2012

Statistical fracture mechanics : an unconventional approach to crack formation in brittle solids

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STATISTICAL FRACTURE MECHANICS: AN UNCONVENTIONAL APPROACH TO CRACK FORMATION IN BRITTLE SOLIDS

by

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A THESIS

Submitted in partial fulfillment of the requirements
for the degree of Master of Science in Engineering
in
The Department of Mechanical and Aerospace Engineering
to
The School of Graduate Studies
of
The University of Alabama in Huntsville

HUNTSVILLE, ALABAMA

2012

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We, the undersigned members of the Graduate Faculty of The University of Alabama in Huntsville, certify that we have advised and/or supervised the candidate on the work described in this thesis. We further certify that we have reviewed the thesis manuscript and approve it in partial fulfillment of the requirements for the degree of Master of Science in Engineering in Mechanical Engineering.

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ABSTRACT

School of Graduate Studies
The University of Alabama in Huntsville

Degree Master of Science in Engineering

College/Dept. Engineering/Mechanical and Aerospace Engineering

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Title Statistical Fracture Mechanics: an Unconventional Approach to Crack Formation in Brittle Solids

The main drawback of using brittle materials in load bearing applications is their tendency to fail unpredictably. Progress in brittle material technology (e.g. ceramic and carbon-carbon composites) made those so attractive for structural applications that one has to accept the inherent unpredictability of failure and to learn how to predict with confidence *probabilities of failure*. Significant research effort concerns (i) describing the probability distributions of critical loads, critical crack lengths, etc. and (ii) finding the probability distribution of times to failure, both for a loaded structural element. This thesis provides a review of certain probabilistic approach to brittle fracture.

The approach under review begins with modeling instantaneous crack advance in a micro-heterogeneous solid. The probability of crack formation between any two points in a two-dimensional solid ("crack propagator") is introduced as the main building block of the approach. Continuum mechanics concepts of a crack, stress concentration, energy release rate, etc., are com-

bined with statistical weakest link ideas in a Griffith type criterion of infinitesimal crack advance through a random field of specific fracture energy. Various probabilities associated with crack formation are expressed as averages, or functional integrals, over the space of virtual crack paths. Unorthodox parameters characterizing material's resistance to fracture are introduced together with experimental methodologies of their evaluation.

The next step introduces time into the model. One of the observed modes of slow crack growth in brittle composites exhibits a Markovian stochastic pattern of a microscopical random jump, followed by a random waiting time, followed by a random jump, and so on. Modeling of this phenomenon begins with relating the waiting times, on physical grounds, to random energy barriers at the arrest points and treating the magnitudes of the jumps within the previously described framework. The resulting description of crack growth as a random process allows one to make lifetime predictions, exhibit scale effect, etc. We describe the first step in this direction that employs only the most elementary, rectilinear version of Crack Propagator.

The author has been a collaborator in working on the second, time incorporating model.

Abstract Approval: Committee Chair ______

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DEDICATION

To my beloved *Candio*
(Candice L. Dalton)

PREFACE

The objective of this Thesis is to present a comprehensive review of an approach which stands aside from the mainstream of statistical modeling of fracture. The approach is essentially based on the concept of an ensemble of macroscopically identical fracture specimens and on averaging over it. Equivalently, an ensemble Ω of virtual crack trajectories is associated with a single specimen; the averaging is then expressed in the form of functional integration over Ω. The approach combines the concepts of weakest link theories with fracture mechanics formalism and models crack propagation through a brittle microheterogeneous solid. The statistics of microheterogeneity, e.g. the population of pre-existing defects, is reflected in a random field of specific fracture energy γ and in the statistical features of Ω. The fracture parameters employed in the approach are: parameters of the pointwise distribution of the γ-field; its correlation distance; and the characteristics of roughness of the fracture surfaces, including their fractal dimension. The probability of crack formation between any two points in a two-dimensional solid (referred to as "crack propagator") is introduced as the main building block of the approach. It is expressed as a functional integral (over the set Ω) of the probability of crack formation along a particular path. In Chapter 1 we are reviewing the works [1]-[3], and we freely borrowed with permission from [1]-[3].

CHAPTER 1

A PROBABILISTIC MODEL OF INSTANTANEOUS CRACK FORMATION

1.1 Introduction

More than 200 years of strength of materials studies brought a number of failure criteria, Those are formulated as critical values of various invariants of stress tensor, strain tensor, of energy density or their combinations. In fact, the number of such criteria proposed up to now is well over 100 [4]. This is a clear indication of a lack of understanding.

The recently developed notion of fracture mechanisms map (FMM) put to rest the naive belief in the existence of a universal criterion of failure in terms of classical continuum mechanics. FMMs reflect the existence of various modes of failure for different ranges of temperature and stress [5]. FMM resembles a phase diagram, suggesting that different failure criteria can be employed for various stress and temperature conditions. The fracture modes are distinguished by the type of microdefects dominating the fracture process. It is difficult to provide a quantitative characterization of fracture micromechanisms so, in practice, FMMs are often equipped with micrographs representing the mechanisms pictorially.

A few decades prior to the development of the concept of FMM, an essential role of microdefects in the triggering of fracture was addressed in the

pioneering work of Griffith [6]. The work remained practically unnoticed for a few decades. In the forties, however, Griffith's ideas became a cornerstone of a new discipline, namely, fracture mechanics. The development of fracture mechanics in the last 40 years, with its emphasis on stress concentration at the crack tip, resulted in a much better understanding of failures. Many attempts have been made to formulate crack initiation and crack instability criteria (toughness) as well as kinetic equations of crack growth in terms of stress intensity factors or energy release rate. However, each formulation has been found to have a rather limited domain of validity. Finally, it became clear that the limitations of the toughness criteria and kinetic equations, expressed in terms of macroscopic parameters, have the same origin as the limitations of classical strength criteria, i.e. the essential role of micro-defects not accounted for by the macroscopic considerations. Indeed, macroscopic fracture, being a critical phenomenon, is extremely sensitive to morphological fluctuations on a microscale.

To simplify the analysis, the authors of [5] distinguish two extreme cases of fracture by the role that the defects play in fracture processes. The modeling of these cases requires essentially different concepts and formalisms. One case is called *cooperative fracture*, since the fracture is mainly controlled by damage formed at the tip of a propagating crack in response to the stress concentration. Crack propagation is then inseparable from the evolution of the damage accompanying the crack, although, in the damage array surrounding the crack tip, defect locations, sizes and orientations are random. The behavior of the

"crowd" is essentially deterministic.

In [1] the authors are concerned with another extreme, which models the propagation of a crack through a pre-existing field of defects. The change in the population of the defects is assumed to be negligible. Thus, the statistics of the pre-existing defects, together with the stress field control, the fracture path as well as the rate of crack growth. For example, 27 identical SEN specimens, made of short fiber-reinforced polyester, were each fatigued under the same loading conditions until ultimate failure [7]. In spite of the (macroscopic) identity of the specimens and test conditions, the observed fracture paths, as well as the crack tip location at the instance of crack instability, were noticeably different as illustrated in Fig. 1.1.

The spatial fluctuation of the microdefect population is directly reflected in the stochastic features of fracture surfaces. This results in the scatter of experimentally observed fracture parameters, such as critical crack length, critical load, etc., as well as in the scale effect. Thus, a probabilistic approach seems to be most adequate under these circumstances. Due to the complexity of the phenomenon, there exists a variety of models addressing the problem on various scales, e.g. [8]–[16]. Nevertheless, the overall state of reliability analysis remains far from satisfactory [17]. The objective of this paper is to present, in a consistent manner, basic concepts and results of an approach, which was initially proposed in [18], [19], and which combines the concepts of WLT (Weakest Link Theory) with fracture mechanics formalism.

1.2 Basic assumptions

Several natural assumptions about the process of *brittle fracture* underlie the development that follows [20], [21]:

1. The crack path is randomly selected from a set $\Omega = \{\omega_1, \omega_2, \ldots, \omega_k, \ldots\}$ of virtual crack paths. (The superposition in Fig. 1.1 is a pictorial representation of Ω. More precisely, it is interpreted as a sampling from Ω.) Statistical characterization of Ω is to be based on fractographic analysis of observed crack trajectories.

2. Crack advance along a particular path consists of a sequence of elementary steps controlled by the Griffith criterion $G_I \geq 2\gamma$. [1]

3. Specific fracture energy γ is a random field that models, on a continuum level, the existence of microdefect population within the material.

Remark.

It would be tempting to introduce specific fracture energy as a random function $\gamma(x, n)$ of a point x and the direction, n, normal to a cut. Together with a crack growth criterion, $\gamma(x, n)$ would determine the stochastic geometry of crack paths, as well as the probability of crack formation along a particular path. The probabilities of various fracture events, e.g. that of ultimate failure, could then be expressed as averages over all of the paths that the crack

[1] It is important to note that the energy release rate of the crack ω with its tip at x, $G_I = G_I[x\,|\omega]$, determined from elastostatic solution should be employed in the condition (b). Indeed, it is known that the ERR for dynamic crack propagation can be written as $G_I^{\mathrm{dyn}} = G_I^{\mathrm{stat}} g(v)$ where v is the crack speed, $g(0) = 1$ and $g(v)$ decreases monotonically with the increase of v. Thus, the condition $G_I^{\mathrm{stat}} \geq 2\gamma$ suffices for the crack extension, at least quasi-statically.

