A modified model for the long-term prediction of tropospheric scintillation from coarse-resolution atmospheric data

James L. Suggs Jr.
A MODIFIED MODEL FOR THE LONG-TERM
PREDICTION OF TROPOSPHERIC SCINTILLATION
FROM COARSE-RESOLUTION ATMOSPHERIC DATA

by

JAMES L. SUGGS, JR.

A THESIS

Submitted in partial fulfillment of the requirements
for the degree of Master of Science
in
The Department of Physics and Astronomy
to
The School of Graduate Studies
of
The University of Alabama in Huntsville

HUNTSVILLE, ALABAMA

2022
In presenting this thesis in partial fulfillment of the requirements for a master’s de-
gree from The University of Alabama in Huntsville, I agree that the Library of this
University shall make it freely available for inspection. I further agree that permis-
sion for extensive copying for scholarly purposes may be granted by my advisor or,
in his/her absence, by the Chair of the Department or the Dean of the School of
Graduate Studies. It is also understood that due recognition shall be given to me
and to The University of Alabama in Huntsville in any scholarly use which may be
made of any material in this thesis.

James L. Suggs, Jr.

James L. Suggs, Jr. Mar 24, 2022 (date)
Submitted by James L. Suggs, Jr. in partial fulfillment of the requirements for the degree of Master of Science in Physics and accepted on behalf of the Faculty of the School of Graduate Studies by the thesis committee.

We, the undersigned members of the Graduate Faculty of The University of Alabama in Huntsville, certify that we have advised and/or supervised the candidate of the work described in this thesis. We further certify that we have reviewed the thesis manuscript and approve it in partial fulfillment of the requirements for the degree of Master of Science in Physics.
ABSTRACT

School of Graduate Studies
The University of Alabama in Huntsville

Degree Master of Science College/Dept. Science/Physics and in Physics Astronomy
Name of Candidate James L. Suggs, Jr.
Title A Modified Model for the Long-Term Prediction of Tropospheric Scintillation from Coarse-Resolution Atmospheric Data

It has been well established that scintillation, or random fades and enhancements of a propagating wave due to turbulence-induced fluctuations of the refractive index, can have deleterious effects on radar and satellite communications systems, especially for frequencies above 10 GHz (3 cm). Kolmogorov theory characterizes turbulence along a given propagation path in terms of the refractive index structure parameter $C_n^2$. Tropospheric scintillation models are typically driven by $C_n^2(z)$ profiles developed from atmospheric measurements/predictions of pressure, temperature, relative humidity, and wind speed/direction along a given propagation path. In this work, high-resolution distributions of outer-scale, wind shear, and squared buoyancy frequency will be analyzed and used to describe an alternative scintillation model. To conclude, the modified method will be invoked on Global Forecast System (GFS) data and the results compared to measurement data and predictions of existing models.
Abstract Approval: Committee Chair

Department Chair

Graduate Dean
ACKNOWLEDGMENTS

I would like to thank Dr. Vaughn and deciBel Research for the support and approval of the effort presented herein. I would like to thank Dr. Gregory for the guidance, and for believing in me. I would like to thank Dr. Pendleton for the mentoring...and the patience. I would like to thank Dr. Herren for the revelations of what I thought I knew about mathematics. I would like to thank Karen and Eli for the consideration and understanding. I would like to thank God for everything.
TABLE OF CONTENTS

List of Figures ix

List of Tables xi

List of Symbols xii

Chapter

1 Introduction 1

2 Basic Theory and Definitions 2

2.1 Scintillation 2

2.2 Characterization of Atmospheric Turbulence 2

2.3 The Refractive Index Structure Constant 12

3 Methodologies and Procedures 14

3.1 Prediction of Tropospheric Scintillation: A Statistical Model 14

3.2 Modifying the Existing Statistical Model 20

3.3 Atmospheric Data 21

3.3.1 High-Resolution Radiosonde Replacement System (RRS) Data 21

3.3.2 Coarse-Resolution Global Forecast System (GFS) Data 22

3.3.3 World Map of Köppen-Geiger Climate Classification 23

3.4 Processing the Atmospheric Data 25
3.4.1 Data Cleansing, Reduction, and Interpolation ................................ 25
3.4.2 Data Smoothing .............................................................................. 26
3.4.3 Scaling Mean Wind Shear and Mean Squared Buoyancy ................ 27
3.5 Building the Distributions of $L_0$, $S$, and $N^2$ ............................. 31
3.6 The Model Functions for $L_0$, $S$, and $N^2$ ...................................... 38
3.7 Determination of Model Function Parameters ................................. 40
3.7.1 Method of Moments ...................................................................... 40
3.7.2 Nonlinear Least Squares Fitting ..................................................... 46
3.8 Generating the $C_n^2$ Profiles ............................................................ 49
3.8.1 Vasseur’s Approach ...................................................................... 50
3.8.2 Monte Carlo Sampling .................................................................... 50
3.9 Calculating $\sigma^2_\chi$ and $\sigma^2_{\sigma^2_\chi}$ ........................................ 52
3.10 Antenna Aperture-Averaging ............................................................ 53
3.11 Computing Cumulative Statistics for Scintillation Variance and Fade 54

4 Results and Discussion ........................................................................ 56

4.1 Comparison of $C_n^2$ Profiles and Scintillation Variance Cumulative Statistics: Modified versus Vasseur Models ................................. 56
4.2 Comparison with Measurement/Additional Model Data ........................ 59

REFERENCES .................................................................................. 63

APPENDIX A: Vasseur’s Appendix A ...................................................... 68

APPENDIX B: Vasseur’s Appendix B ...................................................... 70
# LIST OF FIGURES

<table>
<thead>
<tr>
<th>FIGURE</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1 Kolmogorov cascade theory of turbulence.</td>
<td>4</td>
</tr>
<tr>
<td>2.2 Coordinate systems used in derivation of $\sigma^2_\chi$.</td>
<td>8</td>
</tr>
<tr>
<td>3.1 World Map of Köppen-Geiger Climate Classification.</td>
<td>24</td>
</tr>
<tr>
<td>3.2 Examples of data smoothing. Miami, Florida at 0000 UTC 15 Apr 2015.</td>
<td>26</td>
</tr>
<tr>
<td>3.3 Computed mean wind shear profiles.</td>
<td>29</td>
</tr>
<tr>
<td>3.4 Computed mean squared buoyancy profiles.</td>
<td>30</td>
</tr>
<tr>
<td>3.5 Portion of actual and rearranged potential temperature profiles. Miami, Florida at 0000 UTC 15 Apr 2015.</td>
<td>32</td>
</tr>
<tr>
<td>3.6 Building Distributions of $L_0$, $S$, and $N^2$.</td>
<td>33</td>
</tr>
<tr>
<td>3.7 Distributions of outer scale, wind shear, and squared buoyancy at (4000 → 5000) meters from July through September 2015 at Miami, Florida.</td>
<td>34</td>
</tr>
<tr>
<td>3.8 Distributions of outer scale, wind shear, and squared buoyancy at (6000 → 7000) meters from October through December 2015 at Albuquerque, New Mexico.</td>
<td>35</td>
</tr>
<tr>
<td>3.9 Distributions of outer scale, wind shear, and squared buoyancy at (7000 → 8000) meters from January through March 2015 at Koror, Palau.</td>
<td>36</td>
</tr>
<tr>
<td>3.10 Distributions of outer scale, wind shear, and squared buoyancy at (2000 → 3000) meters from January through March 2015 at Fairbanks, Alaska.</td>
<td>37</td>
</tr>
<tr>
<td>3.11 Cost analysis to determine parameters a, b, c, d, and f.</td>
<td>44</td>
</tr>
</tbody>
</table>
4.1 Annual scintillation variance cumulative statistics at Covenant University. .......................................................... 57
4.2 Seasonal $C_n^2$ profiles at Covenant University. .................. 58
4.3 Monthly scintillation intensity at Covenant University. ............ 61
4.4 Annual scintillation fade cumulative statistics at Covenant University. 61
4.5 Monthly scintillation fade cumulative statistics at Covenant University. 62
4.6 Monthly averaged ground-level temperatures at Covenant University. 62
### LIST OF TABLES

<table>
<thead>
<tr>
<th>TABLE</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1 Scaling Factors for $\langle S(z) \rangle$, and $\langle N^2(z) \rangle$</td>
<td>28</td>
</tr>
<tr>
<td>3.2 Model functions for $L_0(z)$, $S(z)$, and $N^2(z)$</td>
<td>38</td>
</tr>
<tr>
<td>3.3 $\sigma_S$ and $\sigma_{N^2}$ Free Parameter Values</td>
<td>43</td>
</tr>
<tr>
<td>3.4 Constant Parameters: Exponential and Johnson SU PDFs</td>
<td>49</td>
</tr>
<tr>
<td>4.1 Characteristics of ground measurement station</td>
<td>59</td>
</tr>
<tr>
<td>4.2 $\sigma_{\chi}^2$, $\sigma_{\chi^2}^2$, $m$, and $s$</td>
<td>60</td>
</tr>
</tbody>
</table>
## LIST OF SYMBOLS

<table>
<thead>
<tr>
<th>SYMBOL</th>
<th>DEFINITION</th>
</tr>
</thead>
<tbody>
<tr>
<td>⟨·⟩</td>
<td>mean (ensemble average)</td>
</tr>
<tr>
<td>( _1F_1(a; c; z) )</td>
<td>confluent hypergeometric function of the first kind</td>
</tr>
<tr>
<td>( A(\frac{\Delta z}{L}) )</td>
<td>factor in empirical cross-correlation function</td>
</tr>
<tr>
<td>( A(R, r) )</td>
<td>amplitude weighting function</td>
</tr>
<tr>
<td>( \mathcal{A} )</td>
<td>factor related to unnormalized ( P_L P_S P_N^2 )</td>
</tr>
<tr>
<td>( B_n(r, r') )</td>
<td>spatial covariance of refractive index fluctuations</td>
</tr>
<tr>
<td>( C_n^2(z) )</td>
<td>refractive index structure parameter</td>
</tr>
<tr>
<td>( \overline{\langle C_n^2(z) \rangle} )</td>
<td>period average of mean structure parameter</td>
</tr>
<tr>
<td>( \chi )</td>
<td>log-amplitude of the ratio of instantaneous to mean amplitude</td>
</tr>
<tr>
<td>( \chi^2(p) )</td>
<td>objective function of vector ( p )</td>
</tr>
<tr>
<td>( D )</td>
<td>physical antenna diameter</td>
</tr>
<tr>
<td>( D_{\text{eff}} )</td>
<td>effective antenna diameter</td>
</tr>
<tr>
<td>( D_n(\rho) )</td>
<td>refractive index structure function</td>
</tr>
<tr>
<td>( \Delta H )</td>
<td>thin turbulent layer thickness</td>
</tr>
<tr>
<td>( dz, \Delta z )</td>
<td>thickness of the horizontal slab through which ( \langle C_n^2(z) \rangle ) has been calculated</td>
</tr>
<tr>
<td>( \delta )</td>
<td>Johnson SU PDF shape parameter</td>
</tr>
</tbody>
</table>
\( \delta \epsilon \) small dielectric constant perturbation

\( \delta n \) small refractive index perturbation

\( E \) electric field strength

\( E_0 \) unperturbed electric field strength

\( E\{ \cdot \} \) expected value

\( e(z) \) water vapor pressure

\( \epsilon \) dielectric constant

\( \eta \) antenna efficiency

\( G \) gain of a parabolic antenna

\( G(R,r) \) Green function

\( g \) acceleration of gravity

\( g \) antenna averaging factor

\( \Gamma(x) \) gamma function

\( \Gamma_{C^2_n}(z, z + \Delta z, \tau) \) two-dimensional (height and time) cross-correlation function of the refractive index structure parameter

\( \Gamma_r(z, z + \Delta z, \tau) \) empirical cross-correlation function

\( \Gamma_{\sigma^2_n}(\tau) \) autocorrelation function of the scintillation variance

