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Narrative to make the Banach-Tarski Paradox more Approachable

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Narrative to make the Banach-Tarski Paradox more approachable

by

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1 Introduction

In 1924 Stefan Banach and Alfred Tarski published a paper called "Sur la décomposition des ensembles de points en parties respectivement congruentes," in which they introduced the Banach-Tarski Paradox. This paradox states that in the 3 dimensional Euclidean space a solid sphere can be decomposed and rearranged to form two copies of the original sphere. This creates a paradox because intuition tells you that a rotation should not change the volume but due to how the parts are defined the volumes are not well defined.

The purpose of this paper is to take this established proof and add narrative that makes it more approachable for undergraduate students. This paradox is well-known in the field of algebraic geometry and it would benefit students to read through if they are interested in further study in mathematics.

Section 2 covers the relevant background information necessary for the proof. Section 3 will follow along with the proof published by Grzegorz Tomkowicz and Stan Wagon in "The Banach–Tarski Paradox". This section will add details that are assumed to the existing proofs and fill in a few gaps. The goal is to allow students to understand all components of this proof without having to do any outside reading.

2 Background Information

This section states definitions and theorems that will be critical for the proof. The main proof will begin in section 3. This section is intended to fill in gaps in knowledge so that notation and references can be understood.

2.1 Axiom of Choice

Axiom of Choice:

When given a group of sets you can form a new set by choosing one element from each set.

Example:

Let $S = \{S_1, ..., S_n\}$ where S_i is a set for all $i = 1, ..., n$. In other words S is a set of n sets.

By the axiom of choice there exists a new set $S' = \{s_1, ..., s_n\}$ where $s_i \in S_i$ for all $i = 1, .., n$

The example above uses a finite set of sets, but this also applies to infinite sets which is what enables us to prove paradoxes like the Banach-Tarski Paradox.

2.2 Groups

Definition 1. *We call a set G a* ***group*** *under operation $* : G \times G \to G$ if the following hold $\forall a, b, c \in G$,*

1. *[Associativity]:* $(a * b) * c = a * (b * c)$
2. *[Identity]:* $\exists e \in G$ *such that* $a * e = e * a = a$
3. *[Inverse]:* $\exists a^{-1} \in G$ *such that* $a * a^{-1} = e = a^{-1} * a$

Definition 2. *We call a set H a* ***subgroup*** *of a group G if $H \subset G$ and H is also a group.*

Lemma 1. *To show a subset H of a group G is a subgroup of G it is sufficient to show $\forall a \in G$,*

1. *[Identity]:* $\exists e \in H$ *such that* $a * e = e * a = a$
2. *[Inverse]:* $\exists a^{-1} \in H$ *such that* $a * a^{-1} = e = a^{-1} * a$

Proof:

Assume:

1. G is a group
2. $H \subset G$

3. [Identity]: $\exists e \in H$ such that $a * e = e * a = a$
4. [Inverse]: $\exists a^{-1} \in H$ such that $a * a^{-1} = e = a^{-1} * a$

Associativity holds for all elements of G and all elements of H are in G

Thus Associativity holds for H.

Therefore by *Definition* 2 H is a subgroup of G.

Definition 3. *A group G is* ***partitioned*** *by $H_1, .., H_n \subset G$ if,*

1. $H_i \cap H_j = \varnothing \ \forall i \neq j$
2. $\bigcup_{i=1}^{n} H_i = G$

2.3 Functions

Let A and B be sets and $f : A \to B$ be a function,

1. f is **surjective** if $\forall b \in B \ \exists a \in A$ such that $f(a) = b$
2. f is **injective** if $(\forall a_1, a_2 \in B)(a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2))$
3. f is **bijective** if f is injective and surjective.

If A and B are groups,

4. f is a **homomorphism** if $(\forall a_1, a_2 \in A)(f(a_1 * a_2) = f(a_1) * f(a_2))$
5. f is an **ismorphism** if it is an injective homomorphism
6. A and B are **isomorphic** if there exists a function $g : A \to B$ such that g is a surjective isomorphism.

Definition 4. *A set A is* ***countable*** *if there exists an injective function $f : A \to \mathbb{N}$, where $\mathbb{N}$ is the natural numbers.*

Lemma 2. *If for a set A there exists a function $g : A \to \mathbb{N} \times \mathbb{N}$, then A is countable.*

Proof:

Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ where,

$$f(1,1) = 1,\ f(1,2) = 2,\ f(2,1) = 3,\ f(1,3) = 4,\ f(2,2) = 5,\ f(3,1) = 6, ...$$

Thus $g \circ f : A \to \mathbb{N} \times \mathbb{N}$ is injective

Therefore A is countable

Lemma 3. *All homomorphic functions $\phi : G \to H$ map the identity of G to the identity of H.*

Proof: Let e_G and e_H be the identity elements of G and H respectively,

Let $a \in G$,

then $\phi(a) = \phi(ae_G) = \phi(a)\phi(e_G)$

$\Rightarrow e_H = \phi(e_G)$

2.4 Spaces

Vector Spaces:

A group X is a **vector space** where,

a. $\forall x \in X$ x is called a vector.

b. The group operation of X $+$ called vector addition.

c. A field $\mathbb{F}$, like $\mathbb{R}^n$ or $\mathbb{C}^n$, acts on X under scalar multiplication where $\forall \alpha \in \mathbb{F}$ and $\forall x \in X$ $\alpha x \in X$.

d. $\forall a \in \mathbb{F}$ a is called a scalar.

if it satisfies the following:

1. $x + y = y + x$ $\forall a, b \in X$
2. $\alpha(x + y) = \alpha x + \alpha y$ $\forall \alpha \in \mathbb{F}$ and $\forall x, y \in X$
3. $(\alpha + \beta)x = \alpha x + \beta x$ $\forall \alpha, \beta \in \mathbb{F}$ and $\forall x \in X$
4. $\alpha(\beta x) = (\alpha\beta)x$ $\forall \alpha, \beta \in \mathbb{F}$ and $\forall x \in X$
5. $\exists e \in X$ such that $ex = xe = x$ $\forall x \in X$

Definition 5. *For a vector space X a set of vectors $A = \{x_1, ..., x_n : x_i \in X\}$ are **linearly independent** if $a_1x_1 + \cdots + a_nx_n = 0$ where $a_i \in \mathbb{F}$ for $i = 1, ..., n$ if and only if $a_i = 0$ for $i = 1, ..., n$.*

Definition 6. *A set e is a **basis** of a vector space X if $e = \{x_1, ..., x_n : x_i \in X\}$ where the elements are linearly independent and $\forall x \in X$ $x = a_1x_1 + ... + a_nx_n$ where $a_i \in \mathbb{F}$ for $i = 1, ..., n$.*

Eigenvalues and Eigenvectors:

For a vector space X and $x \in X$ a scalar a is an **eigenvalue** of x if $\exists \lambda \in X$ such that $ax = \lambda x$.

Similarly a vector λ is an **eigenvector** of x if $\exists \alpha$ that is a scalar such that $ax = \lambda x$.

Metric Spaces:

A group X is a **metric space** if there exists a function $d : X \times X \to [0, \infty)$, called a metric, such that for any $x, y, z \in X$ the following holds:

1. $d(x, y) = 0$ iff $x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, z) \leqslant d(x, y) + d(y, z)$

Definition 7. $||x|| = d(x, 0)\ \forall x \in X$, *where* X *is a metric space with metric* d.

note: d measures distance between elements of the set X and $||.||$ measures the distance of an element and 0.

Inner Product Space:

A vector space X, is an **inner product space** if $\exists \langle\rangle : X \times X \to \mathbb{F}$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ is called an **inner product** if it satisfies the following:

1. $\langle x, x\rangle \geqslant 0\ \forall x \in X$
2. $\langle x, x\rangle = 0 \Rightarrow x = 0$
3. $\langle \alpha x + y, z\rangle = \alpha\langle x, y\rangle + \langle y, z\rangle\ \forall x, y, z \in X$ and $\alpha \in \mathbb{F}$
4. $\langle x, y\rangle = \overline{\langle y, x\rangle}\ \forall x, y \in X$

Lemma 4. *An inner product space is a vector space.*

Proof:

Let $d(x, y) = \sqrt{\langle x - y, x - y\rangle}$, then

1. $d(x, y) = 0 \iff \sqrt{\langle x - y, x - y\rangle} = 0 \Rightarrow x - y = 0 \Rightarrow x = y$
2. $d(x, y) = \sqrt{\langle x - y, x - y\rangle} = \sqrt{\langle x, x - y\rangle - \langle y, x - y\rangle} = \sqrt{\overline{\langle x - y, x\rangle} - \overline{\langle x - y, y\rangle}} = \sqrt{\overline{\langle x, x\rangle} - \overline{\langle y, x\rangle} - \overline{\langle x, -y\rangle} + \overline{\langle y, y\rangle}} = \sqrt{-\overline{\langle y - x, x\rangle} + \overline{\langle y - x, y\rangle}} = \sqrt{\langle y - x, y - x\rangle} = d(y, x)$
3. $d(x, z) = \sqrt{\langle x - z, x - z\rangle} = \sqrt{\langle x - y + y - z, x - z\rangle} = \sqrt{\langle x - y, x - z\rangle + \langle y - z, x - z\rangle} \leqslant \sqrt{\langle x - y, x - z\rangle} + \sqrt{\langle y - z, x - z\rangle} = d(x, y) + d(y + z)$

Therefore d is a metric and X is a metric space.

Definition 8. *For an inner product space* X, $x_1, x_2 \in X$ *are* ***orthogonal*** *if* $\langle x_1, x_2\rangle = 0$

Definition 9. *For a inner product space* X, *a set of vectors* $\{x_1, ..., x_2\}$ *is* ***orthonormal*** *if all vectors are orthogonal to each other and* $||x_i|| = 1$ *for* $i = 1, ..., n$

Definition 10. *An element* x *of an inner product space* X *is* ***normed vector*** *if* $||x|| = 1$. *A vector* $x' \in X$ *can be normalized by* $\frac{x'}{||x||}$.

Lemma 5. $\mathbb{R}^3$ *is an inner product space with the Euclidean inner product where*

$$\text{for } x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \in \mathbb{R}^3\ \langle x, y\rangle = x_1y_1 + x_2y_2 + x_3y_3,\ ||x|| = \sqrt{\langle x, x\rangle} = x_1^2 + x_2^2 + x_3^2,$$
$$\text{and } d(x, y) = \sqrt{\langle x - y, x - y\rangle} = (x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2$$

Definition 11. $\mathbb{S}^2 = \{x \in \mathbb{R}^3 : ||x|| = r\}$ *for some* $r > 0$ *where* $||.||$ *is defined in Lemma 5.*

Definition 12. $B^3 = \{x \in \mathbb{R}^3 : ||x|| \leqslant r\}$ *for some* $r > 0$ *where* $||.||$ *is defined in Lemma 5.*

Matrix Representation:

The **matrix representation** of a linear map L on a vector space V, with an ordered basis α, into another vector space W, with an ordered basis β is a matrix where each column maps an element α into elements of β as described by the linear mapping.

Also, the elements are ordered meaning the kth column maps the kth element of α into elements of β.

Let v_k be the kth element of basis α and $\{w_1, ..., w_n\} = \beta$, then $L(v_k) = a_1w_1 + ...a_nw_n$ which makes the kth column $\begin{bmatrix} a_1 \\ ... \\ a_n \end{bmatrix}$.

2.5 Equivalence Relation

Relation:

A **relation** is simply a subset of the cross product of two sets, denoted by $R \subset X \times X$, where $(x, y) \in R$ is denoted by xRy.

Equivalence Relation:

A relation $R \subset X \times X$, is an **equivalence relation** on X if the following are true:

1. [Reflexive]: $xRx\ \forall x \in X$
2. [Symmetric]: $xRy \Rightarrow yRx\ \forall x, y \in X$
3. [Transitive]: xRy and $yRz \Rightarrow xRz\ \forall x, y, z \in X$

Equivalence Class:

For an equivalence relation $R \subset X \times X$ and $x \in X$, the **equivalence class** of x is denoted by

$$[x] = \{x' \in X : xRx'\}$$

Theorem 2.1. *Equivalence classes partition the set, meaning for an equivalence relation* $R \subset X \times X$ *every element of* X *is an element of an equivalence class and* $\forall a, b \in X$ $a \in [b] \iff [a] = [b]$

Proof:

Let $x \in X$, then $x \in [x]$.

Thus $\forall x \in X$, x is in an equivalence class.