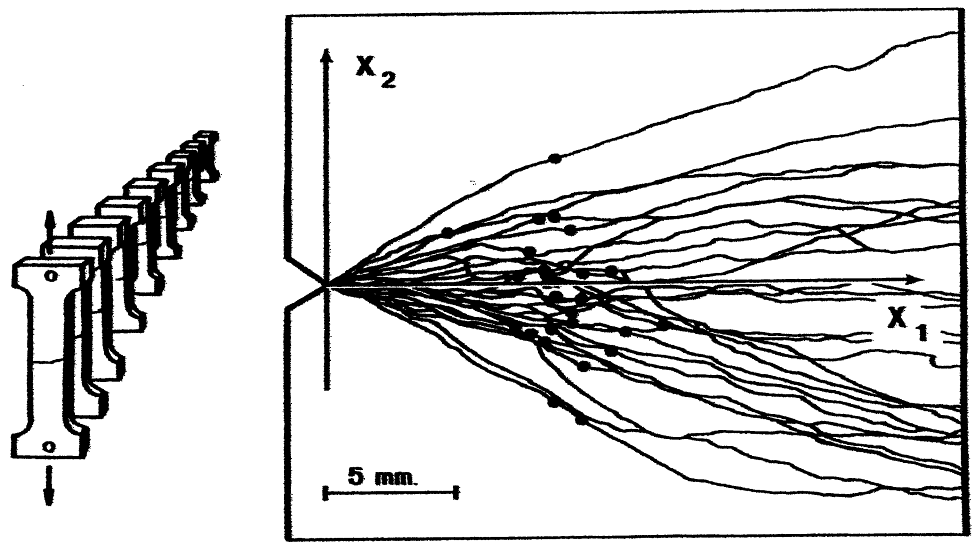


Figure 1.1. The ensemble of 27 macroscopically identical SEN specimens (fatigued to failure under identical conditions) and the superposition of the profilograms of the cracks traced from each individual specimen (only one trajectory took place in each specimen) [7]. Dots indicate the points of transition to unstable propagation (critical points). (reprinted with permission from [1]).

might possibly follow. A serious weakness of this program stems from the need to determine experimentally parameters of the random fracture energy $\gamma(x, n)$*, which is not directly measurable. Indeed, for each point of* x*,* $\gamma(x, n)$ *is an even function on a unit sphere* $|n| = 1$*, whose crudest non-isotropic approximation is quadratic:* $\gamma(x, n) = n \cdot A \cdot n$*, where the symmetric random matrix* $A = A(x)$ *contains three different random entrees in the two-dimensional (2-D) case and six in the three-dimensional (3-D) case. Even for a macroscopically homogeneous solid (A independent of* x*), the first two moments and ¿ covariances for these random entrees amount to 12 empirical constants in the 2-D case (42 in the 3-D case)–an unattractive situation.*

In [18], [19], it was proposed to overcome this difficulty through an entirely different characterization of $\gamma(x, n)$, while retaining the idea of averaging. First, instead of treating the shape of a crack as a consequence of the function $\gamma(x, n)$, it was proposed to use the actual crack trajectories (experimentally observable!) as part of the characterization of $\gamma(x, n)$. Second, it was proposed that only minimal values of $\gamma(x, n)$, with respect to n, play a role, since the crack advances locally in a direction of least resistance (see [21] for the discussion in terms of the energy barrier). Thus, the random function, $\gamma(x, n)$, can be characterized by the roughness of fracture paths and the random field $\gamma(x)$, whose pointwise distribution is one of the minimal value distributions. That γ is non-negative dictates the choice of the Weibull distribution among

the three classical min-value distributions [22]. [2]

1.3 Crack propagator

One of the main building blocks of the formalism is the crack propagator. Let us consider a loaded solid containing a crack to ω with its tip at x (Fig. 1.2). First, in [2] the authors define a conditional propagator, $P(x,\, X \,|\, \omega)$, as the probability that the crack extends from x through another point X along the path ω, i.e. the condition $G_I \geq 2\gamma$ is met at every point of the crack trajectory ω. Thus, the authors of [2] are interested in the probability of the conjunction of the events:

$$P(\underline{x},\, \underline{X} \,|\, \omega) = \lim_{n\to\infty} \mathrm{Prob}\left\{ \bigcap_{k=1}^{n} (G_I[x_i \,|\, \omega] > 2\gamma(x_k)) \right\}, \tag{1.1}$$

where $\underline{x}_k$, $k = 1, 2, \ldots, n$ are points on ω, which subdivide ω it into n small portions, and $G_I[\underline{x}_k \,|\, \omega]$ is the ERR of the crack ω with its tip at $\underline{x}_k$. The right-hand side of Equation (1.1) can be evaluated [23]:

$$P(\underline{x},\, \underline{X} \,|\, \omega) = \exp\left\{ -\int_x^X \mathrm{Prob}[2\gamma(\xi) \geq G_I[\xi \,|\, \omega]] \frac{d\xi}{r} \right\}, \tag{1.2}$$

where r is a correlation distance of the γ-field, a microscopical parameter. Since minimal values of γ are encountered along ω, the Weibull distribution is

[2] An interesting approach to the modeling of random crack trajectories has been presented in [26]. The process of crack propagation is modeled as a sequence of elementary crack increments, each being characterized by a random angle and increment's length. The model provides a phenomenological description of observed fracture trajectories. However, it does not offer a correlation between the random characteristics of the fracture path and critical fracture parameters such as the material's specific fracture energy γ or critical energy release rate G_{IC}.

employed as the pointwise distribution of γ along the crack trajectory:[3]

$$F(\gamma) = \begin{cases} 1 - \exp\left[- \left(\Gamma\left(1 + \frac{1}{\alpha}\right) \frac{\gamma - \gamma_{min}}{\gamma^* - \gamma_{min}} \right)^{\alpha} \right], & \gamma > \gamma_{min} \\ 0, & \gamma \leq \gamma_{min} \end{cases}$$

where $\alpha > 0$, γ^* and γ_{min} are the shape factor, mean and minimal values of the γ-field, respectively, and $\Gamma(\cdot)$ denotes the Γ-function. Our parameterization of the Weibull distribution departs slightly from the conventional one.

If one assumes that the crack trajectories are mutually exclusive (only one trajectory is observed in every individual specimen), then the probability $P(x, X)$ of crack extension from x through X can be written as the total probability:

$$P(\underline{x},\, \underline{X}) = \sum_k P(\underline{x}\,, \underline{X} \,|\, \omega_k)\, \mathrm{Prob}\{\omega_k\}. \tag{1.3}$$

Here $\mathrm{Prob}\{\omega_k\}$ stands for the probability that the crack "chooses" a path ω_k among all virtual paths extending from x to X. $P(x\,,\, X)$ is called crack propagator (CP). In other words, CP is the average over all virtual crack trajectories of the probability of crack formation along a particular trajectory.

For a continuum model, the set f/is uncountable, and the sum in eq. (4)

[3]One may want to look closely at the assumption that the values of γ entering the Griffith criterion at each point x have Weibull distribution. The rationale for choosing Weibull has been that (a) those values represent minimal values of the fracture energy $\gamma(x, n)$ with respect to various orientations n of the infinitesimal fracture surface element, hence a distribution of minimal values supplied by the theory of extreme value statistics should be employed, and (b) the fracture energy is a non-negative quantity, and among the three known families of minimal value distributions, only Weibull distribution has no "tail" towards $-\infty$. However, the values of γ are also bounded from above on physical grounds (for example, $\gamma < E/10$). Therefore, it appears desirable to assign to γ a "minimal value distribution", which is bounded towards both $-\infty$ and $+\infty$ (has no "tails"). There is no such distribution among the three classical families [22]. A suggestion of how to overcome this difficulty is contained in [24], where a new family of distributions is proposed and argued to represent "minimal values".

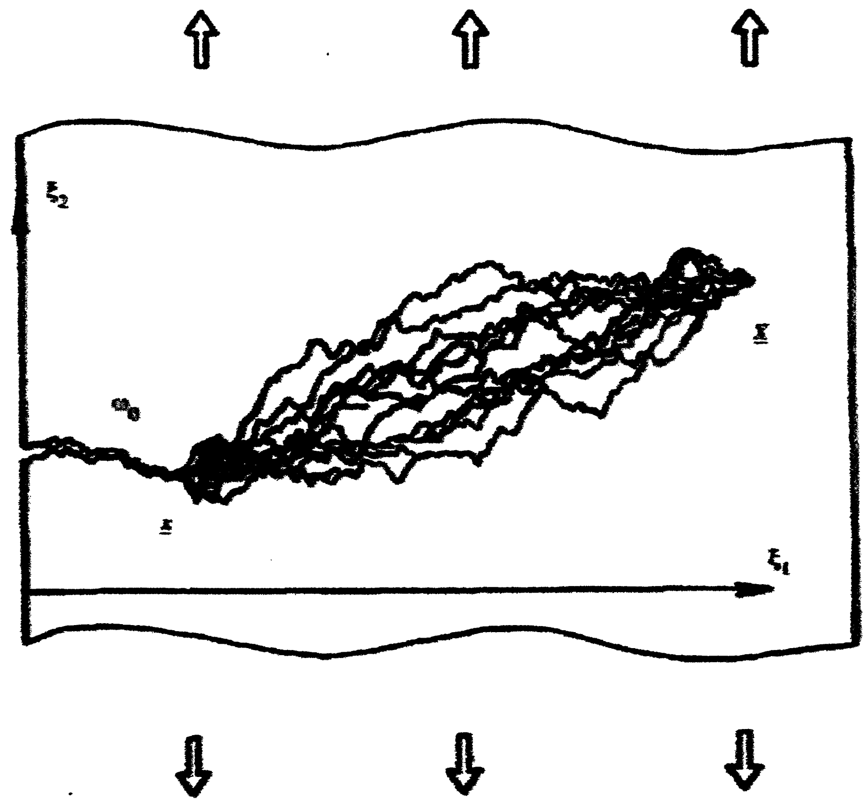


Figure 1.2. A sample set of virtual crack paths leading from point x to point X. (reprinted with permission from [1]).

leads to the concept of a functional integral [19], [21]

$$P(\underline{x}, \underline{X}) = \int_{\Omega} P(\underline{x}\,, \underline{X}\,|\,\omega)\, d\mu(\omega), \tag{1.4}$$

where $d\mu(\omega)$ is a continuum analog of $Prob\{\omega_k\}$ and, intuitively, represents the probability that the random path ω of an actual crack is confined to an infinitesimally narrow corridor $\Delta\Omega(\omega)$ surrounding ω:

$d\mu(\omega) = Prob\{|\omega(\xi) - \omega(\xi_l)| < d\omega(\xi_l), \quad x_l \leq \xi_l \leq X_l\}$ (see Figure 1.3).

