_i}{(1 + N_i)(1 + w_i)(\mu_2 + \xi_2 N_i)}
\]

\[
= \left[ \xi_1 - \frac{1 + g_i}{(1 + N_i)^2} \right] (\mu_2 + \xi_2 N_i) + \frac{w_i}{(1 + N_i)^2 (1 + w_i)^2} \frac{b}{(1 + N_i)(1 + w_i)^2 (\mu_2 + \xi_2 N_i)}
\]

\[
+ \frac{b(\xi_1 \mu_2 - \xi_2 \mu_1) - b\xi_2 \mu_1 w_i - b\xi_1 \xi_2 N_i w_i}{(1 + w_i)^2 (\mu_2 + \xi_2 N_i)}
\]

\[
= \left[ \xi_1 - \frac{1 + g_i}{(1 + N_i)^2} \right] (\mu_2 + \xi_2 N_i) + \left[ \frac{1}{(1 + N_i)^2} - \frac{\xi_2 (\mu_1 + \xi_1 N_i)}{\mu_2 + \xi_2 N_i} \right] \frac{g_i (\mu_2 + \xi_2 N_i)}{1 + w_i}
\]

\[
+ \frac{b(\xi_1 \mu_2 - \xi_2 \mu_1)}{(1 + w_i)^2 (\mu_2 + \xi_2 N_i)}.
\]

Thus, it is evident that

\[
\frac{\det J_i}{w_i (\mu_2 + \xi_2 N_i)} = \xi_1 - \frac{1 + g_i}{(1 + N_i)^2} + \left[ \frac{1}{(1 + N_i)^2} - \frac{\xi_2 (\mu_1 + \xi_1 N_i)}{\mu_2 + \xi_2 N_i} \right] \frac{g_i}{1 + w_i}
\]

\[
+ \frac{b(\xi_1 \mu_2 - \xi_2 \mu_1)}{(1 + w_i)^2 (\mu_2 + \xi_2 N_i)^2}.
\]

Note that, from (2.8), it is easy to verify that

\[
1 + w = (1 + N)Q_2, \quad g = N - w = N - ((1 + N)Q_2 - 1) = (1 + N)(1 - Q_2),
\]

\[
G_1 = N - w = (1 + N)(1 - Q_2) = (1 + N) \left( 1 - \frac{\mu_1 + \xi_1 N}{\mu_2 + \xi_2 N} Q_2 \right).
\]

By direct computation of \( \frac{dQ_2}{dN} \), we can show that

\[
\frac{d}{dN} \left( \frac{\mu_1 + \xi_1 N}{\mu_2 + \xi_2 N} Q_2 \right) = \frac{dQ_2}{dN} = \frac{-1 + \xi_1 (1 + N)^2}{(1 + N)^2} = \xi_1 - \frac{1}{(1 + N)^2}.
\]
Based on $g = G_1$ and $1 + w = (1 + N)Q_2$, we can see that

$$
\frac{-g}{(1 + N)^2} = \frac{(1 + N)(Q_2 - 1)}{(1 + N)^2} = \frac{Q_2 - 1}{1 + N}, \quad \frac{g}{1 + w} = \frac{(1 + N)(1 - Q_2)}{(1 + N)Q_2} = \frac{1 - Q_2}{Q_2}.
$$

Based on our expression for $\frac{dQ_2}{dN}$,

$$
\frac{dG_2}{dN} = Q_2 \frac{\xi_2(\mu_1 + \xi_1 N) - \xi_1(\mu_2 + \xi_2 N)}{(\mu_1 + \xi_1 N)^2} + \left( \frac{1}{(1 + N)^2} \right) \frac{\mu_2 + \xi_2 N}{\mu_1 + \xi_1 N} = \xi_2 + \frac{(1 + N)\xi_2 \mu_1 - \xi_1 \mu_2}{(1 + N)(\mu_1 + \xi_1 N)^2} - \frac{(\mu_2 + \xi_2 N)(\mu_1 + \xi_1 N)}{(1 + N)(\mu_1 + \xi_1 N)^2}.
$$

From this, it can be shown that

$$
\frac{\xi_2 \mu_1 - \xi_1 \mu_2}{(1 + N)(\mu_1 + \xi_1 N)(\mu_2 + \xi_2 N)} = \frac{\mu_1 + \xi_1 N}{\mu_2 + \xi_2 N}
$$

Thus, we can use this result and our values for $\frac{dQ_2}{dN}$, $\frac{-g}{(1 + N)^2}$ to get

$$
\frac{1 + w_i}{w_i(\mu_2 + \xi_2 N_i)} = (1 + w_i) \left[ \frac{dQ_2(N_i)}{dN} + \frac{Q_2(N_i) - 1}{1 + N_i} + \frac{b\xi_1 \mu_2 - b\xi_2 \mu_1}{(1 + w_i)^2(\mu_2 + \xi_2 N_i)^2} \right.
$$

$$
\left. + \frac{1 - Q_2(N_i)}{Q_2(N_i)} \left( \frac{\xi_2 \mu_1 - \xi_1 \mu_2}{(1 + N_i)(\mu_1 + \xi_1 N_i)(\mu_2 + \xi_2 N_i)} - \frac{\mu_1 + \xi_1 N_i dG_2(N_i)}{\mu_2 + \xi_2 N_i} \right) \right]
$$

$$
= (1 + w_i) \left[ \frac{dQ_2(N_i)}{dN} + \frac{Q_2(N_i) - 1}{1 + N_i} + \frac{b\xi_1 \mu_2 - b\xi_2 \mu_1}{(1 + w_i)^2(\mu_2 + \xi_2 N_i)^2} \right.
$$

$$
- \frac{1 - Q_2(N_i)}{Q_2(N_i)} \left( \frac{\mu_1 + \xi_1 N_i}{\mu_2 + \xi_2 N_i} \frac{dG_2(N_i)}{dN} \right)
$$

$$
+ \frac{\xi_2 \mu_1 - \xi_1 \mu_2}{1 + w_i (1 + N_i)(\mu_1 + \xi_1 N_i)(\mu_2 + \xi_2 N_i)} \right] ;
$$

