Investigating a Triality Hypothesis for Quantum Gravity

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Investigating a Triality Hypothesis for Quantum Gravity

Honors Senior Thesis of Lisa Kodgis

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Assuming ten degrees of freedom in the metric for General Relativity, one can argue that the appropriate symmetry group for the representations of particles and fields should be $SO(8)$. This group is unique in the sense that three representations, the vector and the left and right-handed chiral spinors, all have the same number of degrees of freedom. It is suspected that this enhanced symmetry will allow a quantum theory constructed from these representations to be renormalizable. This project will explore the mathematics of representations of Lie Groups and their applications to quantum theory.
Group Theory and Applications

This section will describe important definitions from group theory in some detail. The first part, Group Theoretical Definitions and Theorems, will be abstract definitions. The next part, Applied Group Theory, will describe specific groups and apply some of the abstract definitions.

Group Theoretical Definitions and Theorems

This section will list and explain relevant definitions and theorems. It will begin with the definition of a group. Then it will define homomorphism and other mappings. Finally, it will discuss what a representation is.

§ Definition of a Group

The first object that one must understand in order to understand a group is that of a set, which is a collection of objects. For our purposes, all sets will consist of real numbers. The second item that one must understand is the notion of an operation. An operation is merely a way to take a few objects and rearrange them. Our operations will be binary, meaning that we will only arrange two and only two objects at a time. There is an additional constraint on our operations, which is known as closure. Closure means that the order that we take the objects matters and that we can only have one result of this rearrangement.
With this being said, a group, $G$, is a set with a binary operation, $*$. Take objects from $G$, say $a$, $b$ and $c$. The operation satisfies the following three properties:

1) **Associative**  
\[(a * b) * c = a * (b * c)\]

2) **Existence of Identity**  
\[a * e = e * a = a\]

3) **Existence of Inverses**  
\[a * b = b * a = e\]

Every element has a unique inverse by closure. The *identity* is also unique.

§ **Other Definitions**

A *(proper) subgroup* is also a group, but whose set has less elements than the original group of which it is a subgroup. A *subgroup* shares the operation as its original set. All groups have *generators*. The *generators* are the minimum amount of elements that are needed to reconstruct the group under its operation. The group theory term “generator” will become synonymous with the linear algebra term “basis.”

§ **Homomorphism, Isomorphism, and Automorphism**

There are three types of mappings that are often used in the theory of groups. The mappings that will be discussed here are *homomorphism*, *isomorphism*, and *automorphism*, which is a type of *isomorphism* (Weisstein 2004). The description of the nature of a mapping between two groups will lead into the definition of the three important mappings.
A mapping is a very basic notion. It is simply a rule that tells you which element from one set corresponds to an element of another set. All mappings mentioned here have one restriction, which is that all elements from the input set can only be used once. This means that each element from the input can only be mapped to one element in the output set.

Let G and H be two groups. We know that the group G has an associated set, also called G. In order for G to be a set associated with a group, G must contain at least one element, namely the identity. Since this set is nonempty, one can take two arbitrary elements of G, call the elements a and b.

One can now create a mapping. This mapping will be called f. Symbolically it is written as follows, where G is the input set.

\[ f: G \rightarrow H \]

This mapping takes an element, a, from the set G and finds a corresponding element \( f(a) \) that is in H.

A **homomorphism** is a mapping that preserves algebraic structure (Chesnut 1974). It is commonly stated in the more concrete form below.

\[ f(a \ast b) = f(a) \ast f(b) \]

The algebraic property that the mapping preserves is the binary relation of the group.

A different type of mapping that is important in comparing two groups is known as an **isomorphism**. It has the same symbolic definition as above, but with the additional restriction that the mapping must be **bijective** (Weisstein 2004). So isomorphism is a type of homomorphism (Chesnut, 1974). The concept of a bijective mapping is used extensively throughout mathematics. A
bijection is a mapping that is one-to-one and onto. In order to decide if a mapping is one-to-one or onto, you must look at how the elements in the output set are being mapped. If any element in the output set has only one element from the input set mapped to it, then the mapping is one-to-one. If all elements from the output set are used once and only once, then the mapping is onto.

Finding isomorphisms between groups is a way to better understand groups. A newly discovered group might be hard to work with mathematically. If it is isomorphic to a group that is well known and understood, then all your work toward understanding this new group is done because the isomorphic group contains the same algebraic information. If the isomorphism is from one set to the same set, then it is an automorphism.