Commonly, $d\mu(\omega)$ is referred to as a "probabilistic measure on Ω".

1.4 Evaluation of crack propagator

Representations of the CP in (1.4) lead to three major tasks:

1. the selection of a set of virtual crack trajectories possessing a prescribed roughness together with a probabilistic measure dr(co) on it;
2. the determination of the conditional propagator $P(\underline{x}\,, \underline{X}\,|\,\omega)$;
3. the evaluation of the averages represented by the functional integrals in (1.4).

In the rest of this section we discuss these tasks (also see [8]) in more detail.

1.4.1 Task 1. Selection of the space Ω and the measure $d\mu(\omega)$

The choice of Ω will be universal: the space of all continuous functions $\omega(\xi_l)$ on the segment $x_l \leq \xi_l \leq X_l$ with prescribed values at the end points:

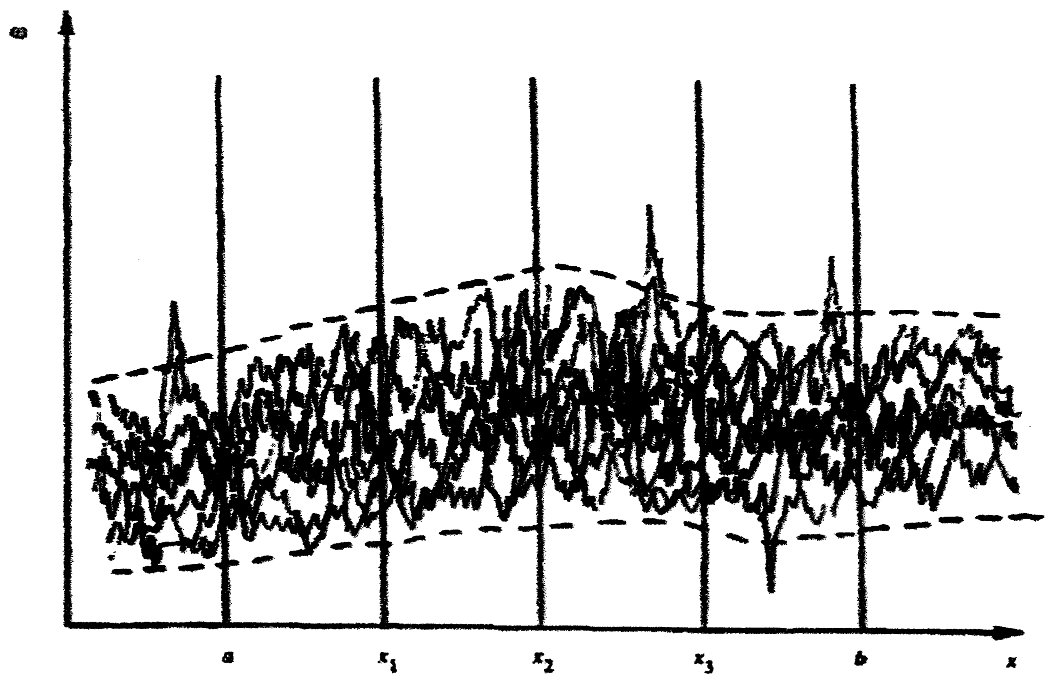


Figure 1.3. A schematic representation of a subset $\Delta\Omega(\omega)$ of Ω as a "narrow corridor" containing ω. (reprinted with permission from [2]).

$\omega(x_l) = x_2$, $\omega(X_l) = X_2$ (the graphs of the functions represent crack paths connecting $\underline{x}$ and $\underline{X}$, see Figure 1.2). The measure is then selected so that it is concentrated on trajectories of certain roughness. In [8] the authors begin with the highest roughness possible for a random continuous path with independent increments, which is characterized by the fractal dimension 3/2. Namely, the authors of [2]-[8] represent crack trajectories as the graphs of one-dimensional Brownian motion $\xi_2 = \omega(\xi_l)$ (ξ_l plays the role of time) and accordingly select $d\mu(\omega)$ to be a "conditional Wiener measure with a parameter D" [25].

$$d\mu^D(\omega) = \text{const}\exp\left\{-\int_{x_l}^{X_l}\frac{2}{D}\left[\frac{d\omega}{d\xi_l}\right]^2 d\xi_l\right\}\prod_{x_l}^{X_l} d\omega(\xi_l) \tag{1.5}$$

with the following normalization

$$\int_\Omega d\mu^D(\omega) = \frac{1}{\sqrt{2\pi D(X_l - x_l)}}\exp\left[-\frac{(X_2 - x_2)^2}{2D(X_l - x_l)}\right]. \tag{1.6}$$

The observed fracture profiles are often smoother than Brownian paths.

1.4.2 Task 2. Evaluation of the conditional propagator $P(\underline{x}, \underline{X}|\omega)$

Equation (1.2) for the conditional CP results from the assumption that the arrest during an infinitesimal crack increment is an unlikely event with probability proportional to the length of the increment and the coefficient of

proportionality is given by

$$\begin{aligned} U[\xi_1 \,|\, \omega] \quad &\equiv \mathrm{Prob}\{2\gamma > G_I[\xi_1 \,|\, \omega]\}/r \\ &= [1 - F(G_I[\xi_1 \,|\, \omega]/2)]/r \\ &= \frac{1}{r}\exp\left[-\left(\Gamma\left(1+\frac{1}{\alpha}\right)\frac{G_I[\xi_1 \,|\, \omega]/2\gamma^* - q}{1-q}\right)^{\alpha}\right]. \end{aligned} \tag{1.7}$$

Here $q = \gamma_{\min}/\gamma^*$ and the last equality is due to (1.3). This means that the occurrence of arrest along ω i.e. $2\gamma > G_I$, is governed by a Poisson process (ξ_1 playing the role of time) with the variable intensity $U[\xi_1 \,|\, \omega]/r$.

Putting (1.2), (1.4) and (1.7) together, the authors get the following expression for the crack propagator:

$$P(\underline{x},\, \underline{X}) = \int_{\Omega}\left\{-\int_{x_1}^{X_1}\exp\left[-\left(\Gamma\left(1+\frac{1}{\alpha}\right)\frac{G_1[\xi_1 \,|\, \omega]/2\gamma^* - q}{1-q}\right)^{\alpha}\right]\frac{d\xi_1}{r}\right\}d\mu^{(D)}(\omega). \tag{1.8}$$

1.4.3 Task 3. Evaluation of the functional integral representing the crack propagator $P(\underline{X}, \underline{x})$

An integral $\int_{\Omega} \ldots d\mu(\omega)$ over an infinite dimensional space of functions Ω is defined as a limit of conventional finite dimensional (multiple) integrals, whose multiplicity tends to infinity. On a conceptual level, each multiple integration is performed over an appropriate finite dimensional subspace of Ω for example, consisting of piecewise linear functions with a prescribed finite number of kinks. When written down, such integration amounts to ordinary repeated integration in several variables (see below). In the case of Wiener

integrals, i.e. when $d\mu(\omega)$ is a Wiener measure, equation (1.5), specifics are as follows [25].

Temporarily, let us denote coordinates in the plane by x, y instead of x_1, x_2. Fix two points (a, b) and (A, B), $a < A$, in the plane and take Ω to be the space of all continuous functions $y = \omega(x)$, $a < x < A$, such that $\omega(a) = b$, and $\omega(A) = B$ (in our context: the space of virtual crack paths beginning at (a, b) and ending at (A, B)). Let $F[\omega]$ be a functional whose argument ω runs through the space Ω. The Wiener integral of $F[\omega]$ over Ω is defined as the following limit of multiple integrals.

Begin with an equipartition $a \equiv x_0 < x_1 < \ldots X_n < X_{n+1} \equiv A$ of the segment $a \leq x \leq A$, with the step $\Delta x = (A - a)/(n + 1)$. For any $y_1, \ldots, y_n$ consider a function $\omega^{(n)}(x)$, which gives a piecewise linear interpolation between the points (a, b), $(x_1, y_1), \ldots,$ (x_n, y_n), (A, B) (i.e. $\omega^{(n)}(x_j) = y_j$, $j = 0, 1, \ldots, n+1$, and $\omega^{(n)}(x)$ is linear for $x_j \leq x \leq x_{j+1}$, $j = 0, 1, \ldots, n$; here $y_0 \equiv b$, $y_{n+1} \equiv B$). Denote $F(y_1, \ldots, y_n) = F[\omega^{(n)}]$. Then the Wiener integral of the functional $F[\omega]$ with respect to the measure $d\mu^{(D)}(\omega)$ given by (1.5) is

$$\begin{aligned} \int_\Omega F[\omega] d\mu^{(D)}(\omega) &= \lim_{n\to\infty} \frac{1}{[2\pi D\Delta x]^{\frac{n+1}{2}}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} F(y_1, \ldots, y_n) \\ &\times \exp\left[-\frac{(y_1 - b)^2}{2D\Delta x} - \frac{(y_2 - y_1)^2}{2D\Delta x} - \ldots - \frac{(y_n - y_{n-1})^2}{2D\Delta x} - \frac{(B - y_n)^2}{2D\Delta x} \right] dy_1 \ldots dy_n. \end{aligned} \tag{1.9}$$

The exponential function, against which $F(y_1, \ldots, y_n)$ is integrated, expresses the product of Gaussian probabilistic weights assigned to each increment of $\omega^{(n)}(x)$. There are various techniques to perform functional integration in

the expression for CP. Those include analytical methods such as reduction to solving of partial or sometimes even ordinary differential equations, as well as numerical methods such as the Monte-Carlo technique. In this section, the analytical treatment is presented.