\( \gamma \) Johnson SU PDF shape parameter

\( H \) thin turbulent layer height
\( h_L \) height of turbulent layer

\( \mathbf{J} \) Jacobian matrix

\( k \) frequency wavenumber

\( \kappa \) Pearson Type III PDF skew parameter

\( \kappa \) spherical wavenumber radius coordinate

\( \kappa_c \) wavenumber cutoff

\( \bm{\kappa} \) wavenumber vector

\( L_0(z) \) outer scale of turbulence

\( L_1 \) constant in empirical cross-correlation function

\( L_2 \) constant in empirical cross-correlation function

\( L_{Ozmidov}(z) \) Ozmidov length

\( L_T(z) \) Thorpe length

\( L_q(x) \) Laguerre polynomial

\( \ell_0(z) \) inner scale of turbulence

\( \lambda \) exponential (inverse), Pearson Type III and Johnson SU PDF scale parameter

\( \lambda \) Levenberg-Marquardt damping parameter

\( \lambda \) wavelength

\( M(z) \) vertical gradient of the refractive index
\( m \) median value of the scintillation variance distribution

\( \mu_k \) \( k^{th} \) central moment

\( \mu'_k \) \( k^{th} \) raw moment

\( N^2(z) \) squared buoyancy frequency

\( n(r, t) \) refractive index field

\( n_d \) number of days of Monte Carlo samples

\( n_s \) number of “successful” Monte Carlo samples

\( \omega \) spherical azimuthal coordinate

\( P_{L_0} \) outer scale probability density function

\( P_S \) wind shear probability density function

\( P_{N^2} \) squared buoyancy frequency probability density function

\( p(\sigma^2) \) long-term probability density function of scintillation variance

\( p(z) \) pressure

\( p_{50(C_n^2)}(z) \) median of mean structure parameter

\( p_{\text{long-term}}(\chi) \) long-term probability density function of scintillation log-amplitude

\( p_{\text{short-term}}(\chi|\sigma^2) \) conditional short-term probability distribution function of \( \chi \), assuming a constant value of \( \sigma^2 \)

\( \Phi_n(\kappa, z) \) spatial power spectrum

\( \phi \) cylindrical azimuthal coordinate
\( \phi(x) \)  \hspace{1cm} \text{Gaussian PDF}

\( \varphi(z) \)  \hspace{1cm} \text{wind direction}

\( \psi \)  \hspace{1cm} \text{spherical polar coordinate}

\( q(z) \)  \hspace{1cm} \text{specific humidity}

\( R(z) \)  \hspace{1cm} \text{humidity contribution to } M^2

\( Ric \)  \hspace{1cm} \text{critical value of the Richardson number}

\( R_i \)  \hspace{1cm} \text{Richardson number}

\( r \)  \hspace{1cm} \text{cylindrical radial coordinate}

\( \rho(z) \)  \hspace{1cm} \text{mean density of dry air in the slab}

\( \rho \)  \hspace{1cm} \text{displacement vector}

\( S(z) \)  \hspace{1cm} \text{wind shear}

\( s \)  \hspace{1cm} \text{standard deviation of } \ln \left( \sigma^2_{\chi} \right)

\( s_{(C_n^2)}(z) \)  \hspace{1cm} \text{parameter of the } \langle C_n^2(z) \rangle \text{ log-normal distribution}

\( \sigma_{(C_n^2)}(z) \)  \hspace{1cm} \text{standard deviation of mean structure parameter}

\( \sigma_{N^2}(z) \)  \hspace{1cm} \text{Vasseur’s Gaussian PDF scale parameter}

\( \sigma_S(z) \)  \hspace{1cm} \text{Vasseur’s Rice PDF scale parameter}

\( \sigma_{y_i} \)  \hspace{1cm} \text{measurement error for datum } y(t_i)

\( \sigma^2_{\sigma^2_\chi} \)  \hspace{1cm} \text{variance of scintillation variance}

\( \sigma^2_{\chi} \)  \hspace{1cm} \text{scintillation variance of the log-amplitude of } \chi
\( \sigma_x^2 \) \hspace{1cm} \text{mean (time-averaged) scintillation variance}

\( T(z) \) \hspace{1cm} \text{temperature}

\( \theta \) \hspace{1cm} \text{elevation angle}

\( \theta(z) \) \hspace{1cm} \text{potential (specific) temperature}

\( v(z) \) \hspace{1cm} \text{wind speed}

\( W \) \hspace{1cm} \text{weighting matrix}

\( x \) \hspace{1cm} \text{square of the ratio between the antenna diameter and the size of the Fresnel zone}

\( \mathbf{x} \) \hspace{1cm} \text{vector } \mathbf{x}

\( \xi \) \hspace{1cm} \text{exponential, Pearson Type III and Johnson SU PDF location parameter}

\( \hat{y}(t; \mathbf{p}) \) \hspace{1cm} \text{model function of an independent variable } t \text{ and a vector of parameters } \mathbf{p}

\( \mathbf{y}^T \) \hspace{1cm} \text{transpose of } \mathbf{y}

\( z \) \hspace{1cm} \text{height (altitude)}

\( z_{\text{eff}} \) \hspace{1cm} \text{effective turbulent path}
To Karen and Eli – the loves of my life.
I learned very early the difference between knowing the name of something and knowing something.

—Richard P. Feynman
CHAPTER 1

INTRODUCTION

In order to develop a foundational framework useful for making connections with the concepts presented in the following models, this work opens with a brief overview of some basic theory and definitions associated with scintillation, atmospheric turbulence, and the refractive index structure constant. Next, an existing method, proposed by Hugues Vasseur, used for computing long-term scintillation statistics from coarse-resolution (20 to 50 meters) radiosonde data is reviewed. Following the review, a method is described for modifying the existing model by fitting probability density functions to the long-term distributions of outer scale, wind shear, and squared buoyancy (frequency) data built from high-resolution radiosonde data spanning a period of one year at each of several locations in order to develop new functions which characterize these distributions in a more general way. From Global Forecast System (GFS) data, $C_n^2(z)$ profiles and annual scintillation fade statistics generated with the modified and Vasseur models will be compared. In conclusion, long-term scintillation fade and intensity will be computed using the modified model, and the results compared to measurement data and predictions from other models.
CHAPTER 2

BASIC THEORY AND DEFINITIONS

2.1 Scintillation

Scintillation is the random fades and enhancements of a propagating signal’s amplitude and phase due to turbulence-induced small-scale fluctuations of the refractive index. Atmospheric turbulence can originate from variations of temperature, humidity, pressure, and wind gradient. In the presence of clouds, these fluctuations are generally enhanced [1], [2]. Experimental evidence suggests that scintillation can be attributed to the turbulent mixing of air masses with varying amounts of water content in and around non-raining stratus and cumulus clouds [3], [4]. Rainy conditions make it difficult to distinguish actual scintillation from rapid signal variations due to rain attenuation. As rain attenuation is typically much more pronounced than scintillation fades, this work will consider only scintillation resulting from clear-weather (possibly cloudy, but not raining) scenarios.

2.2 Characterization of Atmospheric Turbulence

The atmosphere, considered a viscous fluid, has two distinct states of motion – laminar and turbulent. Mixing does not occur in laminar flow where velocity
flow characteristics change in some regular fashion. However, for turbulent flow, the velocity field loses its uniform characteristics as a result of dynamic mixing, thereby forming subflows known as turbulent eddies. Turbulence, being a nonlinear process, is fundamentally governed by the Navier-Stokes equations. The inherent mathematical difficulties associated with solving these equations prompted Kolmogorov to develop a statistical theory that relies greatly on dimensional analysis and other simplifications and approximations [5].

Adoption of the energy cascade theory of turbulence is useful for visualizing (see Figure 2.1) the turbulent structure of the atmosphere [6]. Basically, the theory says that when the wind velocity increases to a point at which the critical Reynolds number is exceeded, local unstable air masses, or “eddies,” are formed with characteristic dimensions slightly less than that of the parent flow. The larger eddies break apart, due to inertial forces, into smaller ones to form a continuum of eddy sizes. Energy is transferred, or cascaded, from a macroscale $L_0$ (outer scale of turbulence) to a microscale $\ell_0$ (inner scale of turbulence). The inertial subrange is defined as the range of eddy sizes bounded above by the outer scale $L_0$ and below by the inner scale $\ell_0$. Eddies are assumed statistically homogeneous and isotropic for scale sizes smaller than $L_0$, with turbulence properties independent of the parent flow. As eddy sizes decrease, the relative amount of energy dissipated by viscous forces increases until the dissipated energy equals the kinetic energy supplied by the parent flow, upon which the Reynolds number is reduced to the order of unity, thereby producing an eddy size defining the inner scale of turbulence $\ell_0$ [7].
For radio waves passing through a turbulent atmosphere, instability of the electric field amplitude is governed by the medium’s dielectric constant variance, and can be characterized via the random wave equation [8],

\[ \nabla^2 E + k^2 [1 + \delta\epsilon(r, t)] E = 0, \]

(2.1)

where it is assumed that the atmosphere is homogeneous in composition, but inhomogeneous in pressure and temperature.
The Rytov approximation can be used to solve equation (2.1) for a small $\delta \epsilon$ perturbation [8].

$$E(R) = E_0(R) \exp \left[ -k^2 \int d^3 r \delta \epsilon(r) A(R, r) \right], \quad (2.2)$$

where

$$A(R, r) = \text{Re} \left[ G(R, r) \frac{E_0(r)}{E_0(R)} \right] \quad (2.3)$$

is the real part of the Green function $G(R, r)$ and the ratio of electric fields [8].

Emphasizing the natural logarithmic of the amplitude fluctuation [8],

$$\chi = \ln \left( \frac{E}{E_0} \right), \quad (2.4)$$

equation (2.2) can be rewritten as [8]

$$\chi = -k^2 \int d^3 r \delta \epsilon(r) A(R, r) < 1 \quad [\text{Np}]. \quad (2.5)$$

Because $\langle \chi \rangle = 0$, the variance of the log amplitude fluctuations is [8]

$$\sigma^2_{\chi} \equiv \langle \chi^2 \rangle - \langle \chi \rangle^2 =$$

$$\langle \chi^2 \rangle = k^4 \int d^3 r \int d^3 r' A(R, r) A'(R, r') \langle \delta \epsilon(r, t) \delta \epsilon(r', t) \rangle \quad [\text{Np}^2]. \quad (2.6)$$

The atmospheric refractive index $n$ is one of the most significant parameters associated with wave propagation through a turbulent atmosphere. The index is
very sensitive to small-scale temperature variations. These fluctuations combined with
turbulent mixing induce a random behavior in the field of atmospheric index of
refraction [7]. Decomposing $n$ into an average value and a small component that is a
stochastic function of position and time, the field may be mathematically expressed as [9]

$$n(r, t) = \langle n(r, t) \rangle + \delta n(r, t), \quad (2.7)$$

where the angle brackets indicate the ensemble average. The fluctuating term $\delta n$, which physically manifests as scintillation, may be modeled as a zero-mean locally stationary process [10].

The dielectric constant can be related to the refractive index as [8]

$$\epsilon = n^2, \quad (2.8)$$

implying that [8]

$$\delta \epsilon = 2 \langle n(r, t) \rangle \delta n. \quad (2.9)$$

Since $\langle n(r, t) \rangle \approx 1$, this relationship becomes [8]

$$\delta \epsilon = 2 \delta n. \quad (2.10)$$

Hence, the spatial covariance of dielectric constant fluctuations in equation (2.6) can be given in terms of the spatial covariance of refractive index fluctuations [8]:

$$\langle \delta \epsilon(r, t) \delta \epsilon(r', t) \rangle = 4 \langle \delta n(r, t) \delta n(r', t) \rangle. \quad (2.11)$$
A three-dimensional Fourier transform connects the spatial covariance of refractive index fluctuations $B_n(r)$ and the spatial power spectrum $\Phi_n(\kappa)$ [8]:

$$B_n(r, r') \equiv \langle \delta n(r, t) \delta n(r', t) \rangle = \int d^3 \kappa \Phi_n(\kappa) \exp[i\kappa \cdot (r - r')] . \quad (2.12)$$

For the atmospheric refractive index $n$, the statistical description of the random field of turbulence-induced fluctuations follows similarly to that given for the random field of turbulent velocities. More specifically, there exists an inertial subrange also bounded above by an outer scale $L_0$ and below by an inner scale $\ell_0$. If the field of velocity fluctuations within the inertial subrange exhibits the properties of statistical homogeneity and isotropy, then these properties are inherited by the field of refractive-index fluctuations within the corresponding inertial subrange [7], i.e.,

$$D_n(\rho) = \langle [\delta n(r, t) - \delta n(r + \rho, t)]^2 \rangle \approx C_n^2 |\rho|^{2/3}, \quad \ell_0 < |\rho| < L_0, \quad (2.13)$$

where $D_n(\rho)$ is the refractive index structure function, and $C_n^2$ is the refractive index structure constant. The approximation is due to Kolmogorov [8].