Let $a, b \in X$,

assume $a \in [b]$

bRa

if $x \in [a]$, then xRa

$\Rightarrow xRb$ by definition of an equivalence relation

$\Rightarrow x \in [b]$

$\Rightarrow [a] \subset [b]$

if $x \in [b]$, then xRb

$\Rightarrow xRa$ by definition of an equivalence relation

$\Rightarrow x \in [a]$

$\Rightarrow [b] \subset [a]$

Thus $[a] = [b]$

assume $[a] = [b]$

$aRa \Rightarrow a \in [a]$

Thus $a \in [b]$

Therefore $a \in [b] \iff [a] = [b]$.

2.6 Free Products

Words:

Suppose:

1. $G = \{G_\alpha\}_{\alpha \in A}$ is a family of groups and 1_α denotes the identity element of the group G_α
2. $w = ((w_1, \alpha_1), \cdots, (w_n, \alpha_n))$ where $w_i \in G_{\alpha_i}$ is a **word** in G with length n.
3. $W(G)$ is the set of all words in G.
4. $w, w' \in W(G)$

Concatenation of w and w' is:

$ww' = ((w_1, \alpha_1), \cdots, (w_n, \alpha_n), (w'_1, \alpha'_1), \cdots, (w'_{n'}, \alpha'_{n'}))$

The **elementary operations** on words are:

1. An **elementary reduction**, this occurs when

   i) $\alpha_i = \alpha_{i+1}$ and is

   $(w_1, \alpha_1), \cdots, (w_i, \alpha_i), (w_{i+1}, \alpha_{i+1}), \cdots, (w_n, \alpha_n) = (w_1, \alpha_1), \cdots, (w_i w_{i+1}, \alpha_{i+1}), \cdots, (w_n, \alpha_n)$

   ii) $w_i = 1_{\alpha_i}$ and is

   $(w_1, \alpha_1), \cdots, (w_{i-1}, \alpha_{i-1}), (w_i, \alpha_i), (w_{i+1}, \alpha_{i+1}), \cdots, (w_n, \alpha_n)$

   $= (w_1, \alpha_1), \cdots, (w_{i-1}, \alpha_{i-1}), (w_{i+1}, \alpha_{i+1}), \cdots, (w_n, \alpha_n)$

2. The inverse of any elementary reduction.
3. The identity operator.

Definition 13. *A word is **fully reduced** if no elementary reductions can be performed.*

Lemma 6. *For every word the reduced word is unique.*

Proof:

Let $w = ((w_1, \alpha_1), \cdots, (w_n, \alpha_n))$ be a word in G with length n.

Since Since the length is finite and each elementary reduction shortens the word, it will either reduce to the empty word or a fully reduced word

Also, the two elementary reductions are independent they can be repeated in any order until $\alpha_i \neq \alpha_{i+1}$ and $w_j \neq 1_{\alpha_j}$ for all $i, j \in \{1, ..., n\}$ and result in the same word that can be no further reduced.

Definition 14. $W(a) = \{$*all words such that their unique fully reduced word begin with* $a^n : n \in \mathbb{N}\}$.

Free Product:

The **free product** of G, is denoted by $\underset{\alpha \in A}{\ast} G_\alpha$ and is the set of equivalence classes of $W(G)$, the set of all words in G, extended by concatenation. This can be described as w and w' share the same reduced word.

In this case the equivalence class is defined by wRw' if w' can be obtained from w with just elementary operations, shown to be an equivalence class in Lemma 7.

Let $w, w' \in \underset{\alpha \in A}{\ast} G_\alpha$, then $w = [((w_1, \alpha_1), \cdots, (w_n, \alpha_n))]$ and $w' = [((w'_1, \alpha'_1), \cdots, (w'_m, \alpha'_m))]$

In Theorem 2.2, it is shown that $\underset{\alpha \in A}{*} G_\alpha$ is a group with the group operation $*$, where $*$ is defined by $[w]*[w'] = [((w_1, \alpha_1), \cdots, (w_n, \alpha_n))] * [((w'_1, \alpha'_1), \cdots, (w'_m, \alpha'_m))] = [((w_1, \alpha_1), \cdots, (w_n, \alpha_n), (w'_1, \alpha'_1), \cdots, (w'_m, \alpha'_m))] = [ww']$.

Lemma 7. *R as defined above is an equivalence relation.*

Proof:

Let $w, w', w'' \in \underset{\alpha \in A}{*} G_\alpha$,.

1. [Reflexive] $w = w$ so wRw, since the identity is an elementary operation.
2. [Symmetric] $wRw' \Rightarrow w'$ can be obtained from w by elementary operations,

   Since the inverse of all elementary operations are elementary operations w can be obtained from w' by elementary operations,

   therefore $w'Rw$.
3. [Transitive] wRw' and $w'Rw'' \Rightarrow w'$ can be obtained from w by elementary operations and w'' can be obtained from w' by elementary operations,

   then w'' can be obtained from w by elementary operations

   therefore wRw''

Thus R is an equivalence relation.

Theorem 2.2. *The free product is a group with the operation $*$, where $*$ is defined by $[w] * [w'] = [ww']$.*

Proof:

$*$ is well defined:

let $w_1, w'_1 \in [w_1]$ and $w_2 \in [w_2]$,

note: need to show $[w_1] * [w_2] = [w'_1] * [w_2]$

$\Rightarrow w_1 = w_2$ by elementary operations

$\Rightarrow w_1 w_2 = w'_1 w_2$ under elementary operations

$\Rightarrow w_1 w_2 \in [w'_1 w_2]$

$\Rightarrow [w_1 w_2] = [w'_1 w_2]$ by Theorem 2.1

thus $[w_1] * [w_2] = [w_1 w_2] = [w'_1 w_2] = [w'_1] * [w_2]$

Therefore $*$ is well-defined.

Suppose $\underset{\alpha \in A}{*} G_\alpha$ the free product of G and let $[w], [w'], [w"] \in \underset{\alpha \in A}{*} G_\alpha$, then $[w] = [((w_1, \alpha_1), \cdots, (w_n, \alpha_n))]$, $[w'] = [((w'_1, \alpha'_1), \cdots, (w'_m, \alpha'_m))]$, and $[w''] = [((w''_1, \alpha''_1), \cdots, (w''_p, \alpha''_p))]$

1. [Closure] $[w]*[w'] = [ww'] = [((w_1, \alpha_1), \cdots, (w_n, \alpha_n), (w'_1, \alpha'_1), \cdots, (w'_m, \alpha'_m))] \in \underset{\alpha \in A}{*} G_\alpha$

2. [Associativity] $([w] * [w']) * [w''] = [ww'] * [w''] = [ww'w''] = [w] * [w'w''] = [w] * ([w'] * [w''])$

3. [Identity] let $e = [()] \in \mathop{*}_{\alpha \in A} G_\alpha$

$[w] * e = [w] * [()] = [w()] = [w] = [()w] = [()] * [w] = e * [w]$

4. [Inverse] $[w] * [((w_n^{-1}, \alpha_n), \cdots, (w_1^{-1}, \alpha_1))] = [((w_1, \alpha_1), \cdots, (w_n, \alpha_n), (w_n^{-1}, \alpha_n), \cdots, (w_1^{-1}, \alpha_1))] = [()] = e = [()] = [(w_n^{-1}, \alpha_n), \cdots, (w_1^{-1}, \alpha_1), (w_1, \alpha_1), \cdots, (w_n, \alpha_n)] = [((w_n^{-1}, \alpha_n), \cdots, (w_1^{-1}, \alpha_1))] * [w]$

then $[w]^{-1} = [((w_n^{-1}, \alpha_n), \cdots, (w_1^{-1}, \alpha_1))] \in \mathop{*}_{\alpha \in A} G_\alpha$

Inclusion Map:

For a set $S \subset G$ define the **inclusion map** $\phi : S \to G$ by $\phi(s) = s \; \forall s \in S$.

Natural Injection:

Define the **natural injection** $\iota_\alpha : G_\alpha \to \mathop{*}_{\alpha \in A} G_\alpha$ by $\iota_\alpha(g) = [((g, \alpha))]$

Note: ι_α embeds G_α in $\mathop{*}_{\alpha \in A} G_\alpha$

Theorem 2.3. *Suppose:*

1. $G = \{G_\alpha\}_{\alpha \in A}$ *represents a family of groups*
2. $\mathop{*}_{\alpha \in A} G_\alpha$ *represents the free product of* G
3. H *is a group*
4. $\phi_\alpha : G_\alpha \to H$ *is a collection of homomorphisms*
5. $\iota_\alpha : G_\alpha \to \mathop{*}_{\alpha \in A} G_\alpha$ *is the natural injection*

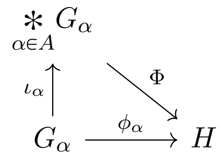
Define $\Phi : \mathop{*}_{\alpha \in A} G_\alpha \to H$ *by*

$$\text{for } g_i \in G_{\alpha_i} \;\; \Phi([(g_1, \alpha_1), \cdots, (g_n, \alpha_n)]) = \phi_{\alpha_1}(g_1) \cdots \phi_{\alpha_n}(g_n)$$

Note: The members of the free product are equivalence classes.

Conclusions:

1. Φ *is well defined*
2. Φ *is a homomorphism*
3. Φ *is the unique homomorphism that goes from* $\mathop{*}_{\alpha \in A} G_\alpha$ *to* H *that ensures commutativity of*



for each $\alpha \in A$*.*

Proof(1):

To show that Φ is well defined it needs to be shown that Φ maps two equal elements of $\ast_{\alpha\in A} G_\alpha$ map to the same element of H. Since equal elements are equal by elementary operations it must be shown for all elementary operations.

Let $g_i \in G_{\alpha_i}$,

assume $\alpha_k = \alpha_{k+1}$

$$\begin{aligned}
&\Phi([(g_1,\alpha_1),\cdots,(g_k,\alpha_k),(g_{k+1},\alpha_{k+1}),\cdots,(g_n,\alpha_n)]) \\
&\quad = \phi_{\alpha_1}(g_1)\cdots\phi_{\alpha_k}(g_k)\phi_{\alpha_{k+1}}(g_{k+1})\cdots\phi_{\alpha_n}(g_n) \\
&\quad = \phi_{\alpha_1}(g_1)\cdots\phi_{\alpha_k}(g_k g_{k+1})\cdots\phi_{\alpha_n}(g_n) \qquad \text{[Equivalent by elementary reduction]} \\
&\quad = \Phi([(g_1,\alpha_1)\cdots(g_k g_{k+1},\alpha_k)\cdots(g_n,\alpha_n)])
\end{aligned}$$

assume $g_k = 1_{\alpha_k}$, where e_{α_k} is the identity element of G_k and let e_H be the identity element of H

$$\begin{aligned}
&\Phi([(g_1,\alpha_1)\cdots(g_{k-1},\alpha_{k-1})(g_k,\alpha_k)(g_{k+1},\alpha_{k+1})\cdots(g_n,\alpha_n)]) \\
&\quad = \Phi([(g_1,\alpha_1)\cdots(g_{k-1},\alpha_{k-1})(e_{\alpha_k},\alpha_k)(g_{k+1},\alpha_{k+1})\cdots(g_n,\alpha_n)]) \\
&\quad =\phi_{\alpha_1}(g_1)\cdots\phi_{\alpha_{k-1}}(g_{k-1})\phi_{\alpha_k}(e_{\alpha_k})\phi_{\alpha_{k+1}}(g_{k+1})\cdots\phi_{\alpha_n}(g_n) \\
&\quad =\phi_{\alpha_1}(g_1)\cdots\phi_{\alpha_{k-1}}(g_{k-1})e_H\phi_{\alpha_{k+1}}(g_{k+1},)\cdots\phi_{\alpha_n}(g_n) \qquad \text{[By Lemma 3]} \\
&\quad =\phi_{\alpha_1}(g_1)\cdots\phi_{\alpha_{k-1}}(g_{k-1})\phi_{\alpha_{k+1}}(g_{k+1})\cdots\phi_{\alpha_n}(g_n) \qquad \text{[Elementary reduction]} \\
&\quad = \Phi([(g_1,\alpha_1)\cdots(g_{k-1},\alpha_{k-1})(g_{k+1},\alpha_{k+1})\cdots(g_n,\alpha_n)])
\end{aligned}$$

These can be repeated for any k so this means that for any elements $[h],[g] \in \ast_{\alpha\in A} G_\alpha$ that are equal by elementary operations $\Phi([h]) = \Phi([g])$.

Thus Φ is well defined.