First, let us simplify the inner integrand in (1.8). The dependence of G_I on a crack trajectory ω is quite complex; no general solution is available. The effect of the overall macroscopic shape of a crack on the ERR can be accounted for by perturbation methods or other means [30]- [32]. The effect of sharp corners on a microscale can be approximated by existing kinked crack solutions [31], [33], since the functional $G_I[\xi_1 \,|\, \omega]$ is highly sensitive to the direction of the crack at the very tip and is practically not affected by small perturbation away from the tip [34]. It follows from an analysis of existing solutions that the short-wave effect dominates. Therefore, we approximate the crack which follows ω to a depth ξ_1, by the crack which follows the mean crack path

$$\bar{\omega}(\xi_1) = \int_{\Omega} \omega(\xi_1) d\mu(\omega) \tag{1.10}$$

to the depth ξ_1 and has a random kink at ξ_1. This leads to the following approximation for the energy release rate associated with the crack formed along ω to a depth ξ_1:

$$G_I[\xi_1 \,|\, \omega] = G_I^0(\xi_1) \left[1 + k_1 \frac{\Delta\omega(\xi_1)}{r} - k_2 \left(\frac{\Delta\omega(\xi_1)}{r} \right)^2 \right], \tag{1.11}$$

where $G_I^0(\xi_1) = G_I^0[\xi_1 \,|\, \omega]$ is a "zeroth approximation" for G_I (the zeroth term in a functional Taylor expansion of G_I about $\bar{\omega}$, $k_i = k_i\,(\underline{x}\,,\, \underline{X})$, $i = 1, 2$, and $\Delta\omega = \omega - \bar{\omega}$. Actually, it is known that, for a Wiener process, $\bar{\omega}$ is the

straight segment connecting $\underline{x}$ and $\underline{X}$, thus $k_i = k_i(\lambda),$, $i = 1, 2$, where the slope $\lambda = (X_2 - x_2)/(X_1 - x_1)$ and the functions $k_i(\lambda)$ can be evaluated on the basis of numerical solutions reported in [33].

This approximation of the functional $G_I[\xi_1 \,|\, \omega]$ by a function of the crack tip coordinates allows one to use a celebrated result of Kac, which reduces the functional integration in (1.8) to solving of a partial differential equation. Indeed, it is known that $P(\underline{x}, \underline{X})$ given by (1.8), with $G_I[\xi_1 \,|\, \omega]$ substituted by $G_I(\xi_1, \omega(\xi_1))$, is the solution of each of the following two boundary value problems [25] , [35] , [36]:

$$\frac{\partial P(\underline{x}, \underline{X})}{\partial X_1} = \frac{D}{2}\frac{\partial^2 P(\underline{x}, \underline{X})}{\partial X_2^2} - \frac{1}{r}U(\underline{X})P(\underline{x}, \underline{X}) \tag{1.12}$$

$$P(\underline{x}, \underline{X})|_{X_1 = x_1} = \delta(X_2, x_2)$$

$$P(\underline{x}, \underline{X}) \to 0, \text{ as } X_2 \to \pm\infty, \ X_1 > x_1$$

and

$$\frac{\partial P(\underline{x}, \underline{X})}{\partial x_1} = -\frac{D}{2}\frac{\partial^2 P(\underline{x}, \underline{X})}{\partial x_2^2} + \frac{1}{r}U(\underline{X})P(\underline{x}, \underline{X}) \tag{1.13}$$

$$P(\underline{x}, \underline{X})|_{x_1 = X_1} = \delta(x_2 - X_2)$$

$$\frac{\partial P(\underline{x}, \underline{X})}{\partial x_2} \to 0, \text{ as } x_2 \to \pm\infty, \ x_1 > X_1,$$

where $U(\underline{\xi})/r$ given by (1.7), expresses the probability density (in ξ_1) of the crack arrest at a point $\underline{\xi}$ and accounts for the stress state (through the energy release rate G_I). Using an apparent analogy, we refer to D as a *crack diffusion coefficient* and to the above choice of Ω and $d\mu(\omega)$ as a *diffusion approximation.* The diffusion approximation is attractive due to the well-developed machinery

available for averaging over sets of Brownian paths.

Equations (1.12) and (1.13) are direct analogs of forward and backward Kolmogoroff equations in diffusion theory. It is recognized that those two equations are two different representations of the same physical phenomenon in diffusion. The situation is different in the modeling of fracture. The above equations correspond to two essentially different problems, namely, crack formation in stable and unstable specimen-loading configurations (see Figure 1.4). The configuration is called stable (unstable), if within the expected range of crack lengths the energy release rate decreases (increases) with the crack extension. The crack arrest is expected for a stable configuration, whereas an avalanche-like, uncontrolled crack propagation occurs in an unstable case.

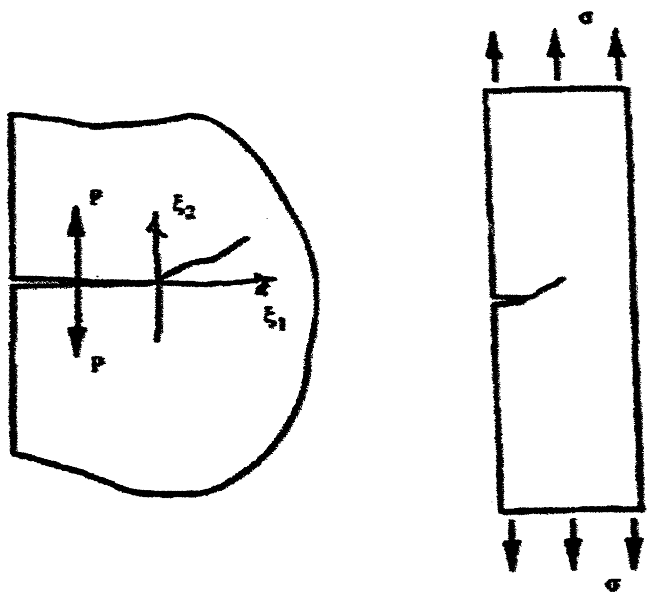
Stable case.

Let us consider a pre-notched specimen (see Figure 1.4(a)) and the probability $P(\underline{X})$ that a crack growing from the notch passes through $\underline{X}$. Apparently, $P(\underline{X})$ is related to the crack propagator (note the position of the coordinate system):

$$P(\underline{X}) = P(\underline{0}, \underline{X}). \tag{1.14}$$

It is immediate from (1.12), that $P(\underline{X})$ is the solution of

$$\begin{aligned} &\frac{\partial P(\underline{X})}{\partial X_1} = \frac{D}{2}\frac{\partial^2 P(\underline{X})}{\partial X_2^2} - \frac{1}{r}U(\underline{X})P(\underline{X}) \\ &P(\underline{X})|_{X_1=0} = \delta(X_2) \\ &P(\underline{X}) \to 0, \ \text{ as } X_2 \to \pm\infty, \ X_1 > 0. \end{aligned} \tag{1.15}$$



(a) (b)

Figure 1.4. Two types of specimen-loading configurations: (*a*) stable; and (*b*) unstable. (reprinted with permission from [2]).

The reduced form of crack propagator, $P(\underline{X})$, yields the probability density $p_a(\underline{X})$ for the crack arrest location. Indeed, the probability $p_a(\underline{X})d\underline{X}_1 d\underline{X}_2$ of the crack being arrested in a vicinity $dX_1 dX_2$ of $\underline{X}$ equals to the product of the probability $P(\underline{X})d\underline{X}_2$ of reaching the vicinity and the conditional probability $U(\underline{X})d\underline{X}_1/r$ of being arrested upon getting there. Hence

$$p_a(\underline{X}) = P(\underline{X})U(\underline{X})/r. \tag{1.16}$$

Note that $p_a(\underline{X})$ can be directly compared to the observed scatter of crack arrest locations (in contrast to the crack propagator).

Unstable case.

This case is illustrated by the failure of a single edge notched specimen shown in Figure 1.4(b). By the specimen's failure we mean a sudden extension of the crack ω_0 (its tip at $\underline{x}$) to the opposite edge. The event of failure is the sum of events consisting of crack extension to an arbitrary point (B, X_2). For the single crack fracture, these events are mutually exclusive. Hence, the probability $P(\underline{x}; \sigma)$ of failure under a given load σ can be expressed through the crack propagator $P(\underline{x}, (B, X_2))$, which represents the probability (density in X_2) of crack extension from $\underline{x}$ to (B, X_2):

$$P(\underline{x}; \sigma) = \int_{-\infty}^{\infty} P(\underline{x}, (B, X_2))dX_2 \tag{1.17}$$

(σ enters the crack propagator through the energy release rate). It is immedi-

ate from (1.13) that $P(\underline{x}) \equiv P(\underline{x};\sigma)$ is the solution of

$$\frac{\partial P(\underline{x})}{\partial x_1} = -\frac{D}{2}\frac{\partial^2 P(\underline{x})}{\partial x_2^2} + \frac{1}{r}U(\underline{x})P(\underline{x}) \tag{1.18}$$

$$P(\underline{x})|_{x_1=B} = 1$$

$$\frac{\partial P(\underline{x})}{\partial x_2} \to 0, \text{ as } x_2 \to \pm\infty, \ \ x_1 < B.$$

The probability $P(\underline{x};\sigma)$ yields some observable distributions. For example, if a sharp notch is produced and the applied load is increased monotonically, then the load σ_c immediately preceding failure (critical load) is a random variable whose distribution function is

$$\begin{aligned} F_\sigma c(\sigma) = \ & \text{Prob}\{\sigma_c \leq \sigma\} \\ = \ & \text{Prob}\{\text{under load } \sigma, \text{ extension of the initial crack to the} \\ & \text{opposite edge along at least one path is guaranteed}\} \\ = \ & P((0,\, l);\, \sigma) \end{aligned} \tag{1.19}$$

(notch is horizontal as in Figure 1.4(b), its length is l).