Based on the $2/3$ power law of the spatial separation, the spatial power spectrum can be modeled as [8]

$$\Phi_n(\kappa, z) = 0.033 C_n^2(z) \kappa^{-11/3}, \quad 0 < \kappa < \infty. \quad (2.14)$$
Consider an overhead source with a point-receiver at $R = 0$. Using cylindrical coordinates \[8\],

\[ E(r) = E_0 \exp(-ikz), \]  
\[ E(R) = E_0, \]  

and

\[ G(R, r) = \frac{1}{4\pi \rho} \exp(ik\rho). \]

\[ \text{Figure 2.2: Coordinate systems describing the displacement vector } \rho \text{ and the wavenumber vector } \kappa. \]
Substituting equations (2.15), (2.16), and (2.17) into equation (2.3), the amplitude weighting function becomes [8]

\[ A(R, r) = \text{Re} \left[ \frac{\exp(ik\sqrt{z^2 + r^2}) E_0 \exp(-ikz)}{4\pi \sqrt{z^2 + r^2}} \right], \quad (2.18) \]

and after invoking the paraxial approximation for small-angle scattering [8], viz.,

\[ \sqrt{z^2 + r^2} = z\sqrt{1 + \frac{r^2}{z^2}} \approx z + \frac{r^2}{2z} \quad \text{(paraxial approximation)}, \quad (2.19) \]

the function simplifies to [8]

\[ A(R, r) = \frac{1}{4\pi z} \cos \left( k \frac{r^2}{2z} \right). \quad (2.20) \]

With the simplified \( A(R, r) \), equations (2.6) and (2.11) yield [8]

\[
\sigma^2_x = \frac{k^4}{4\pi^2} \int_0^\infty \frac{dz_1}{z_1} \int_0^{2\pi} d\phi_1 \int_0^\infty r_1 \, dr_1 \cos \left( k \frac{r_1^2}{2z_1} \right) \int_0^\infty \frac{dz_2}{z_2} \\
\times \int_0^{2\pi} d\phi_2 \int_0^\infty r_2 \, dr_2 \cos \left( k \frac{r_2^2}{2z_2} \right) \langle \delta n(\rho_1) \delta n(\rho_2) \rangle. \quad (2.21)
\]
In spherical wavenumber coordinates (see Figure 2.2), the spatial covariance of the refractive index fluctuations is [8]

\[
\langle \delta n(\rho_1)\delta n(\rho_2) \rangle = \int d^3 \kappa \Phi_n \left( \kappa, \frac{z_1 + z_2}{2} \right) \exp[i\kappa \cdot |\rho_1 - \rho_2|]
\]

\[
= \int_0^\infty \kappa^2 d\kappa \int_0^\pi d\psi \sin \psi \Phi_n \left( \kappa, \frac{z_1 + z_2}{2} \right)
\]

\[
\times \int_0^{2\pi} d\omega \exp \{i[z_1 \kappa \cos \psi + r_1 \kappa \sin \psi \cos(\phi_1 - \omega)]
\]

\[
- i[z_1 \kappa \cos \psi + r_1 \kappa \cos \psi \cos(\phi_2 - \omega)] \}.
\] (2.22)

Substituting equation (2.22) into equation (2.21) gives [8]

\[
\sigma_c^2 = \frac{k^4}{4\pi^2} \int_0^\infty \frac{dz_1}{z_1} \int_0^\infty \frac{dz_2}{z_2} \int_0^\infty r_1 dr_1 \int_0^\infty r_2 dr_2 \int_0^\infty \kappa^2 d\kappa \int_0^\pi d\psi \sin \psi \Phi_n \left( \kappa, \frac{z_1 + z_2}{2} \right)
\]

\[
\times \cos \left( \frac{kr_1^2}{2z_1} \right) \exp(i z_1 \kappa \cos \psi) \cos \left( \frac{kr_2^2}{2z_2} \right) \exp(-i z_2 \kappa \cos \psi) \int_0^{2\pi} d\omega
\]

\[
\times \int_0^{2\pi} \exp[r_1 \kappa \sin \psi \cos(\phi_1 - \omega)] d\phi_1 \int_0^{2\pi} \exp[r_2 \kappa \sin \psi \cos(\phi_2 - \omega)] d\phi_2.
\] (2.23)

Equation (2.23) can be simplified (see [8] for the intermediate steps) to [8]

\[
\sigma_c^2 = 4\pi^2 k^2 \int_0^\infty \kappa d\kappa \Phi_n(\kappa, z) \int_0^\infty \sin^2 \left( \frac{z\kappa^2}{2k} \right) dz.
\] (2.24)
Choosing a thin turbulent layer of thickness $\Delta H$ at fixed altitude $H$ [8], viz.,

$$C_n^2(z) = \begin{cases} 
1 & H < z < H + \Delta H \\
0 & \text{otherwise,}
\end{cases}$$  \hfill (2.25)

and recalling the Kolmogorov power spectrum from equation (2.14), the variance of the log-amplitude fluctuations for the thin-layer $C_n^2$ model can be written as [8]

$$\sigma^2_\chi = 4\pi^2 k^2 \int_0^\infty \kappa \, d\kappa \Phi_n(\kappa, z) \int_0^\infty \sin^2\left(\frac{\kappa z^2}{2k}\right) \, dz$$

$$= 0.563 C_n^2 \Delta H H^{5/6} k^{7/6} \left[\text{Np}^2\right].$$  \hfill (2.26)

Expressing this in decibel units,

$$\sigma^2_\chi = 42.48 C_n^2 \Delta H H^{5/6} k^{7/6} \left[\text{dB}^2\right].$$  \hfill (2.27)

For multiple thin layers (and letting $H \to z$), the variance of the log-amplitude fluctuations can finally be expressed as

$$\sigma^2_\chi = \frac{k^{7/6}}{(\sin \theta)^{11/6}} \sum_z C_n^2(z) z^{5/6} \Delta z \left[\text{dB}^2\right],$$  \hfill (2.28)

where the quantity has been scaled for an oblique path angle $\theta > 5^\circ$. The constraint on the path angle will be a necessary requirement for the two scintillation models presented in this work.
2.3 The Refractive Index Structure Constant

The refractive index structure constant provides a measure of the strength of fluctuations in the refractive index – it represents the amplitude of the spatial correlation of the refractive index fluctuations between two points at unitary distance [9]. The values of $C_n^2$ typically range from $10^{-10}$ (strong) to $10^{-20}$ (weak) m$^{-2/3}$. At a constant height above the ground, and for short periods at a fixed propagation distance, $C_n^2$ is essentially constant. However, the refractive index structure constant (more appropriately parameter, in this scenario) is a function of height above the ground for slanted and vertical paths. In principle, once the $C_n^2(z)$ profile has been determined along a chosen path, the corresponding scintillation effects, for a given stationary period, can be known. For longer, non-stationary periods, a statistical approach is in order. Along a given path, $C_n^2(z)$ will typically be highest near the Earth’s surface, or in the atmospheric boundary layer (ABL), and in regions where stratus or fair weather cumulus clouds are present, since in these layers the turbulence is strongest [11].

The ABL can be described as the layer closest to the Earth’s surface where the interaction and heat exchange with the surface dominate the atmospheric dynamics. As the sun heats the ground during the daytime, convective instability is created by layers of cold air above the warming surface, giving rise to thermal plumes and strong turbulence with $C_n^2$ values in the range of $10^{-13}$ or $10^{-14}$ m$^{-2/3}$. During the nighttime hours, more stable conditions due to warmer air over the colder ground usually result in lower $C_n^2$ values. Around sunrise and sunset, the thermally neutral
condition of almost identical surface and air temperatures result in very low wind speeds and minimum values of $C_n^2$ [7].

In the literature, it has been reported that there is a significant correlation between the occurrence of scintillation and the presence of cumulus clouds along the path of propagation, suggesting that clouds are responsible for at least part of the turbulent activity causing the scintillation. This “turbulent attenuation” is caused by the mixing of air masses with different water contents in and around clouds and precipitation [4]. For cloud tops capped by a temperature inversion with drier air above, there can be a dramatic increase in $C_n^2$ values [1]. While worth noting, the complex (and little understood) relationship between tropospheric scintillation and the presence of clouds along the propagation path will not be (directly) explored in this work.
3.1 Prediction of Tropospheric Scintillation: A Statistical Model

The statistical model reviewed herein is the work of Hugues Vasseur [12]. The reader is directed to the cited work for further details and other references. It is included here for convenience and completeness, as it is directly related to the purposes of this work.

Vasseur’s model describes a method for computing the long-term scintillation statistics from coarse-resolution radiosonde data, i.e., data that has been approximated by linear piecewise interpolations of the significant level data. The height resolution is roughly 20 to 50 meters. Using the theory of propagation through a turbulent medium, large amounts of radiosonde data from a period of one month or more are used to extract the statistical features of the structure parameter profiles, from which the long-term distribution of scintillation effects are inferred. The method is not limited by any particular propagation measurements and is based solely on radiosonde data and theoretical considerations. The method is generally applicable to any slant-path configuration, since the atmospheric dependencies are captured by the radiosonde profiles.
Scintillation effects are characterized by the log-amplitude $\chi$ [dB], or the ratio of the instantaneous amplitude of the observed signal to the mean amplitude, and the scintillation variance $\sigma_\chi^2$ [dB$^2$], that is the variance of the log-amplitude $\chi$. Turbulence-induced scintillation is inherently a stochastic process, and therefore should be discussed in terms of its statistical features. For stationary periods on the order of minutes, the scintillation variance is approximately constant, while the scintillation log-amplitude distribution may be reasonably fitted by a zero-mean Gaussian distribution with variance equal to the mean scintillation variance during the given stationary period. For longer nonstationary periods, the variability of $\sigma_\chi^2$ must be considered. In such cases, the long-term probability density function of scintillation log-amplitude can by modeled by a Gaussian distribution with log-normally distributed variance [12], viz.,

$$p_{\text{long-term}}(\chi) = \int_0^\infty p_{\text{short-term}}(\chi|\sigma_\chi^2) p(\sigma_\chi^2) \, d\sigma_\chi^2,$$

(3.1)

where $p_{\text{short-term}}(\chi|\sigma_\chi^2)$ is the conditional short-term probability distribution function of $\chi$, assuming a constant value of $\sigma_\chi^2$ for the scintillation variance, a Gaussian given by [12]

$$p_{\text{short-term}}(\chi|\sigma_\chi^2) = \frac{1}{\sqrt{2\pi}\sigma_\chi^2} \exp\left(-\frac{\chi^2}{2\sigma_\chi^2}\right),$$

(3.2)
and \( p(\sigma^2) \) is the long-term probability density function of scintillation variance, a log-normal distribution given by [12]

\[
p(\sigma^2) = \frac{1}{\sqrt{2\pi \sigma^2_s}} \exp \left( -\frac{[\ln(\sigma^2/m)]^2}{2s^2} \right),
\]

where \( m \) [dB^2] is the median value of the scintillation variance distribution and \( s \) (dimensionless) is the standard deviation of \( \ln(\sigma^2) \).

For stationary scintillation events, \( \sigma^2 \) can be related to the refractive index structure constant as [12]

\[
\sigma^2 = 42.48 \frac{k^{7/6}}{(\sin \theta)^{11/6}} \int_{\text{height}} C^2_n(z) z^{5/6} \, dz \quad \text{[dB}^2\text{]},
\]

where the frequency wavenumber \( k = 2\pi/\lambda \) [m\(^{-1}\)] with \( \lambda \) [m] the wavelength, \( \theta \) the elevation angle, \( z \) [m] the height above the ground level, and \( C^2_n(z) \) [m\(^{-2/3}\)] the height-dependent refractive index structure parameter.