Proof(2):

Let $[g],[h] \in \ast_{\alpha\in A} G_\alpha$, then

$[g] = [(g_1,\alpha_1),\cdots,(g_n,\alpha_n)]$ for some $g_i \in G_{\alpha_i}$

$[h] = [(h_{n+1},\alpha_{n+1}),\cdots(h_{n+m},\alpha_{n+m})]$ for some $h_i \in G_{\alpha_i}$

Thus,

$$\Phi([g][h]) = \Phi([(g_1,\alpha_1),\cdots,(g_n,\alpha_n)][(h_{n+1},\alpha_{n+1}),\cdots(h_{n+m},\alpha_{n+m})])$$

$= \Phi([(g_1, \alpha_1), \cdots, (g_n, \alpha_n)(h_{n+1}, \alpha_{n+1}), \cdots (h_{n+m}, \alpha_{n+m})])$

$= \phi_{\alpha_1}(g_1) \cdots \phi_{\alpha_n}(g_n)\phi_{\alpha_{n+1}}(h_{n+1}) \cdots \phi_{\alpha_{n+m}}(h_{n+m})$

$= \Phi([(g_1, \alpha_1), \cdots, (g_n, \alpha_n)])\Phi([(h_{n+1}, \alpha_{n+1}), \cdots (h_{n+m}, \alpha_{n+m})])$

$= \Phi([g])\Phi([h])$

Therefore Φ is a homomorphism.

Proof(3):

Suppose Φ' is another homomorphism from $\underset{\alpha \in A}{\ast} G_\alpha$ into H where the figure commutes, then $\Phi' \circ \iota_\alpha = \phi_\alpha \ \forall \alpha \in A$.

Let $h \in G_\alpha$, then

$(\Phi' \circ \iota_\alpha)(h) = \Phi'([h])$

$= \Phi'([(h, \alpha)])$

$= \phi_\alpha(h)$

Let $[g] \in \underset{\alpha \in A}{\ast} G_\alpha$, then

$[g] = [(g_1, \alpha_1), \cdots, (g_n, \alpha_n)]$ for some $g_i \in G_{\alpha_i}$

$\Phi'([g]) = \Phi'(\iota_{\alpha_1}(g_1, \alpha_1), \cdots, \iota_{\alpha_n}(g_n, \alpha_n))$

$\Phi'(\iota_{\alpha_1}(g_1, \alpha_1)) \cdots \Phi'(\iota_{\alpha_n}(g_n, \alpha_n)))$

$= \phi_{\alpha_1}(g_1) \cdots \phi_{\alpha_n}(g_n)$

$= \Phi([g])$

Thus $\Phi' = \Phi$.

2.7 Free Groups:

Suppose:

1. σ represents a mathematical object
2. $F(\sigma) = \{\sigma\} \times \mathbb{Z}$
3. $F(\sigma) \times F(\sigma) \to F(\sigma)$ by $(\sigma, m) \cdot (\sigma, n) = (\sigma, m + n) \ \forall m, n \in \mathbb{Z}$

$(F(\sigma), \cdot)$ is the **free group** generated by σ.

Proof:

Let $x, y, z \in F(\sigma)$, then $x = (\sigma, m)$, $y = (\sigma, n)$, and $z = (\sigma, p)$,

1. [Associativity] $(x \cdot y) \cdot z = ((\sigma, m) \cdot (\sigma, n)) \cdot (\sigma, p) = (\sigma, m+n) \cdot (\sigma, p) = (\sigma, m+n+p) = (\sigma, m) \cdot (\sigma, n+p) = (\sigma, m) \cdot ((\sigma, n) \cdot (\sigma, p)) = x \cdot (y \cdot z)$
2. [Identity] if $e = (\sigma, 0)$,

   then $x \cdot e = (\sigma, m) \cdot (\sigma, 0) = (\sigma, m + 0) = (\sigma, m) = x = (\sigma, m) = (\sigma, 0 + m) =$

$(\sigma, 0) \cdot (\sigma, e) = e \cdot x$

3. [Inverse] let $x^{-1} = (\sigma, -m)$,

   then $x \cdot x^{-1} = (\sigma, m) \cdot (\sigma, -m) = (\sigma, m - m) = (\sigma, 0) = e = (\sigma, 0) = (\sigma, -m + m) = (\sigma, -m) \cdot (\sigma, m) = x^{-1} \cdot x$

Let S represent a set, the free group, derived from the free product, on S is denoted by

$$F(S) = \underset{\sigma \in S}{\ast} F(\sigma)$$

A group G is a free group if $\exists S \subseteq G$ such that $\Phi : F(S) \to G$ generated by the inclusion map, as discussed in Theorem 2.4, $\phi : S \to G$ is a surjective isomorphism.

Natural Injection:

Suppose:

1. S is a set.
2. $F(\sigma)$ represents the free group generated by $\sigma \in S$
3. $F(S)$ is the free group of S

$\iota : S \to F(S)$ given by $\iota(\sigma) = \iota_\sigma((\sigma, 1))_{\sigma \in S}$ is called the **natural injection**.

Theorem 2.4. *Suppose:*

1. S is a set.

2. $F(S)$ denotes the free group of S.

3. $\iota : S \to F(S)$ denotes the natural injection.

Conclusions:

1. For each group H and map $\phi : S \to H$, $\exists$ a unique homomorphism $\Phi : F(S) \to H$ such that

$$\begin{array}{ccc} F(S) & & \\ \iota\uparrow & \searrow^{\Phi} & \\ S & \xrightarrow{\phi} & H \end{array}$$

commutes.

2. Suppose $S \subset H$, then $\forall s_i \in S$ and $\forall z_i \in \mathbb{Z}$, the set of all integers,

$$\Phi(\iota(s_1)^{z_1} \cdots \iota(s_k)^{z_k}) = \phi(s_1)^{z_1} \cdots \phi(s_k)^{z_k}$$

Proof (1):

$\forall \sigma \in S$ define $\phi_\sigma : F(\sigma) \to H$ by

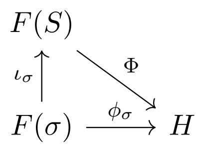
$$\phi_\sigma(\sigma^n) = \phi(\sigma)^n$$

$\phi_\sigma(\sigma^n\sigma^m) = \phi_\sigma(\sigma^{n+m}) = \phi(\sigma)^{n+m} = \phi(\sigma)^n\phi(\sigma)^m = \phi_\sigma(\sigma^n)\phi_\sigma(\sigma^m)$

Therefore ϕ_σ is a homomorphism.

Let $\iota_\sigma : F(\sigma) \to F(S)$ be the free product embedding.

By Theorem 2.3 there exists a unique homomorphism $\Phi : F(S) \to H$ such that



commutes for each $\sigma \in S$.

So,

$$\phi(\sigma) = \phi_\sigma(\sigma^1) = \Phi(\iota_\sigma(\sigma^1)) = \Phi([((\sigma, 1))]) = \Phi(\iota(\sigma)) \ \forall \sigma \in S$$

which means the figure commutes.

Suppose there exists another homomorphism Φ' such that the figure commutes, then

$$(\forall \sigma \in S) \ (\Phi(\iota_\sigma(\sigma)) = \phi_\sigma(\sigma^1) = \phi(\sigma) = \Phi'(\iota_\sigma(\sigma^1)))$$

therefore $\Phi = \Phi'$, which is shown to be unique in Theorem 2.4, so Φ is unique.

Proof (2):

Let $s_i \in S$ and $z_i \in \mathbb{Z} \ \forall i \in \{1, ..., k\}$, then

$$\Phi(\iota(s_1^{z_1}) \cdots \iota(s_k^{z_k})) = \Phi(\iota(s_1^{z_1})) \cdots \Phi(\iota(s_k^{z_k})) = \phi_{s_1}(s_1^{z_1}) \cdots \phi s_k(s_k^{z_k}) = \phi(s_1)^{z_1} \cdots \phi(s_k)^{z_k}$$

Generator:

Suppose $S \subset G$ where G is a group with group operation $*$,

then the subgroup **generated** by S in G, denoted by S, is the intersection of all subgroups of G that contain S.

Lemma 8. *For a group G and set $S \subset G$, $\langle S \rangle = \{s_1^{n_1} \cdots s_k^{n_k} : s_i \in S, i \in \mathbb{Z}\}$*

Proof:

Let H be a subgroup of G such that $\langle S \rangle \subset H$,

Since groups are closed $\{s_1^{n_1} \cdots s_k^{n_k} : s_i \in S, i \in \mathbb{Z}\} \subset H$

Thus $\langle S \rangle = \{s_1^{n_1} \cdots s_k^{n_k} : s_i \in S, i \in \mathbb{Z}\} \subset \langle S \rangle$

Let $s \in \{s_1^{n_1} \cdots s_k^{n_k} : s_i \in S, i \in \mathbb{Z}\}$,

then $s_k^{-n_k} \cdots \cdot_1^{-n_1} \in \{s_1^{n_1} \cdots s_k^{n_k} : s_i \in S, i \in \mathbb{Z}\}$ and $ss_k^{-n_k} \cdots \cdot_1^{-n_1} = e = s_k^{-n_k} \cdots \cdot_1^{-n_1} s$

Also, $e = s_1^{n_1} \cdots s_k^{n_k}$ where $n_i = 0$ for all $i \in \mathbb{Z}$

Thus $\{s_1^{n_1} \cdots s_k^{n_k} : s_i \in S, i \in \mathbb{Z}\}$ is a group containing S.

Therefore $\langle S \rangle \subset \{s_1^{n_1} \cdots s_k^{n_k} : s_i \in S, i \in \mathbb{Z}\}$

Therefore $\langle S \rangle = \{s_1^{n_1} \cdots s_k^{n_k} : s_i \in S, i \in \mathbb{Z}\}$

Lemma 9. *For $S = \{\sigma, \tau\}$, $\langle S \rangle$ is countable.*

Proof:

Let $s \in \langle S \rangle$,

then by Lemma 8 $s = \sigma^{n_1} \tau^{n_2} \cdots \sigma^{n_{i-1}} \tau^{n_i}$

Let $n = \sum_{j=1}^{i} n_j$ and $g : \langle S \rangle : \mathbb{N} \times \mathbb{N}$ by

$g(s) = (n, h_n(s))$ where

$h_n(\sigma^n) = 1$, $h_n(\sigma^{-n}) = 2$,

$h_n(\sigma^{n-1}\tau) = 3$, $h_n(\sigma^{-n+1}\tau) = 4$, $h_n(\sigma^{n-1}\tau^{-1}) = 5$, $h_n(\sigma^{-n+1}\tau^{-1}) = 6$,

$h_n(\sigma^{n-2}\tau^2) = 7, h_n(\sigma^{-n+2}\tau^2) = 8, h_n(\sigma^{n-2}\tau\sigma) = 9, h_n(\sigma^{-n+2}\tau\sigma) = 10, h_n(\sigma^{n-2}\tau\sigma^{-1}) = 11$, $h_n(\sigma^{-n+2}\tau\sigma^{-1}) = 12$, $h_n(\sigma^{n-2}\sigma^2) = 13, ..., h_n(\tau^n) = k - 1$, $h_n(\tau^{-n}) = k$

Thus h_n is injective which implies that g is injective

Therefore by Lemma 2 $\langle S \rangle$ is countable

Definition 15. *The **rank** of a group G is the cardinality, or size, of the smallest set that generates G.*

Free Generator:

$S \subset G$ is a **free generator** if and only if $\Phi : F(S) \to G$ generated by Theorem 2.4 is a surjective isomorphism.

Definition 16. *The **free rank** of a group G is the cardinality, or size, of the smallest set that freely generates G.*

Lemma 10. *If $\iota : S \to F(S)$ is the natural injection, where $F(S)$ denotes the free group of S, then:*

1. *ι is injective.*
2. *$\forall x \in F(S)$ there exists a unique $\{\sigma_1, \cdots, \sigma_k\}$ from S such that $\sigma_{i+1} \neq \sigma_i$ and $x = \sigma_1^{n_1} \cdots \sigma_k^{n_k}$*

Proof(1):

Let $\sigma_1, \sigma_2 \in S$ such that $\sigma_1 \neq \sigma_2$.

$\iota(\sigma_1) = \iota_{\sigma_1}(\sigma_1) = (\sigma_1^1, \sigma_1) \neq (\sigma_2^1, \sigma_2) = \iota_{\sigma_2}(\sigma_2) = \iota(\sigma_2)$

Therefore ι is injective.