Following the terminology common in the diffusion theory, we refer to eq. (1.18) as "backward crack diffusion equation" (unstable case), and to (1.15) as "forward crack diffusion equation" (stable case).

CHAPTER 2

A PROBABILISTIC MODEL OF QUASI-STATIC CRACK GROWTH

2.1 Introduction

In this Chapter, we model an observed mode of slow crack growth in brittle materials, namely, a sequence of microscopic jumps of random length, each one followed by an arrest of random duration. The jump lengths are modeled by a non-homogeneous Poisson process with a space coordinate in place of what ordinarily is time (Sec. 2.2), and the arrest durations are modeled by a homogeneous Poisson process whose intensity is related to random energy barriers at the arrest points and, consequently, depends on the crack arrest location (Sec. 2.4). In Sec. 2.3, the notions of stable and unstable specimen-loading configurations are defined; it is stated that the paper is devoted to the more common unstable case. In Sec. 2.5, crack growth is described as a random process and its transition probability is shown to satisfy a hyperbolic PDE (in contrast to the parabolic Kolmogoroff equations, which govern the transition probabilities in [11]). In Sec. 2.6, illustrative numerical examples are considered. These include prediction how crack depth distribution changes in time and demonstration of the effect of 'microstructure' on life time scatter. The last example in Sec. 2.6 indicates that the model is capable of simulating the Paris law. Section Conclusions contains a brief discussion of future

developments.

2.2 Modeling a crack jump

Let us consider crack formation in a loaded two-dimensional solid. In this paper, we are concerned only with crack propagation along a straight line, say the x-axis (see Fig. 2.1). "Crack jump" stands for an event during which a previously arrested crack instantaneously advances and is then arrested, with its tip at a new location. (The advance is considered instantaneous in comparison to the time scale of the crack growth process under consideration). Material in front of the crack is viewed as a sequence of randomly distributed obstacles. Once initiated, the crack jumps through all of the obstacles it can overcome and is arrested at the first 'insurmountable obstacle' (see below). Thus the length of each jump is random. What happens after the arrest is discussed in Sec. 2.4.

It is technically convenient to begin by introducing the probability that a crack with its tip at x will overcome all of the obstacles between x and an arbitrary $X \geq x$ (and, therefore, the crack arrest would occur to the right of X). This probability will be denoted $[X|x]$. This is a special case of the Crack Propagator that was introduced in Chapter 1 as the probability of instantaneous crack formation between two arbitrary points of a loaded

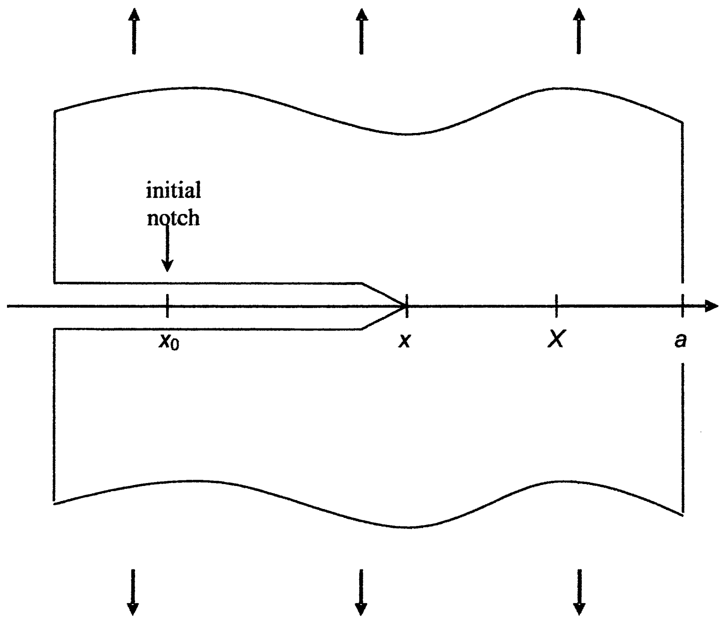


Figure 2.1. Specimen-loading configuration to be considered.

two-dimensional solid. We now proceed to define the needed version of crack propagator.

Consider a solid with a straight crack (Fig. 2.1). We will refer to the x-coordinate of the crack tip as *crack depth.* Assume that the applied loading is symmetrical relative to the crack line to justify rectilinear crack growth along the x-axis. We assume that, in the course of the instantaneous phase, the crack, its tip currently at x, will advance farther by dx if the elastic energy release resulting from crack advancing from x to $x + dx$ would be greater than the energy required for breaking the material between x and $x + dx$. The (linear) "elastic energy release" is $G(x)dx$, where $G(x)$ is considered a known deterministic function, the so-called *energy release rate* (ERR). The "energy required for breaking" is taken to be $2\gamma(x)dx$, where $\gamma(x)$ is the value of a

random field γ at x (when γ is a constant they refer to it as specific fracture energy). Thus, "no arrest at x" means $2\gamma(x) \leq G(x)$.

We make the following assumption about the random field γ:[1]

(a) γ is homogeneous in x,

(b) values of γ at every point are governed by a Weibull distribution,

$$F(\gamma) = \begin{cases} 1 - \exp\left(- \left[\Gamma\left(1 + \frac{1}{\alpha}\right) \frac{\gamma - \gamma_{\min}}{\gamma^* - \gamma_{\min}}\right]^{\alpha}\right) & \text{if } \gamma \geq \gamma_{\min} \\ 0 & \text{if } \gamma < \gamma_{\min}. \end{cases} \tag{2.1}$$

Here the parameters γ^* and $\gamma_{\min}$ are the average and minimal values of γ, respectively, and $\alpha > 0$ is known as a 'shape parameter' – they characterize the scatter of material's strength on a microscale; $\Gamma(\cdot)$ is the Γ-function,

(c) properties of γ on disjoint intervals are independent, and

(d) for any coordinate $x \geq x_0$ and for any number $g \geq \gamma_{\min}$, the probability of γ exceeding g somewhere between x and $x + dx$ is proportional to dx, specifically,

$$\text{Prob}\{\, \gamma > g \text{ somewhere on } (x,\, x + dx] \,\} = (1 - F(g))\frac{dx}{r}, \tag{2.2}$$

where $r > 0$ is a parameter. (When the solid under consideration is finite, one assumes that r is small in comparison to the solid's size.)[2]

Therefore, the probability of the crack being arrested on $(x,\, x + dx]$ is

$$\text{Prob}\{\, 2\gamma > G(x) \text{ somewhere on } (x,\, x + dx] \,\} = \frac{U(x)}{r} dx, \tag{2.3}$$

where $U(x) = 1 - F\left(\frac{G(x)}{2}\right)$.

[1] One can find a discussion of the assumed properties of γ and consequences, 'in physical terms', in [3], [15].

[2] For examples of experimental evaluation of the parameters γ^*, $\gamma_{\min}$, α , and r of the γ-field, see [7], [42].

(In physical terms, one is describing the probability conditional on that either the crack has already passed through x or the initiation at x has occurred.)

For two arbitrary points x and X, $x \leq X$, on the x-axis let us define *crack propagator* (CP), $[X|x]$, as the probability that the crack can advance instantaneously to at least X conditional on that either the crack has already passed through x or the initiation at x has occurred. (One can formally define $[X|x] = 0$ for $x > X$, which would express the assumption that cracks do not cure.) The definition is a one-dimensional case of the notion introduced in [1] and is a cumulative version of the one that was used in [15]. Notice that if x were representing "time", then the above assumptions would mean that the random event of crack arrest is governed by a non-homogeneous Poisson process with "time"-dependent intensity $U(x)/r$. The absence of crack arrest between x and X (conditional, as described above), corresponds, in the Poisson process terminology, to having "zero number of events in "time" interval $(x, X]$". Its probability is therefore expressed by a well known formula (the last equality in Eq.(2.4) below):

$$\begin{aligned}
[X\,|\,x] \quad &= \text{Prob}\{\,\text{no arrest on } (x,\, X]\,\} \\
&= \text{Prob}\{2\gamma \ \leq G_1(x), \text{everywhere on } (x,\, X]\,\} \\
&= \exp\Big\{-\int_x^X \frac{U(\xi)}{r} d\xi\Big\}.
\end{aligned} \tag{2.4}$$

Thus, the explicit expression for CP becomes (for $x \leq X$)

$$[X\,|\,x] = \exp\Bigg\{-\int_x^X \exp\Bigg(-\Bigg[\Gamma\Big(1+\frac{1}{\alpha}\Big)\frac{\frac{G(x)}{2}-\gamma_{\min}}{\gamma^*-\gamma_{\min}}\Bigg]^{\alpha}\Bigg)\frac{d\xi}{r}\Bigg\}. \tag{2.5}$$

The following useful property of CP is obvious from its definition and the independence of properties of the field γ on disjoint intervals (as well, as from the last expression in Eq. (2.4)): for any $x \leq x' \leq X$,

$$[X \mid x'] \, [x' \mid x] = [X \mid x]. \tag{2.6}$$

Remark. Limiting transition to the case of a homogeneous material with no microstructure (constant γ) may be understood physically in more than one way. One possibility is to suppose that $r \to \infty$. This is inconsistent with the assumptions of the model. Indeed, by construction, the model applies to scales large relative to r. A second approach involves letting both r and the variance of γ tend to zero simultaneously. This one is consistent with the model.

2.3 Stable and unstable configurations

In physical terms, one may call a specimen-loading configuration stable, if a crack, once initiated from the notch, will surely be arrested, as opposed to there being a possibility of it reaching the specimen's right edge and breaking the specimen in two. In the latter case, the configuration may be called unstable.

