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Dynamical Systems: Predictability and Chaos

Blane Hollingsworth

THE UNIVERSITY OF ALABAMA IN HUNTSVILLE

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Name of Candidate: Blane Hollingsworth

Department: Mathematics

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BLANE HOLLINGSWORTH

Approved by:

[signature] 12/03/97

Project Advisor Date

M. H. Chang 1/29/98

Department Chair Date

Honors Program Director for Honors Council Date

Dynamical Systems: The Definition of Chaos

I. Noting that the space we are dealing with is $\Re^2$ initially, the first question that springs to mind is,"What is a dynamic system?" Almost every function you have ever dealt with in calculus or the like is a **dynamic system**, all one has to do is iterate a value. **Iteration** is the process of repeatedly computing subsequent values under a continuous application of the function for a particular initial argument. For any function f and for some fixed value x_0:

$$x_0;\ f(x_0);\ f(f(x_0));\ f(f(f(x_0)));\ \ldots$$

is known as a dynamic system.

It becomes cumbersome to write f(f(f(k))), especially when iterating a value hundreds of times, so we adopt a notational convention:

$$f(f(x)) = f^2(x); \quad f(f(f(x))) = f^3(x); \text{ and so on.}$$

Now we can define the **forward orbit** of x as the set of all iterations of x (i.e., x, f(x), $f^2(x)$, ...).

Example 1:
Let $f(x) = 2x$ and $x_0 = 1$. Then

$$O = \{1, 2, 4, 8, 16, 32, \ldots 2^n\} \text{ (for n a natural number)}$$

where O is the forward orbit of f.

A function f is a **homeomorphism** iff f is one-to-one, onto, continuous, and f^{-1} is continuous. The notion of homeomophism is important, since if f is one of them, we can define the **backward orbit** of f as the set of all $f^n(x)$ such that n is a negative integer. The union of the forward orbit and the backward orbit is called the **full orbit**.

The reason we define all of these concepts is to reduce the understanding of the wide concept of dynamic systems to understanding full orbit of a function f and a value x_0. Here number crunching and approximation will not suffice; the accumulation of even the most seemingly insignificant of approximations will completely alter the system eventually, so we must define more to thoroughly describe the appropriate situations and truly understand the "big picture", as it were.

Let x_0 be a fixed value in the domain of a function f. We say that x_0 is a **fixed point** if

$$f(x_0) = x_0.$$

We say x is a **periodic point** if

$$f^n(x) = x \quad \text{(for some natural number n).}$$

We say x has a **period** n. Note that if $f^n(x) = x$, then $f^{2n}(x) = x$ as well, due to its periodicity. Thus, we define **prime period** of x as the least positive n such that x is a periodic point.

Example 2:
$f(x) = -x$. $x = 0$ is a fixed point, and any $x \neq 0$ is a periodic point of period 2.

This example has an important concept associated with it, that of **density**. A well-ordered set D is dense within the space X if $D \cup D'$ (D' is the set of all accumuation points of D) is equal to X.

Example 3:
Consider the natural numbers in $\Re$. Are there any natural numbers between n and n +1 (for n a natural number)? The answer is no, and as a result, the naturals are not dense in $\Re$ because there is a point in $\Re$ that is not an accumulation point of the naturals, for example, $\frac{1}{2}$ is too far away from 1 and 2.

Now consider the rationals in $\Re$. Are there any rationals between x, y (for x, y rational)? The answer is yes, because there exists another rational number a such that

$$x < a < y \quad \text{(for } x < y\text{, of course)}$$

by property of rational numbers. Thus, in the accumulation points (the irrationals), for some point k, k is infintely close to some z in the rationals (as we can get as close as we want by the property exploited above). So we see that the union of the rationals and the irrationals is indeed $\Re$, and we conclude that the rationals are dense in $\Re$.

Note that the rationals, irrationals, and reals are all dense in $\Re$, but the integers and naturals are not, for there does not exist say, an integer between -4 and -5. Also helpful to think about is that every set is dense within itself. The notion of density will prove quite valuable in the defining of chaos later on; it is important to digest it fully.