Proof(2):

Let $x \in F(S)$,

then there exists a unique reduced word $((\sigma_1^{n_1}, \sigma_1), \cdots, (\sigma_k^{n_k}, \sigma_k)) \in x$ where $((\sigma_1^{n_1}, \sigma_1), \cdots, (\sigma_k^{n_k}, \sigma_k)) = \sigma_1^{n_1} \cdots \sigma_k^{n_k}$ and $\sigma_{i+1} \neq \sigma_i$

If there exists another word such that $((\beta_1^{n_1}, \beta_1), \cdots, (\beta_k^{n_k}, \beta_k)) \in x$ where $\beta_{i+1} \neq \beta_i$, then the word is reduced and that means $\sigma_i = \beta_i \ \forall i$ because there is only one reduced word.

Therefore the conclusion holds.

2.8 Orbits

Group Actions:

Let G be a group with identity e,

We call $\alpha : G \times X \to X$ a **left group action** if:

1. $e * x = \alpha(e, x) = x \ \forall x \in X$
2. $g * (h * x) = \alpha(g, \alpha(h, x)) = \alpha(g * h, x) = (g * h) * x \ \forall g, h \in G$ and $\forall x \in X$

We call $\alpha : X \times G \to X$ a **right group action** if:

1. $x * e = \alpha(e, x) = x \ \forall x \in X$
2. $(x * g) * h = \alpha(h, \alpha(g, x)) = \alpha(g * h, x) = x * (g * h) \ \forall g, h \in G$ and $\forall x \in X$

We call α a **group action** and say G acts on X if α is either a left or right action.

α-Invariant:

Suppose $\alpha : G \times X \to X$ is a group action on X.

$S \subset X$ is **α-invariant** if $\alpha(g, p) \in S \ \forall g \in G$ and $p \in S$

Lemma 11. *Suppose:*

1. *$\alpha : G \times X \to X$ is a group action on X*
2. *$S \subset X$ is α-invariant*

then $\alpha : G \times S \to S$ is a group action on S

Proof:

Assume α is a left group action,

1. Let $s \in S \Rightarrow s \in X$, thus

   $e * s = \alpha(e, x)$

2. Let $g, h \in G$ and $s \in S$, then

   $g * (h * s) = \alpha(g, \alpha(h, s)) = \alpha(g * h, s) = (g * h) * s$

   Since S is α-invariant $\alpha(g, S) \subset S$, thus $\alpha : G \times S \to S$

Therefore by definition $\alpha : G \times S \to S$ is a left group action on S

Similarly this can be shown for right group actions

Orbits:

Suppose $\alpha : G \times X \to X$ is a group action on X.

O is an orbit of α if the following are satisfied:

1. O is α-invariant.
2. ($O' \subset O$, $O' \neq O$, and O' is α-invariant) $\Rightarrow O' = \varnothing$

note: 2. means O is the smallest, non-empty, α-invariant set containing any of the values in O.

Fixed Points:

Suppose $\alpha : G \times X \to X$ is a group action on X.

If a group contains no fixed points, then $\alpha(g, x) \neq x$ $\forall g \in G \backslash \{e\}$ and $\forall x \in X$.

Definition 17. *Suppose $\alpha : G \times X \to X$ is a group action on X,*

*then x is a **fixed point** in a group G if $\exists g \in G \backslash \{e\}$ such that $\alpha(g, x) = x$.*

Lemma 12. *Let $\alpha : G \times X \to X$ be a group action on X, let O and O' be orbits of α, and let $E(p) = \{\alpha(g, p) : g \in G\}$*

Then the following are true:

1. *$O = E(p)$ $\forall p \in O$*
2. *$O' = O$ or $O' \cap O = \varnothing$*
3. *$E(p)$ is an orbit of α $\forall p \in X$*
4. *S is α-invariant if and only if S is the union of disjoint orbits of α*

Proof (1):

Let $\alpha(g, p) \in E(p)$, then

by definition of α-invariance $\alpha(g, p) \in)$

thus $E(p) \subseteq O$

Let $g \in G$, then

$\alpha(g, p) \in E(p)$ by definition

thus $E(p)$ is α-invariant

So by definition of orbits $E(p) \subset O \Rightarrow E(p) = \varnothing$ or $E(p) = O$ but we know $E(p)$ is non-empty so $E(p) = O$.

Proof (2):

Let $x \in O \cap O'$

$\Rightarrow (\forall g \in G)(\alpha(g, x) \in O$ and $\alpha(g, x) \in O')$ [Since O and O' are α-invariant]

Thus $\alpha(g, p) \in O \cap O'$ $\forall g \in G$ and $\forall p \in O \cap O'$ meaning $O \cap O'$ is α-invariant

Therefore by definition $O = O \cap O' = O'$ or $O \cap O' = \varnothing$ [Since $O \cap O' \subseteq O$ and $O \cap O' \subseteq O'$]

Proof (3):

Suppose $S \subseteq E(p)$ is α-invariant.

Let $x \in S$ and $z \in E(p)$,

$\Rightarrow \exists g, h \in G$ such that $x = \alpha(g, p)$ and $z = \alpha(h, p)$ [Since $E(p)$ is α-invariant]

$\Rightarrow z = \alpha(hg^{-1}, x) \in S$ [$hg^{-1} \in G$ and S is α-invariant]

Therefore $S = E(p)$ or $S = \varnothing$ so by definition $E(p)$ is an orbit.

Proof (4) ($\Rightarrow$):

Let $p \in S$,

$\Rightarrow E(p) \subseteq S$ [Since S is α-invariant]

$\Rightarrow S = \bigcup_{p \in S} E(p)$

And we know $E(p)$ are disjoint orbits

Proof (4) ($\Leftarrow$):

Let $p \in S$

$\Rightarrow p \in O$ where $O \subset S$ and O is an orbit

$\Rightarrow \alpha(g, p) \in O \subset S$ $\forall g \in G$

Thus S is α-invariant

2.9 Paradoxical

Piecewise Congruent:

$A, B \subset X$ are α-**piecewise congruent**, denoted by $A \sim_\alpha B$ if:

1. G and X are groups
2. $\alpha : G \times X \to X$ is a group action on X
2. There exists partitions $\{A_1, ..., A_n\}$ of A and $\{B_1, ..., B_n\}$ of B such that $(\forall i \leqslant n)(\exists g_i) \in G$ such that $A_i = \alpha(g_i, B_i)$

note: In some texts this is also referred to as G-equidecomposable.

Definition of α-Paradoxical:

E is **α-paradoxical** if

1. G is a group acting on set X with group action α
2. E $\subset$ X
3. $\exists$ disjoint $A, B \subset E$ such that $A \sim_\alpha E$ and $B \sim_\alpha E$

Remark: Defined in 1.2 of Wagon [1]

Lemma 13. *The function $g : X \to X$ defined by $g(x) = \alpha(g, x)$ is a bijective functions for all $g \in G$.*

Proof:

Injective:

Let $x_1, x_2 \in X$ where $x_1 \neq x_2$,

assume $g(x_1) = g(x_2)$, then $\alpha(g, x_1) = \alpha(g, x_1) \Rightarrow \alpha(g^{-1}g, x_1) = \alpha(g^{-1}g, x_1) \Rightarrow \alpha(e, x_1) = \alpha(e, x_1) \Rightarrow x_1 = x_2$ which is a contradiction so $g(x_1) \neq g(x_2)$

Thus g is injective.

Surjective:

Let $y \in X$,

then $y = \alpha(e, y) = \alpha(gg^{-1}, y) = \alpha(g, \alpha(g^{-1}, y)) = g(\alpha(g^{-1}, y))$ and $\alpha(g^{-1}, y) \in X$

Thus g is surjective.

Therefore g is bijective.

2.10 Determinants

Linear Operators:

Let X and Y be vector spaces, $T : X \to Y$ is a linear operator if:

1. X and Y share a common field
2. $T(ax_1 + bx_2) = aTx_1 + bTx_2$ where a and b are scalars and $x_1, x_2 \in X$

Linear Operator Facts:

1. The **nullspace** of T is defined by $\mathcal{N}(T) = \{x \in X : Tx = 0\}$
2. The **range** of T is defined by $\mathcal{R}(T) = \{y \in Y : \exists x \in X$ such that $Tx = y\}$

Tensors:

For a vector space V let $V^k = V \times V \times \cdots \times V$ (k times),

Then $T : V^k \to \mathbb{F}$ is a **tensor** if T is multi-linear.

Let $\mathcal{T}^k = \{T : V^k \to \mathbb{F} : T$ is a tensor$\}$, the set of all tensors.

Let $\dim(T^k) = n$.

$T : V^k \to \mathbb{F}$ is an **alternating tensor** if:

1. $T \in \mathcal{T}^k$
2. $T(x_1, \cdots, x_i, \cdots, x_j, \cdots, x_k) = -T(x_1, \cdots, x_j, \cdots, x_i, \cdots, x_k)$

note: 2 indicates that when two indices are switched the sign is switched.

Let $\Lambda^n = \{T : V^k \to \mathbb{F} : T$ is an alternating tensor$\}$, the set of all alternating tensors.

Tensor Facts:

1. Λ^k is a subgroup of $\mathcal{T}^k$.
2. $\dim(\Lambda^k) = 1$

Determinants:

Let e be a basis of a vector space X, then $\mathbf{det}(X, e)$ is the unique element of Λ^k such that

$$\det(X, e)(e_1, ..., e_n) = 1.$$

For a matrix A let A^i be the ith row of matrix A, then

$$\det A = \det(X, e)(A^1, \cdots, A^n)$$

2.11 Special Orthogonal

Symmetric Groups:

For a set X the **symmetric group**, denoted by $S(X)$, is the set of all bijective functions mapping from X into X, with the group operation being function composition, denoted by $\circ$.

Theorem 2.5. *$S(X)$ is a group.*

Proof:

Suppose X is a metric space, let $S(X)$ be the symmetric group of X and let $g, h, f \in S$,

1. [Closure] $g \circ h \in S(X)$ by definition of function composition
2. [Associativity] Let $x \in X$, $((g \circ h) \circ f)(x) = (g \circ h)(f(x)) = g(h(f(x)) = g((h \circ f)(x)) = (g \circ (h \circ f))(x)$ by definition of function composition.
3. [Identity] Let e be the identity function, then e is bijective and $e \circ g = g \circ e = g$.
4. [Inverse] g is bijective $\Rightarrow g^{-1}$ is a bijective.

Isometry Groups:

Assume:

1. X is a metric space with metric d.
2. $S(X)$ is the symmetric group of X.

Then $I(X) = \{f \in S(X) : d(f(x), f(y)) = d(x, y)\ \forall x, y \in X\}$ is called the **isometry group** of X with the group operation of function composition, denoted by $\circ$.

note: This means that the isometry group is the set of bijective functions that preserve distance.

Theorem 2.6. *The isometry group is a subgroup of the symmetry group.*

Proof:

Suppose I is the isometry group for all bijective isometries mapping from metric space X to X and let $g, h, f \in I$,

1. [Closure Under Group Operation]
   Let $x, y \in X$ $d((f \circ g)(x), (f \circ g)(y)) = d(f(g(x)), f(g(y))) = d(g(x), g(y)) = d(x, y)$
   $\Rightarrow f \circ g \in I(X)$.
2. [Closure Under Inverses]
   Let $x, y \in X$ $d(f^{-1}(x), f^{-1}(y)) = d(f(f^{-1}(x)), f(f^{-1}(y))) = d(x, y)$
   $\Rightarrow f^{-1} \in I(X)$.

Since $I(X) \subset S(X)$ by Lemma 1 $I(X)$ is a subgroup of $S(X)$.

Special Orthogonal:

Suppose:

1. X is a finite dimension vector space.
2. $L(X, X)$ is the set of linear operators on X.

Definitions:

1. $SO(X) = \{T \in L(X, X) : det(T) = 1\}$
2. $SO(X)$ is referred to as the **special orthogonal group** of X.