Several conditions must be satisfied such that (3.4) holds [12]:

1. Turbulence must be well-developed to satisfy the Kolmogorov theory.

2. The incident wave is a plane wave.

3. Scintillation is assumed to be weak.

4. Equation (3.4) consists of the asymptotic approximation of a more complicated relationship valid for \( L_0 \gg \sqrt{\lambda(z/\sin \theta)} \), where \( L_0 \) [m] is the outer scale of turbulence. This condition effectively restricts slant-path angles \( \theta \) to \( > 5^\circ \).
5. Scintillation variance calculated from equation (3.4) characterizes fluctuations received by a hypothetical point receiver, which implies that equation (3.4) must be scaled by an antenna aperture-averaging factor.

Essentially, Vasseur’s method consists of two steps:

1. Statistical characteristics of the $C_n^2(z)$ profile are inferred from analysis of the radiosonde data.

2. The inferred $C_n^2(z)$ vertical profile features are used to predict $m$ and $s$, the parameters that govern the scintillation long-term statistics.

According to Kolmogorov theory [12],

$$C_n^2(z) = a M(z)^2 L_0(z)^{4/3} \left[ m^{-2/3} \right],$$

where $a \approx 2.8$ is a dimensionless constant, $M(z) = dn(z)/dz \; [m^{-1}]$ is the vertical gradient of the refractive index, and $L_0 \; [m]$ is the outer scale of turbulence [12]. The direct calculation of the $C_n^2(z)$ profile from equation (3.5) would require radiosonde data with a vertical resolution on the order of one meter or so, such that the fine-structure and random behavior of the turbulent troposphere might be detected.

As a way to circumvent this issue, Vasseur has proposed a probabilistic approach to estimate the mean structure parameter at height $z$ [12], viz.,

$$\langle C_n^2(z) \rangle = 2.8 M_0(z)^2 \langle R(z) \rangle \int_0^{\infty} L_0^{4/3} P_{L_0} \int_0^{\infty} P_S \int_{-\infty}^{S^{2Ric}} (N^2)^2 P_{N^2} dN^2 dS dL_0,$$

(3.6)
where $M_0(z)$ and $R(z)$ are defined by $M(z) = M_0 N^2 R^{1/2}$ and $R$ represents the humidity contribution to $M^2$. $L_0$, $S$, and $N^2$ are random variables representing the outer scale of turbulence, the wind shear, and the squared buoyancy (frequency), respectively, and characterized by the probability density functions $P_{L_0}$, $P_S$, and $P_{N^2}$. $Ric \approx 0.25$ is the critical value of the Richardson number. The buoyancy frequency, also known as the Brunt-Väisälä frequency, is the oscillation frequency of a parcel that has been vertically displaced in a statically stable environment. The Richardson number is the ratio of squared buoyancy to the squared wind shear. Its critical value is used to characterize the instability of the medium. The expected values calculated using equation (3.6) represent the mean structure constant at height $z$ through an atmospheric slab whose thickness is related to the vertical resolution of the radiosonde profile. The parameters and probability density functions given in equation (3.6) are described in Appendix A. They are derived from basic atmospheric profiles measured by radiosonde:

1. pressure $p(z)$ [hPa]
2. temperature $T(z)$ [K]
3. water vapor pressure $e(z)$ [hPa]
4. wind speed $v(z)$ [m/s]
5. wind (cardinal) direction $\varphi(z)$ [degrees].

Equation (3.6) gives the vertical profile of the mean refractive index structure constant from atmospheric data collected during one radiosonde ascent. Profiles of $\langle C_n^2(z) \rangle$
(where the bar indicates a time average), \( p_{50(C^2_n)}(z) \), and \( \sigma_{(C^2_n)}(z) \) are calculated, referring to the mean, median, and standard deviation, respectively. Provided the distribution is calculated over a period of at least one month, experimental evidence suggests that the long-term probability density functions of \( \langle C^2_n \rangle \) are reasonably approximated by a log-normal distribution, regardless of height \( z \).

From equations (3.4) and (3.6), it is possible to obtain the instantaneous value of the scintillation variance during a single radiosonde ascent by replacing the integral of \( C^2_n(z)z^{5/6}dz \) by a summation of \( \langle C^2_n(z) \rangle z^{5/6}\Delta z \), where \( \Delta z \) represents the thickness of the horizontal slab through which \( \langle C^2_n(z) \rangle \) has been calculated, and summing over all height values for which atmospheric data is available. However, the stated purpose of Vasseur’s model is to predict long-term statistics of scintillation from a large amount of radiosonde data, so that equation (3.4) is used to relate the statistical moments of the scintillation variance to the long-term characteristics of the mean structure parameter profile. The simplest moments are the mean and variance [12], viz.,

\[
\sigma^2_{\chi} = 42.48 \frac{k^{7/6}}{(\sin \theta)^{11/6}} \sum_z \langle C^2_n(z) \rangle z^{5/6} \Delta z \tag{3.7}
\]

and

\[
\sigma^2_{\chi} = \left( \sum_z \sum_{z'} \left[ 42.48 \frac{k^{7/6}}{(\sin \theta)^{11/6}} \Gamma_{C^2_n}(z, z', 0) z^{5/6}z'^{5/6} \Delta z \Delta z' \right] - \sigma^2_{\chi} \right) \tag{3.8}
\]

where \( \Gamma_{C^2_n}(z, z + \Delta z, \tau) = E \{ C^2_n(z, t) C^2_n(z + \Delta z, t + \tau) \} \), \( E \{ \cdot \} \) denoting the expected value is the two-dimensional (height and time) cross-correlation function of
the refractive index structure parameter whose expression is given in Appendix B.

For nonstationary events, $\sigma^2_x$ will not be constant, and therefore $\sigma^2_{\sigma^2_x}$ will be nonzero.

Given that the long-term scintillation variance is log-normally distributed, $m$ and $s$ from (3.3) are given as [12]

\[
m = \frac{\bar{\sigma}^2_x}{\sqrt{\sigma^2_{\sigma^2_x} + \bar{\sigma}^2_x}} \text{ [dB}^2]\]

and

\[
s = \sqrt{\ln \left(1 + \frac{\sigma^2_{\sigma^2_x}}{\bar{\sigma}^2_x}\right)} \quad \text{(dimensionless)}. \tag{3.10}
\]

3.2 Modifying the Existing Statistical Model

This work will investigate Vasseur’s assertion that the long-term statistics of the outer scale $L_0$, wind shear $S$, and squared buoyancy $N^2$ are best represented by uniform, Rician, and Gaussian probability density functions. It will be shown that $L_0$, $S$, and $N^2$ are better characterized by the exponential [13], Pearson Type III [13], and Johnson SU [14] probability density functions, respectively. These new functions have been fitted to distributions of the three atmospheric quantities built from high-resolution radiosonde data spanning a period of one year at each of four locations. It will be evident that these functions are better equipped to represent the observed subtleties in outer scale, wind shear, and squared buoyancy, even across such a limited representation of climatic diversity. The high-resolution data will be used to show that several of the parameters of the new probability density functions can essentially be treated as constants, all the while preserving the afforded generality. It will
not be necessary to require high-resolution data for routine invocation of the modified scintillation prediction model, as characterization of the turbulent fine-structure represented by small-scale variation in $L_0$, $S$, and $N^2$ is effectively captured by the probability density functions, *i.e.*, calculation of the $\langle C_n^2(z) \rangle$ profiles for other locations can be completed using only coarse-resolution atmospheric data, since the pressure, temperature, and specific humidity change over much larger height scales compared with the scale over which $L_0(z)$, $S(z)$, and $N^2(z)$ will vary. The probability density functions will be parameterized in terms of the true atmospheric data.

### 3.3 Atmospheric Data

#### 3.3.1 High-Resolution Radiosonde Replacement System (RRS) Data

High-resolution (about five meters) Radiosonde Replacement System (RRS) data at one-second resolution are available (at the time of this analysis) from


via the FTP server at


The RRS data are available from the beginning of the RRS program in 2005 to the present time. There are almost 100 radiosonde stations in total located across the contiguous United States, Alaska, Hawaii, the Caribbean Islands, and several western Pacific tropical islands. At each radiosonde station, weather balloons are typically launched twice per day, at 0000 UTC and 1200 UTC. Instruments onboard
the balloons measure and record various atmospheric quantities such as pressure, temperature, relative humidity, and horizontal winds. The burst height of the balloons is in the range of 25 to 30 kilometers.

For application to this work, RRS data from the year 2015 have been acquired for the following four locations of climatically diverse classification (see Figure 3.1):

1. Miami, Florida,
2. Albuquerque, New Mexico,
3. Koror, Palau,

These atmospheric measurement data were used to develop new probability density functions which serve as models for the outer scale of turbulence, wind shear, and squared buoyancy.

3.3.2 Coarse-Resolution Global Forecast System (GFS) Data

Coarse-resolution (hundreds of meters) Global Forecast System (GFS) atmospheric prediction data are available (at the time of this analysis) via the secure HTTP server at

https://www.ncei.noaa.gov/data/global-forecast-system/access/historical/analysis/.
The Global Forecast System (GFS) is a National Centers for Environmental Prediction (NCEP) weather forecast model. It is a numerical weather prediction system containing a global computer model and variational analysis run by the U.S. National Weather Service (NWS). The model generates data for dozens of atmospheric and land-soil variables, including temperatures, winds, precipitation, soil moisture, and atmospheric ozone concentration. To improve the accuracy of weather predictions, the system couples four separate models – an atmospheric model, an ocean model, a land/soil model, and a sea ice model. Forecasts for up to 16 days in advance are produced by the GFS model, which is run four times per day at 0000 UTC, 0600 UTC, 1200 UTC, and 1800 UTC.

For purposes of the present work, GFS data from the year 2015 were used to populate the coarse-scale pressure, temperature, relative humidity, and horizontal winds profiles required by the scintillation models.

3.3.3 World Map of Köppen-Geiger Climate Classification

A map of the Köppen-Geiger Climate Classification is presented here for convenient association of the data locations analyzed in this work [15]. The map’s resolution and final scaling make it difficult, if not impossible, to ascertain the climate classification of Koror, Palau – it is designated Af: equatorial, fully humid.
Figure 3.1: World Map of Köppen-Geiger Climate Classification [15].
3.4 Processing the Atmospheric Data

3.4.1 Data Cleansing, Reduction, and Interpolation

Occasionally, a given radiosonde data set will contain states (sets of pressure, temperature, etc., at a given altitude) where some instrument error code/flag is present. The RRS data sets selected for this work have been cleaned to ensure that they are free of such states. Also, again occasionally, the balloon that carries the radiosonde instrument package will burst prematurely, resulting in a shorter than typical data run. Every RRS data set used in this analysis has been validated such that states exist from ground-level up to approximately 10 kilometers.

The GFS data have been reduced to a region (grid of 0.5° × 0.5°) about each high-resolution RRS data set latitude/longitude using the wgrib2 software application found at


The GFS atmospheric quantities given at the four grid-points enclosing a particular RRS data set latitude/longitude have been bilinearly interpolated to that latitude/longitude. Additionally, these quantities have been linearly (logarithmically in the case of pressure) interpolated in altitude to generate values at every 100 meters from 1000 millibars up to about 10 kilometers.
### 3.4.2 Data Smoothing

The measured atmospheric quantities given in the radiosonde data have been “smoothed” in order to remove the inherent instrument quantization error (see Figure 3.2). This step is necessary only to avoid errors in the derivative operations required for some of the calculations [16]. The method selected for this task is that of Savitzky-Golay – least squares calculations are performed by convolution of the data points with properly chosen sets of integers that depend on the desired window-size and order of the convolute [17], [18]. The MATLAB function `smoothdata(….)`, called with the Savitzky-Golay filter option, was used to smooth the various atmospheric quantities required for the present analysis.