§ Representations

Representations are used throughout group theory. A representation of a group is a homomorphism from the group to another group. When one group is a matrix group, then one can apply matrix theory.

Applied Group Theory

This section begins with some basics from linear algebra, next it will define many groups, and finally it will define the matrix representation. There will be a presentation of examples.
§ Matrices and Linear Algebra

This section defines some terms that will be used later to define some groups. The most important definition in linear algebra is that of the identity. The identity is the matrix with all 1's on the main diagonal and all other elements being zero. The identity can be of any dimension. Multiplying a matrix by the identity is equivalent to multiply a number by one. The term trace will become important later when solving the Dirac equation; it is the multiplicative product of all the elements on the main diagonal of a matrix.

A hermitian matrix is one that is equal to its conjugate transpose (Wikipedia 2007a). For our purposes, conjugate will mean to change the sign in front of imaginary terms and transpose means to simply flip the elements of the matrix about its main diagonal. Hermitian matrices describe observable quantities in the theory of quantum mechanics (Cottingham and Greenwood 1998).

There are multiple ways to express if a matrix is unitary, one such way being that the conjugate transpose is equal to the inverse. Another way will be described later in unitary groups (Cottingham and Greenwood 1998).

A basic tool of linear algebra is that of a basis. There are typically multiple bases for a group or vector space. A basis for a vector space is a set, S, that is a subset of the vector space (contains vectors from the space) and satisfies two conditions. First S must be linearly independent. Let $S = \{x_1, x_2, ..., x_n\}$, then the following equation

$$A_1 x_1 + A_2 x_2 + ... + A_n x_n = 0$$
must admit only the trivial solution, i.e., $A_1 = A_2 = ... = A_n = 0$. The second requirement to be a basis is the $S$ spans the vector space. This means that any vector in the space, $x$, can be expressed as a linear combination of the vectors in $S$. This linear combination looks like the following.

$$x = A_1 x_1 + A_2 x_2 + ... + A_n x_n$$

Any vector in the space is reachable through a linear combination of its basis (Hartfiel 2001).

§ Matrix Lie Groups

There are multiple matrix lie groups and they are all subgroups of the general linear group. The set associated with the general linear group is a set of matrices. All matrices in this general linear set must be invertible, and the matrix lie groups have additional constraints on their associated sets (Hall 2003).

There are many ways to determine if a matrix is invertible. The primary way to determine this is to find the inverse of a given matrix; if it has an inverse, then it is invertible. The inverse of a matrix is the also a matrix such that when you multiply a matrix by its inverse, you get the identity matrix.

The General Linear Group

The general linear group over the real numbers, $GL(n;R)$, is the set of all invertible matrices under the operation of matrix multiplication. All matrix lie groups are subgroups of the general linear group (Hall 2003).
Matrix Lie Group Condition

A matrix lie group, \( G \), is a subgroup of the general linear group that has a special condition that if you take a sequence of matrices in \( G \) and that sequence converges to a specific matrix, \( A \), then either \( A \) is in \( G \) or else \( A \) is noninvertible (Hall 2003).

Unitary Group

The unitary group is often abbreviated as \( U(n) \), where the matrices in the set associate with the group are of dimension \( n \times n \). The property that all member matrices of this group have is that they are unitary. This means that the column vectors are orthonormal. In other words, the dot product between a column of the conjugate transpose matrix and a column of the original matrix is 1 if they are the same column and zero otherwise (Hall 2003).

A simple example is \( U(1) \). Since \( n = 1 \), this group contains numbers instead of matrices. These numbers can be complex, so this group can be represented as unit circles. (Wikipedia 2007c)

Special Unitary Group

The special unitary group is composed of unitary matrices with determinant equal to one (Hall 2003). It is denoted by \( SU(n) \). A particularly interesting special unitary group is \( SU(2) \). \( SU(2) \) is often thought to have three generators (Cottingham and Greenwood 1998). These generators pop up a lot throughout
physics. They are known as the **Pauli spin matrices**. These generators also serve as a basis of this group. This group, $SU(2)$, will be discussed later in the section on particle physics. Below are the Pauli Spin Matrices.

$$
\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
$$

Another group that is of importance to particle physics is $SU(3)$. A common basis is the Gell-Mann basis, which is composed of eight hermitian, traceless matrices (Cottingham and Greenwood 1998). Below is the Gell-Mann basis.