II. An example well worthy of consideration is the quadratic family $F_\mu(x) = \mu x(1-x)$. What are the fixed points p of F(x)? Clearly p = 0 works trivially, but one's instinct should flag the importance of μ as a parameter. In fact, $p = \frac{\mu-1}{\mu}$ remains the only other one (barring that $\mu = 0$):

$$\begin{aligned} x = \mu x(1-x) &\Rightarrow x = \mu x - \mu x^2 \\ &\Rightarrow 0 = (\mu-1)x - \mu x^2 \\ &\Rightarrow (\mu-1)x = \mu x^2 \\ &\Rightarrow \mu-1 = \mu x \quad \text{(because } x = 0 \text{ is another solution)} \end{aligned}$$

$$\Rightarrow \frac{\mu-1}{\mu} = x$$

Thus we have found the 2 fixed points of F(x).
We can continue using this example to find the other definitions, but a much richer enterprise lies ahead. Presented next is a relatively deep property. First, a lemma and some propositions must be proven in order to demonstrate this general result, which will be formulated as our main theorem:

<u>Theorem 1</u>: For all $x \in (0, 1)$ and for $1 < \mu < 2$, $\lim_{x \to \infty} F^n_\mu(x) = p$

That is to say that all iterations will eventually converge to the fixed point $\frac{\mu-1}{\mu}$ (hereby referred to as p), under these special parameters.

Proof of Theorem 1: We need some lemmas first:

<u>Lemma 1</u>: $x > p \Rightarrow F(x) > p$ for $x \in (0, \frac{1}{2})$, $x \neq p$, and $1 < \mu < 2$.

Proof: Let $x \in (0, \frac{1}{2})$, $x \neq p$, and $1 < \mu < 2$. Suppose otherwise, then there exists $x > p$ such that $F(x) \leq p$.

$$\Rightarrow \mu x (1- x) \leq p$$
$$\Rightarrow \mu x (x -1) > \frac{1-\mu}{\mu}$$
$$\Rightarrow \mu p (p -1) > \frac{1-\mu}{\mu}$$ (this is the same as saying that $\frac{1-\mu}{\mu^2} > \frac{1-\mu}{\mu}$, and μ's parameters are such as to make it true)

$$\Rightarrow \mu(\mu -1)(\tfrac{\mu-1}{\mu}-1) > 1- \mu$$
$$\Rightarrow \mu^2 - \mu - \mu^2 - (\mu -1) + \mu > 1- \mu$$
$$\Rightarrow 1- \mu > 1 - \mu$$

This is a contradiction.
Thus, $F(x) \leq p \Rightarrow x \leq p$, and the contrapositive gives us what we are looking for:
$x > p \Rightarrow F(x) > p$.

The proof is similar to show that $x < p \Rightarrow F(x) < p$.

<u>Lemma 2</u>: F(x) is increasing on $(0,\frac{1}{2})$ and decreasing on $(\frac{1}{2}, 1)$.
Proof: $F'(x) = \mu - 2\mu x$, and since $1 < \mu < 2$ (this holds for any $\mu > 0$), $\mu - 2\mu x > 0$ when $x \in (0, \frac{1}{2})$, as $2x < 1$.

We see also that $\mu - 2\mu x < 0$ for $x \in (\frac{1}{2}, 1)$ as $2x > 1$.
Thus, by definition of increasing and decreasing, the lemma is proven.

Now a couple of propositions are needed.

Proposition 1: Let F: [0,1] $\mapsto \Re$ be defined by $F(x) = \mu x(1-x)$ with $1 < \mu < 2$.
If $x \neq p$ ($p = \frac{\mu-1}{\mu}$) and $x \in (0, \frac{1}{2})$, then $|F(x) - p| < |x - p|$

Proof: Suppose otherwise. Then there exists b such that

$$| F(b) - p | \geq | b - p|$$

This implies, by def. of F: $|\mu b - \mu b^2 - p| \geq |b - p|$

(1) $\Rightarrow |\mu b - \mu b^2 - p| \geq b - p$ ($|a| \geq a$ for all a.)

Case1: Suppose $b > p$. $\Rightarrow$ $F(b) > p$ (by Lemma)

$$\Rightarrow \mu b - \mu b^2 > p$$
$$\Rightarrow \mu b - \mu b^2 - p > 0$$

Thus, from (1), we have $\mu b - \mu b^2 - p \geq b - p$

$$\Rightarrow \mu b - \mu b^2 \geq b$$
$$\Rightarrow \mu - \mu b \geq 1 \quad (b > 0)$$
$$\Rightarrow \mu - 1 \geq \mu b$$
$$\Rightarrow b \leq \frac{\mu-1}{\mu} = p$$

This contradicts the assumption that $b > p$, thus Proposition 1 holds for $b > p$.
Since $b \neq p$ by initial constraint, we only need consider one other case:

Case 2: Suppose $b < p$. $\Rightarrow F(b) < p$ (by Lemma 1)

$\Rightarrow \mu b - \mu b^2 - p < 0$ (similar to above)