![Figure 3.2: Examples of data smoothing. Miami, Florida at 0000 UTC 15 Apr 2015.](image-url)
3.4.3 Scaling Mean Wind Shear and Mean Squared Buoyancy

Approximating function derivatives via numerical differentiation is tricky. The situation is exacerbated when the differentiation must be performed using sampled data, *i.e.*, the functional form representing the data is not known – there is only a table of values. Additionally, the effect of round-off error on the accuracy of derivatives computed from finite-difference formulas must be considered. Some functions and even sampled data are also inherently “rough,” meaning that undersampled data may not reflect *rapid* variations. [19]. Finally, with sampled data that is taken from digital instruments, quantization is a problem. The quantization situation was described in Section 3.4.2. The remaining issues are now discussed.

It is obvious from equations (A.4) and (A.8) that calculations of mean squared buoyancy and mean wind shear involve the derivatives of tabled (or functions of) atmospheric data. These derivatives were computed using the finite-difference formula

\[ f'(x_0) = \frac{f_1 - f_0}{h} + O(h), \]  

(3.11)

where double-precision values were used in order to reduce the effect of round-off error.

The most troubling of the numerical differentiation issues given above, for the derivative quantities required in this work, is related to undersampled data. For the high-resolution RRS data, the sampling is adequate. However, the GFS predictions are given on a very coarse altitude scale, such that any numerical derivatives
computed with the data will be substantially underresolved when compared to the same quantities derived from the RRS data.

To alleviate this problem, profiles of mean wind shear and mean squared buoyancy were generated using both the GFS and RRS Miami, Florida 2015 data sets. The daily profiles were cost-analyzed in order to determine a “first-order” correction factor $\alpha$ to be used for scaling the GFS-derived quantities. The cost analysis procedure consisted of minimizing the parameterized sum of absolute differences (at each coordinated altitude $z$) of mean wind shear or mean squared buoyancy computed from both the GFS and RRS data sets.

In this context, the cost function that was minimized is expressed as

$$\sum_i |x_{\text{RRS}_i} - \alpha x_{\text{GFS}_i}|,$$

where $x_{\text{RRS}_i}$ and $x_{\text{GFS}_i}$ are vector elements of computed $\langle S(z_i) \rangle$ or $\langle N^2(z_i) \rangle$.

The daily $\alpha$ values were then averaged to produce an overall (applicable to any given day) scaling factor. The analysis suggested that the GFS-derived mean wind shear should be scaled by a factor of about 2.23, and the GFS-derived mean squared buoyancy by a factor of roughly 0.78. See Figure 3.3 and Figure 3.4 for some sample single-day profiles and the applied scaling.

**Table 3.1**: Scaling Factors for $\langle S(z) \rangle$, and $\langle N^2(z) \rangle$.

<table>
<thead>
<tr>
<th>Atmospheric Quantity</th>
<th>Scale Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\langle S(z) \rangle$ (mean wind shear)</td>
<td>2.23</td>
</tr>
<tr>
<td>$\langle N^2(z) \rangle$ (mean squared buoyancy)</td>
<td>0.78</td>
</tr>
</tbody>
</table>
Figure 3.3: Sample single-day computed mean wind shear profiles from January (top-left) through December (bottom-right) 2015 at Miami, Florida (0000 UTC).
Figure 3.4: Sample single-day computed mean squared buoyancy profiles from January (top-left) through December (bottom-right) 2015 at Miami, Florida (0000 UTC).
3.5 Building the Distributions of \( L_0, S, \) and \( N^2 \)

The method described herein for modeling the outer scale profile \( L_0(z) \) is based on the work of Martini et al. [9]. Definitions used in calculating the wind shear profile \( S(z) \), the squared buoyancy profile \( N^2(z) \), and the supporting cast of related atmospheric quantities can be found in most introductory atmospheric thermodynamics texts. For example, see Petty [20].

The chosen method for estimating the outer scale of turbulence profile \( L_0(z) \) is based on the determination of the Thorpe length \( L_T(z) \) from potential temperature profiles [21], [22], where the Thorpe length is the length scale of turbulent overturning in a stratified turbulent medium. It is reasoned that a stable atmosphere will be characterized by a monotonic potential temperature profile, and therefore inversions of the profile signal turbulent mixing. If a point \((\theta(z), z_m)\) on the curve height \( z \) vs. potential temperature \( \theta \) must be moved to \((\theta(z), z_n)\) in order to produce a monotonic profile (see Figure 3.5), a Thorpe displacement is defined as \( z_n - z_m \). Also, by definition, the Thorpe length is the root-mean-square of the Thorpe displacement and is proportional to the Ozmidov length \( L_{Ozmidov}(z) \approx 0.9L_T(z) \), which in turn is proportional to the outer scale length \( L_0(z) \approx L_{Ozmidov}(z)/2.27 \).

Turbulent layers are identified by evaluation of the Richardson number, defined as [9]

\[
R_i = \frac{N^2}{S^2},
\]  

(3.13)
Figure 3.5: Portion of actual and rearranged potential temperature profiles. Miami, Florida at 0000 UTC 15 Apr 2015.

where $N$ is the buoyancy frequency and $S$ is the wind shear. Values of $R_i$ less than 0.25 define an unstable stratification and a developed turbulence.

Distributions of $L_0(z)$, $S(z)$, and $N^2(z)$ have been built using high-resolution radiosonde data from the year 2015 (see Figure 3.6) captured at the four locations of mutually differing climate classification. See Figure 3.7, Figure 3.8, Figure 3.9, and Figure 3.10 for examples of the distributions at various altitudes and periods of the year.
Build distributions of $N^2$ across the 730 datasets from the year 2015 at Miami, FL. Likewise, for $S$ and $L_0$. (The pressures and temperatures are needed for a later calculation.)

**Figure 3.6:** Building Distributions of $L_0$, $S$, and $N^2$.

The focal point of this work is to investigate how Vasseur’s scintillation model might be modified by incorporating observed statistical features of $L_0$, $S$, and $N^2$ associated with various climate classifications into the joint probability density function of equation (3.6) – a function that must adequately convey the turbulence fine-structure necessary for accurately depicting the $C_n^2(z)$ profile.
Figure 3.7: Distributions of outer scale, wind shear, and squared buoyancy at (4000 → 5000) meters from July through September 2015 at Miami, Florida.
Figure 3.8: Distributions of outer scale, wind shear, and squared buoyancy at (6000 → 7000) meters from October through December 2015 at Albuquerque, New Mexico.
Figure 3.9: Distributions of outer scale, wind shear, and squared buoyancy at (7000 → 8000) meters from January through March 2015 at Koror, Palau.
Figure 3.10: Distributions of outer scale, wind shear, and squared buoyancy at (2000 → 3000) meters from January through March 2015 at Fairbanks, Alaska.
3.6 The Model Functions for $L_0$, $S$, and $N^2$

A set of candidate model functions for the outer scale $L_0(z)$, the wind shear $S(z)$, and the squared buoyancy $N^2(z)$ were selected by utilizing Python’s distfit package. The distfit software attempts to fit up to 89 univariate probability density functions to an input sample data set and returns several scoring metrics that suggest how well (or not) each of the functions fit the given data. The sample data input to the distfit software were taken from the distributions of $L_0(z)$, $S(z)$, and $N^2(z)$ which were computed using high-resolution radiosonde data from Miami, Albuquerque, Koror, and Fairbanks during the year 2015. These data were constrained to subsets, or windows, of altitudes and days of the year. The window dimensions were constants of 200 meters by 30 days. The window midpoint was permitted to slide in steps of 100 meters, spanning a range of altitudes from ground-level through 10 kilometers, with curve fitting performed on the sampled data from each window. The window was then slid by 15 days, and the whole process repeated, covering a range of days for the entire year of 2015. The functions found in Table 3.2 were indicated by the distfit software to be good fits to the sampled data.

<table>
<thead>
<tr>
<th>Sampled Data</th>
<th>Probability Density Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_0(z)$ (outer scale)</td>
<td>Exponential</td>
</tr>
<tr>
<td>$S(z)$ (wind shear)</td>
<td>Pearson Type III</td>
</tr>
<tr>
<td>$N^2(z)$ (squared buoyancy)</td>
<td>Johnson SU</td>
</tr>
</tbody>
</table>

Table 3.2: Model functions for $L_0(z)$, $S(z)$, and $N^2(z)$. 

38
The exponential probability density function is given by [13]

\[ P_{L0}(x; \lambda) = \lambda \exp[-\lambda(x - \xi)], \quad (3.14) \]

where \( x \in [\xi, \infty) \) and \( \lambda \in (0, \infty) \).

The Pearson Type III probability density function is given by [13]

\[ P_S(x; \kappa, \xi, \lambda) = \frac{|\beta|}{\lambda \Gamma(\alpha)} \left[ \beta \left( \frac{x - \xi}{\lambda} - \zeta \right) \right]^{\alpha-1} \exp \left[ -\beta \left( \frac{x - \xi}{\lambda} - \zeta \right) \right], \quad (3.15) \]

where \( x \in [\xi - 2\lambda/\kappa, \infty) \), \( \xi \in (-\infty, \infty) \), \( \lambda \in (0, \infty) \), and

\[
\begin{align*}
\beta &= \frac{2}{\kappa} \\
\alpha &= \beta^2 = \frac{4}{\kappa^2} \\
\zeta &= -\frac{\alpha}{\beta} = -\beta
\end{align*}
\quad \kappa \in (0, \infty). \quad (3.16)
\]

The Johnson SU probability density function is given by [14]

\[ P_{N2}(x; \gamma, \delta, \xi, \lambda) = \frac{\delta}{\lambda \sqrt{(x - \xi)^2 + 1}} \phi \left[ \gamma + \delta \log \left( \frac{x - \xi}{\lambda} + \sqrt{\left( \frac{x - \xi}{\lambda} \right)^2 + 1} \right) \right], \quad (3.17) \]

where \( \phi \) is the probability density function of the normal distribution and \( x \in (-\infty, \infty) \), \( \gamma \in (-\infty, \infty) \), \( \delta \in (0, \infty) \), \( \xi \in (-\infty, \infty) \), and \( \lambda \in (0, \infty) \).
3.7 Determination of Model Function Parameters

There are many popular methods to be found in the literature and textbooks for fitting an arbitrary curve to sampled data [23]. For example, three of the most widely used approaches are the method of nonlinear least squares, the method of moments, and the method of maximum likelihood (the *distfit* software mentioned previously uses this method). In this work, the method of moments will be used to relate the parameters of the selected model functions to measured atmospheric quantities such as pressure and temperature.

3.7.1 Method of Moments

It was reasoned that the parameters of the new model functions for $L_0(z)$, $S(z)$, and $N_2(z)$ are functions of atmospheric quantities such as pressure, temperature, *etc.*, as is the case for the parameters of Vasseur’s model functions (see equations (A.9) and (A.12)). Vasseur’s expressions for $\sigma_S(z)$ and $\sigma_{N_2}(z)$ served as a starting point for establishing the connections between the new model function parameters and the various atmospheric quantities. The parameters of the new model functions can be expressed in terms of the atmospheric quantities by equating moments of the new model functions to moments of Vasseur’s model functions (these moments being functions of parameters which are functions of pressure, temperature, *etc.*), and then solving the system of equations for the new parameters.