$$
\lambda_{1J} = \begin{pmatrix} 0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0 \end{pmatrix} \quad \lambda_{2J} = \begin{pmatrix} 0 & -i & 0 \\
i & 0 & 0 \\
0 & 0 & 0 \end{pmatrix} \quad \lambda_{3J} = \begin{pmatrix} 1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 0 \end{pmatrix} \quad \lambda_{4J} = \begin{pmatrix} 0 & 0 & 1 \\
0 & 0 & 0 \\
1 & 0 & 0 \end{pmatrix}
$$

$$
\lambda_{5J} = \begin{pmatrix} 0 & 0 & -i \\
i & 0 & 0 \\
0 & 0 & 0 \end{pmatrix} \quad \lambda_{6J} = \begin{pmatrix} 0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0 \end{pmatrix} \quad \lambda_{7J} = \begin{pmatrix} 0 & 0 & 0 \\
0 & 0 & -i \\
i & 0 & 0 \end{pmatrix} \quad \lambda_{8J} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -2 \end{pmatrix}
$$

You can construct any other matrix in $SU(3)$ using a linear combination of these eight matrices. Or you can construct $SU(3)$ by multiplying different combinations of these eight matrices, including by itself an infinite amount of times.
Orthogonal Group

The **orthogonal group** is a group whose associate set contains only orthogonal matrices. An **orthogonal matrix** is one whose inverse is equal to its transpose (Roland 2002).

Special Orthogonal Group

All elements of the **orthogonal group** have two possibilities for the value of determinant; they are either 1 or -1. The **special orthogonal group** consists of those matrices with determinant equal to 1. This group is denoted by \( \text{SO}(n) \) (Hall 2003).

A particular example of the **special orthogonal groups** is \( \text{SO}(3) \). This group contains 3x3 matrices with determinant equal to one. These matrices perform rotations. If you multiply a set of coordinates by one of these matrices, then you get the position of the rotated coordinates (Cottingham and Greenwood 1998).

Those members of the **orthogonal group** with determinant equal to negative one do not form a group of rotations. This is because if you multiply two matrices with determinant equal to negative one, then you get a matrix with determinant equal to one, which is a member of the **special orthogonal group**. So it does not satisfy closure (Cottingham and Greenwood 1998).
The Lorentz group is of interest to the theory of special relativity. This group is represented as \( O(3;1) \) and is a subgroup of the generalized orthogonal group \( \text{O}(9;1) \). The Lorentz group is represented as \( \text{O}(3;1) \) and is a subgroup of the generalized orthogonal group. This corresponds to \( \text{O}(9;1) \).

\[ e \quad a \]
\[ e \quad a \]
\[ a \quad e \]

Table. Below is the Cayley table for the group of two elements.

Table of the group using the binary operation \( * \). This is known as a Cayley multiplication table. This table displays all possible ways to combine the elements of the group. Below is the group with only two elements, \( e \) and \( a \). Below is the group.

If you have a group \( G \) and you want to find the matrix representation of \( G \), then you simply find a set of matrices that are homomorphic to \( G \). (Chesnut, 1974) A matrix representation is best defined through example. This example:

\[ \begin{array}{c|cc}
    & e & a \\
\hline
    e & e & a \\
    a & a & e \\
\end{array} \]

Based on the rules of multiplication, the identity is \( e \). The above group is isomorphic to the group of permutations of two objects.

§ Matrix Representation

The general relativity tensor. This corresponds to \( \text{O}(9;1) \). The hypotheses assumed ten degrees of freedom in the "\( \text{O} \)" corresponds to time. The hypotheses assumed ten degrees of freedom in the Lorentz group (Hall 2003). Physically, the "\( \text{O} \)" corresponds to the spatial dimensions and the Lorentz group is of interest to the theory of special relativity. This group is represented as \( \text{O}(3;1) \) and is a subgroup of the generalized orthogonal group.
In this group, (1) (2) is the identity. This corresponds to leaving the two objects in the same position. The other permutation, (12) is simply exchanging places of the two objects.