Definition 18. *$SO(\mathbb{R}^3)$, the special orthogonal group of $\mathbb{R}^3$, is denoted by SO_3*

Lemma 14. *$\alpha : SO_3 \times \mathbb{R}^3 \to \mathbb{R}^3$ defined by $\alpha(T, x) = Tx$ $\forall T \in SO_3$ and $\forall x \in \mathbb{R}^3$, is left group action on $\mathbb{R}^3$*

Proof:

1. Let $x \in \mathbb{R}^3$, then

   $ex = \alpha(e, x) = x$
2. Let $T_1, T_2 \in SO_3$ and $x \in \mathbb{R}^3$, then

   $\alpha(g, \alpha(h, x)) = g\alpha(h, x) = g(hx) = (gh)x = \alpha(gh, x)$

Therefore by definition α is a left group action on $\mathbb{R}^3$

Lemma 15. *Suppose:*

1. *X is a finite dimension vector space.*
2. *$S(X)$ is the symmetric group of X.*
3. *$SO(X)$ is the special orthogonal group of X.*

Conclusion:

$SO(X)$ is a subgroup of $S(X)$

Proof:

Suppose:

1. $U \in SO(X)$
2. $n = dim(X)$
3. $e = \{e_1, \cdots, e_n\}$ represents a basis for X

Observations:

1. U is injective $\Rightarrow U \in S(X)$.
2. U is injective iff $\mathcal{N}(U) = \{0\}$.
3. $\mathcal{N}(U) \neq \{0\}$ iff $\exists e_i \in e$ such that $Ue_i = 0$.

Assume $Ue_i = 0$,

$$\det U = \det(X, e)([Ue_1]_e, \cdots, [Ue_i]_e, \cdots, [Ue_n]_e) = \det(X, e)([Ue_1]_e, \cdots, 0, \cdots, [Ue_n]_e) = 0$$

This is a contradiction since $U \in SO(X) \Rightarrow \det U = 1$.

Therefore $Ue_i = 0$ $\forall i$, thus $\mathcal{N}(U) = \{0\}$, which implies U is injective, which implies $U \in S(X)$.

Suppose $U_1, U_2 \in SO(X)$, then

1. $\det U_1U_2 = \det U_1 \det U_2 = 1*1 = 1$

   Therefore $U_1U_2 \in SO(X)$.

2. $\det U_1^{-1} = \det U_1^{-1}*1 = \det U_1^{-1} \det U_1 = \det (U_1^{-1}U_1) = \det I = 1$

   Therefore $U_1^{-1} \in SO(X)$.

Therefore by Lemma 1 $SO(X)$ is a subgroup of $S(X)$.

3 Banach-Tarski Proof

This section generally follows the proof given by Grzegorz Tomkowicz and Stan Wagon with some additions and alternate proofs for added clarity.

There are two basic sections of the proof. The first section begins by proving that a specific type of group, called a free group is paradoxical. The next section shows that free groups act on spheres in a specific manner that passes on their paradoxical nature, making spheres paradoxical.

3.1 Free Groups are Paradoxical

To prove the Banach-Tarski Paradox we begin by proving that free groups of free rank two are paradoxical and discuss what that entails. After showing that free groups we an equivalent form of paradoxical is demonstrated. This form is useful for the next section of the proof.

Theorem 3.1. *A free group F of free rank 2 is α-Paradoxical where $\alpha : F \times F \to F$ is left multiplication.*

Remark: Theorem 1.2 in Wagon [1]

Proof:

Suppose:

1. $\{\sigma, \tau\}$ is a free generator of F
2. $W(a)$ is as defined in Definition 14

Thus, $\forall w \in F \Rightarrow w$ has a unique fully reduced word $w' = \sigma^{n_1}\tau^{n_2}\cdots\sigma^{n_{m-1}}\tau^{n_m}$, shown in Lemma 6

Thus, w' begins with $\sigma^n, \sigma^{-n}, \tau^n, \tau^{-n}$, or $w = e$ where $n \in \mathbb{N}$

Thus, $w \in W(\sigma)$, $w \in W(\sigma^{-1})$, $w \in W(\tau)$, $w \in W(\tau^{-1})$, or $w \in \{e\}$ which are disjoint because a word can only begin with one of $\sigma^n, \sigma^{-n}, \tau^n, \tau^{-n}$, or be the identity.

$\Rightarrow F = \{e\} \cup W(\sigma) \cup W(\sigma^{-1}) \cup W(\tau) \cup W(\tau^{-1})$ where the sets are disjoint.

Also, $\sigma W(\sigma^{-1}) = \{e\} \cup W(\sigma^{-1}) \cup W(\tau) \cup W(\tau^{-1})$

This is because $A = W(\sigma^{-1})$ can be split into two sets $B = \{w :$ the unique fully reduced word of w begins with $\sigma^{-1}\}$ and $\{w :$ the unique fully reduced word of w begins with σ^{-n} where $n \in \mathbb{N}\backslash\{1\}\}$ then $\sigma A = \{e\} \cup W(\tau) \cup W(\tau^{-1})$ because the first σ^{-1} will go away through elementary reductions. $\sigma B = W(\sigma^{-1})$ because through elementary reductions the power of the first σ^{-n} will be reduced by one and include σ^{-1}.

Similarly, $\tau W(\tau)^{-1}) = \{e\} \cup W(\sigma \cup W(\sigma^{-1}) \cup W(\tau^{-1})$

$\Rightarrow W(\sigma) \cup \sigma W(\sigma^{-1}) = F$ and $W(\tau) \cup \tau W(\tau^{-1}) = F$ through disjoint unions

Thus, $W(\sigma) \cup W(\sigma^{-1}) \sim_\alpha F$, $W(\tau) \cup W(\tau^{-1}) \sim_\alpha F$, and $(W(\sigma) \cup W(\sigma^{-1})) \cap (W(\tau) \cup W(\tau^{-1})) = \varnothing$

Therefore, by definition of paradoxical F is α-Paradoxical.

Definition 19. *$A \preccurlyeq_\alpha B$ if $\exists B_1 \subseteq B$ such that $A \sim_\alpha B_1$, where α is a group action on some set $X \supset A, B$.*

Lemma 16. *If $A \sim_\alpha B$, then there is a bijection $h : A \to B$ such that $C \sim_\alpha h(C)$ whenever $C \subset A$, where $\alpha : G \to X$ is a group action on X.*

Proof:

$A \sim_G B \Rightarrow$ there exists partitions $\{A_1, ..., A_n\}$ of A, $\{B_1, .., B_n\}$ of B, and $g_1, ..., g_n \in G$ such that for each $B_i = \alpha(g_i, A_i)$

$\Rightarrow B = \bigcup_{i=1}^{n} \alpha(g_i, A_i)$

$\Rightarrow B = h(A)$ where $h(x) = \alpha(g_i, x)$ when $x \in A_i$ for all $i = 0, ..., n$, which is bijective since $\{A_1, .., A_n\}$ is a partition of A and $h_1, ..., h_n$ are bijective.

Let $C \subset A$

$\Rightarrow \{A_1 \cap C, ..., A_n \cap C\}$ partition C and $\{h(A_1 \cap C), ..., h(A_n \cap C)\}$ partition g(C) and $g_1, ..., g_n \in G$ such that $h(A_k \cap C) = \alpha(g_i, A_k \cap C)$ by definition of h.

Therefore $C \sim_\alpha h(C)$

Lemma 17. *If $A_1 \cap A_2 = \varnothing = B_1 \cap B_2$ and $A_1 \sim_\alpha B_1$ and $A_2 \sim_\alpha B_2$, then $A_1 \cup A_2 \sim_\alpha B_1 \cup B_2$, where α is a group action on some set $X \supset A, B$.*

Proof:

$A_1 \sim_\alpha B_1$ and $A_2 \sim_\alpha B_2 \Rightarrow$ there exists partitions $\{C_1, ..., C_n\}$ of A_1, $\{D_1, .., D_n\}$ of B_1, $\{C_{n+1}, ..., C_{n+m}\}$ of A_2, $\{D_{n+1}, .., D_{n+m}\}$ of B_2, and $g_1, ..., g_{n+m} \in G$ such that for each $C_i = \alpha(g_i, D_i)$

$\Rightarrow \{C_1, ..., C_{n+m}\}$ partitions $A_1 \cup A_2$, $\{D_1, ..., D_{n+m}\}$ partitions $B_1 \cup B_2$ [Since A_1 and A_2 are disjoint and B_1 and B_2 are disjoint]

$\Rightarrow A_1 \cup A_2 \sim_\alpha B_1 \cup B_2$

Lemma 18. *$\sim_\alpha$ is and equivalence relation on $\mathcal{P}(X)$, where α is a group action on X.*

Proof:

Reflexive:

$X = e(X)$ where e is the identity function

$\Rightarrow X \sim_\alpha X$ [since the identity function is bijective]

Symmetric:

$X \sim_\alpha Y \iff$ there exists partitions $\{X_1, ..., X_n\}$ of X, $\{Y_1, .., Y_n\}$ of Y, and $g_1, ..., g_n \in G$ such that for each $Y_i = \alpha(g_i, X_i)$

$\Rightarrow \{Y_1, ..., Y_n\}$ partitions Y and $\{X_1, .., X_n\}$ partitions A, and $g_1^{-1}, ..., g_n^{-1} \in G$ such that for each $X_i = \alpha(g_i^{-1}, Y_i)$

$\iff Y \sim_\alpha X$

Transitive:

$X \sim_G Y$ and $Y \sim_G Z \iff$ there exists partitions $\{X_1, ..., X_n\}$ of X, $\{Y_1, .., Y_n\}$ of Y, and $g_1, ..., g_n \in G$ such that for each $i = 1, ..., n$ $Y_i = \alpha(g_i, X_i)$ and there exists partitions $\{Y_{n+1}, ..., Y_{n+m}\}$ of Y, $\{Z_1, .., Z_m\}$ of Z, and $h_1, ..., h_m \in G$ such that for each $i = 1, ..., m$ $Y_{n+i} = \alpha(h_i, A_i)$

$\Rightarrow \{X_{ij} : i = 1, ..., n$ and $j = 1, ..., m\}$ where $X_{ij} = \alpha(g_i^{-1}, Y_{n+j}) \cap X_i$ partitions X, and $\{Z_{ij} : i = 1, ..., n$ and $j = 1, ..., m\}$ where $z_{ij} = \alpha(h_j, Y_{n+j} \cap Z_i)$ partitions Z and for all $Z_{ij} = \alpha(h_j g_i, X_{ij})$

$\Rightarrow X \sim_\alpha Z$

Lemma 19. *Suppose:*

1. *X is a set*
2. *G is a group*
3. *$\alpha : G \times X \to X$ is group action on X*
4. *$E, F \subset X$*

Conclusion:

If E is α-paradoxical and $E \sim_\alpha F$, then F is α-paradoxical.

Proof:

E α-paradoxical $\Rightarrow \exists$ disjoint $A, B \subset E$ such that $A \sim_\alpha E$ and $B \sim_\alpha E$

Since by Lemma 18 α-paradoxical is an equivalence class $A \sim_\alpha E \sim_\alpha F$ and $B \sim_\alpha E \sim_\alpha F$

By Lemma 16 there exists a bijection $h : E \to F$ such that $h(C) \sim_\alpha C \; \forall C \subset E$

Thus $h(A) \cap h(B) = \varnothing$ and $h(E) = F$ and $h(A) \sim_\alpha A \sim_\alpha F \sim_\alpha B \sim_\alpha h(B)$

$\Rightarrow h(A) \sim_\alpha F \sim_\alpha h(B)$

Therefore by definition of paradoxical F is α-paradoxical

Theorem 3.2. *Banach-Schroeder-Bernstein Partial Order Theorem:*

Suppose G acts on X, with group action α, and $A, B \subset X$. If $A \preccurlyeq_\alpha B$ and $B \preccurlyeq_\alpha A$, then $A \sim_\alpha B$.

Proof:

$A \preccurlyeq_\alpha B$ and $B \preccurlyeq_\alpha A \Rightarrow$ for some $A_1 \subset A$ and $B_1 \subset B$ $A_1 \sim_\alpha B$ and $A \sim_\alpha B_1$

$\Rightarrow$ there exist bijections $f : A \to B_1$ and $g : A_1 \to B$. [By Lemma 16]

Let $C_0 = A \backslash A_1$ and $C_{n+1} = g^-1(f(C_n))$ and let $C = \overset{\infty}{\underset{n=0}{\cup}} C_n$.

$\Rightarrow g(A \backslash C) = g(A \backslash \{A \backslash A_1 \cup \overset{\infty}{\underset{n=1}{\cup}} C_n\}) = g(A_1 \backslash \overset{\infty}{\underset{n=1}{\cup}} C_n) = B \backslash g(\overset{\infty}{\underset{n=0}{\cup}} g^-1(f(C_n)) = B \backslash f(\overset{\infty}{\underset{n=0}{\cup}} C_n) = B \backslash f(C)$ [Possible since f and g are bijective]

$g(A \backslash C) = B \backslash f(C)$ and $g(A \backslash C) \sim_\alpha A \backslash C \Rightarrow A \backslash C \sim_\alpha B \backslash f(C)$ by [Lemma 16]

Lemma 16 $\Rightarrow C \sim_\alpha f(C)$

$\Rightarrow A \backslash C \cup C \sim_\alpha B \backslash f(C) \cup f(C)$ [By Lemma 17]

$\Rightarrow A \sim_\alpha B$

Corollary 3.2.1. *Corollary 3.7 (Wagon)*

A subset E of X is α-paradoxical, where α is a group action on X, iff there are disjoint sets $A, B \subset E$ where $A \cup B = E$ and $A \sim_\alpha E \sim_\alpha B$.