Before getting into the details of the relationships between the parameters of the new model functions and those of Vasseur’s functions, it is important to discuss
the constants appearing in equations (A.9) and (A.12). The value 0.348 appearing in equation (A.12) is a typical value associated with the mean density of dry air in a layer of atmosphere. However, the remaining constants appear to have been derived from some fitting process to observed data. The specific approach used to arrive at the given values is not evident from Vasseur’s referenced paper [24]. For application to the present work, the constants were treated as free parameters to be fitted via cost analysis using high-resolution RRS data from the four geographical locations given earlier. The procedure is outlined here:

1. Compute the sample variance of the wind shear $S$ and squared buoyancy $N^2$ from the high-resolution radiosonde data as functions of altitude and day of year for each location:

$$\text{Var}(X) = \frac{1}{N-1} \sum_{i=1}^{N} (x_i - \bar{x})^2,$$

(3.18)

where

$$\bar{x} = \text{Mean}(X) = \frac{1}{N} \sum_{i=1}^{N} x_i.$$

(3.19)

2. Compute the variances as functions of altitude and day of the year for each location using equations (A.9) and (A.12), where the constants have been replaced by free parameters. Use the values given by Vasseur as a starting point. The signs (positive/negative) are assumed to be correct.
For the Rice distribution, the raw moments are given by [13]

\[
\mu'_{k_S} = \sigma_S^k 2^{k/2} \Gamma(1 + k/2) L_{k/2} \left( -\frac{(S(z))^2}{2\sigma_S^2} \right),
\] (3.20)

where \( L_q(x) \) is a Laguerre polynomial, being related to the confluent hypergeometric function of the first kind [13]

\[
L_q(x) = {}_1F_1(-q; 1; x).
\] (3.21)

Therefore, the variance is found to be

\[
\mu_{2S} = 2\sigma_S^2 + \langle S(z)^2 \rangle - \frac{\pi}{2} \sigma_S^2 {}_1F_1 \left( -\frac{1}{2}, 1, -\frac{(S(z))^2}{2\sigma_S^2} \right),
\] (3.22)

where \( \sigma_S \) with free parameters \( a, b, c, \) and \( d \) is given as

\[
\sigma_S = a L_0^{-b} |\langle N^2(z) \rangle|^{-c} \left( 0.348 \frac{p(z)}{T(z)} \right)^{-d} \left[ s^{-1} \right].
\] (3.23)

For the Gaussian distribution, the relationship is [13]

\[
\mu_{2N^2} = \sigma_{N^2}^2,
\] (3.24)

where \( \sigma_{N^2} \) with free parameter \( f \) is given as

\[
\sigma_{N^2}(z) = f \sigma_S(z) \sqrt{|\langle N^2(z) \rangle|} \left[ s^{-2} \right].
\] (3.25)
3. Iteratively adjust \{a, b, c, d, f\} until the cost function is minimized. For example, a simple cost function chosen for this work is

\[
\sum_i |x_{\text{statistical}_i} - x_{\text{Vasseur}_i}|,
\]

(3.26)

where \(x_{\text{statistical}_i}\) and \(x_{\text{Vasseur}_i}\) are vector elements of computed \(S\) and \(N^2\) variances, taken together, from steps 1 and 2. To be more precise, the one-dimensional vectors \(x_{\text{statistical}}\) and \(x_{\text{Vasseur}}\) consist of elements of both \(S\) and \(N^2\) variances, with each element of the one vector coordinated with an element of the other vector at a given altitude, day of year, and location.

The cost analysis procedure presented above is easily adapted to larger sets of geographical inputs, presumably with the intent of covering a broader spectrum of climate representation. The results of the cost analysis procedure are listed in Table 3.3. Also see Figure 3.11. At least for the limited set of geographical inputs considered in this work, the cost analysis shows that \(\sigma_S\) does not depend on the outer scale \(L_0\).

<table>
<thead>
<tr>
<th>Model</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>f</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vasseur</td>
<td>0.180</td>
<td>0.300</td>
<td>0.250</td>
<td>0.150</td>
<td>1.095</td>
</tr>
<tr>
<td>Modified</td>
<td>0.036</td>
<td>0.000</td>
<td>0.200</td>
<td>0.120</td>
<td>1.534</td>
</tr>
</tbody>
</table>

Table 3.3: \(\sigma_S\) and \(\sigma_{N^2}\) Free Parameter Values
Figure 3.11: Cost analysis to determine parameters a, b, c, d, and f. Red pluses are statistical, blue circles are cost-fitted Vasseur expressions.
The parameters of the Pearson Type III probability density function are related to its mean, variance, and moment coefficient of skewness by [13]:

$$
\mu_{1s}' = \xi, \\
\mu_{2s} = \lambda^2, \\
\frac{\mu_{3s}}{\mu_{2s}^{3/2}} = \kappa.
$$

(3.27) (3.28) (3.29)

The central moments $\mu_{ks}$ can be determined from equation (3.20) by the usual transformation from raw to central moments [25].

The parameters of the Johnson SU probability density function are related to its mean and variance as [14]:

$$
\mu_{1N^2} = \langle N^2(z) \rangle = \xi - \lambda \exp\left(\frac{1}{2\delta^2}\right) \sinh\left(\frac{\gamma}{\delta}\right)
$$

(3.30)

$$
\mu_{2N^2} = \frac{\lambda^2}{2} \left[ \exp\left(\frac{1}{\delta^2}\right) - 1 \right] \left[ \exp\left(\frac{1}{\delta^2}\right) \cosh\left(\frac{2\gamma}{\delta}\right) + 1 \right].
$$

(3.31)

The extraction of $\gamma$ and $\delta$ from equations (3.30) and (3.31) (a system of highly nonlinear equations) was performed by *scipy.optimize.fsolve*. This Python module is a wrapper around Fortran’s MINPACK hybrd and hybrj algorithms [26], [27]. These algorithms implement a modification of the Powell hybrid method [28]. Powell’s method, introduced in 1970 by Michael J. D. Powell, is similar to the Levenberg–Marquardt algorithm (discussed in the following section), except that it uses an explicit trust region. Following each iteration, a given step from the Gauss-Newton
algorithm is used as an update to the current solution if it is within the trust region. Otherwise, the method seeks to minimize the objective function along the direction of steepest descent. If the point of minimization is inside the trust region, the new solution is taken at the intersection of the line joining the minimization point and the Gauss-Newton step and the trust region boundary. If outside, it is truncated to the trust region boundary and taken as the updated solution [29].

The method used to determine the location parameter $\xi$ and rate/scale parameter $\lambda$ of the exponential and Johnson SU model functions does not follow the prescription given above, and is discussed below.

### 3.7.2 Nonlinear Least Squares Fitting

In least squares fitting, the goal is to fit a parameterized curve to a set of distributed data by minimizing an objective function which is expressed as the sum of the squares of the errors between the curve, or rather the model function, and the set of distributed data. For nonlinear applications, the algorithm is iterative in nature, and each pass seeks to reduce the sum of the squares of the errors between the model function and the distribution by well-chosen updates to the model parameters. A number of variations on this theme exists in the literature, some more specialized than others. In this work, the nonlinear least squares fitting was performed using the Levenberg-Marquardt algorithm [30], [31].

The Levenberg-Marquardt method is a combination of two numerical minimization algorithms: the gradient descent method and the Gauss-Newton method. In the gradient descent method, parameter updates occur in the direction of steepest
descent. The Gauss-Newton method assumes that the model function is \textit{locally} quadratic in the parameters. When the parameters are close to their optimal values, the algorithm behaves more like the Gauss-Newton method, and when not, like the gradient descent method.

For a model function \( \hat{y}(t; \mathbf{p}) \) of an independent variable \( t \) and a vector of \( n \) parameters \( \mathbf{p} \) that is fit to a set of \( m \) data points \( (t_i, y_i) \), the objective function can be expressed as [30]

\[
\chi^2(\mathbf{p}) = \sum_{i=1}^{m} \left[ \frac{y(t_i) - \hat{y}(t_i; \mathbf{p})}{\sigma_{y_i}} \right]^2 \tag{3.32}
\]

\[
= (\mathbf{y} - \hat{\mathbf{y}}(\mathbf{p}))^T \mathbf{W} (\mathbf{y} - \hat{\mathbf{y}}(\mathbf{p})) \tag{3.33}
\]

\[
= \mathbf{y}^T \mathbf{W} \mathbf{y} - 2 \mathbf{y}^T \mathbf{W} \hat{\mathbf{y}} + \hat{\mathbf{y}}^T \mathbf{W} \hat{\mathbf{y}}, \tag{3.34}
\]

where \( \sigma_{y_i} \) is the measurement error for datum \( y(t_i) \), and \( \mathbf{W} \) is a \textit{weighting} matrix.

For the gradient descent method, the parameter update \( \mathbf{h} \) to \( \mathbf{p} \), such that \( \mathbf{p} \rightarrow \mathbf{p} + \mathbf{h} \), that moves the parameters in the direction of steepest descent is given by [30]

\[
\mathbf{h}_{\text{gd}} = \alpha \mathbf{J}^T \mathbf{W} (\mathbf{y} - \hat{\mathbf{y}}), \tag{3.35}
\]

where the positive scalar \( \alpha \) determines the length of the step in the steepest-descent direction and the \( m \times n \) Jacobian matrix \( \mathbf{J} = [\partial \hat{\mathbf{y}} / \partial \mathbf{p}] \) represents the local sensitivity of the function \( \hat{y} \) to variation in the parameters \( \mathbf{p} \).
For the Gauss-Newton method, the parameter update $h$ that minimizes $\chi^2$ results in the normal equations [30]

$$[J^T W J] h_{gn} = J^T W (y - \hat{y}).$$

(3.36)

By introducing a *damping parameter* $\lambda$, the Levenberg-Marquardt algorithm adaptively varies the parameter updates between the gradient descent update and the Gauss-Newton update, as given by [30]

$$[J^T W J + \lambda I] h_{lm} = J^T W (y - \hat{y}),$$

(3.37)

where small values of $\lambda$ result in a Gauss-Newton update, and large values of $\lambda$ result in a gradient descent update.

The values of $\lambda$ are normalized to the values of $J^T W J$ in Marquardt’s relationship [30]:

$$[J^T W J + \lambda \text{diag} (J^T W J)] h_{lm} = J^T W (y - \hat{y}).$$

(3.38)

It is typical to terminate the Levenberg-Marquardt algorithm after a specified number of iterations, or whenever there is convergence in one of the gradients, the parameters, or the reduced $\chi^2 \nu \equiv \chi^2 / (m - n + 1)$ [30].

The exponential, Pearson Type III, and Johnson SU probability density functions were fit to the distributions of $L_0$, $S$, and $N^2$ from Section 3.5 using the Levenberg-Marquardt algorithm. The resulting parameter values were found to be in good agreement with those values produced with the method of moments approach.
The location parameter $\xi$ and rate/scale parameter $\lambda$ of the exponential and Johnson SU model functions were determined by computing the mode values of $\xi$ and $\lambda$ (across all four geographical locations) taken from the windowing process described in Section 3.6. The distributions of $\xi$ and $\lambda$ were narrowly (small variance) distributed about their modes, and for purposes of this work, taken to be constants. The parameters are listed in Table 3.4.

**Table 3.4:** Constant Parameters: Exponential and Johnson SU PDFs.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Exponential</th>
<th>Johnson SU</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\xi$</td>
<td>3.0 m</td>
<td>$-5.0 \times 10^{-6}$ s$^{-2}$</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>0.025 m$^{-1}$</td>
<td>$4.2 \times 10^{-5}$ s$^{-2}$</td>
</tr>
</tbody>
</table>

### 3.8 Generating the $C_n^2$ Profiles

To generate the mean $C_n^2$ at height $z$, (3.6) is evaluated with the new model functions (with parameters as determined in the previous sections) and $M_0(z)$ and $R(z)$ as given by equations (A.1) and (A.2), respectively. There are many algorithms to choose from when numerically integrating equation (3.6), each with its own accuracy, computational efficiency, ease of implementation, *etc.* Some examples include the family of Newton-Cotes (quadrature) formulas with the trapezoidal rule, Simpson’s rule, and Gaussian quadrature being common examples [19]. Often, numerical integration of complex functions over complicated domains is best handled by Monte Carlo integration [32], [31]. As will be discussed in an upcoming section, a slight variation of this concept provides a method for not only evaluating equation (3.6),
but also for producing the $\overline{\sigma^2_\chi}$ and $\sigma^2_{\sigma^2_\chi}$ quantities needed for the calculation of $m$ and $s$ in equations (3.9) and (3.10).

3.8.1 Vasseur’s Approach

Vasseur states that his approach to solving the three-dimensional integral in equation (3.6) is through repeated one-dimensional integrations via Romberg’s method of order 10 [33]. Equation (3.6) was initially approached in this work just as Vasseur has proposed, thereby producing reasonable results for estimations of the mean structure parameter at height $z$. However, for reasons not fully understood, Vasseur’s equation (3.8) occasionally generates negative values for the variance of the scintillation variance, $\sigma^2_{\sigma^2_\chi}$, which is unacceptable. Further analysis of this issue was deemed outside the scope of this work. To avoid this problem, an alternative approach was devised by which the mean scintillation variance and variance of the scintillation variance, $\overline{\sigma^2_\chi}$ and $\sigma^2_{\sigma^2_\chi}$, are computed by the method of Monte Carlo sampling.