This means that in order for (1) to hold,

$\mu b - \mu b^2 - p \leq b - p$ (by property of absolute value)

$$\Rightarrow \mu b - \mu b^2 \leq b$$
$$\Rightarrow \mu - \mu b \leq 1 \quad (b > 0)$$
$$\Rightarrow \mu - 1 \leq \mu b$$
$$\Rightarrow b \geq \frac{\mu-1}{\mu} = p$$

Since this contradicts the supposition that $b < p$, we have considered all cases and found none of them to hold, thus, our original supposition was false, and we conclude

$$| F(x) - p | < | x - p|$$

A much stronger result, however, is instrumental in proving our original property:

Proposition 2:

For $x \neq p$, $|F(x) - p| \leq k(x)\,|x - p|$ for $0 < k(x) < 1$.

By the Mean Value Theorem, there exists $c \in (x, p)$ such that

$F'(c) = \frac{|F(x) - p|}{|x - p|}$ for $x < c < p$ or $p < c < x$, whichever the case may be.

By Lemma 2, we can say that, if we consider $x > p \in (0, \frac{1}{2})$, $F(x) - p = |F(x) - p|$, because $F(x)$ is increasing over that interval, and obviously $x - p = |x - p|$ as well. If $x < p$ over the same interval, $F(x) - p < 0$ (because $F(p) = p$ and F is increasing) and $x - p < 0$ trivially, so their quotient will retain the same sign if we take the absolute values of both (i.e., the negative signs for each will cancel). This takes care of our first interval. Similarly, for $x \in (\frac{1}{2}, 1)$, $F(x)$ is always decreasing, so $F(x) - p$ and $x - p$ again are the same as if they had absolute values signs around them. In either case:

$|x - p|\,(F'(c)) = |F(x) - p|$.

Thus we can always take $k(x)$ to be $F'(c)$, and the inequality holds, for $0 < F'(c) < 1$: $F'(c) = 0$ means that $|F(x) - p| = 0$. But this means that $x = p$, a contradiction, so

$F'(x) > 0$.

$F'(c) = 1$ means that $|F(x) - p| = |x - p|$. But this means that $x = p$, so $F'(c) \neq 1$.
In fact, $F'(c) < 1$, as a direct consequence of Proposition 1.
Thus, indeed $0 < F'(c) < 1$, and so taking $k(x)$ to be $F'(c)$ is legal. This completes our proof.

With this information, we claim that this inequality is a weak form of a contraction mapping, as $k(x)$ is a function of x rather than a constant. We are interested in the orbit of $F(x)$, so now another important lemma should be proven:

Lemma 3: For all $x \in (0, \frac{1}{2})$, $F^n(x) \in (0, \frac{1}{2})$.

Proof: By induction, we first show if x is in the interval, then $F(x)$ is $F(x) = \mu x - \mu x^2$ is subtracting an element from $(0, \frac{\mu}{2})$ (namely, μx) by one from $(0, \frac{\mu}{4})$ (namely, μx^2), so the resultant element $F(x)$ must lie in at least $(0, \frac{\mu}{4})$. But by μ's constraints, $\frac{\mu}{4} < \frac{1}{2}$, so $F(x)$ is in the interval $(0, \frac{1}{2})$. Now suppose that $F^n(x) \in (0, \frac{1}{2})$. If $F^{n+1}(x) \in (0, \frac{1}{2})$, then the lemma is proven.

$F^{n+1}(x) = F(F^n(x))$; call $F^n(x) = y$ for simplicity. We know by assumption that y is in $(0, \frac{1}{2})$, and we know that for any such y, $F(y)$ is in $(0, \frac{1}{2})$ as well. Thus, $F^{n+1}(x)$ is indeed in the interval, and our lemma is proven.

From here, we claim without proof that upon iteration of F(x) and k(x), recalling that $0 < k(x) < 1$ and the above lemma, that we can extend this inequality to subsequent iterations:

$$|F^n(x) - p| \leq k^n(x)\,|x - p|$$

Thereby forcing the right side to contract to 0 as n approaches infinity. Then it becomes clear that indeed:

$$\lim_{n\to\infty} F^n(x) = p$$

This example serves to demonstrate the richness and complexity of such a dynamical system, but now we abandon it (with intent to return) to discuss the topic of chaos.

III. Chaos is a concept which can be given in many different settings, however, ours is mostly a topological one, and thus chaos will be defined in such a context after a few preliminary definitions:

A **topologically transitive** function $f: J \to J$ means that for any two open sets U, V in J there exists $k > 0$ such that $f^k(U) \cap V \neq \emptyset$.