Note: If you can show $A \sim_\alpha E \backslash A \sim_\alpha E$ then E is α-paradoxical.

Proof:

Let $E \subset X$ be α-paradoxical $\iff \exists A, B \subset E$ that are disjoint and $A \sim_\alpha E \sim_\alpha B$

$B \subset E \backslash A$ and $E \sim_\alpha B \Rightarrow E \preccurlyeq_\alpha E \backslash A$

$E \backslash A \sim_\alpha E \backslash A$ and $E \backslash A \subset E \Rightarrow E \backslash A \preccurlyeq_\alpha E$

Thus by Theorem 3.2 $A \sim_\alpha E \sim \alpha E \backslash A$

Therefore E is α-paradoxical

Corollary 3.2.1 applied to Free Groups of free rank 2

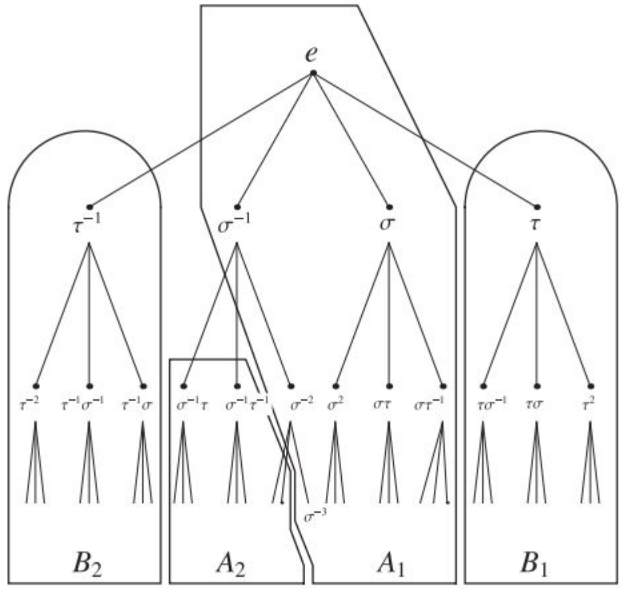


Figure 1: Figure 3.2 Wagon [1]

We have already demonstrated that F is F-paradoxical (Theorem 3.1) so by Corollary 3.2.1 there must exist A and B such that $A \cup B = F$ and $A \sim_F F \sim_F B$. Figure 1 demonstrates four groups that satisfy $A \cup B = F$ where $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$. We have previously established that $W(\sigma) \cup \sigma W(\sigma^-1) = F$. Similarly since $B_1 = W(\tau)$ and $B_2 = W(\tau^-1)$, $B_1 \cup \tau B_2 = F$. This means $B \sim_G F$. Based on the selection of A_2 in the figure $\sigma A_2 = A_2 \cup B$. So $A_1 \cup \sigma A_2 = F$. Thus $A \sim_F F$. So with the selection of A and B based on figure 1 $A \cup B = F$ and $A \sim_F F \sim_F B$.

3.2 Spheres are Paradoxical

By using that α-paradoxical, as proven in the previous section, we can prove that spheres are paradoxical. This is the formation of the Banach-Tarski Paradox. This requires specifying that paradoxical groups can transfer their paradoxical nature if they act in a specific manner. Then we must prove that there is a subgroup in SO_3 that is a free group of free rank two that acts in that manner upon SO_3.

Proposition 1. *If G is β-paradoxical and acts upon X without nontrivial fixed points with group action α, then X is α-paradoxical.*

Where β is multiplication, $\beta(g_1, g_2) = g_1 g_2 \; \forall g_1, g_2 \in G$ and if it is a left group action or $\beta(g_1, g_2) = g_2 g_1 \; \forall g_1, g_2 \in G$ if it is a right group action.

note: This means if we can show any paradoxical group G acts upon a set X without nontrivial fixed points, then X is α-paradoxical, where $\alpha : G \times X \to X$ is group action on

X

Remark: Theorem 1.10 of Wagon [1]

Proof:

Assume:

1. $\alpha : G \times X \to X$ be a group action on X, where $\alpha(g, x) \neq x$ for all $g \in G \backslash \{e\}$
2. If α is a left group action $\beta : G \times G \to G$ is a left group action on G, where $\beta(x, y) = xy$
3. If α is a right group action $\beta : G \times G \to G$ is a right group action on G, where $\beta(x, y) = yx$
4. G is β-paradoxical
5. $M \subset X$ such that M contains exactly one element from each orbit of X

   Note: This is assumed to be possible by the Axiom of Choice
6. $H^* = \{\alpha(g, m) : g \in H \text{ and } m \in M\}$ for any $H \subset G$

Let $G_1, G_2 \subset G$ be disjoint,

Assume $G_1^* \cap G_2^* \neq \varnothing$,

$\Rightarrow \exists g_1 \in G_1,\ g_2 \in G_2$ and $m_1, m_2 \in M$ such that $\alpha(g_1, m_1) = \alpha(g_2, m_2)$

$\Rightarrow m_1 = m_2$ [Since the orbits are disjoint (Lemma 12) and since each element of M is in a different orbit]

$\Rightarrow \alpha(g_1, m_1) = \alpha(g_2, m_1)$

$\Rightarrow \alpha(g_2^{-1} g_1, m_1) = \alpha(e, m_1) = m_1$

$\Rightarrow g_2^{-1} g_1 = e$ [Since G acts upon X without nontrivial fixed points]

$\Rightarrow g_1 = g_2$

$\Rightarrow \{g_1\} \subset G_1 \cap G_2$

This is a contradiction of the assumption, therefore $G_1^* \cap G_2^* = \varnothing$ for all disjoint $G_1, G_2 \subset G$.

$(G_1 \cup G_2)^* = \{\alpha(g, m) : g \in G_1 \cup G_2 \text{ and } m \in M\} = \{\alpha(g, m) : g \in G_1 \text{ and } m \in M\} \cup = \{\alpha(g, m) : g \in G_2 \text{ and } m \in M\} = G_1^* \cup G_2^*$

Therefore $G_1^* \cup G_2^* = (G_1 \cup G_2)^*$ disjointly

G is β-paradoxical $\iff$ $\exists$ disjoint $A, B \subset G$ such that $A \sim_\beta G$ and $B \sim_\beta G$

$\iff$ for some m,n $\exists g_1, ..., g_m, h_1, ..., g_m \in G$ and disjoint $A_1, ..., A_m, B_1, ..., B_n \subset G$ such that $G = \bigcup_i \beta(g_i, A_i) = \bigcup_j \beta(h_j, B_j)$, let $G_{A,i} = \beta(g_i, A_i)$ and $G_{B,i} = \beta(h_i, B_i)$

$\Rightarrow \{G_{A,1}, ..., G_{A,n}\}$ and $\{G_{B,1}, ..., G_{B,n}\}$ partition X and are disjoint

$\alpha(g_i, A_i^*) = \{\alpha(a, x) : a \in A_i, x \in M\}) = \{\alpha(g_i a, x) : a \in A_i, x \in M\}$

$= \{\alpha(\beta(g_i, a), x) : a \in A_i, x \in M\} = \{\alpha(g, x) : g \in G_{A,i}, x \in M\} = G_{A,i}^*$

Similarly, $\alpha(h_j, B_j^*) = G_{B,j}^*$

Thus, $X = \bigcup_i \alpha(g_i, G^*_{A,i}) = \bigcup_j \alpha(h_j, G^*_{B,j})$

Since the above are disjoint unions and $\bigcup_i G^*_{A,i} \cap \bigcup_j G^*_{B,j} = \varnothing$, X is by definition α-paradoxical.

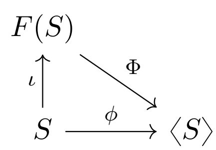
Theorem 3.3. *For any group G, $S \subset G$ is a free generator of $\langle S \rangle$ if for any fully reduced word $\iota(s_1)^{n_1}...\iota(s_k)^{n_k}$ from $\{F(\sigma)\}_{\sigma \in S}$ where $s_1^{n_1}...s_n^{n_k} = e$ implies that $n_i = 0$ $\forall i$, where $F(S)$ is the free group of S and ι is the natural injection.*

Proof:

Suppose:

1. For any fully reduced word $\iota(s_1)^{n_1}...\iota(s_k)^{n_k}$ from $\{F(\sigma)\}_{\sigma \in S}$ $s_1^{n_1}...s_k^{n_k} = e \iff n_i = 0$ $\forall i$
2. $\phi : S \to \langle S \rangle$ be the inclusion map
3. Φ be generated by conclusion 1 of *Theorem* 2.4

$\Phi : F(S) \to \langle S \rangle$ such that



To show that S is a free generator it must be shown that Φ is injective and surjective.

Injective:

Let $w_1, w_2 \in F(S)$ such that $\Phi(w_1) = \Phi(w_2)$ and $w_{r_1} = s_1^{n_1}...s_k^{n_k}$ and $w_{r_2} = s_1'^{n_1'}...s_j'^{n_j'}$ be the unique reduced words of w_1 and w_2 respectively.

Thus $[w_1] = [s_1^{n_1}...s_k^{n_k}]$ and $[w_2] = [s_1'^{n_1'}...s_j'^{n_j'}]$

$\Rightarrow \Phi(w_1) = (\Phi \circ \iota)(s_1)^{n_1} \cdots (\Phi \circ \iota)(s_k)^{n_k} = \phi(s_1)^{n_1} \cdots \phi(s_k)^{n_k} = (s_1)^{n_1} \cdots (s_k)^{n_k}$ and

$\Phi(w_2) = (\Phi \circ \iota)(s_1')^{n_1'} \cdots (\Phi \circ \iota)(s_j')^{n_j'} = \phi(s_1')^{n_1'} \cdots \phi(s_j')^{n_j'} = (s_1')^{n_1'} \cdots (s_j')^{n_j'}$

$\Rightarrow s_1^{n_1} \cdots s_k^{n_k} = s_1'^{n_1'} \cdots (s_j')^{n_j'}$

$\Rightarrow s_j'^{-n_j'} \cdots s_1'^{-n_1'} s_1^{n_1} \cdots s_k^{n_k} = e$

Cases:

1. $\iota(s_j')^{-n_j'} \cdots \iota(s_1')^{-n_1'} s_1^{n_1} \cdots \iota(s_k)^{n_k}$ is a reduced word, then $n_i = 0$ and $n_i' = 0$, hence $w_1 = e = w_2$
2. $\iota(s_j')^{-n_j'} \cdots \iota(s_2')^{-n_2'} \iota(s_1)^{n_1' - n_1} s_2^{n_2} \cdots \iota(s_k)^{n_k}$ is a reduced word, then $n_1 = n_1'$ and $n_i = 0$ and $n_i' = 0$ for $i > 1$, hence $w_1 = s_1^{n_1} = s_1'^{n_1} = w_2$
3. ...

4. $\iota(s_1)^{n_1'-n_1}\cdots\iota(s_k)^{n_k-n_k}$ is a reduced word, then $n_i = n_i'$, hence $w_1 = s_1^{n_1}\cdots s_k^{n_k} = w_2$

So $w_1 = w_2$

Thus Φ is injective.

Surjective:

Let $g \in \langle S \rangle$

$\Rightarrow g = s_1^{n_1}...s_k^{n_k}$ for some $n_i \in \mathbb{Z}$ and $s_i \in S$ $\forall i \in 1,...,k$ [By Lemma 8]

$\Phi(\iota(s_1^{n_1})...\iota(s_k^{n_k})) = (\Phi\circ\iota)(s_1^{n_1})...(\Phi\circ\iota)(s_k^{n_k})) = s_1^{n_1}...s_k^{n_k} = g$

Thus Φ is surjective.