The regular representation of this group looks like this (Chesnut 1974).

\[
\begin{array}{c|cc}
\cdot & (1) & (12) \\
\hline
(1) & (1) & (12) \\
(12) & (12) & (1) \\
\end{array}
\]

\[
D(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad D(a) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

We can see that this preserves the algebraic properties. From the definition of homomorphism and using the first Cayley table, \(D(e) \ast D(a) = D(a \ast e)\). This property is preserved by the representation.

\[
D(e) D(a) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = D(e \ast a) = D(a)
\]

\[
D(a) D(e) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = D(a \ast e) = D(a)
\]

\[
D(e) D(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = D(e \ast e) = D(e)
\]

\[
D(a) D(a) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = D(a \ast a) = D(e)
\]

We can also extend this example to define the direct product representation.
If the representations can be put in the block form above, then it is a \textit{reducible representation} (Chesnut 1974).

\[
D(a) \times D(e) = \begin{pmatrix} D(a) & 0 \\ 0 & D(e) \end{pmatrix}
\]

\textbf{Particle Physics with Group Theory}

This section will describe elementary particles, which is a future goal of this project. The next section will contain information about what theories are currently in use and their purpose. Finally, it will describe the standard model of particle physics, which is the best approximate theory we have.

\textbf{§ Particles}

The goal of this project is to construct a theory that describes all elementary particles. This section of the paper will catalogue these particles. There are two major classes of subatomic particles. They are the fermions and the bosons. This classification is based on \textit{spin}. \textit{Spin} is short for \textit{spin angular momentum}, which is an intrinsic quality of particles. The fermions are half integer spin particles. Bosons have integer values of spin.

A good overview of particles relative to their mass is given in the picture below, though it is not a complete list of particles. The figure shows the six quarks (top, bottom, charm, strange, down, up), which are thought to constitute
most fermions. The figure also shows leptons, which do not have constituent particles.

The leptons come in three generations (just nomenclature). The first generation is the electron and the electron neutrino, the second generation is the muon and muon neutrino and the third generation is the tau and the tau neutrino particles. The figure also shows all the messenger particles of the forces; they are the photon, gluon, W, Z, and the hypothetical Higgs particle, which is thought to be the source of mass.

Fig 1: Relative masses of selected particles (Anonymous undated)

§ Theories and Unification

There are multiple correct theories within the realm of physics. Each have specific ranges of size, energy, and speed that for which they can make predictions. A goal of physics is unify these theories. The theory that is used to
describe subatomic phenomena is known as quantum mechanics. The theory that deals with astronomical bodies or things that are moving at speeds approaching that of light are the theories of special and general relativity.

§ The Standard Model

The standard model is an incomplete theory that summarizes the current state of particle physics. Symbolically this theory is stated as $SU(3) \times SU(2) \times U(1)$. These are the matrix lie groups as stated earlier, but in this context they are known as gauge groups. It is thought that there is a more fundamental group of which all of these gauge groups are subgroups (Schwarz undated).

Four fundamental forces are recognized in modern physics. They are the strong, weak, electromagnetic, and gravitational forces. The gauge groups describe three of these forces, with the exception of gravity. Here are the Feynman diagrams that display their associated bosons.

![Feynman diagrams of the electromagnetic, weak, and strong interactions.](image)

Fig 2: Basic force interactions (Anonymous undated)
The boson for Electromagnetic force is the photon, which is shown above as the exchange particle for the scattering of electrons. The bosons for the Weak force are the $W^+$, $W^-$, and $Z$ particles, only the $W^-$ is shown above. The Strong force acts on a property of quarks known as color. Gluons exchange this color information as shown in the left diagram. The right diagram has been disproved. The dotted line symbolizes a weak interaction, but the pion is not a boson, though this was once suspected.

$SU(3)$ describes the strong interaction. The gauge groups $U(1) \times U(3)$ describes what is referred to as the electroweak interaction. This is the partial unification of the electromagnetic and the weak forces. (Schwarz undated) But it is also known that $SU(2)$ and $SO(3)$ are related by this electroweak theory (Cottingham and Greenwood 1998).

§ Dirac Equation and Spinors

Two representations of $SO(8)$ are the left and right spinors. The source of these spinors is solutions to the Dirac equation. The Dirac equation is the equation that describes the motion of spin-$1/2$ particles that is consistent with quantum mechanics and special relativity (Wikipedia 2007b). This project is first concerned with the form of the Dirac equation where mass is zero.

$$i\hbar \gamma^\mu \partial_\mu \psi = 0$$

We choose this form first because it one of the simpler forms of this equation.
Conclusion

It is from these three representations that one can construct the quantum theory using techniques of quantum field theory that are beyond the scope of this paper. One must first understand groups and representations. This paper presented a brief introduction to these subjects and displayed their relevance to physics.
References