The idea is that for any point and any neighborhood around it, eventually the iterations of that point will move to some other neighborhood in the set. (arbitrarily small).

$f: J \to J$ has **sensitive dependence on initial conditions** if there exists $\delta > 0$ such that, for any $x \in J$ and any neighborhood N(x) (of x), there exists $y \in N(x)$ and $n \geq 0$ such that $|f^n(x) - f^n(y)| > \delta$.

Basically, for any two points in the same neighborhood, eventual iteration under f will cause the two points to become separated by at least δ. It is important to note that this is not a property of all points in N(x), just for one point, and that one point is all we need. Sensitive dependence on initial conditions is a problem when actually performing numeric computation; the infinitesimal difference between them can become quite large under many iterations, and disguise the true nature of the orbit.

Now, we can define chaos:
$f: V \to V$ is said to be **chaotic** on V if the following 3 conditions hold:

1) periodic points are dense in V
2) f is topologically transitive
3) f has sensitive dependence on initial conditions.

In general, one can see that the chaotic mapping has a degree of regularity, i.e., that periodic points are dense in V, but the sensitive dependence causes unpredictability, and we cannot separate out any more of V to consider (to break the problem down into something easier, to wit) due to the topological transitivity.

Consider the following "tent map" T(x):

$$T(x) = \begin{cases} 2x & \text{if } 0 \leq x < \frac{1}{2} \\ 2(1 - x) & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

We can see how this map resembles F(x), which we earlier studied in depth. In fact, F(x) is chaotic, but the proof involves much, much more preliminary information, which is not our goal. Our goal is contributing to the understanding of chaos, and demonstrating the richness of dynamical systems. By proving that T(x) is chaotic instead of F(x), one gets an easier example of something that is chaotic, and the similar geometries between the two should give some kind of intuition that indeed F(x) is chaotic.

<u>Theorem 2</u>: T(x) is chaotic over the interval [0, 1].

Proof: Let $\delta > 0$. For $x \in [0, \frac{1}{2}]$, let $y \in N(x)$.

$$|T(x) - T(y)| = |2x - 2y| = 2|x - y|$$

But since y is in N(x), y is separated by x (over the metric $d(x,y) = |x - y|$) by any $\epsilon > 0$. If we take $\delta = \epsilon$, then

$$2|x - y| = 2\epsilon = 2\delta > \delta$$

Thus, for n = 1, $|T(x) - T(y)| > \delta$ is true when $\delta = \epsilon$, so such a δ exists.
We are not quite done, for there is also a special case where $x = \frac{1}{2}$ and $y > \frac{1}{2}$:

$$|T(x) - T(y)| = |2x - 2(1- y)| = 2|x + y - 1|$$

But x + y is the same thing as saying $\frac{1}{2} + (\frac{1}{2} + \epsilon)$ (for $y \in N(x)$) which is $1 + \epsilon$. Thus, we can select our $\delta = \epsilon$ and yield the same result. Thus, T(x) has sensitive dependence on initial conditions over the interval $[0, \frac{1}{2}]$.

Next, it must be shown that T(x) is topologically transitive.
The procedure for proving this is the case is to take arbitrary intervals and show that they map to I, so any open set A will map to I, and thus, any other open set B in I will intersect A in at least one point.
We start with those of the form (0, a). For $a < \frac{1}{2}$, $T^n(0, a) = (0, 2^n a)$ such that $2^n a < \frac{1}{2}$ and n is the largest natural number for which this holds. This means that

$$(0, \tfrac{1}{2}) \subset T^{n+1}(0, a) = (0, 2^{n+1} a).$$

$$\begin{aligned} \text{So } T^{n+2}(0, a) &= T^{n+2}(0, \tfrac{1}{2}) \cup T^{n+2}(\tfrac{1}{2}, a) \\ &= (0, 1) \cup T^{n+2}(\tfrac{1}{2}, a) \\ &= (0, 1). \end{aligned}$$

We see that if $a \geq \frac{1}{2}$, we need only iterate once to yield the same result.
Thus, any open set of the form (0, a), when iterated, will necessarily become (0, 1).
Since our space is [0, 1], this guarantees that any other open set will intersect the iterations of (0, a).

Next, we consider open sets of the form (b, 1).

$$\text{But } T(b, 1) = (2(1-1), 2(1-b))$$

call $a = 2(1-b)$. $\qquad = (0, a)$

So we have a form which was previously shown, and thus all open sets of the form (1, b) can be iterated to intersect any open set in [0, 1].