Therefore S freely generates $\langle S \rangle$

Theorem 3.4. *$\sigma = \frac{1}{7}\begin{bmatrix} 6 & 2 & 3 \\ 2 & 3 & -6 \\ -3 & 6 & 2 \end{bmatrix}$ and $\tau = \frac{1}{7}\begin{bmatrix} 2 & -6 & 3 \\ 6 & 3 & 2 \\ -3 & 2 & 6 \end{bmatrix}$ are matrix representations of free generators in SO_3 using the standard basis.*

Remark: Follows Theorem 2.1 of Wagon [1]

Proof:

To prove this we will show that $\sigma^{n_1}\tau^{n_2}\cdots\sigma^{n_{k-1}}\tau^{n_k} = e \iff n_i = 0\ \forall i$ which implies σ and τ are free generators through Theorem 3.3.

This can be shown with simplified matrices $M_\sigma = \begin{bmatrix} 6 & 2 & 3 \\ 2 & 3 & -6 \\ -3 & 6 & 2 \end{bmatrix}$ and $M_\tau = \begin{bmatrix} 2 & -6 & 3 \\ 6 & 3 & 2 \\ -3 & 2 & 6 \end{bmatrix}$ since $\sigma^{n_1}\tau^{n_2}\cdots\sigma^{n_{k-1}}\tau^{n_k} = (\frac{1}{7})^{(\sum_{i=1}^{k} n_i)} M_\sigma^{n_1}M_\tau^{n_2}\cdots M_\sigma^{n_{k-1}}M_\tau^{n_k}$.

We will look specifically at the first vector of $M_\sigma^{n_1}M_\tau^{n_2}\cdots M_\sigma^{n_{k-1}}M_\tau^{n_k}$, and we want to show that it will never be $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, since the matrix reprsentation of $e = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. The vector can be reduced using modulo 7, where v mod 7 means every element of v has undergone modulo 7, because if x mod $7 \neq 0$, then $x \neq 0$. An example is $\begin{bmatrix} 5 \\ 17 \\ -2 \end{bmatrix}$ mod $7 = \begin{bmatrix} 5 \\ 3 \\ 5 \end{bmatrix}$.

This ability to look at the result with modulo 7 holds because (ab) mod $n = (a$ mod $n)(b$ mod $n)$ mod n and $(a+b)$ mod $n = (a$ mod $n + b$ mod $n)$ mod n which implies for matrices A and B (AB) mod $n = (A$ mod $n)(B$ mod $n)$ mod n and in this case we are looking at $M_\sigma^{n_1}M_\tau^{n_2}\cdots M_\sigma^{n_{k-1}}M_\tau^{n_k}$ mod $7 = (M_\sigma^{n_1}$ mod $7)(M_\tau^{n_2}$ mod $7)\cdots(M_\sigma^{n_{k-1}}$ mod $7)(M_\tau^{n_k}$ mod

7) mod 7.

Define four sets of vectors as follows:

$$V_\sigma = \left\{ \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 2 \\ 4 \end{bmatrix} \right\} \; V_\sigma^- = \left\{ \begin{bmatrix} 3 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 6 \\ 4 \\ 5 \end{bmatrix} \right\} \; V_\tau = \left\{ \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ 4 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix} \right\} \; V_\tau^- = \left\{ \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \\ 2 \end{bmatrix} \right\}$$

Vector Characteristics:

1. $\forall v \in V_\sigma \cup V_\tau \cup V_\tau^-$, $M_\sigma v \bmod 7 \in V_\sigma$
2. $\forall v \in V_\sigma^- \cup V_\tau \cup V_\tau^-$, $M_{\sigma^-} v \bmod 7 \in V_\sigma^-$
3. $\forall v \in V_\tau \cup V_\sigma \cup V_\sigma^-$, $M_\tau v \bmod 7 \in V_\tau$
4. $\forall v \in V_\tau^- \cup V_\sigma \cup V_\sigma^-$, $M_{\tau^-} v \bmod 7 \in V_\tau^-$

Example calculations:

$v \in V_\sigma$:

$$M_\sigma \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 & 2 & 3 \\ 2 & 3 & -6 \\ -3 & 6 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 26 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 26 \\ -3 \\ 1 \end{bmatrix} \bmod 7 = \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix} \in V_\sigma$$

$$M_\sigma \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 & 2 & 3 \\ 2 & 3 & -6 \\ -3 & 6 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 41 \\ 16 \\ 12 \end{bmatrix}, \begin{bmatrix} 41 \\ 16 \\ 12 \end{bmatrix} \bmod 7 = \begin{bmatrix} 6 \\ 2 \\ 4 \end{bmatrix} \in V_\sigma$$

$$M_\sigma \begin{bmatrix} 6 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 & 2 & 3 \\ 2 & 3 & -6 \\ -3 & 6 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 52 \\ -6 \\ 2 \end{bmatrix}, \begin{bmatrix} 52 \\ -6 \\ 2 \end{bmatrix} \bmod 7 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \in V_\sigma$$

Therefore $\forall v \in V_\sigma$, $M_\sigma v \bmod 7 \in V_\sigma$

This can be repeated for all other vector characteristics.

Simplified Matrices First Column Characteristics:

Starting with M_σ the first column is $\begin{bmatrix} 6 \\ 2 \\ -3 \end{bmatrix}$ which is reduced to $\begin{bmatrix} 6 \\ 2 \\ 4 \end{bmatrix} \in V_\sigma$ through modulo 7.

So, the first column of M_σ^1 is in V_σ

Also, by vector characteristic 1, the first column of M_σ^2 is in V_σ because $M_\sigma^2 = M_\sigma M_\sigma$

and the first column of $M_\sigma \in V_\sigma$.

Thus, by vector characteristic 1 and induction, the first column of $M_\sigma^{n_1}$ is in V_σ for all $n_1 > 0$.

Similarly you can show that the first column of $M_\sigma^{n_1} \in V_\sigma^-$ for all $n_1 < 0$.

Then looking at $M_\sigma^{n_1} M_\tau^{n_2}$ this satisfies 3 if $n_2 > 0$ and 4 if $n_2 < 0$, thus $M_\sigma^{n_1} M_\tau^{n_2}$ is in V_τ or V_τ^- respectively.

Next looking at $M_\sigma^{n_1} M_\tau^{n_2} M_\sigma^{n_3}$ this satisfies 1 if $n_3 > 0$ and 2 if $n_3 < 0$, thus $M_\sigma^{n_1} M_\tau^{n_2} M_\sigma^{n_3}$ is in V_σ or V_σ^- respectively.

Through induction you can repeat this to get that if any $n_i \neq 0$, then the modulo reduced first column of $M_\sigma^{n_1} M_\tau^{n_2} \cdots M_\sigma^{n_{k-1}} M_\tau^{n_k}$ is in $V_\sigma \cup V_\sigma^- \cup V_\tau \cup V_\tau^-$

Thus $\sigma^{n_1}\tau^{n_2}\cdots\sigma^{n_{k-1}}\tau^{n_k} = e \iff n_i = 0\ \forall i$

Therefore by Theorem 3.3 σ and τ are free generators in SO_3.

Lemma 20. *Rodrigues' rotation formula*

Suppose:

1. ∂B_3 denotes the boundary of the unit ball in $\mathbb{R}^3$ defined by $\partial B_3 = \{x \in \mathbb{R}^3$ such that $||x|| = 1\}$

2. $R(n,\theta) : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $R(n,\theta)x = cos(\theta)x + \langle n, x\rangle(1 - cos(\theta))n + sin(\theta)(n \times x)$ $\forall x \in \mathbb{R}^3$

Conclusion:

$$R \in SO_3 \iff \exists n \in \partial B_3 \text{ and } \exists \theta \in R \text{ such that } R = R(n,\theta)$$

Proof: $(\Rightarrow)$

Let $R \in SO_3$, then

1 is an eigenvalue

Let $n \in \mathbb{R}^3$ be a normalized eigenvector of 1 and e be an orthonormal basis for $\mathbb{R}^3$ such that $e_1 = n$, then

$$[R]_e = \begin{bmatrix} 1 & 0 & 0 \\ 0 & R_{22} & R_{23} \\ 0 & R_{32} & R_{33} \end{bmatrix}$$

Since the columns are orthonormal they all lie on ∂B_3

$$\Rightarrow \exists \theta_2, \theta_3 \in \mathbb{R}^3 \text{ such that } [R]_e = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\theta_2 & -sin\theta_3 \\ 0 & sin\theta_2 & cos\theta_3 \end{bmatrix}$$

$\Rightarrow$ by orthogonality $0 = -cos\theta_2 sin\theta_3 + sin\theta_2 cos\theta_3 = sin(\theta_2 - \theta_3$

$\Rightarrow \theta_2 = \theta_3 + k\pi$

Since $R \in SO_3$ $det(R) = 1 \Rightarrow k$ is even

Thus,

$$[R]_e = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\theta_2 & -sin\theta_2 \\ 0 & sin\theta_2 & cos\theta_2 \end{bmatrix} = [R(n, \theta_2]_e$$

Therefore, $\exists n \in \partial B_3$ and $\exists \theta \in R$ such that $R = R(n, \theta)$

Proof ($\Leftarrow$)

Let e be an orthonormal basis such that $e_1 = n$, then

$$[R(n,\theta)]_e = cos(\theta)I + (1 - cos\theta)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + sin\theta \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\theta & -sin\theta \\ 0 & sin\theta & cos\theta \end{bmatrix}$$

$$\Rightarrow det(R(n,\theta)) = det([R(n,\theta)]_e) = cos^2\theta + sin^2\theta = 1$$

Therefore $R(n, \theta) \in SO_3$

Theorem 3.5. *The Hausdorff Paradox:*
There exists a countable subset $D \subset \mathbb{S}^2$, such that $\mathbb{S}^2 \backslash D$ is α-paradoxical, where $\alpha : SO_3 \times \mathbb{S}^2 \to \mathbb{S}^2$ is a group action on $\mathbb{S}^2$, where $\mathbb{S}^2$ is defined in Definition 11.

Remark: Theorem 2.3 of Wagon [1]

Proof:

Suppose:

1. $S = \{\sigma, \tau\}$ from Theorem 3.4
2. $\alpha : SO_3 \times \mathbb{R}^3 \to \mathbb{R}^3$ defined by $\alpha(T, x) = Tx$ $\forall T \in SO_3$ and $\forall x \in \mathbb{R}^3$, is left group action on $\mathbb{R}^3$ by Lemma 14
3. I represents the identity element of SO_3
4. $R(n, \theta) \in SO_3$ as defined in Lemma 20

By Theorem 3.1 $\langle S \rangle$ is paradoxical.

Let $U \in SO_3 \backslash \{I\}$,

If $Ux = x$, then U is an eigenvector with eigenvalue 1.

By Lemma 20 $\exists n \in \partial B^3$ and $\exists \theta \in \mathbb{R}$ such that $U = R(n, \theta)$ and since $cos(\theta)x + \langle n, x\rangle(1 - cos\theta)n$ lies in the span of xn and $sin\theta(n \times x)$ is perpendicular to xn, $xx = \pm n$

note: if $\theta = 0$, then $n = e$

Thus, each $U \in SO3\backslash\{I\}$ has two fixed points

By Lemma 9 $\langle S\rangle$ is countable, hence $\langle S\rangle\backslash\{I\}$ is countable,

thus, $D = \{n \in \mathbb{S}^2 : Un = n, U \in \langle S\rangle\backslash\{I\}\}$ is countable.

Define $\alpha' : \langle S\rangle \times \mathbb{S}^2\backslash D \to \mathbb{S}^2\backslash D$ by $\alpha'(U, x) = Ux\ \forall U \in \langle S\rangle$ and $\forall x \in \mathbb{S}^2\backslash D$

$\langle S\rangle \subset SO3 \Rightarrow Ux \in \mathbb{S}^2\backslash D$

note: to show $\mathbb{S}^2\backslash D$ is α-invariant it suffices to show that any element $U \in \langle S\rangle\backslash D$ doesn't have the characteristic of $Ux = x$

Let $x \in \mathbb{S}^2$ suppose $\exists U_1, U_2 \in \langle S\rangle\backslash\{I\}$ such that $U_1x = U_2(U_1x) = (U_1U_2)x$, then

$(U_2U_1)I = (U_1)I \Rightarrow U_2 = I$,

The same can be shown for U_1 so under α nothing from $\mathbb{S}^2\backslash D$ maps to D

Thus $U_1x \in \mathbb{S}^2\backslash D$ so $\mathbb{S}^2\backslash D$ is α-invariant

Thus, α' is a left group action on $\mathbb{S}^2$ by Lemma 11

D contains all nontrivial fixed points thus, $\langle S\rangle \subset SO_3$ acts on $\mathbb{S}^2\backslash D$ with the only fixed point being I, thus no nontrivial fixed points.