Let a denote the x-coordinate of the right edge of the specimen under consideration, see Fig. 2.1 (we include the possibility of $a = \infty$.) Formally, we will call a specimen-loading configuration *stable*, if the probability of a crack

reaching a is zero for any crack depth x, i.e. if $[a|x] = 0$ for all $x \geq x_0$; otherwise we will call the configuration *unstable.*[3]

In [15], stable configurations were considered. In this paper, we are interested in unstable configurations, which are more common.

Notice that, for a given $x \geq x_0$, $[s|x]$ as a function of s monotonically decreases from $[x|x] = 1$ to $[a|x] > 0$. The distribution function for crack arrest depths conditional on the crack initiation occurring at x is

$$F(s\,|x) = \begin{cases} 1 - [s\,|x], & \text{if } x < s < a \\ 1, & \text{if } a \leq x. \end{cases} \tag{2.7}$$

For an unstable configuration, $F(\cdot\,|x)$ has a jump at a whose size is $[a\,|x] > 0$. From now on, unless the opposite is stated, let us assume $a < \infty$ (finite width specimen). Under this assumption one can consider the first two moments of the crack arrest distribution.

The mean value $m_1(x)$ of the crack arrest depth conditional on the crack initiation occurring at x is

$$m_1(x) = \int_x^a s\,dF(s\,|x) = x + \int_x^a [s\,|x]ds \tag{2.8}$$

(the second equality follows from Eq. (2.7) and integration by parts). The second moment of the crack arrest depth is

$$m_2(x) = \int_x^a s^2\,dF(s\,|x) = x^2 + 2\int_x^a s\,[s\,|x]ds \tag{2.9}$$

(again, the second equality follows from Eq. (2.7) and integration by parts).

[3]Stable configurations occur, for example, when a half-infinite solid is loaded through prescribed displacements. Unstable configurations commonly occur in finite specimens (unless one is compressed for large crack depths) and in semi-infinite solid loaded through applied forces.

2.4 Crack arrest duration

Let us make the following assumption about the (random) duration of crack arrest.

$$\text{Prob}\{\text{initiation from } x \text{ during } dt\} = \Lambda(\gamma^*,\, G(x))\frac{dt}{\tau} \tag{2.10}$$

Here τ (a characteristic time scale for the submicroscopic processes) is introduced explicitly to render the function $\Lambda(\gamma^*, G)$ dimensionless. In the absence of other relevant dimensional constants, $\Lambda(\gamma^*, G)$ has to be a function of γ^*/G. One might expect that Λ depend on the average relative energy barrier $(2\gamma^* - G)/G$. The dependence should be then such that a small barrier results in large values of Λ (short arrest) and a large barrier results in small values of Λ (long arrest), e.g. $\Lambda = G/(2\gamma^* - G)$. Actual choice of the function Λ is to be dictated by physical considerations outside of the scope of this paper. In Sec. 2.6, we make a choice for the sake of illustrative numerical examples.

Evidently, we have assumed that, for a crack arrested at x, its subsequent initiation is governed by a Poisson process in time with the time-independent intensity $\Lambda(x)/\tau$. Thus, in particular, the probability that initiation does not happen between times t_1 and $t_2 \geq t_1$ is

$$\begin{aligned} &\text{Prob}\left\{ \begin{array}{c} \text{no crack initiation} \\ \text{between } t_1 \text{ and } t_2 \geq t_1 \end{array} \middle| \begin{array}{c} \text{the crack was arrested} \\ \text{at } x \text{ prior to } t_1 \end{array} \right\} \\ &\qquad = \exp\left(-\tfrac{\Lambda(x)}{\tau}(t_2 - t_1)\right). \end{aligned} \tag{2.11}$$

2.5 Distribution of crack arrest locations as a function of time

Let $\xi(t)$ denote the position of the crack tip at a time t. Since we model crack jumps as instantaneous, there is ambiguity in this description of $\xi(t)$. Namely, if t is an instant of a jump, then the crack tip position at t is not uniquely defined. Let us define $\xi(t)$ then as the *rightmost* position of the crack tip at t. Thus, the random process $\xi(t)$ has monotonically growing piecewise constant realizations. Also, notice that the statement $\xi(t) = x$ implies that, at t, the crack tip is at x *and* the crack is in the state of arrest.

Denote by $P(x,t)$ the probability that, at a time t, the crack depth is less than or equal x, assuming that, at $t = 0$, the crack tip was at x_0 in the state of arrest,

$P(x,\, t) = \text{Prob}\{\xi(t) \le x \,|\, \xi(0) = x_0\}.$

We proceed to derive a PDE for $P(x,t)$. Let $x \ge x_0$ and $t \ge 0$. Assuming that $\xi(t) \le x$, there are two and only two mutually exclusive possibilities to have $\xi(t + dt) \le x$:

(a) $\xi(t) = x_0$ (in the state of arrest) and, during $[t, t + dt]$, it did *not* happen that crack initiation occurred and was followed by an instantaneous crack jump to a depth greater than x; the probability of this is

$$P(x_0, t)\left(1 - \frac{\Lambda(x_0)dt}{\tau}[x \,|\, x_0]\right);$$

(b) $\xi(t) \in (x',\, x'+dx']$ for some $x' \in (x_0,\, x]$ and, during $[t,\, t+dt]$, it did *not*

happen that crack initiation occurred and was followed by an instantaneous crack jump to a depth greater than x; the probability of $\xi(t) \in (x',\, x'+dx']$ is $P(x'+dx',\, t) - P(x',\, t) = P_x(x',\, t)dx'$,[4] the probability of crack initiation that is followed by an instantaneous crack advance beyond x is $\frac{\Lambda(x')dt}{\tau}[x\,|\,x']$; since, for non-intersecting intervals $(x',\, x'+dx']$, the events $\xi(t) \in (x',\, x'+dx']$ are mutually exclusive, then the probability of the event under consideration is

$$\int_{x_0}^{x} P_x(x',\, t)\left(1 - \frac{\Lambda(x')dt}{\tau}[x\,|\,x']\right)dx'.$$

Thus,

$$\begin{aligned}
P(x,\, t+dt) = &\quad P(x_0,\, t)\left(1 - \frac{\Lambda(x_0)dt}{\tau}[x\,|\,x_0]\right) + \int_{x_0}^{x} P_x(x',\, t)\left(1 - \frac{\Lambda(x')dt}{\tau}[x\,|\,x']\right)dx' \\
= &\quad -P(x_0,\, t)\,\frac{\Lambda(x_0)dt}{\tau}[x\,|\,x_0] + P(x_0,\, t) + \int_{x_0}^{x} P_x(x',\, t)dx' \\
&\quad -\int_{x_0}^{x} P_x(x',\, t)\frac{\Lambda(x')dt}{\tau}[x\,|\,x']dx' \\
= &\quad -P(x_0,\, t)\,\frac{\Lambda(x_0)dt}{\tau}[x\,|\,x_0] + P(x,\, t) - \int_{x_0}^{x} P_x(x',\, t)\frac{\Lambda(x')dt}{\tau}[x\,|\,x']dx'\,.
\end{aligned} \tag{2.12}$$

From Eq. (2.6), we have $[x\,|\,x'] = [x\,|\,x_0]/[x'\,|x_0]$. Substituting this into the integral above we get

$$\begin{aligned}
P(x,\, t+dt) = -P(x_0,\, t)\,\frac{\Lambda(x_0)dt}{\tau}[x\,|\,x_0] + P(x,\, t) \\
-[x\,|\,x_0]\int_{x_0}^{x}\frac{P_x(x',t)}{[x'\,|\,x_0]}\,\frac{\Lambda(x')dt}{\tau}\,[x\,|\,x']dx'
\end{aligned}$$

Move $P(x,t)$ to the left hand side, divide by dt, and factor out $[x\,|\,x_0]$ to get

$$P_t(x\,,t) = -[x\,|\,x_0]\left(P(x_0,\, t)\frac{\Lambda(x_0)}{\tau} + \int_{x_0}^{x}\frac{P_x(x',t)}{[x'\,|x_0]}\,\frac{\Lambda(x')}{\tau}[x\,|\,x']dx'\right). \tag{2.13}$$

[4] Subscripts stand for partial derivatives. Here, $P_x \equiv \frac{\partial P}{\partial x}$.

Take $\partial/\partial x$ of both sides, use that $\frac{\partial}{\partial x}[x \,|\, x_0\,] = -[x \,|\, x_0\,]\,\frac{U(x)}{r}$ (see Eq. (2.4)), and, finally, use Eq. (2.13) to substitute the expression in parentheses by $-P_t\,(x\,,t)/[x \,|\, x_0\,]$ to obtain

$$P_{tx}\,(x\,,t) + \frac{U(x)}{r} P_t\,(x\,,t) + \frac{\Lambda(x)}{\tau}\,P_x\,(x\,,t) = 0. \tag{2.14}$$

In addition, we have the initial condition $P(x, 0) = 1$, $x \geq x_0$ (at $t = 0$ the crack tip is at x_0) and the boundary condition

$$P(x_0\,,t\,) = \exp\left(-\frac{\Lambda(x_0)}{\tau}\,t\right), \quad t \geq 0 \tag{2.15}$$

(the probability that, by the time t, crack growth has not yet begun, i.e. no initiation at x_0 has occurred; c.f. Eq. (2.11)).

2.6 Illustrative numerical examples

In this section, we consider a (vertical) strip in uniform tension applied far from the crack path. If the remote tension stress is σ and the strip width is a, then [43], [44]

$$G(x) = \frac{2\sigma^2 a}{E}\,\beta \sec^2\beta\,\frac{\tan\beta}{\beta}\left[0.752 + 2.02\,\frac{2}{\pi}\,\beta + 0.37(1-\sin\beta)^3\right]^2, \quad \beta = \frac{\pi x}{2a}, \tag{2.16}$$

where E is the Young's modulus. We also use

$$\Lambda(x) = 2.0\left(\frac{G(x)}{2\gamma^*}\right)^{0.2}, \tag{2.17}$$

$\gamma^* = 4{,}000, \quad \gamma_{\min} = 0, \quad \alpha = 2, \quad r/a = 0.005, \quad \tau = 1, \quad 2\sigma^2 a/E = 32.$

2.6.1 Evolution of crack depth distribution

In the specimen, each crack propagates to the left-to-right with time. So, the crack depth distribution should move to the right with time.