3.8.2 Monte Carlo Sampling

It is a simple, straightforward procedure to produce (pseudo-)random numbers, or draws, from common probability distributions, such as uniform and Gaussian, using widely available algorithms and software applications. Oftentimes, however, a given pursuit requires randomly-generated numbers taken from complicated (even multivariate) distributions. There are several popular methods for generating random deviates drawn from probability density distributions, such as the transformation,
rejection, and Box and Müller methods. In this work, the rejection method was the one of choice due to its sufficiently simple implementation.

As a trivial example [31], suppose random deviates are to be drawn from

\[ P(x) = 1 + ax^2 \quad x \in [-1, 1]. \]  \hspace{1cm} (3.39)

First, generate a random deviate \( x' \) uniformly distributed on the domain of \( P(x) \), i.e., between \(-1\) and 1, and a second random deviate \( y' \) uniformly distributed between 0 and \((1 + a)\), the allowed range of \( P(x) \). It is easy to see that \( x' \) and \( y' \) must be given by

\[ x' = -1 + 2r_i \]  \hspace{1cm} (3.40)

and

\[ y' = (1 + a)r_{i+1}, \]  \hspace{1cm} (3.41)

where \( r_i \) and \( r_{i+1} \) are successively generated random values of \( r \) drawn from the uniform distribution.

A result is counted if the point \((x', y')\) falls between the \( x \)-axis and the curve \( P(x) \), i.e., if \( y' < P(x') \), and discarded otherwise. After a sufficiently large number of draws, the entire area between the \( x \)-axis and the \( P(x) \) will be uniformly populated by this process and the selected samples will be the values of \( x' \) drawn from the distribution \( P(x) \).
3.9 Calculating $\sigma^2_\chi$ and $\sigma^2_{\chi_i}$

Application of the Monte Carlo sampling process to equation (3.6) with

$$P(x) \rightarrow P(L_0, S, N^2) = A P_{L_0} P_S P_{N^2}$$

(3.42)

yields samples of $C^2_n(z, L_0, N^2(S))$ given by

$$C^2_n(z, L_0, N^2(S)) = 2.8 M_0(z)^2 \langle R(z) \rangle L_0^{4/3} \left( N^2(S) \right)^2.$$  (3.43)

The drawn $L_0$, $S$, and $N^2$ are constrained to the volume given by the integration limits in equation (3.6), and the factor $A$ is related to the fact that $P_{L_0} P_S P_{N^2}$ is not normalized on the integration domain since the upper limit of $N^2$ is truncated at $S^2Ric$.

It is evident that from $n_s$ “successful” samples

$$\langle C^2_n(z) \rangle = \frac{1}{n_s} \sum_{i=1}^{n_s} C^2_{n_i}(z, L_0, N^2(S)),$$  (3.44)

which consequently implies the integration in equation (3.6).

With $n_s$ samples of $C^2_n(z, L_0, N^2(S))$ from $n_d$ days, the mean scintillation variance and the variance of the scintillation variance are given by

$$\sigma^2_\chi = \frac{1}{n_s n_d} \sum_{i=1}^{n_s n_d} \sigma^2_{\chi_i}$$  (3.45)
and
\[ \sigma_{\chi}^2 = \frac{1}{nsn_d - 1} \sum_{i=1}^{nsn_d} \left( \sigma_{\chi i}^2 - \sigma_{\chi}^2 \right)^2, \quad (3.46) \]

where
\[ \sigma_{\chi i}^2 = 42.48 \frac{k^{7/6}}{(\sin \theta)^{11/6}} \sum_z C_n^2(z, L_0, N^2(S)) z^{5/6} \Delta z. \quad (3.47) \]

### 3.10 Antenna Aperture-Averaging

In the literature, theoretical scintillation results are most often given in terms of a hypothetical “point-receiver.” However, a finite aperture receiver will have significant effects on the scintillation intensity and temporal characteristics, imposing a high wavenumber cutoff \( \kappa_c \) on the spectrum of refractivity fluctuations. When the transverse correlation length of the scintillation is smaller than the antenna diameter, this low-pass filtering, which is effectively an averaging effect, becomes significant [34].

Assuming a classical Kolmogorov spectrum, the antenna averaging factor \( g \) for an effective antenna diameter \( D_{\text{eff}} \) and wavenumber \( k \) is given by Haddon and Vilar [34] as
\[ g^2 = 3.8637 (x^2 + 1)^{11/12} \sin \left[ \frac{11}{6} \tan^{-1} \left( \frac{1}{x} \right) \right] - 7.0835 x^{5/6}, \quad (3.48) \]

where \( x = 0.0584 (D_{\text{eff}}^2 k/z_{\text{eff}}) \) is the square of the ratio between the antenna diameter and the size of the Fresnel zone.

This smoothing function depends primarily upon the ratio between the effective antenna diameter \( D_{\text{eff}} \) and the maximum Fresnel zone size occurring within the scattering zone, even though a slant path \( C_n^2(z) \) will generally be path dependent.
From basic antenna theory, the gain of a parabolic antenna is given as [35]

\[ G = 10 \log_{10} \left[ \eta \left( \frac{\pi D}{\lambda} \right)^2 \right], \]  

(3.49)

where \( \eta \) is the antenna efficiency, \( D \) is the physical antenna diameter, and \( \lambda \) the wavelength. This implies that the antenna efficiency is given as

\[ \eta = \left( \frac{\lambda}{\pi D} \right)^2 10^{G/10}. \]  

(3.50)

The effective antenna diameter is [36]

\[ D_{\text{eff}} = \sqrt{\eta} D. \]  

(3.51)

The effective path length is given by [36]

\[ z_{\text{eff}} = \frac{2 h_L}{\sqrt{\sin^2 \theta + 2.35 \times 10^{-4} + \sin \theta}} \quad \text{[m]}, \]  

(3.52)

where \( \theta \) is the antenna elevation angle and \( h_L = 1000 \) meters is the height of the turbulent layer.

### 3.11 Computing Cumulative Statistics for Scintillation Variance and Fade

Long-term scintillation statistics are typically represented by cumulative distributions of scintillation variance and fade. An analytical expression can be found
for the cumulative distribution of scintillation variance by integrating equation (3.3):

\[
cdf(\sigma^2_\chi) = \int_{\sigma^2_\chi}^{\infty} \frac{1}{\sqrt{2\pi}\sigma^2_\chi s} \exp\left(-\frac{[\ln (\sigma^2_\chi/m)]^2}{2s^2}\right) \, d\sigma^2_\chi
\]

\[
= \frac{1}{2} \text{erfc}\left[\frac{\ln (\sigma^2_\chi/m)}{\sqrt{2} s}\right],
\]

(3.53)

where \text{erfc}(\cdot) is the complementary error function. Equation (3.53) gives the probability that scintillation variance exceeds \(\sigma^2_\chi\).

Equation (3.1) may be numerically integrated to give a cumulative distribution which describes the probability that scintillation fade exceeds the log-amplitude \(\chi\):

\[
cdf(\chi) = \int_{\chi}^{\infty} \int_{0}^{\infty} \frac{1}{\sqrt{2\pi}\sigma^2_\chi s} \exp\left(-\frac{\chi'^2}{2\sigma^2_\chi s}\right) \frac{1}{\sqrt{2\pi}\sigma^2_\chi s} \exp\left(-\frac{[\ln (\sigma^2_\chi/m)]^2}{2s^2}\right) \, d\sigma^2_\chi \, d\chi'.
\]

(3.54)
CHAPTER 4

RESULTS AND DISCUSSION

4.1 Comparison of $C_n^2$ Profiles and Scintillation Variance Cumulative Statistics: Modified versus Vasseur Models

As a means of establishing a control case for the modified scintillation model, seasonal $C_n^2$ profiles and annual scintillation variance cumulative statistics have been generated from GFS data at latitude 6.7° N and longitude 3.23° E (Covenant University, Ota, Ogun State, Nigeria) for the year 2015 using atmospheric inputs of predicted pressure, temperature, relative humidity, and wind speed/direction at 1200 + 0300 UTC (1500 UTC) for each day of available data using both the modified and Vasseur models. Figure 4.1 and Figure 4.2 suggest that the modified model is capable of producing reasonable outputs when compared to those of Vasseur’s model. A more thorough study involving other locations, periods, and times of day would be beneficial for building confidence in the new model.

Given the similarities in the generated results of the modified and Vasseur models, why bother with the new model? As previously mentioned, the comparisons offered here are associated with a given year, a specific time of day, and a single location. The two models might produce inconsistent results for other scenarios.
There is still much investigation to be done in this regard. However, a substantial advantage of the new model is that the probability density functions are more general than those given by Vasseur. From observations at various altitudes and periods, the distributions of the outer scale are not uniformly distributed, and an exponential model is a good starting point for representing what was actually observed. Also from these observations, there was more than ample suggestion of an asymmetry in the distributions of squared buoyancy – structure that cannot be modeled with Vasseur’s Gaussian distribution. With additional analysis of larger sets of climate data, acceptance of the modified functions as alternatives to those of Vasseur might be justified.

![Graph showing scintillation variance cumulative statistics](image)

**Figure 4.1**: Annual scintillation variance cumulative statistics at Covenant University, Ota, Ogun State, Nigeria 1500 UTC 2015.
Figure 4.2: $C_n^2$ profiles of Jan - Mar, Apr - Jun, Jul - Sep, and Oct - Dec (left to right, top to bottom) at Covenant University, Ota, Ogun State, Nigeria 1500 UTC 2015.
4.2 Comparison with Measurement/Additional Model Data

Scintillation fade and amplitude data have been generated with the modified scintillation model using GFS data at latitude 6.7° N and longitude 3.23° E for the year 2015. Table 4.2 lists the computed $\sigma_\chi^2$, $\sigma_{\sigma_\chi}^2$, and associated $m$ and $s$ values. In Figure 4.3, Figure 4.4, and Figure 4.5, the fade and amplitude data are compared to the results from four other models [4], [36], [37], [38] and measured data [39], [40], [41] at the same location and period. Akinwumi et al. have excluded days with observed precipitation from their analysis and presentation of measured scintillation data [39], [40], [41].

In accordance with the procedure discussed in Section 3.10, the data [42] given in Table 4.1 yield an antenna averaging factor $g$ of approximately 0.96. The $\sigma_\chi^2$ and $m$ quantities have been scaled by $g^2$, and the $\sigma_{\sigma_\chi}^2$ quantities by $g^4$.  

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Specification</th>
</tr>
</thead>
<tbody>
<tr>
<td>Satellite</td>
<td>Astra 2E/2F/2G at 28.2° E</td>
</tr>
<tr>
<td>Beacon Frequency</td>
<td>12.245 GHz</td>
</tr>
<tr>
<td>Signal polarization</td>
<td>Vertical</td>
</tr>
<tr>
<td>Antenna elevation</td>
<td>59.9°</td>
</tr>
<tr>
<td>Site altitude</td>
<td>74 m</td>
</tr>
<tr>
<td>Antenna diameter</td>
<td>0.9 m</td>
</tr>
<tr>
<td>Antenna height</td>
<td>5.9 m</td>
</tr>
<tr>
<td>Antenna gain</td>
<td>60 dB</td>
</tr>
<tr>
<td>Latitude</td>
<td>6.7° N</td>
</tr>
<tr>
<td>Longitude</td>
<td>3.23° E</td>
</tr>
</tbody>
</table>
Table 4.2: $\sigma_{x^2}$, $\sigma_{\sigma_x^2}$, $m$, and $s$.