Finally, we must consider an intermediate interval (a, b), where $a \neq 0$ and $b \neq 1$.
Let c, d be real numbers in [0, 1] and let a < b (without loss of generality):

<u>Case 1</u>: $b < \frac{1}{2}$.

$\Rightarrow \quad a < \frac{1}{2}$

Thus, there exists a smallest natural number n such that $T^n(a, b) \not\subset (0, \frac{1}{2})$, as a and b are doubled until $b > \frac{1}{2}$. Note that this creates 2 subcases:

Case 1 (i) : $a \leq \frac{1}{2}$ (after n iterations of (a, b) as specified above).

$\Rightarrow (a, b) = (a, \frac{1}{2}) \cup [\frac{1}{2}, b)$

$\Rightarrow \quad T^{n+1}(a, \frac{1}{2})$ is of the form (c, 1)

So $T^{n+1}(a, b)$ will be of the form (d, 1) and this reduces to a previous case.

Case 1 (ii): $a > \frac{1}{2}$ (which means $b > \frac{1}{2}$ as well)

$\Rightarrow T^n(a, b) \subset (\frac{1}{2}, 1)$.

So $T^{n+1}(a, b) = (2 (1-b), 2 (1-a)) = (c, d)$

$T^{n+1}(a, b)$ can reduce to Case 1 (i) when $c \leq \frac{1}{2}$ and $d > \frac{1}{2}$; if this is the case then we are done.
If both c and d are still greater than $\frac{1}{2}$, we can iterate again and refer to Case 1 (ii).
If c and d are both less than $\frac{1}{2}$, we can refer to Case 1.
So the question arises, how are we guaranteed that eventually we will get to Case 1 (i), and not just repeat ourselves infinitely?
Suppose the contrary is the case, that we bounce infinitely back and forth without ever containing the point $\frac{1}{2}$ in any iterated interval. Then some interval (a, b) is necessarily a subset of $(0, \frac{1}{2})$, and $T((a,b)) = (2a, 2b)$ is a subset of $(\frac{1}{2}, 1)$.
But $2a = 1-b$ and $2b = 1-a$, for we are over the interval [0, 1] and we can never include $\frac{1}{2}$.

If this is the case, then solving for a and b, we find $a = b = \frac{1}{3}$, a contradicition, as this does not form an interval. Thus, we will eventually get Case 1 (i), and Case 1 is done.

Case 2: $b > \frac{1}{2}$ and $a \leq \frac{1}{2}$.
This is covered by Case 1 (i). Note that if ever b or $a = \frac{1}{2}$, the iteration of (a, b) is of the form (c, 1), and we are done.

Case 3: $a > \frac{1}{2}$

$$\Rightarrow b > \frac{1}{2}$$

This case reduces to Case 1 (ii).

Thus, we have exhausted all possible cases of the intermediate interval (a, b), and along with the analysis of (0,a) and (b, 1), we conclude that T(x) is topologically transitive.

Now all that remains to prove is that periodic points are dense within T(x).

One can generate the periodic points of period n $T^n(x)$ with the following formula, for fixed n:

$$\{0\} \cup \{ \frac{\{2^k\}}{2^n+1} \text{ and } \frac{\{2^k\}}{2^n-1} \text{ such that } k \in \mathbb{N} \text{ and } 0 \leq k \leq b, \text{ where } \frac{2^b}{2^n \pm 1} < 1\}$$

(note: $2^n-1 = 0$, for n =1, so do not consider the term when n = 1)

For T(x), we get $\{0, \frac{2}{3}\}$ for our set of fixed points. For $T^2(x)$, we get $\{0, \frac{2}{5}, \frac{2}{3}, \frac{4}{5}\}$ as point with period 2, and so on.
It becomes evident that each subsequent n yields a broader partition in the sense that each new periodic point of n+1 has more points inbetween, as the denominators increase in number. Each point of the partition of [0, 1] is also inbetween the previous ones, and each point is rational. Due to the density of the rationals in $\mathbb{R}$, it becomes clear that indeed the periodic points of T(x) are dense over [0,1], hence we have shown the final axiom of chaos to hold.
Thus, T(x) is chaotic over the interval [0, 1].

One can see now the complexitites involved in so seemingly simple a function. Comparing the graphs of T(x) and F(x) hopefully give enough intution to believe that F(x) is chaotic for many of the same reasons.
Overall, this paper began with an presumed foundation of undergraduate mathematics, and with some preliminary definitions and examples, has demonstrated the deep nature of simple (and not so simple!) dynamical systems, and contributed to defining and understanding the nature of chaos.