Furthermore, a crack is very unlikely to get arrested much beyond the location where $G = 2\gamma^*$, as G is an increasing function. Thus, one may expect the crack depth distributions to 'bunch up' towards this location, as time increases.

Finally, the probability of failure increases with time. This probability is represented by the atomic (delta-functional) part of the crack depth distribution concentrated at the right end of the specimen, i.e. at $x/a = 1$. Since, at any time, the total probability of the crack depth being between x_0 and a is one, then the area uder the continuous part of the crack depth distribution has to decrease with time. Strictly speaking, there is another atomic prbability cocentrated at $x = x_0$, but this one rapidly (exponentially) decreases with time and can be ignored.

Figure 2.2 below shows evolution of crack depth distribution with time. This evolution conforms to the above expectations.

2.6.2 Distribution of time to failure

In this sub-section, we illustrate the 'effect of microstructure', namely of the value of r, on a macroscopically observable distribution, namely that of

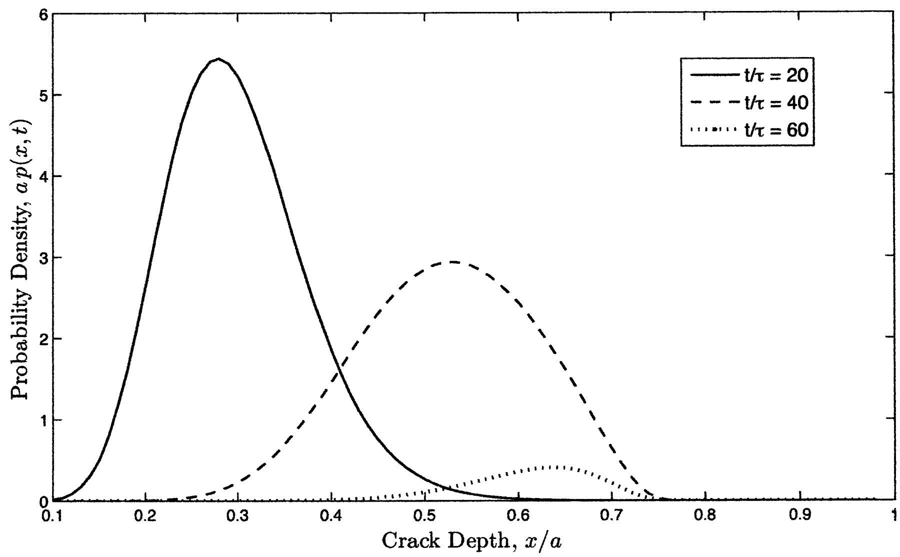


Figure 2.2. Probability densities of crack depth at various times. Not shown are the two atomic probabilities, one at $x = x_0$ and another at $x = a$, $P(x_0, t)$ and $1 - P(a, t)$, respectively.

time to failure. By *time to failure* we mean the (random) time it takes for the crack to reach the end of the specimen, i.e. $x = a$. Evidently, the probability that failure occurs before time t is $P(a, t)$. Figure 2.3 shows the probability density function, $f(t) = P_t\,(a, t)$, for two different values of r.

2.6.3 Average crack speed vs. ERR

In this sub-section, we examine whether the model is capable of imitating the relation, known as Paris law. This relation – between the average crack speed and the amplitude of the 'stress intensity factor' (see below) – is commonly observed under fatigue conditions.

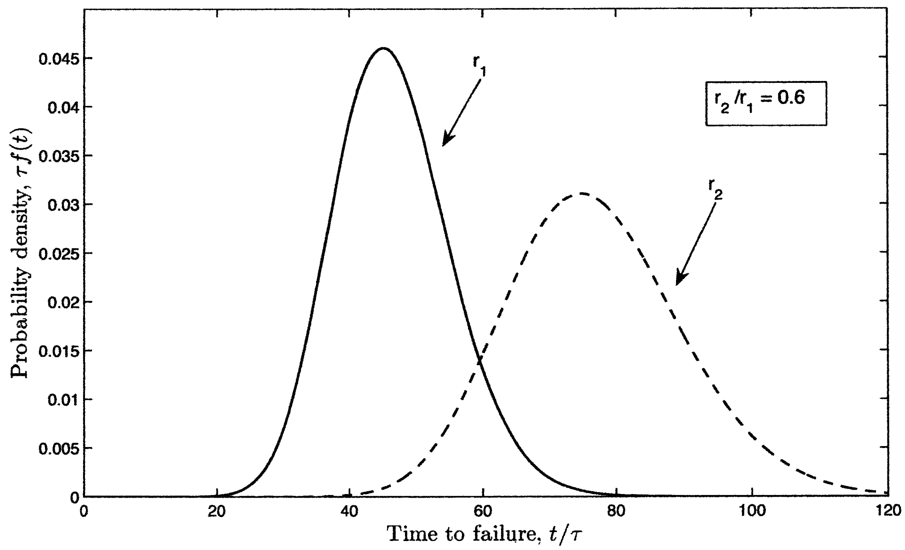


Figure 2.3. Probability density $f(t)$ of time to failure for two values of r: $r_1/a = 0.05$ and $r_2/a = 0.03$.

Notice that, in the scope of the model, if one considers any realization of a (random) growing crack, then the speed of the crack tip takes one of two values, zero or infinity. Thus, the notion of instantaneous crack speed, $v(x)$, is meaningless 'realization-wise', and so the average crack speed, $\hat{v}(x)$, cannot be defined as a straightforward average of $v(x)$.

Let us define *average crack propagation speed,* $\hat{v}(x)$, $x_0 < x < a$, as follows.[5] Let $T(x)$ denote the (random) time at which the crack tip either arrives at or flies by x, $x_0 < x < a$ and let $\hat{T}(x)$ denote the average value of $T(x)$.

[5]As a motivation, consider the case of a deterministic particle moving along the x-axis with its coordinate $x = x(t)$ smoothly increasing with t. Let $t(x)$ denote the time, at which the particle reaches x, and let $v(x)$ denote the velocity of the particle when the particle is at x. Then, $t(x)$ being the inverse of $x(t)$, one has $v(x) = x'(t(x)) = 1/t'(x)$.

Evidently, the distribution function for $T(x)$ is given by

$$F_{T(x)}(t) = \text{Prob}\{T(x) \leq t\} = \text{Prob}\{\text{the crack tip at } x \text{ at } t \text{ or earlier}\}$$
$$= \text{Prob}\{\text{at } t, \text{ the crack tip is at some } x' > x\} = 1 - P(x,t). \quad (2.18)$$

Therefore,

$$\hat{T}(x) = \int_0^\infty t\, dF_{\hat{T}(x)}(t) = -\int_0^\infty t\, P_t\,(x,t)dt. \quad (2.19)$$

Finally, define

$$\hat{v}(x) = \frac{1}{\hat{T}'(x)}. \quad (2.20)$$

The second main character in the Paris law, the first one being $\hat{v}(x)$, is "stress intensity factor", K_I. This is one of the three coefficients that enter asymptotic expression for the stress field around a crack tip (the other two, K_{II} and K_{III}, equal zero in the case under consideration). The definition(s) can be found, for example, in [43]. In our situation $K_I = \sqrt{EG}$, where $G = G(x)$ is given by Eq. (2.16). For cyclic loading, let $K_{\max}$ and $K_{\min}$ denote the K_I-values that correspond to the highest and lowest values of the applied cyclic load, respectively (both $K_{\max}$ and $K_{\min}$ depend on the crack length). Denote $\Delta K_I = K_{\max} - K_{\min}$.

Paris law is the following observation, statistical in nature. Under cycling loading, crack speed v is proportional to a power of the amplitude of the stress intensity factor ΔK_I over a wide range of the values of ΔK_I.[6] It is common to refer to the v-vs.-ΔK_I relation as 'linear in log-log scale'. Also of interest is

[6] In certain situations, it was found that restating Paris Law in terms of ERR is actually beneficial [45].

that the commonly observed dependence of $\ln \hat{v}$ on $\ln(\Delta K_I)$ is 'steeper' outside of the (approximately) linear range.

In Fig. 2.4, we exhibit the relation between $\ln \hat{v}$ and $\ln \Delta(K_I)$ predicted by the model (after finding $\hat{v} = \hat{v}(x)$ and excluding x from $\hat{v}(x)$ and $\Delta(K_I(x))$; the expression for $\Delta(K_I(x))$ is based on Eq. (2.16)).

We find it encouraging that Fig. 2.4 is qualitatively similar to a typically observed (sigmoidal) relationship between the crack speed and the amplitude of the stress intensity factor.

It is desirable to see if the model can be made to fit experimental Paris Law type data. I was able to find comprehensive data for metals only, for which the mechanism of fracture is far from brittle. Moreover, the experiments did not report enough for me to be able to determine all of the parameters of the model. So, the following is a very crude attempt to match model generated $\log(\hat{v})$ vs. $\log(\Delta K_I)$ plot to the experimental data.