Therefore, by Proposition 1 $\mathbb{S}^2\backslash D$ is α-paradoxical.

Theorem 3.6. *If D is a countable subset of $\mathbb{S}^2$, then $\mathbb{S} \sim_\alpha \mathbb{S}\backslash D$, where $\alpha : SO_3 \times \mathbb{S}^2 \to \mathbb{S}^2$ is a group action on $\mathbb{S}^2$, where $\mathbb{S}^2$ is defined in Definition 11.*

Proof:

Suppose:

1. $\alpha : SO_3 \times \mathbb{R}^3 \to \mathbb{R}^3$ defined by $\alpha(T, x) = Tx\ \forall T \in SO_3$ and $\forall x \in \mathbb{R}^3$, which is left group action on $\mathbb{R}^3$ by Lemma 14
2. $z \in \mathbb{S}^2\backslash D$, define $n = \frac{z}{r}$ where r is as defined in Definition 11
3. $R(n, \theta) \in SO_3$ as defined in Lemma 20
4. $A = \{\theta \in [0, 2\pi) : \exists k \in \mathbb{N}$ and $\exists x \in D$ such that $R(n, k\theta)x \in D\}$
5. e is the identity element of SO_3

A is countable $\Rightarrow \exists \phi \in [0, 2\pi)\backslash A$

By definition $R(n, \phi)^k(D) \cap D = \varnothing$

$\Rightarrow R(n, \phi)^{k_2-k_1}D \cap D = \varnothing\ \forall k_1, k_2 \in \mathbb{N}$ such that $k_2 > k_1$

Assume $\exists k_1, k_2 \in \mathbb{N}$ such that $k_1 < k_2$ and $x \in R(n, \phi)^{k_1}D \cap R(n, \phi)^{k_2}(D)$

$\Rightarrow \exists d_1, d_2 \in D$ such that $R(n,\phi)^{k_1} d_1 = R(n,\phi)^{k_2} d_2$

$\Rightarrow R(n,\phi)^{k_2-k_1} d_2 = d_1$ which contradicts the previous conclusion

Thus $R(n,\phi)^{k_1} D \cap R(n,\phi)^{k_2} D = \varnothing$

Define: $D^* = D \cup \bigcup_{k\in\mathbb{N}} R(n,\phi)^k D$

$\Rightarrow R(n,\phi)D* = \bigcup_{k\in\mathbb{N}} R(n,\phi)^k D = D^* \backslash D$

$\Rightarrow e\mathbb{S}^2\backslash D^* \cup R(n,\phi)D^* = \mathbb{S}^2\backslash D^* \cup D^*\backslash D$

$\Rightarrow \mathbb{S}^2\backslash D^* \cup D^* \sim_\alpha \mathbb{S}^2\backslash D^* \cup D^*\backslash D$

Also, $\mathbb{S}^2\backslash D^* \cup D^* = \mathbb{S}^2$ and $\mathbb{S}^2\backslash D^* \cup D^*\backslash D = \mathbb{S}^2 \cup \neg D^* \cup D^* \cup \neg D = \mathbb{S}^2 \cup \neg D = \mathbb{S}^2\backslash D$

Therefore $\mathbb{S}^2 \sim_\alpha \mathbb{S}^2\backslash D$

Corollary 3.6.1. *The sphere $\mathbb{S}^2$ is α-paradoxical, where $\alpha : SO_3 \times \mathbb{S}^2 \to \mathbb{S}^2$ is a group action and $\mathbb{S}^2$ is defined in Definition 11.*

Remark: Corollary 2.9 in Wagon [1]

Proof:

By Theorem 3.5 $\exists$a countable subset $D \subset \mathbb{S}^2$ such that $\mathbb{S}^2\backslash D$ is α-paradoxical.

By Theorem 3.6 $\mathbb{S}^2\backslash D \sim_\alpha \mathbb{S}^2$.

Therefore by Lemma 19 $\mathbb{S}^2$ is α-paradoxical.

Theorem 3.7. *The Banach-Tarski Paradox:*
Any solid ball B^3, as defined in Definition 12, is β-paradoxical, where $\beta : H \times \mathbb{R}^3 \to \mathbb{R}^3$ is a left group action defined by $\beta(f,x) = f(x)$ $\forall f \in H = \{T_2 \circ R \circ T_1 : T_i \in T(\mathbb{R}^3), R \in SO_3\}$ and $\forall x \in \mathbb{R}^3$.

Proof:

Suppose:

1. $\mathbb{S}^2$ is a sphere, as defined in Definition 11
2. B^3 is a solid ball, as defined in Definition 12
3. $\alpha : SO_3 \times \mathbb{R}^3 \to \mathbb{R}^3$ defined by $\alpha(T,x) = Tx$ $\forall T \in SO_3$ and $\forall x \in \mathbb{R}^3$, which is left group action on $\mathbb{R}^3$ by Lemma 14
4. $I(\mathbb{R}^3)$ is the isometry group of $\mathbb{R}^3$
5. $T(\mathbb{R}^3)$ denotes the set of translations on $\mathbb{R}^3$
6. $H = \{T_2 \circ R \circ T_1 : T_i \in T(\mathbb{R}^3), R \in SO_3\}$
7. $\beta : H \times \mathbb{R}^3 \to \mathbb{R}^3$ be the left group action defined by $\beta(f,x) = f(x)$ $\forall f \in H$ and $\forall x \in \mathbb{R}^3$

By Corollary 3.6.1 $\mathbb{S}^2$ is α-invariant,

thus there exists disjoint sets $E^*, F^* \subset \mathbb{S}^2$ such that $E^* \sim_\alpha \mathbb{S}^2$ and $F^* \sim_\alpha \mathbb{S}^2$

Let $E = \{sx : x \in E^*, s \in (0,1]\}$ and $F = \{sx : x \in F^*, s \in (0,1]\}$

thus $E \cap F = \varnothing$ since E^* and F^* are disjoint and $E, F \subset B^3\backslash\{0\}$

$\Rightarrow E \sim_\alpha B^3\backslash\{0\}$ and $F \sim_\alpha B^3\backslash\{0\}$

Thus $B^3\backslash\{0\}$ is α-paradoxical

H is a subgroup of $I(\mathbb{R}^3)$

Thus $B^3\backslash\{0\}$ is β-paradoxical

Suppose $\theta' \in$ is irrational and $\theta = \theta'\pi \in [0, 2\pi)$ and let $k_1\theta = \theta + k_2\pi$ for some $k_1, k_2 \in \mathbb{Z}$,

$\Rightarrow \theta' = \frac{\theta}{\pi}$ and $\theta = \frac{k_2\pi}{k_1 - 1}$

$\Rightarrow \theta' = \frac{k_2}{k_1 - 1} \in \mathbb{Q}$

This contradicts the assumption that θ' is irrational so rotations of θ won't return to the same spot, since for any two rotations ϕ and ϕ' they go to the same spot iff $\phi = \phi' + 2k\pi$ for some $k \in \mathbb{Z}$.

Suppose:

1. $p = (0.5r, 0, 0)$, where r is from Definition 12
2. $T_1x = x + p$, and $T_2x = x - p$
3. $R \in SO_3$, where R represents a rotation of angle θ about the z-axis
4. $S = T_1 \circ R \circ T_2$
5. $K = \{S^n 0 : n \in \mathbb{N}\}$
6. $K_0 = K \cup \{0\}$

note: $S^n = T_1 \circ R^n \circ T_2$ because $T_1 \circ T_2 = e$, where e is the identity of SO_3

By definition of H, $S \in H$,

$S(K_0) = \{S^{n+1}0 : n \in \mathbb{N}\} \cup \{S0\} = \{S^n 0 : n \in \mathbb{N}\} = K$

$\Rightarrow K_0 \sim_\beta S(K_0)$ and $B^3\backslash K_0 \sim_\beta B^3\backslash K_0$

$\Rightarrow K_0 \cup (B^3\backslash K_0) \sim_\beta S(K_0) \cup B^3\backslash K_0$

Also, $K_0 \cup (B^3\backslash K_0) = B^3$ and $S(K_0) \cup B^3\backslash K_0 = K_0\backslash\{0\} \cup B^3\backslash K_0 = B^3\backslash\{0\}$

Thus $B^3 \sim_\beta B^3\backslash\{0\}$

Therefore since $B^3\backslash\{0\}$ is β-paradoxical by Lemma 19 B^3 is β-paradoxical

4 Conclusions

4.1 Summary

The proof relies on two key theorems, that if a paradoxical group acts on another group without fixed points, then both groups are paradoxical and that if a group is equidecomposible to a paradoxical group, then the group is also paradoxical.

The proof begins by demonstrating that free groups of free rank 2 are paradoxical. Then, it demonstrates that for the group SO_3, as defined in Definition 18 contains a free group of free rank 2. Next, the proof shows that the free group from SO_3 acts without nontrivial fixed points on the sphere $\mathbb{S}^2$, as defined in Definition 11, with the removal of a countable set D, thus $\mathbb{S}^2 \backslash D$ is paradoxical. Then, it shows that $\mathbb{S}^2 \backslash D$ and $\mathbb{S}^2$ are equidecomposible, making $\mathbb{S}^2$ paradoxical. Next, this is expanded to the ball minus the origin, denoted $B^3 \backslash \{0\}$, since it is a collection of spheres. Finally, it shows that $B^3 \backslash \{0\}$ is equidecomposible to B^3, making B^3 paradoxical.

Figure 2: Demonstration of Banach-Tarski Paradox [3]

What this paradox is saying is that given a sphere, like in Figure 2, the ball can be separated into two groups, which through a series of translations and rotations on sections of those groups can each form the original ball. Thus duplicating the ball. To show that a ball is paradoxical the proof uses a left group action defined by $\beta(f, x) = f(x)$ $\forall f \in H = \{T_2 \circ R \circ T_1 : T_i \in T(\mathbb{R}^3), R \in SO_3\}$ and $\forall x \in \mathbb{R}^3$. What this group action is doing is applying a translation, a rotation, and then another translation. While not discussed in depth in this paper it is worth noting that these sets undergoing this process will be very jagged to the point of being physically impossible to reproduce.

4.2 Significance

This paradox is significant for demonstrating the results that can occur by accepting the Axiom of Choice. Without this axiom Proposition 1 is impossible to prove, which makes the entire paradox impossible to prove. The Banach-Tarski Paradox is not the most complex paradox to fall out of this and the translations and rotations used to create this axiom are not physically possible. However, this paradox is significant because of these two facts. The sphere is an easily understood mathematical set so it is an incredible example of the effect mathematical paradoxes can have. This contradiction between the mathematical and physical shown through paradoxes like the Banach-Tarski Paradox have sparked much debate over what axioms to base the concepts of mathematics upon. In this case the debate is centralized on the Axiom of Choice. While mathematicians agree that the Banach-Tarski Paradox is significant, there are cases like Lebesque measures where the axiom of choice is critical for maintaining the function of these measures. In many cases modified versions of the Axiom of Choice are used to avoid paradoxs like the Banach-Tarski paradox. The effect of the axiom of choice is discussed in further detail in section 15 of Wagon [1].

A Symbols

$\cup$ - union

$\cap$ - intersection

note: $\bigcap_{i=1}^{n} A_i = A_1 \cap \cdots \cap A_n$ and $\bigcup_{i=1}^{m} B_i = B_1 \cup \cdots \cup B_m$

$\vee$ - or

$\wedge$ - and

$\neg$ - not

$\Rightarrow$ - implies

note : $a \Rightarrow b$ is equivalent to if a, then b or a implies b

$\exists$ - there exists

$\forall$ - for all

: - such that

$\rightarrow$ - maps to

$\in$ - in, or an element of

$\subset$ - is a subset

$\backslash$ - set subtraction

example: $A \backslash B$ is $A \cap \neg B$

$\mathbb{C}$ - set of all complex numbers

$\mathbb{N}$ - set of all normal numbers

note: $\mathbb{N} = \{1, 2, ...\}$

$\mathbb{R}$ - set of all real numbers

$\mathbb{Q}$ - set of all rational numbers

$\mathbb{Z}$ - set of all integers

References

[1] Grzegorz Tomkowicz and Stan Wagon *The Banach–Tarski Paradox*, Cambridge: Cambridge University Press, second edition, 2016.

[2] Francis Edward Su *The Banach-Tarski Paradox*, The Banach-Tarski Paradox, 1990.

[3] Esham, Benjamin D. *Banach-Tarski Paradox*, Wikipedia, September 16, 2007. https://en.wikipedia.org/wiki/Banach%E2%80%93Tarski_paradox.