<table>
<thead>
<tr>
<th></th>
<th>$\sigma_{x^2}$ [dB$^2$]</th>
<th>$\sigma_{\sigma_x^2}$ [dB$^4$]</th>
<th>$m$ [dB$^2$]</th>
<th>$s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Annual</td>
<td>$4.02 \times 10^{-3}$</td>
<td>$3.12 \times 10^{-5}$</td>
<td>$2.35 \times 10^{-3}$</td>
<td>1.04</td>
</tr>
<tr>
<td>Jan - Mar</td>
<td>$5.48 \times 10^{-3}$</td>
<td>$6.00 \times 10^{-5}$</td>
<td>$3.16 \times 10^{-3}$</td>
<td>1.05</td>
</tr>
<tr>
<td>Apr - Jun</td>
<td>$5.26 \times 10^{-3}$</td>
<td>$3.62 \times 10^{-5}$</td>
<td>$3.47 \times 10^{-3}$</td>
<td>0.91</td>
</tr>
<tr>
<td>Jul - Sep</td>
<td>$1.79 \times 10^{-3}$</td>
<td>$4.67 \times 10^{-6}$</td>
<td>$1.14 \times 10^{-3}$</td>
<td>0.95</td>
</tr>
<tr>
<td>Oct - Dec</td>
<td>$3.58 \times 10^{-3}$</td>
<td>$1.55 \times 10^{-5}$</td>
<td>$2.41 \times 10^{-3}$</td>
<td>0.89</td>
</tr>
<tr>
<td>January</td>
<td>$3.54 \times 10^{-3}$</td>
<td>$6.26 \times 10^{-5}$</td>
<td>$1.44 \times 10^{-3}$</td>
<td>1.34</td>
</tr>
<tr>
<td>February</td>
<td>$7.09 \times 10^{-3}$</td>
<td>$6.49 \times 10^{-5}$</td>
<td>$4.69 \times 10^{-3}$</td>
<td>0.91</td>
</tr>
<tr>
<td>March</td>
<td>$6.15 \times 10^{-3}$</td>
<td>$4.10 \times 10^{-5}$</td>
<td>$4.26 \times 10^{-3}$</td>
<td>0.86</td>
</tr>
<tr>
<td>April</td>
<td>$6.95 \times 10^{-3}$</td>
<td>$5.12 \times 10^{-5}$</td>
<td>$4.84 \times 10^{-3}$</td>
<td>0.85</td>
</tr>
<tr>
<td>May</td>
<td>$5.88 \times 10^{-3}$</td>
<td>$4.68 \times 10^{-5}$</td>
<td>$3.84 \times 10^{-3}$</td>
<td>0.92</td>
</tr>
<tr>
<td>June</td>
<td>$3.36 \times 10^{-3}$</td>
<td>$7.78 \times 10^{-6}$</td>
<td>$2.59 \times 10^{-3}$</td>
<td>0.72</td>
</tr>
<tr>
<td>July</td>
<td>$2.44 \times 10^{-3}$</td>
<td>$8.67 \times 10^{-6}$</td>
<td>$1.55 \times 10^{-3}$</td>
<td>0.95</td>
</tr>
<tr>
<td>August</td>
<td>$1.05 \times 10^{-3}$</td>
<td>$6.49 \times 10^{-7}$</td>
<td>$8.29 \times 10^{-4}$</td>
<td>0.68</td>
</tr>
<tr>
<td>September</td>
<td>$2.00 \times 10^{-3}$</td>
<td>$3.52 \times 10^{-6}$</td>
<td>$1.46 \times 10^{-3}$</td>
<td>0.79</td>
</tr>
<tr>
<td>October</td>
<td>$4.58 \times 10^{-3}$</td>
<td>$2.04 \times 10^{-5}$</td>
<td>$3.26 \times 10^{-3}$</td>
<td>0.82</td>
</tr>
<tr>
<td>November</td>
<td>$4.66 \times 10^{-3}$</td>
<td>$1.47 \times 10^{-5}$</td>
<td>$3.59 \times 10^{-3}$</td>
<td>0.72</td>
</tr>
<tr>
<td>December</td>
<td>$1.23 \times 10^{-3}$</td>
<td>$3.12 \times 10^{-6}$</td>
<td>$6.99 \times 10^{-4}$</td>
<td>1.06</td>
</tr>
</tbody>
</table>

The scintillation fade and intensity calculated with the modified scintillation prediction model using atmospheric data for the year 2105 at Covenant University compare well with the same quantities from the referenced models. Compared to the measured values, the present modified model scintillation predictions are low for the months of June through September; however, their correlation with the measured temperatures at Covenant University during the same year (see Figure 4.6) is encouraging [43]. With additional input data and further analysis, the modified model presented in this work could be useful for the prediction of long-term scintillation from coarse-resolution atmospheric data.
**Figure 4.3**: Monthly scintillation intensity at Covenant University, Ota, Ogun State, Nigeria 1500 UTC 2015 [39].

**Figure 4.4**: Annual scintillation fade cumulative statistics at Covenant University, Ota, Ogun State, Nigeria 1500 UTC 2015 [40].
Figure 4.5: Monthly scintillation fade cumulative statistics (1%) at Covenant University, Ota, Ogun State, Nigeria 1500 UTC 2015 [41].

Figure 4.6: Monthly averaged ground-level temperatures at Covenant University, Ota, Ogun State, Nigeria 2015 from Davis weather station and Atmospheric Infrared Sounder (AIRS) [43].
REFERENCES


APPENDICES
APPENDIX A

VASSEUR’S APPENDIX A

\[(Abridgment \ of \ Vasseur’s \ Appendix \ A \ [12])\]

\[M_0(z) = -77.6 \times 10^{-6} \frac{p(z)}{gT(z)} \quad (A.1)\]

where \(g = 9.81 \text{ [m/s}^2\text{]}, \) the acceleration of gravity.

\[\langle R(z) \rangle = \left[ 1 + 1.55 \times 10^4 \frac{q(z)}{T(z)} - \frac{1.55 \times 10^4}{2} \frac{g \frac{\partial q(z)}{\partial z}}{\langle N^2(z) \rangle T(z)} \right]^2 \quad (A.2)\]

where

\[q(z) = 0.6225 \frac{e(z)}{p(z)} \quad \text{(dimensionless)} \quad \text{is the specific humidity} \quad (A.3)\]

and

\[\langle N^2(z) \rangle = g \frac{\partial \ln \theta}{\partial z}(z) \quad \text{[s}^{-2}\text{]} \quad \text{is the mean of the squared buoyancy frequency,} \quad (A.4)\]

where

\[\theta = T(z) \left( \frac{1000}{p(z)} \right)^{0.2858} \quad \text{[K]} \quad \text{is the specific temperature.} \quad (A.5)\]
\( P_{L_0} \) is a uniform distribution between \( L_{0_{\text{min}}} = 3 \) [m] and \( L_{0_{\text{max}}} = 100 \) [m]:

\[
P_{L_0} = \begin{cases} 
\frac{1}{L_{0_{\text{max}}} - L_{0_{\text{min}}}} & \text{if } L_{0_{\text{min}}} \leq L_0 \leq L_{0_{\text{max}}} \\
0 & \text{otherwise.}
\end{cases}
\]  

(A.6)

\( P_S \) is a Rice distribution (assuming both vector shear components are Gaussian):

\[
P_S = \frac{S}{\sigma_S^2(z)} \exp \left( -\frac{S^2 + \langle S(z) \rangle^2}{2\sigma_S^2(z)} \right) I_0 \left( \frac{S\langle S(z) \rangle}{\sigma_S^2(z)} \right),
\]  

(A.7)

where

\[
\langle S(z) \rangle = \sqrt{\left( \frac{\partial v \cos \varphi}{\partial z} (z) \right)^2 + \left( \frac{\partial v \sin \varphi}{\partial z} (z) \right)^2} \quad [s^{-1}]
\]  

(A.8)

is the mean wind shear and

\[
\sigma_S = 0.18 L_0^{-0.3} \langle N^2(z) \rangle^{0.25} \rho(z)^{-0.15} \quad [s^{-1}],
\]  

(A.9)

where

\[
\rho(z) = 0.348 \frac{p(z)}{T(z)} \quad [\text{kg/m}^3].
\]  

(A.10)

\( P_{N^2} \) is a Gaussian distribution:

\[
P_{N^2} = \frac{1}{\sqrt{2\pi} \sigma_{N^2}(z)} \exp \left[ -\frac{(N^2 - \langle N^2(z) \rangle)^2}{2\sigma_{N^2}^2(z)} \right],
\]  

(A.11)

where

\[
\sigma_{N^2}(z) = \sqrt{\frac{6}{5}} \sigma_S(z) \sqrt{\langle N^2(z) \rangle} \quad [s^{-2}].
\]  

(A.12)
APPENDIX B

VASSEUR’S APPENDIX B

(Abridgment of Vasseur’s Appendix B [12])

Presentation of the theoretical development leading to the expression of the variance of the scintillation variance $\sigma_{\chi}^2$ summarized in (3.8).

Using the relationship

$$\sigma_{\chi}^2 = E \left\{ (\sigma_{\chi}^2)^2 \right\} - \sigma_{\chi}^2,$$

(B.1)

the problem is reduced to evaluating the expected value of the square scintillation variance, which is related to the autocorrelation function of $\sigma_{\chi}^2$ as

$$E \left\{ (\sigma_{\chi}^2)^2 \right\} = \Gamma_{\sigma_{\chi}^2}(\tau = 0),$$

(B.2)

where

$$\Gamma_{\sigma_{\chi}^2}(\tau) = E \left\{ \sigma_{\chi}^2(t) \sigma_{\chi}^2(t + \tau) \right\}$$

(B.3)

is the autocorrelation function of the scintillation variance.
From (3.4), note that the scintillation variance is given as the spatial integral of the structure parameter profile, therefore its autocorrelation function depends on the two-dimensional (height and time) cross-correlation function of the refractive index structure parameter, viz.,

\[
\Gamma_{\sigma^2}(\tau) = \frac{42.48}{(\sin \theta)^{11/6}} \int_z \int_{z'} \Gamma_{C_n^2}(z, z', \tau) z^{5/6} z'^{5/6} \, dz \, dz'.
\] (B.4)

Radiosonde data, lacking adequate resolution in space and time, cannot be used to derive the two-dimensional cross-correlation function. However, it can be related to the long-term characteristics of the mean structure parameter profile via an expression developed by Ravard and Chevrier [44] on the basis of a model proposed by Hufnagel [45] as

\[
\Gamma_{C_n^2}(z, z + \Delta z, \tau) = p_{50(C_n^2)}(z) p_{50(C_n^2)}(z + \Delta z) \exp \left( \frac{s_{(C_n^2)}^2(z) + s_{(C_n^2)}^2(z + \Delta z)}{2} \right) \exp[\Gamma_r(z, z + \Delta z, \tau)],
\] (B.5)

where \( s_{(C_n^2)}(z) \) is a parameter of the \( \langle C_n^2(z) \rangle \) log-normal distribution, viz.,

\[
s_{(C_n^2)}(z) = \frac{1}{2} \ln \left( \frac{\langle C_n^2(z) \rangle^2 + \sigma_{(C_n^2)}^2(z)}{p_{50(C_n^2)}^2(z)} \right),
\] (B.6)
and

\[ \Gamma_r(z, z + \Delta z, \tau) = \frac{s_{(C_n^2)}(z) s_{(C_n^2)}(z + \Delta z)}{2} \]

\[ \cdot \left[ A \left( \frac{\Delta z}{L_1} \right) \exp\left(-\tau/5\right) + A \left( \frac{\Delta z}{L_2} \right) \exp\left(-\tau/80\right) \right], \quad (B.7) \]

where

\[ A \left( \frac{\Delta z}{L} \right) = \begin{cases} 
  1 - \frac{\mid \Delta z \mid}{L} & \text{if} \mid \Delta z \mid < L \\
  0 & \text{otherwise}
\end{cases} \quad (B.8) \]

with \( L_1 = 100 \) [m] and \( L_2 = 2000 \) [m].

Note that the calculation of the cross-correlation function \( \Gamma_{C_n^2}(z, z + \Delta z, \tau) \) for \( \Delta z = 0 \) and \( \tau = 0 \), from equations (B.5) and (B.7), yields the expected result of \( \langle C_n^2(z) \rangle^2 + \sigma_{C_n^2}^2(z)^2 \) / \( \sigma_{C_n^2}^2 \).

The long-term variance of the scintillation variance \( \sigma_{C_n^2}^2 \) is derived from the long-term statistical characteristics of the mean structure parameter profile. The integrals in equation (B.4) are replaced by summations over the available mean values of the integrand through horizontal slabs in order to compute equation (3.8).