Since it follows from 2.10 that ($b = \frac{g_i}{1 + w_i}$),

$$
\frac{1 + w_i}{w_i(\mu_2 + \xi_2 N_i)} = (1 + w_i) \left[ \frac{dQ_2(N_i)}{dN} + \frac{Q_2(N_i) - 1}{(1 + N_i)^2} \right]
$$

$$
- \frac{1}{\mu_2 + \xi_2 N_i} \frac{G_2(N_i) - 1}{1 - \frac{\mu_1 + \xi_1 N_i G_2(N_i)}{\mu_2 + \xi_2 N_i}}
$$

$$
= (1 + w_i) \left[ \frac{\mu_1 + \xi_1 N_i}{\mu_2 + \xi_2 N_i} \frac{dG_2(N_i)}{dN} + \frac{\mu_1 + \xi_1 N_i}{\mu_2 + \xi_2 N_i} \frac{G_2(N_i) - 1}{1 + N_i} \right]
$$

$$
+ \frac{\mu_1 + \xi_1 N_i}{\mu_2 + \xi_2 N_i} \frac{dG_2(N_i)}{dN} - 1 \frac{1 - \frac{\mu_1 + \xi_1 N_i G_2(N_i)}{\mu_2 + \xi_2 N_i}}{G_2(N_i)} \frac{dG_2(N_i)}{dN} \right].
$$
After expanding and using the identity derived for \((1 + w)\),

\[
\frac{(1 + w_i) \det J_i}{w_i(\mu_2 + \xi_2 N_i)} = (1 + w_i) \frac{\xi_1 \mu_2 - \xi_2 \mu_1}{(\mu_2 + \xi_2 N_i)^2} G_2(N_i) \\
+ \frac{\mu_1 + \xi_1 N_i}{\mu_2 + \xi_2 N_i} \frac{dG_2(N_i)}{dN}(1 + N_i) \left[ 2 \frac{\mu_1 + \xi_1 N_i}{\mu_2 + \xi_2 N_i} G_2(N_i) - 1 \right] \\
+ \frac{\mu_1}{\mu_2 + \xi_2 N_i} G_2(N_i) \left[ \frac{\mu_1 + \xi_1 N_i}{\mu_2 + \xi_2 N_i} G_2(N_i) - 1 \right] .
\]

The facts that \(G_1 = N - w\), \(Q_1 = \frac{\mu_1 + \xi_1 N}{\mu_2 + \xi_2 N} G_2\) show that, through direct computation

\[
\frac{d(G_1(N_i)G_2(N_i))}{dN} = \frac{d}{dN} \left[ (1 + N_i)(1 - \frac{\mu_1 + \xi_1 N_i}{\mu_2 + \xi_2 N_i} G_2(N_i)) G_2(N_i) \right] \\
= \left[ 1 - \frac{\mu_1 + \xi_1 N_i}{\mu_2 + \xi_2 N_i} G_2(N_i) \right] G_2(N_i) \\
+ (1 + N_i) \left[ 1 - 2 \frac{\mu_1 + \xi_1 N_i}{\mu_2 + \xi_2 N_i} G_2(N_i) \right] \frac{dG_2(N_i)}{dN} \\
- (1 + N_i) \frac{\xi_1 \mu_2 - \xi_2 \mu_1}{(\mu_2 + \xi_2 N_i)^2} G_2^2(N_i) .
\]

and since \(\frac{\mu_1 + \xi_1 N_i}{\mu_2 + \xi_2 N_i}(1 + w) = (1 + N)G_2\),

\[
\frac{\mu_1 + \xi_1 N_i}{\mu_2 + \xi_2 N_i}(1 + w_i) \frac{\xi_1 \mu_2 - \xi_2 \mu_1}{(\mu_2 + \xi_2 N_i)^2} G_2(N_i) = (1 + N_i) \frac{\xi_1 \mu_2 - \xi_2 \mu_1}{(\mu_2 + \xi_2 N)^2} G_2^2(N_i) .
\]

Finally, if we use the above identity, we can show that

\[
- \frac{\mu_1 + \xi_1 N_i}{\mu_2 + \xi_2 N_i} \frac{d(G_1(N_i)G_2(N_i))}{dN} = \frac{\mu_1 + \xi_1 N_i}{\mu_2 + \xi_2 N_i} G_2(N_i) \left[ \frac{\mu_1 + \xi_1 N_i}{\mu_2 + \xi_2 N_i} G_2(N_i) - 1 \right] \\
+ \frac{\mu_1 + \xi_1 N_i}{\mu_2 + \xi_2 N_i} (1 + N_i) \left[ 2 \frac{\mu_1 + \xi_1 N_i}{\mu_2 + \xi_2 N_i} - 1 \right] \frac{dG_2(N_i)}{dN} \\
+ (1 + w_i) \frac{\xi_1 \mu_2 - \xi_2 \mu_1}{(\mu_2 + \xi_2 N_i)^2} \\
= \frac{(1 + w_i) \det J_i}{w_i(\mu_2 + \xi_2 N_i)} .
\]

Thus, we can finally verify the result in (2.7),

\[
\det J_i = - \frac{w_i(\mu_1 + \xi_1 N_i) \frac{d(G_1(N_i)G_2(N_i))}{dN}}{1 + w_i} .
\]
References