The data is taken from [49]. In the experiments described therein, compact tension specimens made of AISI 1018 steel were fatigued. The crack length was recorded as a function of the number of cycles, $x = x(N)$. This yielded dx/dN and its dependence on x. On the other hand stress intensity factor $K_I(x)$ for compact tension specimen is known (for example, see Table 2.4 in [50]),

$$K_I = \frac{P}{b\sqrt{a}} f\left(\frac{x}{a}\right),$$

$$f\left(\frac{x}{a}\right) = \frac{2+\frac{x}{a}}{(1-\frac{x}{a})^{3/2}}\left[0.886 + 4.64\left(\frac{x}{a}\right) - 13.32\left(\frac{x}{a}\right)^2 + 14.72\left(\frac{x}{a}\right)^3 - 5.6\left(\frac{x}{a}\right)^4\right],$$

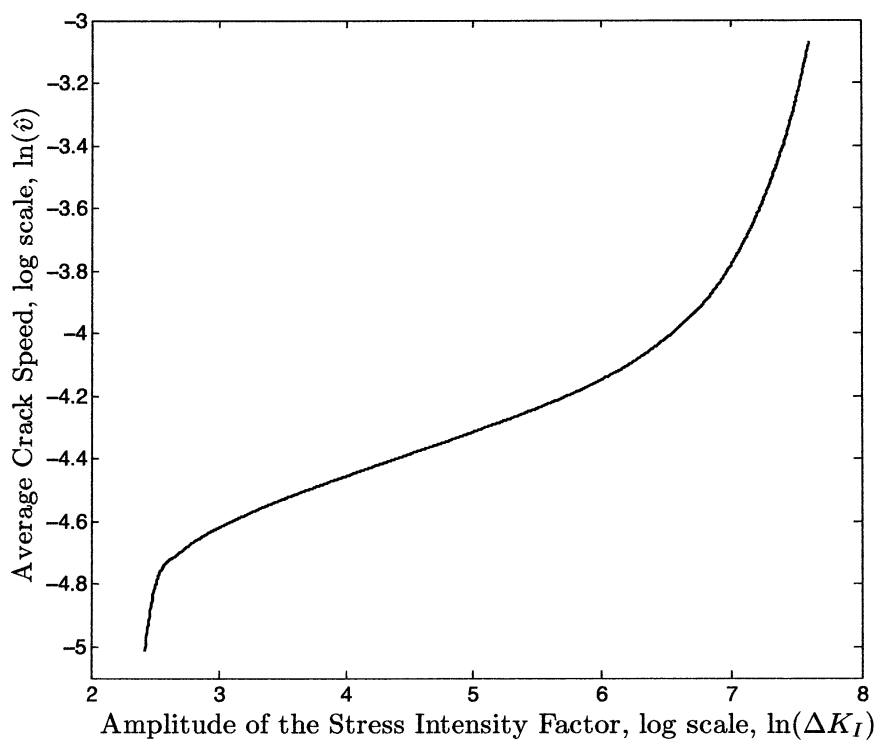


Figure 2.4. Dependence of $\ln(\hat{v})$ on $\ln(\Delta K_I)$ predicted by the model.

where P is the applied load, b is the specimen thickness and a is the specimen width. The values in [49] are $P = 4000\,\mathrm{N}$, $b = 6.35\,\mathrm{mm}$, $a = 63.5\,\mathrm{mm}$.

Having the known experimental dependence of dx/dN on x and the theoretical dependence of ΔK on x, one eliminates x and gets the dependence of dx/dN on ΔK. This dependence is represented by circles in Fig. 2.5 (see Fig.1 in [49]). The solid line in Fig.2.5 represents 'prediction' of the model.

$\gamma_{\min}$ is the smallest value that the fracture energy can have locally; in the absence of experimental data, the simplest choice is $\gamma_{\min} = 0$ (which corresponds to the presence of micro-cracks, voids or, say, very weak inter-grain boundaries), so I used $\gamma_{\min} = 0$.

γ^* is the average value of the fracture energy for the material; that is the kind of value one finds in literature. For example, γ^* is about 1 $\frac{\mathrm{J}}{\mathrm{m}^2}$ for glass [51], 100 $\frac{\mathrm{J}}{\mathrm{m}^2}$ for concrete. For ductile materials such as steel, the plastic dissipation dominates and γ^* is about 500 $\frac{\mathrm{J}}{\mathrm{m}^2}$.

Unfortunately, my code is very sensitive to the values of γ^*. I started my simulation towards experimental results (Fig. 2.5) with $\gamma^* = 600$ $\frac{\mathrm{J}}{\mathrm{m}^2}$ but with the given load (and stress intensity factor range) the process of the crack growth did not develop. I reduced γ^* step-by-step to 250 $\frac{\mathrm{J}}{\mathrm{m}^2}$.

α, known as "shape parameter", affects the shape of the Weibull distribution, e.g. its skewness, and has no clear physical meaning (the only way I understand how to get it is from curve fitting).

As a random field, $\gamma(x)$ is also characterized by a correlation distance r (the values of γ are independent at distanced greater than r). Parameter r

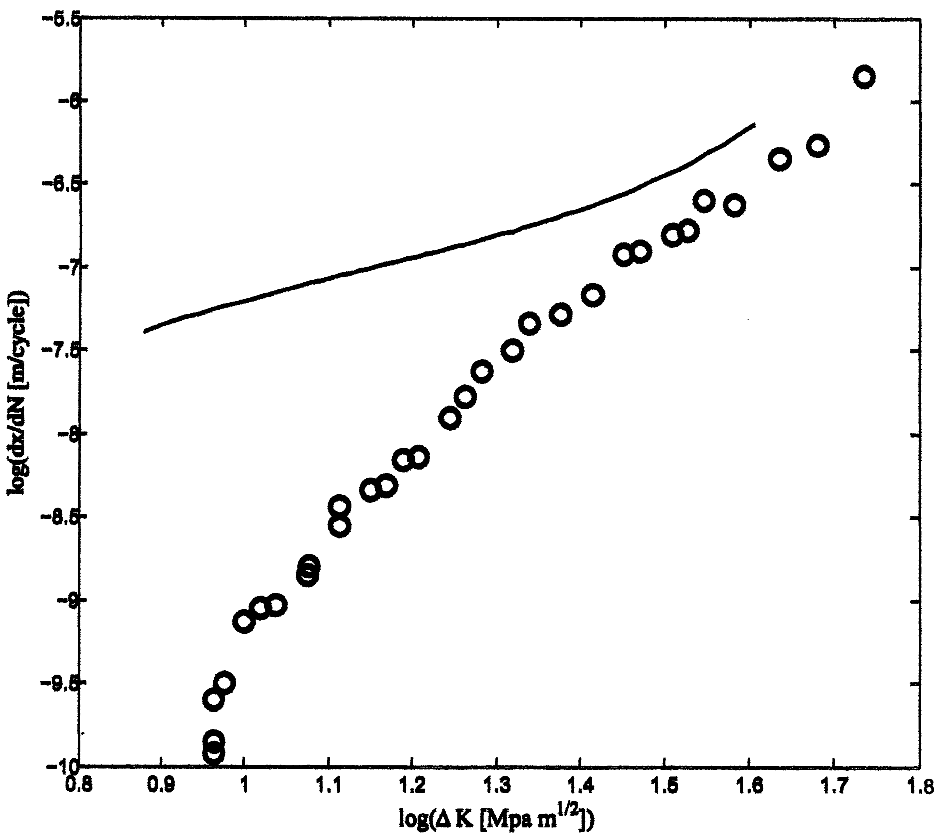


Figure 2.5. Dependence of $\log(dx/dN)$ vs $\log(\Delta K_I)$. Experimental data is from [49].

has to be inferred either from microstructure of the material (several times the grain size) or from morphology of fracture surfaces. I do not think there is full clarity on that issue. I used $r/a = 0.003$, which is not that realistic for metals. The numerical scheme is also sensitive to r and I could not reduce r very much because of numerical instability.

Not surprisingly, the agreement between the model and the experimental data is very poor. However, I would like to point out two qualitatively reasonable features.

(a) The values of average crack speed are predicted for a purely brittle material (fracture energy = energy to create new crack surface), whereas in steel energy is also pumped into a process zone around the crack tip. Thus, for the same values of the stress intensity factor, one should expect faster crack growth in a brittle material, and this is what happens in Fig. 2.5.

(b) As the crack in a steel specimen grows, the relative-to-crack-length size of the process zone decreases, thus making crack formation appear 'more brittle'. So, one may expect the predicted values of crack speed to come closer to the experimental values, which does happen in Fig. 2.5.

Overall, the "agreement" in Fig. 2.5 is "qualitative", at best. At least partially, we attribute this to data coming from not the type of material the model attemts to desribe and from the inability, at present, to determine experimentally the parameters of the γ-field for the material under investigation.

CONCLUSIONS AND FUTURE WORK

The proposed model has two aspects: (i) instantaneous formation of a random crack and (ii) slow crack growth consisting of a sequence of jumps of random lengths and arrests of random durations. The first aspect is developed for 'almost rectilinear' cracks, and, under simplest assumptions, it leads to a parabolic PDE [1]. The second aspect is developed above for a rectilinear crack growth and, under simplest assumptions, leads to a hyperbolic PDE, as outlined in Sec. 2.5.

It would be desirable to leave behind the simplifying, highly restrictive, assumptions that allowed one to derive pretty equations (for CP and $P(x,t)$), so as to make accessible more realistic problems. To do so one needs to decide on the nature of the random crack trajectories, i.e. choosing the space Ω of random paths that a crack may potentially follow. In one of such choices, Ω consists of partially smoothed Brownian paths (their fractional integrals) [46].

Another possibility is to modify the approach to random crack formation found in [47] (from fragmentation to a single crack scenario). One may then proceed numerically by generating a finite ensemble of crack paths, Ω_N, find and tabulate G-values along each path ω from Ω_N, and evaluate and tabulate CP (given by a formula of the type of Eq. (2.5)).

Finally, one would compute $P(x,t)$ (more generally, $P(\vec{r},t)$) by solving an integral-differential equation similar to Eq. (2.13). Finally, one may employ

another minimal value distribution in place of Weibull, Eq. (2.1). Since the range of values of γ is bounded, the distribution introduced in [48] may be more adequate.

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